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Dynamique fortement couplée et intégrabilité dans l'extension supersymétrique maximale de la théorie de Yang-Mills

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Résumé

Dans cette thèse, nous étudions le régime fortement couplé de l'extension supersymétrique maximale de la théorie de Yang-Mills. Cette théorie est tenue pour intégrable, dans la limite planaire, ce qui offre l'opportunité de calculer plusieurs observables pour des valeurs arbitraires de la constante de couplage. Plus précisement, en admettant l'integrabilité de l'operateur de dilatation et en utlisant les équations de l'ansatz de Bethe qui lui sont associées, nous dérivons les expressions à couplage fort de certaines dimensions d'échelle de la théorie. Nous comparons nos résultats avec les prédictions issues de la théorie des cordes duale et discutons la validité de la correspondance AdS/CFT au niveau de nos observables.

Mots-clés: Théorie de Yang-Mills, Dimensions anormales, Intégrabilité, Chaîne de spins, Couplage fort, Correspondance AdS/CFT, Théorie des cordes.

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List of Publications

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Strongly Coupled Dynamics and Integrability in Maximally Supersymmetric Yang-Mills Theory

Abstract

In this thesis, we analyse the strongly coupled regime of the maximally supersymmetric extension of the four-dimensional Yang-Mills theory. This theory is believed to be integrable in the planar limit which offers the possibility of computing various quantities for arbitrary values of the coupling constant. Namely, assuming the complete integrability of the dilatation operator and making use of the associated set of all-loop asymptotic Bethe ansatz equations, we derive the strong coupling expressions of various scaling dimensions in the planar gauge theory. Applying the AdS/CFT correspondence, we compare our results with predictions coming from the dual string theory description and test the gauge/string duality for the quantities under consideration.

Keywords: Yang-Mills theory, Anomalous dimensions, Integrability, Spin chain, Strong coupling, AdS/CFT correspondence, String theory.

Contents

Chapter 1

Introduction

Understanding the dynamics of gauge theories at strong coupling is one of the outstanding problems in the theory of strong interactions – Quantum Chromodynamics (QCD). At present, there exists a lot of evidences that Yang–Mills theories should admit a complimentary description via yet to be identified string theories [1, 2, 3, 4, 5, 6, 7]. The latter operate in terms of collective degrees of freedom - Faraday lines - which are more appropriate to tackle the strong-coupling dynamics of four-dimensional gauge theories. The string dynamics should describe the excitations of the gauge theory flux tube.

The most prominent and thoroughly studied to date example of the gauge/string duality is the correspondence between the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills (SYM) theory and the type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [8, 9, 10, 11]. The four-dimensional $\mathcal{N} = 4$ SYM theory has a vanishing beta function and no dimensionful parameter. It is a non-confining gauge theory, with gauge group $SU(N_c)$, and it defines a conformal field theory (CFT), for any value of the Yang-Mills coupling g^2_{YM} and of the number of color N_c. For this reason, the duality above is also known as the AdS/CFT correspondence. The two (dimensionless) parameters of the string theory, namely the string coupling g_s and the (effective) string tension $\sqrt{\lambda}$, are related to those of the gauge theory as

$$g_s \sim g^2_{YM}\,, \qquad\qquad \sqrt{\lambda} \sim g\,, \tag{1.0.1}$$

where $g^2 \equiv g^2_{\mathrm{YM}} N_c/16\pi^2$ is the ‘t Hooft coupling constant. It follows that in the ’t Hooft planar limit, $N_c \to \infty$ with $g^2 \equiv g^2_{\mathrm{YM}} N_c/16\pi^2$, the string theory becomes free and only depends on the string tension $\sqrt{\lambda}$, which can take arbitrary values. The two-dimensional Lagrangian describing propagation of a single string in the curved background is uniquely fixed by the symmetries [12] up to a multiplicative factor $\propto \sqrt{\lambda}$.

The gauge/string duality establishes the correspondence between scaling dimensions of composite operators in the $\mathcal{N} = 4$ SYM theory and the energies of the string excitations in the dual string picture. In the (planar) gauge theory, the scaling dimensions are defined as eigenvalues of the dilatation operator, that can be computed perturbatively at weak coupling $g^2 \ll 1$. In the string theory, the calculation of the spectrum of energies is obtained after quantizing the string σ-model – a problem that still awaits for its solution. In the regime of string states carrying large quantum numbers, it can be tackled by semiclassical methods [13, 14, 15] as soon as $\sqrt{\lambda} \sim g \gg 1$, which is a necessary condition to tame the quantum fluctuations of the string. To make definite comparisons between the gauge and string theory computations, it is therefore desirable to develop non-perturbative tools on both sides of the correspondence.

An exciting recent development is the emergence of integrability on both sides of the correspondence. On the string side, integrable structures appear at the classical level [16, 17, 18] while the quantum integrability of the full $AdS_5 \times S^5$ world-sheet theory remains an open question (see [19] for a recent survey of quantum integrability of the light-cone string σ-model). On the gauge theory side, this symmetry is not seen at the level of the classical Lagrangian and manifests itself through remarkable integrability properties of the dilatation operator of the gauge theory [20, 21, 22], in planar limit. The gauge/string duality hints that this integrability should exist in the gauge/string theory for arbitrary values of the coupling constant.

The above mentioned integrable structures are not specific to $\mathcal{N} = 4$ SYM theory and they are also present in generic gauge theories [23, 24, 25] including QCD, though for a restricted class of observables. Indeed, the integrability symmetry has been first discovered in studies of high-energy asymptotics of scattering amplitudes in planar QCD [26, 27]. Later, similar integrable structures have been observed in the spectrum of one-loop anomalous dimensions, in the sector of the so-called maximal-helicity Wilson operators [28, 29, 22]. It was found that the one-loop dilatation operator in this sector can be mapped, in the planar limit, into the Hamiltonian of the Heisenberg spin chain, with the spin operators in the chain sites defined by the generators of the 'collinear' $SL(2,\mathbb{R})$ subgroup of the full conformal group [30]. The Heisenberg magnet is a quantum integrable spin chain and its Hamiltonian can be diagonalized by means of the Bethe ansatz. It follows that its energy spectrum, or equivalently the spectrum of one-loop anomalous dimensions, can be obtained by solving a set of algebraic equations, the so-called Bethe ansatz equations.

In planar $\mathcal{N} = 4$ SYM theory, the integrability of the one-loop dilatation operator extends to all operators and it can be mapped into an integrable Heisenberg super-spin-chain with spin operators belonging to the superconformal algebra $\mathfrak{psu}(2,2|4)$ [21]. Its diagonalization leads to a set of nested Bethe ansatz equations for the complete spectrum of one-loop anomalous dimensions [21]. A great deal of activity has been devoted recently to the test of higher loop integrability in the $\mathcal{N} = 4$ theory [31, 32, 33, 34, 35, 36, 37] and significant evidence has been gathered supporting its presence in various closed subsectors specific to this particular theory (see [38] for a review). It culminated in the proposal of all-loop (asymptotic) Bethe ansatz equations for the spectrum of anomalous dimensions of (infinitely long) operators [39, 40, 41, 42, 43, 44]. In this thesis, we will make use of these equations to compute the strong coupling expressions of distinguished observables in the gauge theory and then make comparison with the predictions coming from the dual string theory. We stress that these equations have been conjectured and not proved from first principles. However, they are based on a 'minimal' set of assumptions underlying the general approach to deal with models believed to be completely integrable. They are therefore well constrained and cannot be deformed at will.

We will consider the $\mathfrak{sl}(2)$ sector spanned by Wilson operators built from L copies of the same complex scalar field and N light-cone components of the covariant derivative. The anomalous dimensions in this sector are non-trivial functions of the 't Hooft coupling constant g and of the quantum numbers of the Wilson operators – the twist L and the Lorentz spin N. Significant simplification occurs in the limit when the Lorentz spin grows exponentially with the twist [45],

$L \sim \log N$ with $N \to \infty$. In this limit, the anomalous dimensions scale logarithmically with N for arbitrary coupling and the minimal anomalous dimension $\delta\Delta_{\rm min}$ has the following scaling behavior [45, 46, 47]

$$\delta\Delta_{\rm min} = \Big[2\Gamma_{\rm cusp}(g) + \epsilon(g,j)\Big] \log N + \dots\,, \qquad (1.0.2)$$

where $j = L/\log N$ is an appropriate scaling variable and ellipses denote terms of order $O(\log^0 N)$. Here, the coefficient in front of $\log N$ is split into the sum of two functions in such a way that $\epsilon(g,j)$ carries the dependence on the twist and it vanishes for $j = 0$. The first term inside the square brackets in (1.0.2) has a universal, twist independent form [48, 49]. It involves the function of the coupling constant known as the cusp anomalous dimension. This anomalous dimension was introduced in [50] as describing the scale dependence of Wilson loops with a light-like cusp on the integration contour [51]. The cusp anomalous dimension plays a distinguished role in $\mathcal{N} = 4$ theory and, in general, in four-dimensional Yang-Mills theories since, aside from the logarithmic scaling of the anomalous dimension (1.0.2), it also controls infrared divergences of scattering amplitudes [50, 52], Sudakov asymptotics of elastic form factors [53], gluon Regge trajectories [54] etc.

According to (1.0.2), the asymptotic behavior of the minimal anomalous dimension is determined by two independent functions, $\Gamma_{\rm cusp}(g)$ and $\epsilon(g,j)$. At weak coupling, these functions are given by series in powers of g^2 and the first few terms of the expansion can be computed in perturbation theory [51, 49, 55, 56, 57, 58].[1] The AdS/CFT correspondence permits to obtain the strong coupling expressions of $\Gamma_{\rm cusp}(g)$ and $\epsilon(g,j)$ from the semiclassical expansion of the energy of a folded spinning string [14, 15, 45, 59, 60, 46, 61].

The Bethe ansatz approach to computing these functions to one-loop order at weak coupling was developed in [62, 45, 49]. With the help of the of all-loop (asymptotic) Bethe ansatz equations [41, 43, 44], it was extended to higher loops in [63, 44, 47] leading to integral equations for $\Gamma_{\rm cusp}(g)$ and $\epsilon(g,j)$, the so-called Beisert-Eden-Staudacher (BES) and Freyhult-Rej-Staudacher (FRS) equations, respectively. They are valid in the planar limit for arbitrary values of the scaling parameter j and of the coupling constant g.

For the cusp anomalous dimension, the solution to the BES equation at weak coupling is in agreement with the most advanced explicit four-loop perturbative calculation [51, 49, 55, 56, 57, 58] in the gauge theory. The BES equation can also be analyzed at strong coupling as done in [64, 65, 66, 67, 68, 69, 70, 71]. It was found that the cusp anomalous dimension admits an expansion in $1/g$ [69, 70]

$$\Gamma_{\rm cusp}(g) = \sum_{k=-1}^{\infty} c_k/g^k\,, \qquad (1.0.3)$$

with coefficients being determined order by order by the BES equation. They start as

$$c_{-1} = 2\,, \qquad c_0 = -\frac{3\log 2}{2\pi}\,, \qquad c_1 = -\frac{\rm K}{8\pi^2}\,, \qquad (1.0.4)$$

where K is the Catalan's constant. On the string theory side of the AdS/CFT correspondence, the relation (1.0.3) should follow from the semiclassical expansion of the energy of a folded spinning

[1]Strictly speaking, the scaling function $\epsilon(g,j)$ is not accessible by a direct gauge theory perturbative calculation, in distinction to the cusp anomalous dimension. The scaling function is computed in the gauge theory with the help of integrability, that is by applying the Bethe ansatz approach.

string. In the right-hand side of (1.0.3), the leading contribution $\sim 2g$ should correspond to the classical energy and c_k should describe $(k+1)$-th loop correction ($k \geqslant 0$). Explicit two-loop stringy calculation [14, 15, 60] yields a result which is in a perfect agreement with (1.0.3) and (1.0.4), providing a remarkable verification of the AdS/CFT correspondence.

In this thesis, we will explain how the BES equation can be solved at strong coupling and we will recover the expansion (1.0.3) with the coefficients (1.0.4). We will also consider non-perturbative $\sim \mathrm{e}^{-2\pi g}$ corrections to (1.0.3), which are tied to the non-Borel summability of the series (1.0.3).

Concerning the scaling function $\epsilon(g, j)$, entering (1.0.2), an interesting proposal was put forward in [46] that it can be found exactly at strong coupling in terms of the non-linear O(6) bosonic sigma model embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string σ-model. More precisely, using the dual description of Wilson operators as folded strings spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and taking into account the one-loop stringy corrections to these states [59], it was argued that the scaling function $\epsilon(g, j)$ should be related to the energy density $\varepsilon_{\mathrm{O}(6)}(\rho)$ in the ground state of the O(6) model corresponding to the charge density $\rho \equiv j/2$,

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{\epsilon(g, j) + j}{2} \,. \tag{1.0.5}$$

This relation should hold at strong coupling and for $j \sim m$, where m is the mass gap of the O(6) model. To leading order, the mass gap is found in string theory as [46]

$$m = k\, g^{1/4}\, \mathrm{e}^{-\pi g}\, [1 + O(1/g)] \,, \qquad\qquad k = 2^{3/4}\pi^{1/4}/\Gamma(\tfrac{5}{4}) \,. \tag{1.0.6}$$

The O(6) sigma model is an exactly solvable theory [72] and the dependence of $\varepsilon_{\mathrm{O}(6)}(\rho)$ on the mass scale m and the charge density ρ can be found exactly with the help of the (zero-temperature) thermodynamic Bethe ansatz (TBA) equations. Together with (1.0.5) and (1.0.6), it allows to determine the scaling function $\epsilon(g, j)$ at strong coupling in the regime $j \sim m \sim \mathrm{e}^{-\pi g}$.

In this thesis, we will establish the relation between scaling function and O(6) energy density, Eq. (1.0.5), in both planar $\mathcal{N} = 4$ SYM theory at strong coupling, by solving the FRS equation in the relevant regime [73, 74, 75], and in string theory, by following the proposal of [46] and making use of the results of [15, 45, 59, 61, 76]. We will construct the mass scale m, Eq. (1.0.6), on both sides of the AdS/CFT correspondence and make comparison between the two expressions.

The thesis is organized as follows. In Chapter 2, we start with introducing the Wilson operators of the $\mathfrak{sl}(2)$ sector and developing the Bethe ansatz approach to the computation of their one-loop anomalous dimensions. Then we focus on the large spin limit of the minimal anomalous dimension and derive an integral equation determining the one-loop cusp anomalous dimension and scaling function. In Chapter 3, we consider the BES equation for the all-loop cusp anomalous dimension and analyze its solution at strong coupling, computing both perturbative and leading non-perturbative contribution. In Chapter 4, we analyze the string scaling function obtained from the semiclassical expansion of the energy of a folded spinning string. We show that it is possible to interpret the string scaling function as the energy density of a charged excited state in the O(6) model. From this matching, we extract the two-loop expression of the mass gap of the O(6) model in terms of the coupling constant. Then we perform a similar analysis starting from the FRS equation on the gauge theory side. At strong coupling and within a suitable regime for the scaling variable j, we prove that this equation for the scaling function can be cast into the TBA equations for the O(6) model. In this way, we are able to compute the gauge theory prediction for the mass gap for the O(6) model, that we compare with the string theory result. The Chapter 5 contains concluding remarks.

Chapter 2

Wilson Operators and Integrability

In this chapter, we will study properties of Wilson operators belonging to the so-called $\mathfrak{sl}(2)$ sector. These are gauge-invariant, single-trace, local operators carrying a Lorentz spin N and a twist L. We will be interested in computing the spectrum of their scaling dimensions, obtained by diagonalizing the dilatation operator D in the $\mathfrak{sl}(2)$ sector.

The dilatation operator of the gauge theory is written as a sum of a tree-level and anomalous contribution, denoted D_0 and δD respectively,

$$D = D_0 + \delta D\,. \tag{2.0.1}$$

It corresponds to the splitting of the scaling dimension into canonical and anomalous dimension,

$$\Delta = \Delta_0 + \delta\Delta\,. \tag{2.0.2}$$

The action of D_0 on twist L and spin N Wilson operators is already diagonal, with eigenvalue $\Delta_0 = N + L$. The action of δD is more involved and its diagonalization is a non-trivial problem.

In the planar limit and to one-loop order in the weak coupling expansion, we will see that the dilatation operator δD can be mapped into the Hamiltonian H of a $\mathfrak{sl}(2)$ Heisenberg spin chain. Namely, we will find that

$$\delta D = 2g^2 H + O(g^4)\,, \tag{2.0.3}$$

where $g^2 = g_{\rm YM}^2 N_c/16\pi^2$ is the 't Hooft coupling constant. The Heisenberg magnet is a completely integrable system and its Hamiltonian H can be diagonalized by means of the algebraic Bethe ansatz. As we shall see, it implies that the spectrum of one-loop anomalous dimensions can be found by solving a set of algebraic equations, the so-called Bethe ansatz equations.

In the large spin limit $N \to \infty$, we will solve the Bethe ansatz equations for the minimal anomalous dimension $\delta\Delta_{\rm min}$ and verify the logarithmic scaling

$$\delta\Delta_{\rm min} = 2\Gamma_{\rm cusp}(g)\log N + \ldots\,, \tag{2.0.4}$$

which holds for any value of L [49]. In this way, we will find the one-loop expression of the cusp anomalous dimension $\Gamma_{\rm cusp}(g) = 4g^2 + O(g^4)$. We will also consider the regime of large twist obtained in the generalized scaling limit $L \sim \log N \gg 1$ and will explain how the result (2.0.4) gets modified.

Finally, we will report on the all-loop asymptotic Bethe ansatz equations that are believed to diagonalize the all-loop, asymptotic, planar dilatation operator in the $\mathfrak{sl}(2)$ sector [40, 41, 44].

2.1 Wilson Operators

2.1.1 Definition

In $\mathcal{N}=4$ SYM theory, the $\mathfrak{sl}(2)$ sector is spanned with Wilson operators made out of L copies of a given complex scalar field $\mathcal{Z}(0)$ - evaluated at the origin - and an arbitrary number N of light-cone covariant derivatives $D=n^\mu D_\mu$, with $n^2=0$. They carry two global conserved charges, the Lorentz spin N and the twist L (or R-charge).[1]

The generic expression for twist L and spin N Wilson operators reads

$$\mathcal{O}_{\{k_m\}}(0) \equiv \mathrm{tr}\big[D^{k_1}\mathcal{Z}(0)\dots D^{k_L}\mathcal{Z}(0)\big] \ , \tag{2.1.1}$$

with $\{k_m\}$ a set of L positive integers satisfying $N=k_1+\ldots+k_L$. The trace in (2.1.1) is taken over the (implicit) color indices carried by the operators $D^{k_i}\mathcal{Z}(0)$ viewed as $N_c\times N_c$ traceless matrices, with N_c the number of color. It ensures that the composite operator $\mathcal{O}_{\{k_i\}}(0)$ is gauge invariant. As a result, we note that the operator $\mathcal{O}_{\{k_i\}}(0)$ is defined up to a cyclic permutation $k_1\to k_2\to\ldots\to k_L\to k_1$.

In order to analyse the properties of the Wilson operators with given twist L but arbitrary Lorent spin N, it is convenient to introduce a non-local light-cone operator that serves as a generating function for them. It is given by

$$\mathcal{O}(nz_1,\dots,nz_L) \equiv \mathrm{tr}\big[\mathcal{Z}(nz_1)[z_1,z_2]\dots\mathcal{Z}(nz_L)[z_L,z_1]\big] \ , \tag{2.1.2}$$

where $\{z_m\}$ is a set of abscissae along the light-cone direction specified by n. The symbol $[z_m,z_{m+1}]$ in (2.1.2) stands for a Wilson line, connecting the space-time points nz_m and nz_{m+1}, that ensures the gauge invariance of the non-local operator (2.1.2). We will not need its explicit expression since we can get rid of it by assuming the light-cone gauge $D=\partial$, equivalent to $[z_m,z_{m+1}]=1$. The operator (2.1.2) then simplifies to

$$\mathcal{O}(nz_1,\dots,nz_L) \equiv \mathrm{tr}\big[\mathcal{Z}(nz_1)\dots\mathcal{Z}(nz_L)\big] \ . \tag{2.1.3}$$

One easily verifies that the Taylor expansion of the non-local operator (2.1.3) generates all local Wilson operators (2.1.1) with arbitrary number of derivatives $D=\partial$.

The Wilson operators (2.1.1), or the light-cone operator (2.1.3), suffer from the UV divergences that plague the theory and are enhanced by the product of fundamental fields taken at the same space-time point, or along a light-cone direction. It is then necessary to regularize the theory and renormalize these operators in order to give them a proper meaning. In the following, we will always assume that we are dealing with operators that have been already renormalized.

2.1.2 Symmetries

The Wilson operators fall into multiplets of the conformal group, which is an exact symmetry of $\mathcal{N}=4$ SYM theory. Infinitesimal conformal transformations in 3+1 dimensions define the algebra $\mathfrak{so}(4,2)$ whose 15 generators correspond to 4 translations P_μ, 6 Lorentz rotations $M_{\mu\nu}=-M_{\nu\mu}$, 1

[1] Our terminology is borrowed from QCD where similar operators first appeared in the description of the deeply inelastic scattering [77, 78].

dilatation D and 4 special conformal transformations K_μ. Only few of them leaves the light-cone direction n invariant and acts non-trivially on the light-cone operator (2.1.3). They produce the algebra of the so-called collinear conformal subgroup [30], generated by

$$S_z = -\tfrac{1}{2}\left(D + n^\mu \bar{n}^\nu M_{\mu\nu}\right)\,, \qquad S_+ = -\tfrac{1}{2}\bar{n}^\mu K_\mu\,, \qquad S_- = n^\mu P_\mu\,, \tag{2.1.4}$$

where $\bar{n}$ is a light-like vector, $\bar{n}^2 = 0$, satisfying $\bar{n}^\mu n_\mu = 1$. Notice that the operator $M \equiv n^\mu \bar{n}^\nu M_{\mu\nu}$ in (2.1.4) measures the Lorentz spin.

One can easily verify that the operators (2.1.4) satisfy the $\mathfrak{sl}(2)$ commutation relations

$$[S_z, S_\pm] = \pm S_\pm\,, \qquad [S_+, S_-] = 2S_z\,. \tag{2.1.5}$$

The first equality in (2.1.5) is an immediate consequence of the fact that $n^\mu P_\mu$, respectively $\bar{n}^\mu K_\mu$, has dimension and Lorentz spin both equal to 1, respectively -1. Namely,

$$[D, n^\mu P_\mu] = [M, n^\mu P_\mu] = n^\mu P_\mu\,, \qquad [D, \bar{n}^\mu K_\mu] = [M, \bar{n}^\mu K_\mu] = -\bar{n}^\mu K_\mu\,.$$

The last identity in (2.1.5) is due to the following commutation relation of $\mathfrak{so}(4,2)$

$$[K_\mu, P_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}\,. \tag{2.1.6}$$

Finally, we note that the $\mathfrak{sl}(2)$ generators (2.1.4) commutes with the (internal) $\mathfrak{u}(1)$ R-charge giving the twist of the Wilson operators.

To make use of the $\mathfrak{sl}(2)$ symmetry (2.1.4), we need to know how the operators $S_{z,\pm}$ act on the light-cone operator $\mathcal{O}(nz_1, \ldots, nz_L)$, see Eq. (2.1.3). This is directly relevant to our problem of finding the spectrum of anomalous dimensions since the anomalous part of the dilatation operator $\delta D = D - D_0$ is invariant under the $\mathfrak{sl}(2)$ transformations[2]

$$[\delta D, S_{z,\pm}] = 0\,. \tag{2.1.7}$$

As for the dilatation operator, the operators $S_{z,\pm}$ can be written as

$$S_{z,\pm} = S_{z,\pm\,0} + \delta S_{z,\pm}\,, \tag{2.1.8}$$

with $\delta S_{z,\pm} = O(g^2)$. Here, the tree-level operators $S_{z,\pm\,0} = \lim_{g\to 0} S_{z,\pm}$ satisfy $\mathfrak{sl}(2)$ commutation relations and act canonically on $\mathcal{O}(nz_1, \ldots, nz_L)$. Plugging the decomposition (2.1.8) into Eq. (2.1.7), we find that

$$[\delta D, S_{z,\pm\,0}] = O(g^4)\,, \tag{2.1.9}$$

where we took into account that $\delta D = O(g^2)$. The identity, above, implies that the one-loop dilatation operator is invariant under the tree-level $\mathfrak{sl}(2)$ transformations generated by $S_{z,\pm\,0}$. These classical $\mathfrak{sl}(2)$ transformations are enough for our purpose, since we will not consider (explicitely) higher-loop contributions to δD. To simplify the notations, we will drop the subscript 0 in the following, $S_{z,\pm\,0} \to S_{z,\pm}$.

[2]This is because the operators $S_{z,\pm}$ generate an exact symmetry of the $\mathcal{N} = 4$ SYM theory. They are thus physical observables with zero anomalous dimension, which is equivalent to (2.1.7).

The (classical) spin operators $S_{z,\pm}$ act on the fundamental field $\mathcal{Z}(nz)$ as [30]

$$\begin{aligned} S_z \cdot \mathcal{Z}(nz) &= -\left(z\partial_z - s\right)\mathcal{Z}(nz)\,, \\ S_+ \cdot \mathcal{Z}(nz) &= -\left(z^2\partial_z - 2sz\right)\mathcal{Z}(nz)\,, \\ S_- \cdot \mathcal{Z}(nz) &= \partial_z \mathcal{Z}(nz)\,, \end{aligned} \tag{2.1.10}$$

where $s = -1/2$. Expanding on both sides of (2.1.10) in powers of z, we obtain that

$$\begin{aligned} S_z \cdot \partial^k \mathcal{Z}(0) &= (s-k)\partial^k \mathcal{Z}(0)\,, \\ S_+ \cdot \partial^k \mathcal{Z}(0) &= k(2s+1-k)\partial^{k-1}\mathcal{Z}(0)\,, \\ S_- \cdot \partial^k \mathcal{Z}(0) &= \partial^{k+1}\mathcal{Z}(0)\,, \end{aligned} \tag{2.1.11}$$

where $k = 0, \ldots, \infty$. The relations (2.1.11) show that the set of local operators $\{\partial^k \mathcal{Z}(0)\}$ map into a basis of states $\{S_-^k \, |s\rangle\}$ for the $s = -1/2$ highest weight, irreducible representation of the $\mathfrak{sl}(2)$ algebra. This $\mathfrak{sl}(2)$ module, denoted V_s in the following, is non-compact. It has a unique highest weigth state $|s\rangle$, with defining property $S_+ \, |s\rangle = 0$, and an infinite sequence of descendants $S_-^k \, |s\rangle$, with $k = 1, \ldots, \infty$.

When acting on the operator $\mathcal{O}(nz_1, \ldots, nz_L)$, the $\mathfrak{sl}(2)$ generators $S_{z,\pm}$ decompose into the sum

$$S_{z,\pm} = \sum_{m=1}^{L} S_{z,\pm}^{(m)}\,, \tag{2.1.12}$$

where $S_{z,\pm}^{(m)}$ are the local spin operators acting on $\mathcal{Z}(nz_m)$ in $\mathcal{O}(nz_1, \ldots, nz_L)$, see Eq. (2.1.3). After Taylor expanding $\mathcal{O}(nz_1, \ldots, nz_L)$, we obtain that the Wilson operators (2.1.1) can be mapped into states in the (reducible) $\mathfrak{sl}(2)$ module

$$V = V_{s_1} \otimes \ldots \otimes V_{s_L}\,, \tag{2.1.13}$$

with $s_1 = \ldots = s_L = s = -1/2$. A generic state in V can be written as

$$S_-^{(1)k_1} \ldots S_-^{(L)k_L}\Omega\,, \tag{2.1.14}$$

where $\Omega = |s_1\rangle \otimes \ldots \otimes |s_L\rangle$ has the defining property to be annihilated by all local spin generators $S_+^{(m)}$, with $m = 1, \ldots, L$. Note that it satisfies $S_z \Omega = Ls\,\Omega$ and corresponds to the primary operator[3] tr $\left[\mathcal{Z}(0)^L\right]$. For the Wilson operator $\mathcal{O}_{\{k_m\}}(0)$, see Eq. (2.1.1), we have the mapping

$$\mathcal{O}_{\{k_m\}}(0) \longleftrightarrow \frac{1}{L} \sum_{\pi \in \text{cycl.perm.}} S_-^{(1)k_{\pi(1)}} \ldots S_-^{(L)k_{\pi(L)}}\Omega\,, \tag{2.1.15}$$

where a sum over the L cyclic permutations of $\{k_m\}$ is taken, in order to match the invariance of $\mathcal{O}_{\{k_m\}}(0)$ under these transformations. It follows that the relevant states in V have the property to be eigenstates of the shift operator U with eigenvalue 1. The shift operator, acting on V, is defined by

$$U S_{z,\pm}^{(m)} = S_{z,\pm}^{(m+1)} U\,, \qquad U\Omega = \Omega\,. \tag{2.1.16}$$

Note that the dilatation operator D can be defined as acting on V. Since it commute with the shift operator U, the unwanted states in V, that is those that are not eigenstates of U with eigenvalue 1, can be projected out at the end of day.

[3] Primary operator $\leftrightarrow$ highest weight state.

2.1.3 One-Loop Dilatation Operator

As said previously, the Wilson operators (2.1.1), or the light-cone operator (2.1.3), have to be renormalized to make sense. Once done, they have acquired a dependence on the renormalization scale μ, introduced by the regularization and/or renormalization procedure. This dependence on μ is controlled by the Callan-Symanzik (renormalization group) equation, that takes the simple form

$$\mu\frac{\partial}{\partial\mu}\mathcal{O} = -\delta D\cdot\mathcal{O}\,, \tag{2.1.17}$$

in the $\mathcal{N}=4$ SYM theory. Here, $\mathcal{O}$ stands for a local, or light-cone, renormalized operator and δD does not depend on μ. The Callan-Symanzik equation (2.1.17) gives a mean to define and compute the action of δD on $\mathcal{O}$, in the gauge theory.

The action of δD on a generic (renormalized) operator is not diagonal per se, that is $\delta D\cdot\mathcal{O}$ is not necessarily proportional to $\mathcal{O}$. This is because the operator $\mathcal{O}$ does not renormalize multiplicatively, in general, but, instead, mixes with several operators. This mixing is not completely arbitrary, however, and it should respect the symmetries of the theory preserved by the regularization. In particular, operators with different Lorent spin and twist do not mix. The same conclusion applies for operators with different canonical dimension, due to the absence of mass scale in $\mathcal{N}=4$ SYM theory. Moreover, in the planar limit, the subspace of single-trace operators is invariant under renormalization. One can deduce from these properties that the $\mathfrak{sl}(2)$ sector is closed under renormalization.

In the $\mathfrak{sl}(2)$ sector, δD can be found, at weak coupling, in the form of an expansion in powers of g^2 starting as $\delta D = 2g^2H + O(g^4)$, where H is a g independent operator. In the planar limit, H only involves nearest-neighbor interactions and reads

$$H = \sum_{m=1}^{L} H_{m\,m+1}\,, \tag{2.1.18}$$

when applied to twist L operators. We assume periodic boundary conditions, $m+L=m$, in agreement with the periodic nature of single-trace operators. Explicit expression for the density operator $H_{m\,m+1}$ in (2.1.18) can be easily obtained in the light-cone formalism [24]. The operator $H_{m\,m+1}$ then acts on the two adjacent fields $\mathcal{Z}(nz_m)\mathcal{Z}(nz_{m+1})$ inside the trace of the light-cone operator (2.1.3) as [24]

$$\begin{aligned}H_{m\,m+1}\cdot\mathcal{Z}(nz_m)\mathcal{Z}(nz_{m+1}) = \int_0^1\frac{d\alpha}{\alpha}\Big[&2\mathcal{Z}(nz_m)\mathcal{Z}(nz_{m+1}) - \bar{\alpha}^{-2s-1}\mathcal{Z}(\bar{\alpha}nz_m+\alpha nz_{m+1})\mathcal{Z}(nz_{m+1})\\ &- \bar{\alpha}^{-2s-1}\mathcal{Z}(nz_m)\mathcal{Z}(\bar{\alpha}nz_{m+1}+\alpha nz_m)\Big]\,,\end{aligned} \tag{2.1.19}$$

with $\bar{\alpha}=1-\alpha$ and $s=-1/2$. We see that the effect of the operator $H_{m\,m+1}$ is to displace the fields along the light-cone direction n.

After some algebra, one can verify that the operator H, given in (2.1.18) and (2.1.19), commutes with the (classical) $\mathfrak{sl}(2)$ transformations (2.1.12). This invariance can be made manifest with the help of an alternative representation for H, which can be obtained as follows. Given the $\mathfrak{sl}(2)$ invariance of H, acting on V, see Eq. (2.1.13), and the expression (2.1.12), one easily deduces that the density operator $H_{m\,m+1}$, acting on $V_{s_m} \otimes V_{s_{m+1}}$ inside V, commutes with the spin generators

$$S^{(m)}_{z,\pm} + S^{(m+1)}_{z,\pm} \,. \tag{2.1.20}$$

All operators in the module

$$V_{s_m} \otimes V_{s_{m+1}} \cong V_s^{\otimes 2} \cong \sum_{k=0}^{\infty} V_{2s-k} \,, \tag{2.1.21}$$

with $s = -1/2$, are uniquely specified by the eigenvalues of $S^{(m)}_z + S^{(m+1)}_z$ and of the quadratic Casimir C. The latter is given by[4]

$$C \equiv (S^{(m)}_\alpha + S^{(m+1)}_\alpha)^2 \equiv J_{m\,m+1}(J_{m\,m+1} + 1) \,, \tag{2.1.22}$$

with the invariant spin operator $J_{m\,m+1}$ acting on $V_{2s-k} \subset V_{s_m} \otimes V_{s_{m+1}}$, see Eq.(2.1.21), as

$$J_{m\,m+1} V_{2s-k} = (2s-k) V_{2s-k} \,. \tag{2.1.23}$$

Since the density operator $H_{m\,m+1}$ commutes with (2.1.20), it is a function of $J_{m\,m+1}$ only,

$$H_{m\,m+1} = f(J_{m\,m+1}) \,. \tag{2.1.24}$$

It remains to evaluate the function f, above. To this end, we make use of the expansion

$$\mathcal{Z}(nz_m)\mathcal{Z}(nz_{m+1}) = \sum_{k=0}^{\infty} (z_m - z_{m+1})^k \mathcal{O}_{(k)}(0) + \text{descendants} \,, \tag{2.1.25}$$

where $\mathcal{O}_{(k)}(0)$ is the primary operator of the spin $(2s-k)$ irreducible module appearing in the right-hand side of (2.1.21). Due to the $\mathfrak{sl}(2)$ invariance of $H_{m\,m+1}$, the action of $H_{m\,m+1}$ on $\mathcal{O}_{(k)}(0)$ is diagonal, and the descendants in (2.1.25) are mapped into descendants. Then, plugging the expansion (2.1.25) into (2.1.19), one immediately finds that

$$H_{m\,m+1} \cdot \mathcal{O}_{(k)}(0) \equiv f(2s-k)\mathcal{O}_{(k)}(0) = 2\big[\psi(k-2s) - \psi(1)\big]\, \mathcal{O}_{(k)}(0) \,, \tag{2.1.26}$$

where ψ is the logarithmic derivative of the Euler gamma-function, or psi-function. By the $\mathfrak{sl}(2)$ invariance, the result, above, lifts uniquely to

$$H_{m\,m+1} = 2\big[\psi(-J_{m\,m+1}) - \psi(1)\big] \,, \tag{2.1.27}$$

which is the desired representation. It was obtained in [33] in $\mathcal{N}=4$ SYM theory, but it also applies in QCD [28, 29, 22]. In the latter case, however, the $\mathfrak{sl}(2)$ spin s of the elementary module V_s is equal to $-3/2$ instead of $-1/2$, because the $\mathfrak{sl}(2)$ sector in QCD is defined by taking product of (maximal helicity) gluon fields instead of scalar fields.

We conclude that the one-loop dilatation operator, in the $\mathfrak{sl}(2)$ sector of twist L Wilson operators, can be written as

$$\delta D = 2g^2 H + O(g^4) = 4g^2 \sum_{m=1}^{L} \big[\psi(-J_{m\,m+1}) - \psi(1)\big] + O(g^4) \,, \tag{2.1.28}$$

with $J_{m\,m+1}$ the invariant spin operator on $V_{s_m} \otimes V_{s_{m+1}}$ inside $V = V_{s_1} \otimes \ldots \otimes V_{s_L}$, $(s_1 = \ldots = s_L = -1/2)$.

---

[4]An implicit summation over $\alpha = x, y, z$ is assumed in Eq. (2.1.22) with the $\mathfrak{sl}(2)$ generators $S^{(m)}_{x,y}$ defined as $S^{(m)}_x = (S^{(m)}_+ + S^{(m)}_-)/2$ and $S^{(m)}_y = (S^{(m)}_+ - S^{(m)}_-)/2i$, and similarly for $S^{(m+1)}_{x,y}$.

2.1.4 Cusp Anomalous Dimension

The spectrum of anomalous dimensions $\{\delta\Delta\}$ is obtained by solving the eigenvalue problem

$$\delta D \cdot \mathcal{O}(0) = \delta\Delta\, \mathcal{O}(0)\,, \tag{2.1.29}$$

where $\mathcal{O}(0)$ stands for a local Wilson operator. The equation, above, can be solved at given Lorentz spin N and twist L, i.e. assuming that

$$\mathcal{O}(0) \;=\; \sum_{k_1+\ldots+k_L=N} \Psi_{k_1,\ldots,k_L} \mathrm{tr}\left[\partial^{k_1}\mathcal{Z}(0)\ldots\partial^{k_L}\mathcal{Z}(0)\right]\,, \tag{2.1.30}$$

where $\Psi_{k_1,\ldots,k_L}$ plays the role of a wave function. Note also that, thanks to the $\mathfrak{sl}(2)$ invariance of δD, it is enough to consider primary operators, i.e. $S_+ \cdot \mathcal{O}(0) = 0$.

In the particular case $L = 2$, there is only one primary operator at a given Lorentz spin N.[5] Its one-loop anomalous dimension can be found from Eq. (2.1.28) as [22, 79]

$$\delta\Delta = 8g^2(\psi(N+1) - \psi(1)) + O(g^4)\,. \tag{2.1.31}$$

It leads to the asymptotic behavior

$$\delta\Delta = 8g^2 \log N + O(\log^0 N) + O(g^4)\,, \tag{2.1.32}$$

at large Lorentz spin, $N \gg 1$. According to [48, 49, 46], the logarithmic scaling of the twist-two anomalous dimension is valid for arbitrary values of the coupling constant g and it is controlled by the cusp anomalous dimension $\Gamma_{\mathrm{cusp}}(g)$. Namely,

$$\delta\Delta = 2\Gamma_{\mathrm{cusp}}(g) \log N + O(\log^0 N)\,, \tag{2.1.33}$$

at large N. From (2.1.32), we deduce that $\Gamma_{\mathrm{cusp}}(g) = 4g^2 + O(g^4)$, in agreement with [51, 50].

At higher twist, $L \geqslant 3$, one finds several primary operators with the same Lorentz spin N. The graph of their anomalous dimensions, in function of N, forms a band delimited by the minimal and maximal anomalous dimension trajectories. According to [49], both the minimal and maximal anomalous dimension should scale logarithmically at large N such that

$$\delta\Delta_{\min} \sim 2\Gamma_{\mathrm{cusp}}(g) \log N \leqslant \delta\Delta \leqslant \delta\Delta_{\max} \sim L\Gamma_{\mathrm{cusp}}(g) \log N\,. \tag{2.1.34}$$

We note, in particular, that the minimal anomalous dimension has the *universal* (twist-independent) scaling behavior ($N \gg 1$)

$$\delta\Delta_{\min} = 2\Gamma_{\mathrm{cusp}}(g) \log N + O(\log^0 N)\,. \tag{2.1.35}$$

The verification of the scaling behavior (2.1.35), to one-loop order and for an arbitrary twist L, is not in general as straightforward as in the twist-two case, Eqs. (2.1.31) and (2.1.32).[6] The reason is that solving the equation (2.1.29), with $\mathcal{O}(0)$ as in (2.1.30) and δD given by (2.1.28), amounts to diagonalizing a matrix whose size grows with N and L. This problem becomes rapidly intractable when N (and/or L) get large. In the following, we will explain how it can be solved thanks to the integrability of the one-loop dilatation operator (2.1.28).

[5] The Lorentz spin N is here assumed to be even, since twist-two Wilson operators with odd Lorentz spin are either $\mathfrak{sl}(2)$ descendants or vanish due to the trace in (2.1.30).

[6] An exception is the twist-three case for which explicit expression for the minimal anomalous dimension is known [21]. One has $\delta\Delta_{\min} = 8g^2(\psi(N/2+1) - \psi(1)) + O(g^4) \sim 8g^2 \log N + O(g^4)$.

2.2 Integrability

In this section, we will identify the one-loop dilatation operator in the $\mathfrak{sl}(2)$ sector with the Hamiltonian of the homogeneous $\mathrm{XXX}_{-1/2}$ Heisenberg spin chain. This model has been known since the seminal paper of Bethe [80] to be completely integrable and it can be solved with the algebraic Bethe ansatz [81, 82].[7] It means that it is possible to construct a complete family of operators, with the property to commute among themselves and with the Hamiltonian of the spin chain. The spectral problem for the Hamiltonian is then reduced to the simultaneous diagonalization of the family of conserved charges, and it is achieved by a set of algebraic equations, the so-called Bethe ansatz equations. Their solutions determine the spectrum of spin-chain energies, or equivalently of one-loop anomalous dimensions. The material for this section is taken from [81, 82].

Spin Chain Mapping

The vector space V, see Eq. (2.1.13), is naturally identified with the Hilbert space of a quantum spin chain, with L sites and with spin variables at each site in the representation $s = -1/2$ of $\mathfrak{sl}(2)$. Due to the periodicity and shift invariance of the one-loop dilatation operator, or equivalently of H, Eq. (2.1.28), the spin chain is closed and homogeneous.

The spin chain Hamiltonian H is bounded from below and it takes minimal value when evaluated on the state Ω, with defining property $S_+^{(m)}\Omega = 0$ for $m = 1, \dots, L$. This state is called the ferromagnetic vacuum of the spin chain and it is unique. Above the vacuum Ω, we have excited states constructed by acting on Ω with the lowering operators $S_-^{(m)}$, with $m = 1, \dots, L$, see Eq. (2.1.14). The fundamental excitation of the spin chain, obtained by acting on Ω with one of the lowering operators, is called a magnon. Acting N times on Ω with several lowering operators, we obtain the subspace of N-magnon states, corresponding to Wilson operators carrying Lorentz spin N. We recall that only the states invariant under the shift operator U do correspond to Wilson operators. Finding the one-loop spectrum of anomalous dimensions is therefore equivalent to the eigenvalue problem

$$H\Phi = E\Phi \,, \qquad U\Phi = \Phi \,, \tag{2.2.1}$$

where E is the spin-chain energy of the eigenstate $\Phi \in V$. Moreover, thanks to the $\mathfrak{sl}(2)$ invariance of H, it is sufficient to consider the highest weight states in V, i.e. $S_+\Phi = 0$.

We will now prove that H, given in (2.1.28), coincides with the Hamiltonian of the Heisenberg spin chain and used the remarkable integrability symmetry of this model to solve (2.2.1).

[7] The original Heisenberg spin chain is a model of magnet with spins variables in the (compact) representation $s = 1/2$ of $\mathfrak{sl}(2) \cong \mathfrak{su}(2)$. Here we are interested in an algebraic generalization, preserving integrability, with spins variables in the (non-compact) representation $s = -1/2$. The algebraic Bethe ansatz is a particularly well-suited framework to address this issue. It originates from an attempt to quantize two-dimensional field theories solvable by the classical inverse method, assuming a spacial discretization as a regulator of UV divergences, and it receives inspiration from the Baxter's analysis of solvable two-dimensional lattice models. It was mainly developed by Faddeev, Sklyanin and their collaborators, and it is reviewed in [81, 82]. We also refer the reader to [83] for historical remarks and to [84] for an introduction to several aspects of integrability.

2.2.1 R-Matrix and Yang-Baxter Equation

R-Matrix

The integrability of the Heisenberg spin chain is tied to the existence of a R-matrix satisfying the Yang-Baxter equation. The R-matrix is an operator acting on a tensor product of two $\mathfrak{sl}(2)$ modules and depending on the spectral parameter u (in $\mathbb{C}$),

$$R_{12}(u) : V_{s_1} \otimes V_{s_2} \to V_{s_1} \otimes V_{s_2} \,, \tag{2.2.2}$$

where the spin labels, s_1 and s_2, are a priori distinct and arbitrary. The R-matrix is invariant under the $\mathfrak{sl}(2)$ transformations on $V_{s_1} \otimes V_{s_2}$,

$$\left[R_{12}(u), S_\alpha^{(1)} + S_\alpha^{(2)}\right] = 0 \,, \tag{2.2.3}$$

with $\alpha = x, y, z$, and, as said before, it solves the Yang-Baxter equation. This equation holds as an identity on the space $V_{s_1} \otimes V_{s_2} \otimes V_{s_3}$ and reads

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \,, \tag{2.2.4}$$

where R_{12} acts on $V_{s_1} \otimes V_{s_2}$, R_{13} on $V_{s_1} \otimes V_{s_3}$, etc. Note that a R-matrix, solution to the Yang-Baxter equation, is determined up to a multiplication with an arbitrary (scalar) function of u, a translation of u and a set of discrete operations, like $u \to -u$ for instance.

To begin with, we look for the R-matrix acting on $V_{s_1} \otimes V_{s_2}$ with $s_1 = s_2 = s$, which will be specialized to $s = -1/2$ when needed. This solution was constructed in [85] and it reads

$$R_{12}(u) = P_{12} r_{12}(u) \,, \tag{2.2.5}$$

where P_{12} is the permutation operator on $V_{s_1} \otimes V_{s_2}$ and

$$r_{12}(u) = \frac{\Gamma(-2s+iu)}{\Gamma(-2s-iu)} \frac{\Gamma(-J_{12}-iu)}{\Gamma(-J_{12}+iu)} \,, \tag{2.2.6}$$

with Γ the Euler's Gamma function. Here J_{12} is the invariant spin operator on $V_{s_1} \otimes V_{s_2}$, introduced before and related to the quadratic Casimir as $C = J_{12}(J_{12}+1)$. We note that the R-matrix, Eqs. (2.2.5) and (2.2.6), can be inverted $R(u)^{-1} = R(-u)$ for generic values of u.

Fundamental Transfer Matrix

Starting with the R-matrix (2.2.5), we will now construct a family of commuting observables acting on the spin-chain Hilbert space $V = V_{s_1} \otimes \ldots \otimes V_{s_L}$, with $s_1 = \ldots = s_L = s$. The first step is to introduce the (fundamental) monodromy matrix on $V_{s_0} \otimes V$, with $s_0 = s$, as

$$T_0(u) = R_{0L}(u) R_{0L-1}(u) \ldots R_{01}(u) \,. \tag{2.2.7}$$

It is a $\mathfrak{sl}(2)$ invariant operator on $V_{s_0} \otimes V$,

$$\left[T_0(u), S_\alpha^{(0)} + S_\alpha\right] = 0 \,, \tag{2.2.8}$$

where $\alpha = x, y, z$, and $S_\alpha^{(0)}$, respectively S_α, are the spin generators on V_{s_0}, respectively V. As a consequence of the Yang-Baxter equation for the R-matrix, the monodromy matrix $T_0(u)$ satisfies the RTT relation on $V_{s_0} \otimes V_{s_{0'}} \otimes V$, with $s_{0'} = s_0 = s$,

$$R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)R_{00'}(u-v)\,. \tag{2.2.9}$$

Now we define a spin-chain observable $t(u)$, the so-called (fundamental) transfer matrix, by taking the trace over the space V_{s_0} of the monodromy matrix (2.2.7),

$$t(u) = \mathrm{tr}_0\left[R_{0L}(u)R_{0L-1}(u)\dots R_{01}(u)\right]\,. \tag{2.2.10}$$

The transfer matrix is $\mathfrak{sl}(2)$ and cyclically invariant,

$$[t(u), S_\alpha] = [t(u), U] = 0\,, \tag{2.2.11}$$

where U is the spin-chain shift operator. It is normalized as

$$t(u=0) = U\,, \tag{2.2.12}$$

and, thanks to its dependence on the spectral parameter u, it can be used to generate a family of spin-chain Hamiltonians $\{Q_r\}$. The latter are conventionally defined as

$$\log t(u) = -i \sum_{r\geqslant 1} u^{r-1} Q_r\,. \tag{2.2.13}$$

The first Hamiltonian coincides with the (quasi-)momentum operator $Q_1 = i\log U$ and the second one is, up to a constant, the Hamiltonian H of the Heisenberg spin chain. Indeed, making use of Eqs. (2.2.10), (2.2.5) and (2.2.6), we find

$$Q_2 = i\partial_u \log t(u)_{u=0} = i\sum_{m=1}^{L} \dot{r}_{m\,m+1}(u=0) = 2\sum_{m=1}^{L}\left[\psi(-J_{m\,m+1}) - \psi(-2s)\right]\,, \tag{2.2.14}$$

and the comparison with Eq. (2.1.28) shows that

$$H = Q_2 + 2L(\psi(-2s) - \psi(1))\,. \tag{2.2.15}$$

The higher Hamiltonians $\{Q_r, r \geqslant 3\}$ are not of a nearest-neighbor type and have local multi-spin interactions.

The remarkable property of the transfer matrix is that it commutes with itself for different values of the spectral parameter,

$$t(u)t(v) = t(v)t(u)\,. \tag{2.2.16}$$

This is an immediate consequence of the RTT relation (2.2.9). Indeed, inverting one of the R matrices in (2.2.9), taking the trace over the space $V_{s_0} \otimes V_{s_{0'}}$ and using the cyclicity of the trace, we arrive at (2.2.16). From the commutation relation (2.2.16), we conclude that the family of Hamiltonians $\{Q_r\}$ is Abelian. The higher Hamilonians $\{Q_r, r \geqslant 3\}$ are thus conserved charges, since they commute with Q_2, and they generate hidden symmetries of the Heisenberg spin chain. It remains to explain how they can be diagonalized.

Auxiliary Transfer Matrix

The fundamental transfer matrix is not the only generating function of conserved charges, neither the simplest one to diagonalize. A better candidate can be found by introducing an auxiliary vector space V_a supporting the (two-dimensional) spin 1/2 representation of $\mathfrak{sl}(2) \cong \mathfrak{su}(2)$. It allows us to define the auxiliary monodromy matrix $T_a(u)$ acting on $V_a \otimes V$ as

$$T_a(u) = R_{aL}(u) \ldots R_{a1}(u)\,, \tag{2.2.17}$$

where $R_{am}(u)$ is the R-matrix on $V_a \otimes V_{s_m}$, also called the Lax operator. For later convenience, we note that the auxiliary monodromy matrix can be viewed as a two-by-two matrix,

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{2.2.18}$$

with entries being operators on the spin-chain Hilbert space V. The matrix elements of $T_a(u)$ can be found with the help of the explicit expression for the Lax operator [81]

$$R_{am}(u) = u + i\sigma_\alpha S_\alpha^{(m)}\,, \tag{2.2.19}$$

where σ_α, $\alpha = x, y, z$, are the Pauli matrices acting on V_a. As a two-by-two matrix, the Lax operator can be written as

$$R_{am}(u) = \begin{pmatrix} u + iS_z^{(m)} & iS_-^{(m)} \\ iS_+^{(m)} & u - iS_z^{(m)} \end{pmatrix}. \tag{2.2.20}$$

Taking the trace, over the auxiliary space V_a, of the monodromy matrix $T_a(u)$, we define the auxiliary transfer matrix as

$$t_{\text{aux}}(u) = \text{tr}_a\big[R_{aL}(u) \ldots R_{a1}(u)\big] = A(u) + D(u)\,, \tag{2.2.21}$$

acting on V. Notice that $t_{\text{aux}}(u)$ is a polynomial of u with degree L.

The monodromy operator (2.2.17) satisfies the RTT relation on $V_a \otimes V_{a'} \otimes V$, where $V_{a'} \cong V_a$. Namely,

$$R_{aa'}(u-v)T_a(u)T_{a'}(v) = T_{a'}(v)T_a(u)R_{aa'}(u-v)\,, \tag{2.2.22}$$

whith $R_{aa'}$ the R-matrix acting on $V_a \otimes V_{a'}$. This R-matrix is the simplest (non-trivial) solution to the Yang-Baxter equation, and it is known as the Yang R-matrix. It reads [81]

$$R_{aa'}(u) = u + iP_{aa'}\,, \tag{2.2.23}$$

where $P_{aa'}$ is the permutation operator on $V_a \otimes V_{a'}$. Inverting one of the Yang R-matrix (for $u - v \neq \pm i$) in the RTT relation (2.2.22) and taking the trace over $V_a \otimes V_{a'}$, we deduce that

$$t_{\text{aux}}(u)t_{\text{aux}}(v) = t_{\text{aux}}(v)t_{\text{aux}}(u)\,. \tag{2.2.24}$$

The auxiliary transfer matrix thus generates an Abelian family of spin-chain observables. Moreover, the auxiliary transfer matrix $t_{\text{aux}}(u)$ is conserved, because it commutes with the fundamental transfer matrix $t(v)$, which contains the Heisenberg Hamiltonian. This is the consequence of the RTT relation on $V_a \otimes V_{s_0} \otimes V$

$$R_{a0}(u-v)T_a(u)T_0(v) = T_0(v)T_a(u)R_{a0}(u-v)\,. \tag{2.2.25}$$

We have just learnt that the two transfer matrices, $t_{\text{aux}}(u)$ and $t(v)$, commute with each other. It implies that they can be diagonalized simultaneously. Then, a strategy to find the eigenvalues of $t(v)$ is to first work out the eigenstates of $t_{\text{aux}}(u)$. It can be done by means of the algebraic Bethe ansatz, as we will now see.

2.2.2 Algebraic Bethe Ansatz

Eigenstates

To understand how to find the eigenstates of the auxiliary transfer matrix

$$t_{\text{aux}}(u) = A(u) + D(u)\,, \tag{2.2.26}$$

we may observe that we have already at hand a set of distinguished operators, namely the components of the monodromy matrix (2.2.18). The latter have definite commutation relations with the spin-chain $\mathfrak{sl}(2)$ generators S_α, $\alpha = x, y, z$, which follows from the $\mathfrak{sl}(2)$ invariance of $T_a(u)$ on $V_a \otimes V$,

$$\big[T_a(u), \tfrac{1}{2}\sigma_\alpha + S_\alpha\big] = 0\,. \tag{2.2.27}$$

Combining Eqs. (2.2.27) and (2.2.18), we find, for instance, that

$$\begin{aligned} &\big[S_z, A(u)\big] = 0\,, && \big[S_z, B(u)\big] = -B(u)\,,\\ &\big[S_z, C(u)\big] = C(u)\,, && \big[S_z, D(u)\big] = 0\,. \end{aligned} \tag{2.2.28}$$

We first note that $t_{\text{aux}}(u) = A(u) + D(u)$ commutes with S_z, which means that we can look for eigenstates of $t_{\text{aux}}(u)$ at a given value of S_z, say $Ls - N$. In other words, eigenstates of $t_{\text{aux}}(u)$ can be found with a given number N of magnons. More generally, one can easily show that $t_{\text{aux}}(u)$ is $\mathfrak{sl}(2)$ invariant, which means that it is sufficient to consider highest weight states. Then, we see that the operator $B(u)$ decreases by one unit the value of S_z. It follows that $B(u)\Omega$ is a one-magnon state, with Ω the spin-chain vacuum. Moreover, since $B(u)$ depends on the (arbitrary) spectral parameter u, we expect this state to be generic enough for our problem.[8] Similarly, we construct a N-magnon state as $B(u_1)\ldots B(u_N)\Omega$, and, as we shall see, all N-magnon eigenstates of $t_{\text{aux}}(u)$ can be found in this form, for particular values of $u_1, \ldots, u_N$.

To decide if a state $B(u_1)\ldots B(u_N)\Omega$ is eigenstate of $t_{\text{aux}}(u)$, we will make use of the algebra for the components $A(u), \ldots, D(u)$. The latter follows from the RTT relation (2.2.22) for the auxiliary monodromy matrix. This equation contains sixteen algebraic identities, three of them which we single out for our procedure. They read

$$\big[B(u), B(v)\big] = 0\,, \tag{2.2.29}$$

[8] Note that an arbitrary one-magnon state is not necessarily of the form $B(u)\Omega$ for some u, but it can be found as a linear combination of such states. Therefore, what we are assuming here is that it is sufficient to consider states as $B(u)\Omega$ to find all one-magnon eigenstates of $t_{\text{aux}}(u)$.

and

$$A(u)B(v) = \frac{u-v-i}{u-v}B(v)A(u) + \frac{i}{u-v}B(u)A(v)\,, \qquad (2.2.30)$$

$$D(u)B(v) = \frac{u-v+i}{u-v}B(v)D(u) - \frac{i}{u-v}B(u)D(v)\,. \qquad (2.2.31)$$

Note that to prevent the right-hand side of Eqs. (2.2.30) and (2.2.31) from becoming singular, we will always assume that the spectral parameters are all distinct from each other.

Let us first analyse the situation for the vacuum state of the spin chain, Ω. When acting on Ω, the monodromy matrix takes a triangular form

$$T_a(u)\Omega = \begin{pmatrix} (u+is)^L\Omega & B(u)\Omega \\ 0 & (u-is)^L\Omega \end{pmatrix}. \qquad (2.2.32)$$

This is an immediate consequence of the definition of $T_a(u)$ as a product of Lax operators, Eqs. (2.2.17) and (2.2.20), and of the defining property of Ω to be annihilated by $S_+^{(m)}$, with $m = 1,\dots,L$. The vacuum state Ω is therefore an eigenstate of the transfer matrix with eigenvalue

$$t_{\text{aux}}(u)\Omega = \big(A(u)+D(u)\big)\Omega = \big[(u+is)^L + (u-is)^L\big]\Omega\,. \qquad (2.2.33)$$

As we said above, we look for higher excited eigenstates as

$$\Phi(\{u_i\}) = B(u_1)\dots B(u_N)\Omega\,. \qquad (2.2.34)$$

Thanks to Eq. (2.2.29), the state $\Phi(\{u_i\})$ does not depend on the order in which the product of $B(u_1)$, ..., $B(u_N)$ is taken. Therefore, we can assume, without loss of generality, that $u_1 > \ldots > u_N$. We exclude the case of coinciding spectral parameters for the reason given before, see remark after Eq. (2.2.31) and Eqs. (2.2.39), (2.2.40) below. The state $\Phi(\{u_i\})$ is an eigenstate of the operator S_z with eigenvalue

$$S_z\Phi(\{u_i\}) = (Ls-N)\Phi(\{u_i\})\,. \qquad (2.2.35)$$

However, it is not necessarily a highest weight state because $S_+\Phi(\{u_i\})$ does not vanish for generic values of the spectral parameters $\{u_i\}$.

To see if $\Phi(\{u_i\})$ is an eigenstate of the auxiliary transfer matrix (2.2.26), we apply Eqs. (2.2.30) and (2.2.31) and we obtain the actions of $A(u)$ and $D(u)$ on $\Phi(\{u_i\})$ as

$$A(u)\Phi(\{u_i\}) = \alpha(u,\{u_i\})\Phi(\{u_i\}) + \sum_{k=1}^{N} \beta_k(u,\{u_i\})B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)\,B(u)\Omega\,, \qquad (2.2.36)$$

$$D(u)\Phi(\{u_i\}) = \delta(u,\{u_i\})\Phi(\{u_i\}) + \sum_{k=1}^{N} \gamma_k(u,\{u_i\})B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)\,B(u)\Omega\,, \qquad (2.2.37)$$

where the hat on $B(u_k)$ means the omission of the corresponding operator. The spectral coefficients $\alpha, \delta, \beta_k, \gamma_k$ are given by

$$\alpha(u,\{u_i\}) = (u+is)^L \prod_{j=1}^{N} \frac{u-u_j-i}{u-u_j}\,, \qquad \delta(u,\{u_i\}) = (u-is)^L \prod_{j=1}^{N} \frac{u-u_j+i}{u-u_j}\,, \qquad (2.2.38)$$

and

$$\beta_k(u,\{u_i\}) = \frac{i}{u-u_k}(u_k+is)^L \prod_{j\neq k}^{N} \frac{u_k-u_j-i}{u_k-u_j}\,, \tag{2.2.39}$$

$$\gamma_k(u,\{u_i\}) = -\frac{i}{u-u_k}(u_k-is)^L \prod_{j\neq k}^{N} \frac{u_k-u_j+i}{u_k-u_j}\,. \tag{2.2.40}$$

Combining Eqs. (2.2.36) and (2.2.37) and assuming that two products of creation operators are linearly independent for distinct sets of spectral parameters, we find that $\Phi(\{u_i\})$ is an eigenstate of $t_{\text{aux}}(u) = A(u) + D(u)$ if the set of equations ($k = 1, \dots, N$)

$$\beta_k(u,\{u_i\}) + \gamma_k(u,\{u_i\}) = 0\,, \tag{2.2.41}$$

is satisfied. Using Eqs. (2.2.39) and (2.2.40), we obtain the system of algebraic equations known as Bethe ansatz equations,

$$\left(\frac{u_k-is}{u_k+is}\right)^L = \prod_{j\neq k}^{N} \frac{u_k-u_j-i}{u_k-u_j+i}\,. \tag{2.2.42}$$

Spectral parameters $\{u_k\}$ satisfying (2.2.42) are called Bethe roots and the corresponding eigenstates $\Phi(\{u_i\})$ are Bethe states. The eigenvalue of $t_{\text{aux}}(u)$ corresponding to the Bethe state $\Phi(\{u_i\})$ can be read directly from Eqs. (2.2.36), (2.2.37) and (2.2.38),

$$\begin{aligned} t_{\text{aux}}(u)\Phi(\{u_i\}) &= t_{\text{aux}}(u,\{u_i\})\Phi(\{u_i\}) \\ &= \left[(u+is)^L \prod_{j=1}^{N} \frac{u-u_j-i}{u-u_j} + (u-is)^L \prod_{j=1}^{N} \frac{u-u_j+i}{u-u_j}\right]\Phi(\{u_i\})\,. \end{aligned} \tag{2.2.43}$$

We verify that the poles of the expression in square brackets, above, cancel when the Bethe ansatz equations (2.2.42) are fulfilled, in agreement with the polynomiality of the auxiliary transfer matrix. Moreover, a Bethe state $\Phi(\{u_i\})$ is a highest weight, i.e.

$$S_+\Phi(\{u_i\}) = 0\,. \tag{2.2.44}$$

Later, by examining the solutions to the Bethe ansatz equations (2.2.42), we will argue that their number correctly matches the degeneracy of highest weight states made out of N magnons.

We succeeded in finding the eigenstates of the auxiliary transfer matrix. We found that an eigenstate is characterized by a set of Bethe roots satisfying the Bethe ansatz equations (2.2.42). The Bethe roots play the role of quantum numbers and should encode all information about the eigenstate. In particular, they determine completely the eigenvalue of the auxiliary transfer matrix, Eq. (2.2.43). We will now see that this is also the case for the fundamental transfer matrix.

Energy Spectrum

The evaluation of the fundamental transfer matrix $t(u)$ on a Bethe state $\Phi(\{u_i\})$ requires some algebra, that is explained in [82].[9] The outcome is that

$$t(u)\Phi(\{u_i\}) = t(u,\{u_i\})\Phi(\{u_i\}) = \left[\prod_{k=1}^{N} \frac{u-u_k-is}{u-u_k+is} + O\left(u^L\right)\right]\Phi(\{u_i\})\,. \qquad (2.2.45)$$

From this result and making of use ot the expansion (2.2.13), we obtain the eigenvalue of the higher conserved charges,

$$Q_r\Phi(\{u_i\}) = q_r\Phi(\{u_i\})\,, \qquad (2.2.46)$$

with

$$q_r = \frac{i}{r-1}\sum_{k=1}^{N}\left[(u_k-is)^{1-r} - (u_k+is)^{1-r}\right], \qquad (2.2.47)$$

for $r \leqslant L$. In particular, we find that the Heisenberg Hamiltonian $H = Q_2 + \text{constant}$, see Eq. (2.2.15), has the eigenvalue

$$(H - E_{\text{vac}})\Phi(\{u_i\}) = E\,\Phi(\{u_i\})\,, \qquad (2.2.48)$$

with

$$E = \sum_{k=1}^{N} E_k = -\sum_{k=1}^{N} \frac{2s}{u_k^2+s^2}\,, \qquad (2.2.49)$$

and the vacuum energy is

$$E_{\text{vac}} = 2L(\psi(-2s) - \psi(1))\,. \qquad (2.2.50)$$

Notice that, for $s = -1/2$, the energy E is positive definite and that the vacuum energy vanishes, $E_{\text{vac}} = 0$.

We conclude, recalling the identity between the one-loop (planar) dilatation operator and the Hamiltonian of the $\text{XXX}_{-1/2}$ Heisenberg spin chain, $\delta D = 2g^2H + O(g^4)$, that the spectrum of anomalous dimensions, of Wilson operators carrying twist L and Lorentz spin N, is given by

$$\delta\Delta = 2g^2\sum_{k=1}^{N}\frac{1}{u_k^2+\frac{1}{4}} + O\left(g^4\right)\,, \qquad (2.2.51)$$

where the Bethe roots $\{u_i\}$ are solutions to the Bethe ansatz equations, Eqs. (2.2.42) with $s = -1/2$. We require, furthermore, the condition $U\Phi(\{u_i\}) = \Phi(\{u_i\})$ to be fulfilled, as was already explained. Using $U = t(0)$ and Eq. (2.2.45) with $s = -1/2$, it turns into

$$\prod_{k=1}^{N}\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}} = 1\,. \qquad (2.2.52)$$

[9] Strictly speaking, it is explained in [82] how to construct the eigenvalue of the fundamental, or spin s, transfer matrix from the auxiliary, or spin 1/2, one, for $s > 0$, that is for an arbitrary-finite dimensional representation. Here we assume that the result obtained in this way can be continued to $s = -1/2$.

Interlude: Factorized Scattering

Looking back at Eq. (2.2.49), we note that, remarkably enough, the energy E for a N-magnon Bethe state, with roots $\{u_k\}$, is simply given by the sum of the individual energies $\{E_k\}$. The same is true for the higher conserved charges (2.2.47), and, in particular, for the total momentum $p = q_1$,

$$p = \sum_{k=1}^{N} p_k = i \sum_{k=1}^{N} \log\left(\frac{u_k + is}{u_k - is}\right) . \tag{2.2.53}$$

Then, to each root u_k we can associate a momentum p_k, as above, or reciprocally with $u_k = -s\cot(p_k/2)$. In terms of the momentum p_k, the energy of an individual magnon E_k, with rapidity (i.e. Bethe root) u_k, reads

$$E_k = -\frac{2}{s}\sin^2\left(\frac{p_k}{2}\right) . \tag{2.2.54}$$

The energy of the Bethe state $\Phi(\{u_k\})$ is thus identical to the one of a system of free particles with momenta $\{p_k\}$ and dispersion relation (2.2.54). However, the Heisenberg spin chain is not a free theory, and its energy spectrum is not the one of a theory of free magnons propagating over a periodic lattice of L sites. This is reflected in the fact that the momenta $\{p_k\}$ do not satisfy free quantization conditions on a closed spin chain of length L. The latter would read

$$e^{ip_k L} \equiv \left(\frac{u_k - is}{u_k + is}\right)^L = 1 . \tag{2.2.55}$$

Instead, the momenta $\{p_k\}$ are solutions to the Bethe ansatz equations (2.2.42), that can be written as

$$e^{ip_k L} \prod_{j\neq k}^{N} S(p_k, p_j) = 1 , \tag{2.2.56}$$

where

$$S(p_k, p_j) = \frac{\cot\left(\frac{p_k}{2}\right) - \cot\left(\frac{p_j}{2}\right) - \frac{i}{s}}{\cot\left(\frac{p_k}{2}\right) - \cot\left(\frac{p_j}{2}\right) + \frac{i}{s}} = \frac{u_k - u_j + i}{u_k - u_j - i} . \tag{2.2.57}$$

We note that if $S(p_k, p_j)$ in (2.2.56) were equal to one, the magnons would be free. Therefore, the phase $S(p_k, p_j)$, Eq. (2.2.57), accounts for the interaction between the magnons. Indeed, $S(p_k, p_j)$ is the S-matrix for the scattering of a magnon, with momentum p_k, off a magnon, with momentum p_j, both of them propagating over a spin chain of infinite length.[10] The equations (2.2.56) then reflect the remarkable property that the N-body scattering decomposes into a product of two-by-two scatterings. This can be understood as follows. The Bethe ansatz equations (2.2.56) can be thought of as the conditions of periodicity for the N-body wave function on a closed spin chain of length L. Heuristically, when one of the N magnons, with momentum p_k, is brought once around the spin chain, the wave function for the system is multiplied by a phase, due to the propagation and to the scattering off the others magnons, with momenta $\{p_j, j \neq k\}$. Since the scattering factorizes, this phase is precisely the left-hand side of (2.2.56), and, for a periodic wave function, it should be 1, or equivalently Eqs. (2.2.56).

[10]Note that the scattering of two magnons over a 1D lattice is, for kinematical reasons, necessarily diffractionless: $\{p_k^{\rm in}, p_j^{\rm in}\} = \{p_k^{\rm out}, p_j^{\rm out}\} = \{p_k, p_j\}$ and the two-by-two S-matrix is just a phase.

The factorized scattering property is intimately tied to the existence of higher conserved charges, see discussion in [86] for instance, and it is an alternative way of thinking of integrability. The original derivation by Bethe [80] of his equations (for the compact $s = 1/2$ XXX Heisenberg spin chain) was based on a ansatz for the N-body wave function implementing precisely the factorized scattering hypothesis. This approach is known as the coordinate Bethe ansatz. It has been used in [40] to diagonalize the Heisenberg Hamiltonian for $s = -1/2$ and shown to reproduce the equations (2.2.42) obtained from the algebraic Bethe ansatz. As stressed in [40], the coordinate Bethe ansatz is particularly well-suited to diagonalize the higher-loop dilatation operator and, then, compute the higher-loop deformation of the equations (2.2.42) in the gauge theory.[11] The outcome is a set of all-loop (asymptotic) Bethe ansatz equations [41] that are of the type (2.2.56) but with $S(p_k, p_j)$ now being the all-loop 2-body S-matrix proposed in [42, 43, 44]. These equations are complemented with the dispersion relation that generalizes (2.2.54) to all loops [39, 42]. They will be given and discussed at the end of this chapter.

2.3 Large Spin Limit

In this section, we will analyze the large spin limit $N \to \infty$ of the Bethe ansatz equations of the $\mathrm{XXX}_{-1/2}$ Heisenberg spin chain. We whall see that the latter set of discrete equations simplifies in this limit, and can be conveniently replaced by an integral equation.[12] Solving this equation, for the ground-state distribution of Bethe roots at large N, we will recover the logarithmic scaling for the minimal anomalous dimension, with the correct one-loop expression of the cusp anomalous dimension. The generalized scaling limit $L \sim \log N \gg 1$ [45, 47] will be also considered. More material about the large spin limit of the $s = -1/2$ Bethe ansatz equations can be found in [62, 88, 45, 63, 47].

2.3.1 Bethe Ansatz Equations

We start with few remarks about the solutions to the Bethe ansatz equations, in view of understanding how to characterize the ground-state distribution of Bethe roots, associated with the minimal anomalous dimension.

Counting of the Solutions

The Bethe ansatz equations read

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^{N} \frac{u_k - u_j - i}{u_k - u_j + i}\,, \tag{2.3.1}$$

[11] The algebraic Bethe ansatz cannot be applied in that case, because it is still unknown how to deform the R-matrix at higher loops.

[12] In essence, our analysis of the large spin limit is similar to the treatment of the thermodynamic limit of the Bethe ansatz equations for a gas of bosons with δ-function interaction [87]. Nevertheless, it is important to stress that the large spin limit is not the thermodynamic limit of the $s = -1/2$ Bethe ansatz equations, which would correspond to $N, L \to \infty$ with N/L kept fixed (see [95] for this limit and [45] for a discussion of the different regimes of high-twist high-spin minimal anomalous dimension).

where the Bethe roots u_k, $k = 1, \ldots, N$, are all distincts by construction. We assume, furthermore, that all the roots are real and finite,[13] and that they have been decreasingly ordered, $+\infty > u_1 > u_2 > \ldots > u_N > -\infty$, to avoid any overcounting. We would like to uncover the main features of the solutions to Eq. (2.3.1), in order to understand how to characterize the ground-state distribution of Bethe roots. We start by rewriting the Bethe ansatz equations, Eqs. (2.3.1), as follows

$$\left(\frac{1-2iu_k}{1+2iu_k}\right)^L = (-1)^{L+N-1} \prod_{j=1}^{N} \frac{1+iu_k-iu_j}{1-iu_k+iu_j}\,. \tag{2.3.2}$$

Note that the product in Eq. (2.3.2) is taken over all the roots, including the root u_k, in distinction with Eq. (2.3.1). We now take the logarithm on both sides of the equation (2.3.2), choosing the principal branch, and obtain

$$2\pi n_k = iL\log\left[\frac{1-2iu_k}{1+2iu_k}\right] - i\sum_{j=1}^{N} \log\left[\frac{1+iu_k-iu_j}{1-iu_k+iu_j}\right]. \tag{2.3.3}$$

Here, the mode numbers n_k belong to $\mathbb{Z}$ or to $(2\mathbb{Z}+1)/2$, for $L+N$ odd or even, respectively. It is convenient to introduce an 'off-shell' observable $H(u)$, called the counting function, that is defined, for u an arbitrary real number, by

$$H(u) \equiv iL\log\left[\frac{1-2iu}{1+2iu}\right] - i\sum_{j=1}^{N} \log\left[\frac{1+iu-iu_j}{1-iu+iu_j}\right]. \tag{2.3.4}$$

Then, the equation (2.3.3) becomes equivalent to

$$2\pi n_k = H(u_k)\,. \tag{2.3.5}$$

Note that $H(u)$ depends on the distribution of Bethe roots, through the second term on the right-hand side of Eq (2.3.4), and it is a real function of u, under our general assumptions. The counting function $H(u)$ has nice properties, worth to emphasize. For instance, we find its asymptotic values at infinity as

$$H(\pm\infty) = \pm\pi(L+N)\,, \tag{2.3.6}$$

where we took care of our choice of determination for the logarithm. Moreover, $H(u)$ is an increasing function on the real $u-$axis, since its derivative is positive there,

$$H'(u) = \frac{4L}{1+4u^2} + \sum_{j=1}^{N} \frac{2}{1+(u-u_j)^2}\,. \tag{2.3.7}$$

Observing Eq. (2.3.5) and taking into account the ordering of roots, we conclude that integers n_k are also ordered and bounded,

$$H(\infty) > 2\pi n_1 > \ldots > 2\pi n_N > H(-\infty)\,. \tag{2.3.8}$$

[13] Sending to infinity some of the Bethe roots amounts to consider $\mathfrak{sl}(2)$-descendants. The reality of the solutions to the $s = -1/2$ Bethe ansatz equations is discussed in [62, 88]. It contrasts with the existence of complex solutions in the (compact) $s = 1/2$ case (see [81] for instance). Physically, the difference between the two situations is that the interaction between $s = -1/2$ magnons is repulsive, while it permits formation of bound states in the $s = 1/2$ case, associated to complex rapidities.

This implies that the mode numbers n_k should belong to the following set

$$n_k \in \mathfrak{S} = \left\{ \frac{L+N-1}{2}, \frac{L+N-3}{2}, \dots, \frac{3-L-N}{2}, \frac{1-L-N}{2} \right\}, \tag{2.3.9}$$

independently of the parity of $L+N$. There is a further restriction that the two outermost values of the set (2.3.9) are forbidden. Indeed, considering the largest root u_1, corresponding to the largest number n_1, we get from Eq. (2.3.4) the inequality

$$H(u_1) < iL \log \left[\frac{1-2iu_1}{1+2iu_1} \right] + (N-1)\pi < (L+N-1)\pi \,, \tag{2.3.10}$$

or equivalently, using Eq. (2.3.5),

$$n_1 < \frac{L+N-1}{2} \,. \tag{2.3.11}$$

Similarly for the smallest root u_N, corresponding to the smallest number n_N, we find that

$$n_N > \frac{1-L-N}{2} \,. \tag{2.3.12}$$

Therefore, we conclude that the mode numbers n_k should belong to

$$n_k \in \widehat{\mathfrak{S}} = \left\{ \frac{L+N-3}{2}, \frac{L+N-5}{2}, \dots, \frac{5-L-N}{2}, \frac{3-L-N}{2} \right\}, \tag{2.3.13}$$

leaving $N+L-2$ possible values for n_k. Given a particular distribution of N (distinct) mode numbers n_k in $\widehat{\mathfrak{S}}$, we will assume that there exists a unique distribution of roots $\{u_k\}$ satisfying (2.3.5). This implies that the number of (real) solutions to the Bethe ansatz equations (2.3.1) is given by the binomial coefficient C^N_{N+L-2}. This counting agrees with the number of primary operators with Lorentz spin N, i.e. with $\mathfrak{sl}(2)$ spin $(Ls-N)$, in V,

$$V \cong V_s^{\otimes L} \cong \sum_{N=0}^{\infty} V_{Ls-N}^{\oplus d_N} \,, \qquad\qquad d_N = C^N_{N+L-2} \,. \tag{2.3.14}$$

In particular, we verify that for twist-two operators ($L=2$) there is only one anomalous dimension for a given Lorentz spin N. To obtain the exact counting of primary operators in the gauge theory, one still has to remove from the spin-chain Hilbert space V all the states which are not cyclically invariant. This is enforced by the condition (2.2.52) that can be written as

$$\frac{1}{L} \sum_{k=1}^{N} n_k \in \mathbb{Z} \text{ or } (2\mathbb{Z}+1)/2 \,, \tag{2.3.15}$$

for N even or odd, respectively.

Ground-State Distribution

We will now discuss how to characterize the ground-state distribution of Bethe roots, following [88, 45, 44, 47]. Let us first introduce some terminology, that we will use in this section. We argued before that to any solution of the Bethe ansatz equations there is corresponding distribution of

N mode numbers n_k in $\widehat{\mathfrak{S}}$. Equivalently, we could replace it by a distribution of $(L-2)$ holes $\{\hat{n}_l, l = 1, \ldots, L-2\} \subset \widehat{\mathfrak{S}}$, which is the complementary of $\{n_k\}$ in $\widehat{\mathfrak{S}}$.[14] with the help of the counting function $H(u)$, Eq. (2.3.4), we may associate a rapidity $\hat{u}_l$ to each of these holes as

$$2\pi\hat{n}_l = H(\hat{u}_l)\,, \tag{2.3.16}$$

mimicking this way the relation between the mode number n_k and the Bethe root u_k. The hole rapidities have a nice interpretation. Indeed, due to their definition given above, they are solution to the equation

$$\left(\frac{\hat{u}_l + \frac{i}{2}}{\hat{u}_l - \frac{i}{2}}\right)^L = -\prod_{k=1}^{N} \frac{\hat{u}_l - u_k - i}{\hat{u}_l - u_k + i}\,, \tag{2.3.17}$$

or equivalently

$$t_{\rm aux}(\hat{u}_l, \{u_k\}) = \left(\hat{u}_l + \tfrac{i}{2}\right)^L \prod_{k=1}^{N} \frac{\hat{u}_l - u_k + i}{\hat{u}_l - u_k} + \left(\hat{u}_l - \tfrac{i}{2}\right)^L \prod_{k=1}^{N} \frac{\hat{u}_l - u_k - i}{\hat{u}_l - u_k} = 0\,, \tag{2.3.18}$$

where $t_{\rm aux}(u, \{u_k\})$ is the eigenvalue of the auxiliary transfer matrix $t_{\rm aux}(u)$, evaluated on the Bethe states $\Phi(\{u_k\})$, see Eq. (2.2.43). Therefore, the hole rapidities are roots of $t_{\rm aux}(u, \{u_k\})$. We recall that by construction $t_{\rm aux}(u, \{u_k\})$ is a polynomial of u of degree L. Among the L roots of this polynomial, we identified $(L-2)$ roots as corresponding to hole rapidities. The two missing roots correspond to the two outermost elements of the set $\mathfrak{S}$, that we have excluded from our consideration because they could not be associated to Bethe roots. These remarks allows us to make a contact with an off-shell formulation of the Bethe ansatz equations, known as the Baxter equation. It reads

$$(u + is)^L\, Q(u - i) + (u + is)^L\, Q(u + i) = t_{\rm aux}(u, \{u_j\})Q(u)\,, \tag{2.3.19}$$

where $Q(u) = \prod_{j=1}^{N}(u - u_j)$ is the Baxter polynomial and $s = -1/2$ for scalar operators. It is clear that the Baxter equation implies the Bethe ansatz equations, since for $u = u_k$ the right-hand side of (2.3.19) vanishes and we recover (2.3.1). Moreover, dividing on both sides of (2.3.19) by $Q(u)$, we immediately verify that the Baxter equation correctly reproduces the expression (2.2.43) for the eigenvalue of the auxiliary transfer matrix. One of the advantages of the Baxter formulation of the Bethe ansatz equations is that it is sometimes simpler to characterize $Q(u)$ than the complete set of Bethe roots, especially when the number of roots is large. In the particular twist-two case, it is even possible to solve exactly the Baxter equation for any value of N [27]. As shown in [62, 88, 45, 89], the Baxter equation can be used also to unravel several aspects of the spectrum of anomalous dimensions, including the large spin limit of the minimal anomalous dimension at arbitrary twist. Nevertheless, we will not pursue this direction, and will follow instead the analysis of [44, 47] for convenience with the higher-loop treatment analysed in the following chapter.[15]

So let us come back to the problem of the determination of the ground-state distribution of Bethe roots. We expect the ground state to be non-degenerate. We thus assume an even

[14] Note that for $L = 2$ there are no holes in $\widehat{\mathfrak{S}}$ and all positions in $\widehat{\mathfrak{S}}$ are occupied.

[15] Let us mention however that an all-loop asymptotic Baxter equation for planar $\mathcal{N} = 4$ SYM theory has been proposed in [90] and that it has been used both to study the all-loop large spin limit of minimal anomalous dimension [91] and to solve exactly the $L = 2$ case in a perturbative expansion in the coupling constant [92, 93].

distribution of roots u_k [88, 45], or equivalently of mode numbers n_k, and to respect the cyclicity condition (2.3.15) we choose an even Lorentz spin N. To minimize the (one-loop) anomalous dimension given by

$$\delta\Delta = 2g^2 \sum_{k=1}^{N} \frac{4}{1+4u_k^2} + O\left(g^4\right) , \tag{2.3.20}$$

we will try to take the roots $\{u_k\}$ as large as possible. So let us understand the constraints put on the shape of the distribution by the Bethe equations. We recall that two consecutive roots have to be placed such that the counting function $H(u)$ varies between them by a multiple of 2π. Now we look at the equation (2.3.4) and we see that $H(u)$ is expressed as a sum of two terms. The first term is centered around the origin $u \sim 0$ and looks like an external potential for the roots. The second term, on the other hand, only depends on the relative distances between the roots. Moreover, it is not difficult to see that the first term attracts and bounds the roots close to the origin $u \sim 0$, while the second term generates instead a repulsive interaction between them. It follows that the typical distance between two consecutive roots is smaller when they are close to the origin $u \sim 0$ than to the edges of the distribution. The only freedom that we have to increase the typical distance between roots around the origin, and thus decrease their density there, is to allow for hole rapidities to sit between them. The best we can do is then to put $(L-2)$ hole rapidities in-between the smallest positive root $u_{N/2}$ and the largest negative one $u_{N/2+1}$. That configuration with all the hole rapidities centered around the origin and surrounded by the Bethe roots should correspond to the minimal anomalous dimension [45].[16]

2.3.2 Large Spin Integral Equation

As we have seen, the ground-state distribution of roots is symmetric, supported on the interval $D = D_- \cup D_+$, with $D_+ = -D_- = [a, c] = [u_{N/2}, u_1]$, and characterized by the condition that the $(L-2)$ hole rapidities lie on $[-a, a]$. For generic values of the twist L and the Lorentz spin N, it is not possible to find the explicit distribution of roots satisfying these conditions and solving the Bethe ansatz equations. Nevertheless, some simplifications occur when N and/or L get large. Here we are interested in the limit $N \to \infty$ with L kept fixed, for which we have a large number of roots but a given number of hole rapidities. In that limit, we expect the distribution of roots to be dense[17] and we will thus attempt to substitute to the discrete Bethe ansatz equations a continuum integral equation.

Let us start introducing a new function $\rho(u)$ defined in terms of the counting function $H(u)$, Eq. (2.3.4), as

$$H(u) = 2\pi \int_0^u dv\, \rho(v) \,. \tag{2.3.21}$$

Given the properties of $H(u)$ (computed for the ground state), the function $\rho(u)$ is smooth, positive and symmetric $\rho(u) = \rho(-u)$. Note also that $\rho(u)$ is defined for an arbitrary real value of u. The reason to introduce $\rho(u)$ is that, once integrated, it permits to interpolate the counting of roots (magnons and/or holes) inside some interval. It simply follows from the relation between

[16] Note that, for kinematical reasons, we still have two hole rapidities that are bigger in magnitude than any Bethe roots. As said before, they correspond to the two outermost values of $\mathfrak{S}$.

[17] At least away from $u \sim \pm c$.

$H(u_k)$ and/or $H(\hat{u}_l)$ and the mode numbers n_k and/or $\hat{n}_l$. Thus we refer to $\rho(u)$ as a density distribution. We recall that the counting function $H(u)$ depends on the Bethe roots $\{u_k\}$ and reads explicitely as

$$H(u) = iL\log\left[\frac{1-2iu}{1+2iu}\right] - i\sum_{k=1}^{N}\log\left[\frac{1+iu-iu_k}{1-iu+iu_k}\right]. \tag{2.3.22}$$

To derive an integral equation for $\rho(u)$, we would like to replace the sum over $\{u_k\}$ in the expression above by an integral over the domain D, where the roots condense. To achieve this goal, one could for instance make use of the Euler-Maclaurin summation formula. In this way, one would obtain

$$H(u) = iL\log\left[\frac{1-2iu}{1+2iu}\right] - i\int_D dv\,\rho(v)\log\left[\frac{1+iu-iv}{1-iu+iv}\right] + \ldots, \tag{2.3.23}$$

where dots stand for boundary terms, which depend on u, c and a. However, not all of these corrections are negligible, in particular those associated with the lower-edge parameter a. To circumvent this difficulty, we will complete the sum in (2.3.23) by extending it to the $(L-2)$ hole rapidities $\{\hat{u}_l\}$, which lie inside the interval $[-a, a]$ for the ground state. Doing so and applying the Euler-Maclaurin formula, we obtain

$$H(u) = iL\log\left[\frac{1-2iu}{1+2iu}\right] + i\sum_{l=1}^{L-2}\log\left[\frac{1+iu-i\hat{u}_l}{1-iu+i\hat{u}_l}\right] - i\int_{-c}^{c} dv\,\rho(v)\log\left[\frac{1+iu-iv}{1-iu+iv}\right] + \ldots. \tag{2.3.24}$$

Here the dots stand for corrections that are suppressed as $1/(\rho(c)c^2)$, when $c \to \infty$ with $u < c$. The quantity $\rho(c) = H'(c) = H'(u_1)$ is bounded from below $\rho(c) \geqslant 2$, see Eq. (2.3.7), and as shown in [88, 45] the parameter c is large when $N \gg 1$. Therefore, we can safely neglect the boundary terms in the large spin limit as far as we assume $u < c$. As we shall see, the logarithmic scaling originates from the accumulation of roots around $u \sim 0$ [88, 45], and the contribution from $u \sim c$ to the anomalous dimension is well suppressed, so we will no longer take care of the dots in (2.3.24). Now, differentiating on both sides of (2.3.21) and using (2.3.24), we can deduce an integral equation for the function $\rho(u)$ that reads

$$2\pi\rho(u) = \frac{4L}{1+4u^2} - \sum_{l=1}^{L-2}\frac{2}{(u-\hat{u}_l)^2+1} + 2\int_{-c}^{c} dv\,\frac{\rho(v)}{(u-v)^2+1}. \tag{2.3.25}$$

A similar equation has been obtained in [47], and for $L = 2$, when there is simply no sum over hole rapidities in the right-hand side of (2.3.25), it coincides with the equation of [63]. The equation (2.3.25) determines $\rho(u)$ with an implicit dependence on the twist L, the parameter c and the set of hole rapidities $\{\hat{u}_l\}$. The latter are fixed, by definition, by the relation $H(\hat{u}_l) = 2\pi\hat{n}_l$, that can be written as

$$\int_0^{\hat{u}_l} dv\,\rho(v) = \hat{n}_l\,. \tag{2.3.26}$$

Once the solution to (2.3.25) is known, the relation above can be thought of as a set of effective Bethe ansatz equations that determine the hole rapidities. The edge-parameter c can be eliminated in favor of the Lorentz spin N by the normalization condition $H(c) - H(-c) = 2\pi(N + L - 3)$, or equivalently

$$\int_{-c}^{c} dv\,\rho(v) = N + L - 3\,. \tag{2.3.27}$$

Finally, by using the same strategy as before to convert the sum over the Bethe roots into an integral, the formula for the anomalous dimension reads

$$\delta\Delta = 2g^2 \int_{-c}^{c} dv \, \frac{\rho(v)}{u^2 + \frac{1}{4}} - 2g^2 \sum_{l=1}^{L-2} \frac{1}{\hat{u}_l^2 + \frac{1}{4}} + \dots \,, \tag{2.3.28}$$

where the dots include terms suppressed by $1/c^2$ or higher-loop $O(g^4)$ corrections.

Without solving the equation (2.3.25), we note that if the hole rapidities were not taken into account, or if they were all larger than the Bethe roots, we would find that the twist-L solution is given simply by rescaling the twist-two solution such that[18]

$$2\,\rho(u; L) = L\,\rho(u; L = 2)\,. \tag{2.3.29}$$

Then, since the anomalous dimension of twist-two operators scales at large Lorentz spin as $\delta\Delta_{L=2} = 2\,\Gamma_{\text{cusp}}(g)\log N$, we would have that $\delta\Delta_L = L\,\Gamma_{\text{cusp}}(g)\log N$, which is the behavior of the maximal anomalous dimension but not the desired result. Therefore, while the consideration of the hole rapidities may look like a (subleading) discretization effect in the large N limit, their role appears in fact to be essential for the correct description of the minimal anomalous dimension. As we shall see, their introduction ensures that the logarithmic scaling $\delta\Delta_{\min} = 2\,\Gamma_{\text{cusp}}(g)\log N$ is independent of the twist L, as it should be [49].

2.3.3 Large Spin Solution

Let us discuss now the large c solution to (2.3.25). To simplify our problem, we decompose it by writing the density distribution $\rho(u)$ as the sum

$$\rho(u) = \frac{L}{2}\rho_0(u) + \rho(u, \{\hat{u}_l\})\,. \tag{2.3.30}$$

Here the first term $\rho_0(u)$ satisfies the twist-two equation [63]

$$2\pi\rho_0(u) = \frac{8}{1 + 4u^2} + 2\int_{-c}^{c} dv \, \frac{\rho_0(v)}{(u - v)^2 + 1}\,, \tag{2.3.31}$$

while the second term accounts for the dependence on the holes rapidities and is solution to

$$2\pi\rho(u, \{\hat{u}_l\}) = -\sum_{l=1}^{L-2} \frac{2}{(u - \hat{u}_l)^2 + 1} + 2\int_{-c}^{c} dv \, \frac{\rho(v, \{\hat{u}_l\})}{(u - v)^2 + 1}\,. \tag{2.3.32}$$

Note that due to the minus sign in front of the inhomogeneous term in the right-hand side of (2.3.32), the function $\rho(u, \{\hat{u}_l\})$ is negative. Instead, the function $\rho_0(u)$ and the total density $\rho(u)$ are positive. Keeping u fixed and assuming $c \to \infty$, the integrals over v in (2.3.31) and (2.3.32) can be extended to the full real axis and the equations (2.3.31) and (2.3.32) are solved by the Fourier transform. In this way, we construct the solutions $\rho_0(u)$ and $\rho(u, \{\hat{u}_l\})$ up to corrections of order $O(1/c)$ for $|u| \ll c$. They read

$$\rho_0(u) = \frac{2}{\pi}\int_0^{\infty} dt \; \mathrm{e}^{-t/2}\left(\frac{\cos(ut)}{1 - \mathrm{e}^{-t}} - \frac{1}{t}\right) + C_0\,, \tag{2.3.33}$$

[18] The normalization condition (2.3.27) would be different however. But the right-hand side of (2.3.27) would still scale as N at large N, which is sufficient for the validity of our discussion.

$$\rho(u,\{u_l\}) = -\frac{1}{\pi}\sum_{l=1}^{L-2}\int_0^\infty dt\ \mathrm{e}^{-t}\left(\frac{\cos((u-\hat{u}_l)t)}{1-\mathrm{e}^{-t}} - \frac{1}{t}\right) - \sum_{l=1}^{L-2} C(\hat{u}_l)\,, \tag{2.3.34}$$

in agreement with the findings of [47]. Here, the constants C_0 and $C(\hat{u}_l)$ depend a priori on c and are left undetermined. They reflect the vanishing at $t = 0$ of the (multiplicative) kernel $(1 - \mathrm{e}^{-t})$ in Fourier space, leading to a δ-function ambiguity $\sim C\delta(t)$. The integrals over t in (2.3.33) and (2.3.34) can be done exactly and the solutions $\rho_0(u)$ and $\rho(u,\{u_l\})$ may be expressed in terms of the Euler psi-function. Namely, we get

$$\rho_0(u) = -\frac{1}{\pi}\big(\psi\big(\tfrac{1}{2}+iu\big) + \psi\big(\tfrac{1}{2}-iu\big) + 2\log 2\big) + C_0\,, \tag{2.3.35}$$

$$\rho(u,\{u_l\}) = \frac{1}{2\pi}\sum_{l=1}^{L-2}\big(\psi(1+i(u-\hat{u}_l)) + \psi(1-i(u-\hat{u}_l))\big) - \sum_{l=1}^{L-2} C(\hat{u}_l)\,. \tag{2.3.36}$$

Looking at the expressions (2.3.35) and (2.3.36), we may be a little bit surprised not to find any sign of the expected logarithmic scaling nor of its property to be independent of the twist L.[19] In fact, this information is hidden in the coefficients C_0 and $C(\hat{u}_l)$, which turn out to scale as $\log c$, when $c \gg 1$, and therefore dominate the contributions coming from the psi-functions in (2.3.35) and (2.3.36). A simple way to understand why the coefficients C_0 and $C(\hat{u}_l)$ cannot be of order $O(c^0)$ is to look at the solutions (2.3.35) and (2.3.36) for $1 \ll |u| (\ll c)$. In that case, we may apply the asymptotic behaviors of (2.3.35) and (2.3.36), given respectively by

$$\rho_0(u) = -\frac{\log u^2 + 2\log 2}{\pi} + C_0 + O(1/u^2)\,, \tag{2.3.37}$$

$$\rho(u,\{u_l\}) = \frac{(L-2)\log u^2}{2\pi} - \sum_{l=1}^{L-2} C(\hat{u}_l) + O(1/u^2)\,, \tag{2.3.38}$$

to reveal that the coefficients C_0 and $C(\hat{u}_l)$ should grow at large c in order to restore the positivity of $\rho_0(u)$, and, respectively, the negativity of $\rho(u,\{u_l\})$. To verify it and uncover the logarithmic scaling of the minimal anomalous dimension, we will look at the solutions to (2.3.31) and (2.3.32) in the complementary regime $c \to \infty$ with $\bar{u} = u/c$ fixed, and match their expressions at $\bar{u} \sim 0$ against (2.3.37) and (2.3.38). In this regime, the equations (2.3.31) and (2.3.32) turn into singular integral equations of the Riemann-Hilbert type that can be solved explicitely [88, 45, 63, 47]. Their solutions read [88]

$$\rho_0(\bar{u}) = \frac{1}{\pi}\log\left(\frac{1+\sqrt{1-\bar{u}^2}}{1-\sqrt{1-\bar{u}^2}}\right), \tag{2.3.39}$$

$$\rho(\bar{u},\{u_l\}) = -\frac{L-2}{2\pi}\log\left(\frac{1+\sqrt{1-\bar{u}^2}}{1-\sqrt{1-\bar{u}^2}}\right). \tag{2.3.40}$$

We see from the expression above that we completely lost the information about the distribution of hole rapidities, except for their overall number. This is due to the fact that the rescaled hole rapidities $\{\hat{u}_l/c\}$ vanish at large c. Moreover, we verify that the total density (2.3.30) is independent of L and coincides with the twist-two solution (2.3.39), $\rho(\bar{u}) = \rho_0(\bar{u})$. The solutions (2.3.39)

[19] We recall that L multiplies $\rho_0(u)$ in the expression for the total density $\rho(u)$, see Eq.(2.3.30).

and (2.3.40) are reliable at large c for $|\bar{u}| > 1/c$. Indeed, for $|\bar{u}| \sim 1/c$ the fine structure of the hole rapidities emerges, and accounts in particular for the fact that the densities $\rho(\bar{u}, \{u_l\})$ is not picked around $\bar{u} = 0$, as suggested by (2.3.40), but instead around $\bar{u} = \hat{u}_l/c$, in agreement with (2.3.36). Moreover, the singularity of (2.3.39) and (2.3.40) at $\bar{u} = 0$ is resolved by a $\sim \log c$ behavior. We can now determine the arbitrary constants C_0 and $C(\hat{u}_l)$ of (2.3.35) and (2.3.36). Namely, matching the asymptotic behaviors of (2.3.39) and (2.3.40) for $\bar{u} \ll 1$, given respectively by

$$\rho_0(\bar{u}) = -\frac{\log u^2}{\pi} + \frac{2\log(2c)}{\pi} + O(\bar{u}^2)\,, \tag{2.3.41}$$

$$\rho(\bar{u}, \{u_l\}) = \frac{(L-2)\log u^2}{2\pi} - \frac{(L-2)\log(2c)}{\pi} + O(\bar{u}^2)\,, \tag{2.3.42}$$

with our previous findings, Eqs. (2.3.37) and (2.3.38), we get

$$C_0 = \frac{2\log(2c) + 2\log 2}{\pi}\,, \qquad C(\hat{u}_l) = \frac{\log(2c)}{\pi}\,, \tag{2.3.43}$$

This two values complete our construction of the large c solution.

We are now ready to compute the minimal anomalous dimension given by (2.3.28). We first eliminate c in favor of N with the help of the normalization condition (2.3.27). We note that the integral in (2.3.27) receives a dominant contribution from intermediate values of $\bar{u}$ when c is large. It means that it can be computed with the expressions (2.3.39) and (2.3.40) for $\rho(\bar{u})$. It leads to

$$2c = N + \dots\,, \tag{2.3.44}$$

where dots stand for terms suppressed as compared to $N \gg 1$. The situation is different for the anomalous dimension (2.3.28). It receives dominant contribution from u fixed at large c, and it can thus be computed with the expressions (2.3.33) and (2.3.34) for $\rho(u)$, supplemented with the values of the constants given in (2.3.43). We find this way that

$$\begin{aligned} \delta\Delta_{\text{min}} &= 2g^2 \int_{-\infty}^{\infty} du\, \frac{\rho(u)}{u^2 + \frac{1}{4}} - 2g^2 \sum_{l=1}^{L-2} \frac{1}{\hat{u}_l^2 + \frac{1}{4}} + \dots \\ &= 8g^2 \log N - 8g^2\psi(1) + 2g^2 \sum_{l=1}^{L-2} \Big[\psi\big(\tfrac{1}{2} + i\hat{u}_l\big) + \psi\big(\tfrac{1}{2} - i\hat{u}_l\big) - 2\psi(1)\Big] + \dots\,, \end{aligned} \tag{2.3.45}$$

in agreement with the results of [45, 47]. Here the dots stand for $1/N$-suppressed corrections or higher-loop contributions. We conclude from (2.3.45) that the minimal anomalous dimension scales logarithmically at large Lorentz spin $N \gg 1$. Moreover, we verify that it supports the twist-independent form

$$\delta\Delta_{\text{min}} = 2\Gamma_{\text{cusp}}(g) \log N + \dots, \tag{2.3.46}$$

with the one-loop expression for the cusp anomalous dimension given by $\Gamma_{\text{cusp}}(g) = 4g^2 + \dots$. This results agrees with the known one-loop computation of the cusp anomalous dimension [51, 49], obtained by a direct evaluation of the vev of a cusped Wilson loop in gauge theory.

All along our analysis, we have assumed that the hole rapidities were small. We can now check that it is effectively the case by looking at the equations for them. We recall that they read

$(l = 1, \ldots, L-2)$

$$\int_0^{\hat{u}_l} dv\, \rho(v) = \hat{n}_l\,, \tag{2.3.47}$$

where the mode numbers $\{\hat{n}_l\}$ fill the set $\{(L-3)/2, (L-5)/2, \ldots, -(L-3)/2\}$. Since we found that the density distribution scales as $\rho(u) \sim 2(\log N)/\pi$ at large N, it immediately follows from the relation above that the hole rapidities are small $\hat{u}_l \sim 1/\log N$. Similarly, the minimal magnon rapidity a is small since it satisfies

$$\frac{L-1}{2} = \int_0^a dv\, \rho(v) \simeq a\, \rho(0) \sim \frac{2a}{\pi} \log N\,. \tag{2.3.48}$$

Thus the gap in the distribution of Bethe roots closes in the large N limit. We note however that a possibility to keep this gap open is to assume that the twist L is large and grows logarithmically with N. This limit is interesting to probe the vicinity of the logaritmic scaling and we will now consider it in more details.

2.3.4 Generalized Scaling Limit

The generalized scaling limit is obtained by keeping fixed the scaling variable $j = L/\log N$ in the limit $N, L \to \infty$ [45, 47]. Then, we find from (2.3.48) that a gap forms as $a \sim \pi j/4$ when j starts to grow. Moreover, since the typical distance between the hole rapidities is of order $1/\log N$, their distribution over the interval $[-a, a]$ is dense. We can therefore repeat our analysis above to obtain an integral equation for the (renormalized) hole density distribution $\rho_{\rm h}(u) \equiv \rho(u)/\log N$ with $u \in [-a, a]$. Namely, taking into account our previous findings for the density $\rho(u)$, see Eqs (2.3.30), (2.3.35), (2.3.36) and (2.3.43), and applying the Euler-Maclaurin formula to the sums over hole rapidities, we find that the relation

$$\frac{2\pi}{\log N}\rho(u) = 4 - j\Big[\psi\big(\tfrac{1}{2}+iu\big) + \psi\big(\tfrac{1}{2}-iu\big)\Big] + \int_{-a}^{a} dv\, \rho_{\rm h}(v)\Big[\psi(1+i(u-v)) + \psi(1-i(u-v))\Big]\,, \tag{2.3.49}$$

holds true at large N and fixed j, for any value of u. For $u \in [-a, a]$ we have $\rho_{\rm h}(u) \equiv \rho(u)/\log N$ leading to the hole equation

$$2\pi\rho_{\rm h}(u) = 4 - j\Big[\psi\big(\tfrac{1}{2}+iu\big) + \psi\big(\tfrac{1}{2}-iu\big)\Big] + \int_{-a}^{a} dv\, \rho_{\rm h}(v)\Big[\psi(1+i(u-v)) + \psi(1-i(u-v))\Big]\,. \tag{2.3.50}$$

This equation agrees with that of [47] up to an inessential rescaling of the hole density $\rho_{\rm h}(u)$. After solving the equation (2.3.50) for $\rho_{\rm h}(u)$, one can eliminate the parameter a in favor of $j = L/\log N$ with the help of the normalization condition

$$\int_{-a}^{a} dv\, \rho_{\rm h}(v) = j\,. \tag{2.3.51}$$

We find moreover that the minimal anomalous dimension still scales logarithmically with N but receives corrections depending on j. They can be introduced as [45, 47]

$$\delta\Delta_{\rm min} = \Big[2\Gamma_{\rm cusp}(g) + \epsilon(g, j)\Big] \log N\, + \ldots, \tag{2.3.52}$$

where we absorbed the contribution of the holes into the scaling function $\epsilon(g,j)$. Here the dots stand for corrections of order $O(\log^0 N)$ at fixed j. The one-loop expression for the scaling function $\epsilon(g,j)$ follows from (2.3.45) and reads

$$\epsilon(g,j) = 2g^2 \int_{-a}^{a} dv\, \rho_{\rm h}(v) \left[\psi\big(\tfrac{1}{2}+iv\big) + \psi\big(\tfrac{1}{2}-iv\big) - 2\psi(1) \right] + O(g^4)\,, \tag{2.3.53}$$

in agreement with [47].

To find a more explicit formula for $\epsilon(g,j)$ one needs to solve the hole equation (2.3.50). At small j, or equivalently at small a, the solution can be found as a series obtained by iterating the inhomogeneous term on the right-hand side of the hole equation [47]. The resulting expansion for the scaling function runs in integer powers of j [45, 47]

$$\epsilon(g,j) = \epsilon_1(g)\, j + \epsilon_2(g)\, j^2 + \epsilon_3(g)\, j^3 + \dots\,, \tag{2.3.54}$$

with the one-loop results

$$\epsilon_1(g) = 8g^2 \log 2 + O(g^4)\,, \qquad \epsilon_2(g) = 0 + O(g^4)\,, \qquad \epsilon_3(g) = -\frac{7}{12}\pi^2 g^2 \zeta_3 + O(g^4)\,, \tag{2.3.55}$$

where $\zeta_z = \zeta(z)$ is the Riemann zeta-function. At large j, the analysis of the hole equation is more difficult. It was performed in [94] up to next-to-next-to-leading order. The one-loop result reads explicitely as

$$\epsilon(g,j) = 8g^2 \left[-1 + \frac{1}{j} + O(1/j^2) \right] + O(g^4)\,. \tag{2.3.56}$$

The first term in the the square brackets above exactly compensates the contribution of the cusp anomalous dimension in (2.3.52), and we find that the one-loop anomalous dimension scales as [95]

$$\delta\Delta_{\rm min} = \frac{8g^2 \log^2 N}{L} + \dots\,, \tag{2.3.57}$$

when $1 \ll \log N \ll L$. We stress however that even if L is quite large when j is large, it is always assumed to be much smaller than N for the consistency of our approach. If L were comparable with $N \gg 1$, we would have to revise the construction of the magnon density distribution $\rho(\bar{u})$ for rescaled rapidities $\bar{u} = u/c \sim u/N \sim O(1)$. This is so because (rescaled) hole rapidities $\hat{u}_l/c$ would not be sent to zero if $L \sim N \gg 1$ and, therefore, the density $\rho(\bar{u})$ would no longer be given by the same twist-independent expression $\rho(\bar{u};L) = \rho(\bar{u}, L=2)$. The cancellation in (2.3.57) of the cusp anomalous dimension against the leading scaling function at large j is a manifestation of this lost of universality. In conclusion, the result (2.3.57) holds for $1 \ll \log N \ll L \ll N$ only [45].

The previous remark points toward the existence of another regime when $L \sim N \gg 1$. It can be investigated by taking the scaling limit $N, L \to \infty$ with $\alpha \equiv L/N$ kept fixed and arbitrary. This regime has been analysed in [95] to one loop order in the gauge theory and to leading order at strong coupling in the dual string theory. When $\alpha \ll 1$, it was found that

$$\delta\Delta_{\rm min} = \frac{8g^2 \log^2 (1/\alpha)}{L} \sim \frac{8g^2 \log^2 N}{L}\,, \tag{2.3.58}$$

in agreement with (2.3.57). This is satisfying since the limits $\alpha \to 0$ and $j \to \infty$ overlap. When $\alpha \gg 1$, the anomalous dimension is small and exhibits the BMN scaling

$$\delta\Delta_{\rm min} = \frac{8\pi^2 g^2}{\alpha L} = \frac{8\pi^2 g^2 N}{L^2}\,. \tag{2.3.59}$$

At intermediate α, the one-loop anomalous dimension can be found as $\delta\Delta_{\min} = g^2 F(\alpha)/L$, up to subleading corrections suppressed by higher powers of $1/L$. Here $F(\alpha)$ is independent of the coupling constant and expressed parametrically in terms of some elliptic functions. Remarkably enough, the AdS/CFT correspondence can be verified immediately in the α-regime. As shown in [95], one gets $F_{\text{gauge}}(\alpha) = F_{\text{string}}(\alpha)$ for arbitrary α, with $F_{\text{string}}(\alpha)$ extracted from the energy of a semiclassical spinning string $E - N - L = g^2 F(\alpha)/L$ carrying both a large spin N and momentum L [14, 15]. This direct matching of the gauge ($g^2 \ll 1$) and string ($g \gg 1$) theory results in the α-regime, without the recourse of interpolating between weak and strong coupling, is an illustration of the BMN correspondence [13].[20] The direct matching does not apply however when the anomalous dimension scales logarithmically with N [14, 15].[21] In that case, to verify the AdS/CFT correspondence, one has to compute radiative corrections to the cusp anomalous dimension and scaling function, resum the weak coupling expansion and take the strong coupling limit in order to compare with the string theory. As we shall see, this program became recently feasible thanks to conjectured all-loop Bethe ansatz equations for the $\mathfrak{sl}(2)$ sector of the gauge theory [40, 41, 44].

2.3.5 Concluding Remarks

We see from all these results that the structure of the large-spin minimal anomalous dimension is quite involved and depends crucially on whether L is small or large as compared to $\log N$ and/or N [45]. When $L \sim \log N \gg 1$, the minimal anomalous dimension has a logarithmic growth $\sim \log N$, controlled by the cusp anomalous dimension and the scaling function. Our analysis demonstrated how both observables can be obtained from the solution to the Bethe ansatz equations. The analysis of [95] completes the picture for $L \sim N \gg 1$ and it is found that the anomalous dimension no longer scales logarithmically with N but instead is suppressed as $1/L$.

In the following section, we will report on the deformation of the one-loop Bethe ansatz equations that incorporates the higher-loop corrections of the gauge theory. Given their explicit expressions [41, 44], one can generalize the large N analysis, above, in order to compute the cusp anomalous dimension and scaling function to all loops. It leads to integral equations for the all-loop density distributions of Bethe roots and hole rapidities, constructed in [63, 44, 47]. In the next chapters, we will solve these equations at strong coupling and make comparison with string theory. We will always consider the vicinity of the logarithmic regime corresponding to $j = L/\log N$ fixed and 'small'.[22] We will see that even under this restriction the analysis remains subtle at strong coupling.

2.4 All-Loop Asymptotic Bethe Ansatz Equations

In this section, we present the all-loop asymptotic Bethe ansatz, in the $\mathfrak{sl}(2)$ sector, conjectured in [40, 41, 44].

[20] Strictly speaking, the result of [95] ??? [96] is a generalization of the BMN scaling hypothesis that originally assumes that N is kept fixed when L is taken to be large.

[21] Except if $j \to \infty$.

[22] We will precise later what we mean by small when the coupling gets large.

These equations are most naturally written in terms of a deformed spectral parameter $x(u)$ defined as [39]

$$2x(u) = u + \sqrt{u^2 - (2g)^2}\,. \tag{2.4.1}$$

Here u is the usual spin-chain spectral parameter and $x(u)$ has a cut along the real interval $u^2 < (2g)^2$. We note that $x(u) \simeq u$ if $|u| \gg 2g$ and more particularly if $g \sim 0$. Moreover, we see that the weak coupling expansion of the deformed spectral parameter $x(u)$, assuming u fixed and away from the cut, runs in integer powers of g^2.

Equations

In the $\mathfrak{sl}(2)$ sector, the all-loop asymptotic Bethe ansatz equations [40, 41, 44] read

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp 2i\theta(u_k, u_j)\,, \tag{2.4.2}$$

for a N magnons Bethe state, $k = 1, \ldots, N$. Here, we have introduced the notations $x_k^\pm \equiv x^\pm(u_k) \equiv x(u_k \pm i/2)$, and $\theta(u_k, u_j)$ stands for the so-called dressing phase. We will see later that the dressing phase vanishes up to three-loop in $\mathcal{N} = 4$ SYM theory [44], while it is a leading-order effect at strong coupling in order to correctly reproduce the spectrum of semiclassical states in type IIB superstring theory [98, 40].

Putting the dressing phase to zero, $\theta(u_k, u_j) = 0$, it is immediate to verify that the equations (2.4.2) reproduce the Bethe ansatz equations of the Heisenberg spin chain, in the one-loop approximation $g = 0$ and $x^\pm = u \pm i/2$. Moreover, the equations (2.4.2) were shown to diagonalize the mixing matrix up to two-loop, by explicit gauge theory computations [63, 37] and algebraic construction [35]. They were furthermore proved to be consistent with the known three-loop spectroscopy of anomalous dimensions (see for instance [40]).

Lastly, the equations (2.4.2) are supplemented with the cyclicity condition

$$\prod_{k=1}^{N} \frac{x_k^+}{x_k^-} = 1\,. \tag{2.4.3}$$

Note that for a state with an even distribution of Bethe roots $\{u_k\}$ (and an even Lorentz-spin N), they are automatically satisfied, due to the property $x^+(-u) = -x^-(u)$.

Dressing Phase

The dressing phase $\theta(u_k, u_j)$ admits a bilinear expansion over the infinite tower of conserved charges [98, 40, 99], that can be parameterized as

$$\theta(u_k, u_j) = \sum_{r\geqslant 2} \sum_{s\geqslant r+1} g^{r+s-2}\, c_{r,s}(g)\big(q_r(u_k)q_s(u_j) - q_s(u_k)q_r(u_j)\big)\,. \tag{2.4.4}$$

Here, the all-loop expressions for the charges are given by [39] ($r \geqslant 2$)

$$q_r(u) = \frac{i}{r-1}\left(x^+(u)^{1-r} - x^-(u)^{1-r}\right)\,, \tag{2.4.5}$$

and the coupling-dependent expansion coefficients $c_{r,s}(g)$ all vanish, except for the infinite sequence $s = r + 1 + 2n$ with $n \in \mathbb{N}$.

At strong coupling, the dressing-phase coefficients are given, up to next-to-leading order, by

$$c_{r,s}(g) = g\,\delta_{r+1,s} - \frac{2(r-1)(s-1)}{\pi(s-r)(s+r-2)} + \dots\,, \tag{2.4.6}$$

where dots stand for two- and higher-loop quantum string contributions, suppressed by powers of $1/g$. The first term in the right-hand side of (2.4.6) was obtained in [98, 40] by discretizing the finite-gap equations of the classical string theory [18, 97].[23] One-loop quantum-string correction to the 'classical' dressing-phase, the second term in the right-hand side of (2.4.6), was first discussed and partially computed in [100] by confronting the prediction of the Bethe ansatz equations with an explicit string theory computation for the one-loop energy shift of a spinning string. The complete one-loop expression was constructed in [101] in a similar way. It passed checks [102, 109] and was also explicitely constructed by a direct one-loop quantization of the finite-gap equations in [103]. An all-order $1/g$ expansion for the coefficients $c_{r,s}(g)$, which turns out to be asymptotic, was proposed in [43], by exploiting the crossing symmetry equation, argued in [104] to constrain the functional dependence of the dressing phase.

An exact representation for the dressing-phase coefficients, which 'resums' the asymptotic series (2.4.6), was proposed in [44]. It reads explicitely as

$$c_{r,s}(g) = 2(-1)^n(r-1)(s-1)\int_0^\infty \frac{dt}{t}\,\frac{J_{r-1}(2gt)J_{s-1}(2gt)}{\mathrm{e}^t - 1}\,, \tag{2.4.7}$$

still with $r \geqslant 2$, $s = r+1+2n$ and $n \in \mathbb{N}$. Here, $J_k(t)$ is the k-th Bessel function of the first kind. The latter is a holomorphic function of t, with small t behavior given by $J_k(t) \propto t^k$. It follows immediately that the coefficient $c_{r,s}(g)$ scales at weak coupling as

$$c_{r,s}(g) \sim g^{r+s-2}\,. \tag{2.4.8}$$

Moreover, due to the fact that the Bessel functions have a definite parity, the weak coupling expansion of the coefficients $c_{r,s}(g)$ runs in g^2, as expected from the gauge theory point of view. Combining the weak coupling scaling (2.4.8), the overall factor g^{r+s-2} in Eq. (2.4.4) and the fact that the charges are of order $O(g^0)$, we conclude that the dressing phase starts at four-loop. Namely, we find

$$\theta(u_k, u_j) = 4\zeta_3 g^6\big(q_2(u_k)q_3(u_j) - q_3(u_k)q_2(u_j)\big) + \dots\,, \tag{2.4.9}$$

which is suppressed by g^6 as compared to the $O(g^0)$ contribution of the one-loop Bethe ansatz equations - and so corresponds effectively to a four-loop correction in the gauge theory perturbative expansion. The result $c_{2,3}(g) = 4\zeta_3 g^3 + \dots$ was verified against a direct four-loop gauge theory calculation in [105]. An alternative (integral) representation for the dressing phase was found in [106] and used in [107, 108] to prove that it is the minimal solution to the crossing symmetry equation [104].

[23] The overall factor g^{r+s-2}, that appears in front of the expansion coefficients $c_{r,s}(g)$ in Eq. (2.4.4), may look suspicious at first sight, since it seems to indicate that the strong coupling of $\theta(u_k, u_j)$ simply does not exist, if $c_{r,s}(g) \sim g\,, \forall (r,s)$. However, string semiclassical states are described by a distribution of large Bethe roots $\{u_k\}$, scaling as $u_k \sim g$, $g \to \infty$. It is therefore convenient to first rescale the rapidities as $u_k \propto g\,\hat{u}_k$ before to take the strong coupling limit. After this rescaling has been done, the charge $q_r(u)$ acquires an overall factor g^{1-r}, which combines with g^{1-s} coming from $q_{s-1}(u)$ to cancel the factor g^{r+s-2} in front of $c_{r,s}(g)$.

All-loop Scaling Dimension

The contribution to the all-loop anomalous dimension of an individual magnon, carrying a rapidity u, is directly related to the conserved charge $q_2(u)$, as in the one-loop approximation, and reads explicitely as

$$\delta\Delta(u) = 2g^2 q_2(u) = 2ig^2 \left(\frac{1}{x^+(u)} - \frac{1}{x^-(u)} \right) . \tag{2.4.10}$$

For a state formed of N magnons, with a set of rapidities $\{u_k\}$, the anomalous dimension is simply given by the sums of the individual contributions, that is

$$\delta\Delta = 2g^2 \sum_{k=1}^{N} q_2(u_k) . \tag{2.4.11}$$

The spectrum of all-loop anomalous dimensions, for twist L and spin N Wilson operators, is then obtained by plugging into the relation (2.4.11) the Bethe roots $\{u_k\}$ that are solutions to the all-loop asymptotic Bethe ansatz equations (2.4.2).

Introducing the momentum p defined as

$$p = \lim_{r\to 1} q_r(u) = -i \log \left(\frac{x^+(u)}{x^-(u)} \right) , \tag{2.4.12}$$

the scaling dimension of a conformal operator, carrying a Lorentz spin N, admits the representation

$$\Delta = L + N + \delta\Delta = L + \sum_{k=1}^{N} \sqrt{1 + 16g^2 \sin^2 \left(\frac{p_k}{2} \right)} . \tag{2.4.13}$$

Here, the twist L measures the all-loop scaling dimension of the ferromagnetic vacuum, built out of L scalar fields $\mathcal{Z}(0)$ and no light-cone derivatives acting on. Indeed, this state belongs to a short multiplet of the superconformal algebra $\mathfrak{psu}(2,2|4)$ and its scaling dimension is protected from radiative corrections to all-loop (see [38] for instance). From the relation (2.4.13), we read the all-loop magnon dispersion relation as [39]

$$E = \sqrt{1 + 16g^2 \sin^2 \left(\frac{p}{2} \right)} . \tag{2.4.14}$$

Here, the periodicity in p reflects the discrete nature of the spin chain. Its all-loop dependence was derived in [42] from the residual symmetry algebra of the $\mathcal{N} = 4$ SYM theory, in the limit of infinite spin-chain length ($L \to \infty$).

At strong coupling, we may discriminate between three different regimes. Rescaling the momentum $p \to p/2g$ and expanding in powers of $1/g$, we find the relativistic (massive) dispersion relation

$$E = \sqrt{1 + p^2} . \tag{2.4.15}$$

It corresponds to the regime of perturbative excitations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring σ-model, quantized in the light-cone gauge [13, 19]. On the other hand, had we kept p fixed and taken the strong coupling limit of (2.4.14), we would enter a regime of solitonic excitations with dispersion relation

$$E = 4g \left| \sin^2 \left(\frac{p}{2} \right) \right| . \tag{2.4.16}$$

Semiclassical string solution satisfying the dispersion relation above have been constructed in [109] (see also [19] for the light-cone gauge construction), and called giant magnon. It is a rather peculiar property of type IIB superstring theory on $AdS_5 \times S^5$ that the fundamental (asymptotic) particles interpolate between (perturbative) plane-wave solutions and (non-linear) solitonic solutions of the equations of motion.[24] The third regime, the so-called near-flat space regime, lies in between the two previous ones and it is characterized by $p \sim 1/g^{1/2}$ and $E \sim g^{1/2}$. It was introduced in [110] and further discussed in [67]. Note finally that, as for the gauge theory, the all-loop dispersion relation (2.4.14) can be understood from the residual symmetry algebra of the light-cone superstring theory [109, 111, 19].

Asymptoticity and Wrapping Effect

The all-loop asymptotic Bethe ansatz equations (2.4.2) can be written equivalently as

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L \left(\frac{1 + g^2/(x_k^-)^2}{1 + g^2/(x_k^+)^2}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\right)^2 \exp 2i\theta(u_j, u_k)\,. \tag{2.4.17}$$

It simply follows from substituting into Eq. (2.4.2) the relations

$$\frac{x_k^+}{x_k^-} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \left(\frac{1 + g^2/(x_k^-)^2}{1 + g^2/(x_k^+)^2}\right), \qquad \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} = \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\right), \tag{2.4.18}$$

that are easily derived from the identity

$$x(u) + \frac{g^2}{x(u)} = u\,. \tag{2.4.19}$$

Taking into account that the dressing phase vanishes up to three-loop at weak coupling, one easily verify that the equations (2.4.17) reduce to the Bethe ansatz equations of the $XXX_{-1/2}$ Heisenberg spin chain (2.3.1) when $g \to 0$ (one-loop approximation). Beyond one-loop order, the set of all-loop asymptotic Bethe ansatz equations (2.4.17) can be solved at weak coupling by assuming that the Bethe roots $\{u_k\}$ admit and expansion in powers of g^2,

$$u_k = u_k(g) = u_k^{(0)} + g^2 u_k^{(1)} + \ldots\,, \tag{2.4.20}$$

where $\{u_k^{(0)}\}$ is solution to the $XXX_{-1/2}$ Bethe ansatz equations (2.3.1). In principle, starting from a particular solution $\{u_k^{(0)}\}$, one can solve iteratively the equations (2.4.17) for $\{u_k^{(n)}\}$ with n arbitrarily large. Plugging the obtained expansion (2.4.20) for $\{u_k\}$ in the formula (2.4.10) and expanding at weak coupling, one obtains the perturbative expansion of the anomalous dimension up to order $O(g^{2(n+1)})$.

[24]This situation turns out to be possible due to the noncompatibility of the light-cone gauge fixing, that removes unphysical excitations from the spectrum of the theory, with the relativistic invariance (see [19] and references therein).

More generally, one could expect to resum the expansion (2.4.20) and/or directly solve Eqs. (2.4.17) for the exact distribution of roots $\{u_k(g)\}$ at arbitrary coupling g. However, even if it is possible, it would not mean that the corresponding prediction for the all-loop scaling dimension of a given twist L operator is exact. The reason is that the all-loop Bethe ansatz equations (2.4.17) are only asymptotic. Namely, they diagonalize to all-loop the dilatation operator, in the $\mathfrak{sl}(2)$ sector of Wilson operators carrying an infinitely large twist L only. It turns out, indeed, that the all-loop Bethe ansatz equations (2.4.17) do not incorporate correctly contributions to the dilatation operator that wrap around the (single-trace) operator [39, 112, 41]. In spin-chain language, it means that as soon as the range of the interaction between magnons becomes comparable with the spin-chain length L, the predictions based on (2.4.17) can no longer be trusted. In perturbative gauge theory, and in the planar limit, the range of the interaction increases as the loop order. Therefore, for a twist L operator, the expansion (2.4.20) obtained from solving the equations (2.4.17) is reliable up to order $O(g^{2(L-2)})$ only. It follows that the anomalous dimension of a twist L operator is correctly determined by the all-loop Bethe ansatz equations (2.4.17) up to order $O(g^{2(L-1)})$.[25]

Fortunately, we are interested in the large N limit of the all-loop equations (2.4.17), from which we would like to extract the cusp anomalous dimension by computing the minimal anomalous dimension. As said before, in this regime, the minimal anomalous dimension scales logarithmically with N and is independent on the twist L. This universality of the large spin limit allows us to consider, as a starting point, a Wilson operator carrying an arbitrarily large twist. It follows that the cusp anomalous dimension can be computed to any desired order in the weak coupling expansion, and, eventually, to any value of g after resummation. We note also that a similar conclusion applies for the scaling function obtained in the generalized scaling limit $N, L \sim \infty$ with $j = L/\log N$ kept fixed. In that case, one explicitely assumes L to be large such that the validity of the all-loop Bethe ansatz prediction is guaranteed.

We conclude that the all-loop Bethe ansatz equations predict the cusp anomalous dimension and scaling function for an arbitrary value of the coupling constant. As in the one-loop case considered previously, these two observables can be obtained by solving the integral equations for the density distributions of magnon and hole rapidities. These equations were derived from (2.4.17) in [63, 44, 47]. In the following two chapters, we will analyze them at both weak and strong coupling.

[25] For a very short operator ($L = 2, 3$), it is possible, thanks to the superconformal symmetry, to find a representative of the operator, in the same supermultiplet, which is longer enough for the all-loop Bethe ansatz equations to be valid up to three loops at least. Since operators in same supermultiplet have same anomalous dimension, the all-loop Bethe ansatz equations are valid to at least three-loop order for any operator [41].

Chapter 3

BES Equation

In this chapter, we will analyse the all-loop integral equation that controls the logarithmic scaling of high-spin anomalous dimensions. This equation was proposed by Beisert, Eden and Staudacher in [44, 63], and derived from the all-loop asymptotic Bethe ansatz equations, discussed before. Solving the BES equation determines the cusp anomalous dimension for an arbitrary value of the coupling constant. At weak coupling, the BES equation was solved perturbatively as an expansion in g^2 [44], and the cusp anomalous dimension was found as

$$\Gamma_{\rm cusp}(g) = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - \left(\frac{292\pi^6}{315} + 32\zeta_3^2\right)g^8 + \dots\, . \tag{3.0.1}$$

It agrees with explicit one-loop [51], two-loop [49, 55], three-loop [56, 57] and four-loop [58] $\mathcal{N} = 4$ SYM computations. A numerical solution to the BES equation was proposed in [64]. It demonstrated that the cusp anomalous dimension is a smooth function of the coupling constant, interpolating between $\Gamma_{\rm cusp}(g) \sim 4g^2$ at weak coupling and $\Gamma_{\rm cusp}(g) \sim 2g$ at strong coupling. The latter behavior perfectly matches the expected string-theory result, obtained either from the classical energy of a rotating string carrying a large angular momentum [14], or from the area scaling of a cusped extremal suface [113, 114]. The analysis of [64] allowed furthermore an estimate of the strong coupling expansion of the cusp anomalous dimension, in agreement with higher-loop string theory prediction [15, 115, 60, 116]. Analytically, the strong-coupling solution was first found to leading order [65, 66, 67, 68], and then in the form of a perturbative expansion in $1/g$ [69, 70]. The outcome of this analysis is that the strong coupling expansion of the cusp anomalous dimension reads

$$\Gamma_{\rm cusp}(g) = 2g - \frac{3\log 2}{2\pi} - \frac{\mathrm{K}}{8\pi^2 g^2} - \dots\, , \tag{3.0.2}$$

with K the Catalan's constant. The expansion above perfectly agrees with the classical [14, 113, 114], one-loop [15, 115] and two-loop [60, 116] string-theory computations. Altogether, the expansions (3.0.1), (3.0.2) and their numerical interpolation provide a quite remarkable verification of the AdS/CFT correspondence. Let us mention also, as an additionnal check of the consistency of the Bethe ansatz approach, that both the weak-coupling and strong-coupling results, Eqs. (3.0.1) and (3.0.2), have been obtained from analysis based on the (quantum string) Bethe ansatz equations [137, 138] and/or the all-loop Baxter equation [91, 90].

Our goal in this chapter is to derive the expansion (3.0.2) directly from the BES equation. Our analysis will go along the lines of [71], based in turn on [69, 70]. In addition, we would like also to address the question of the summability of the series (3.0.2). Indeed, it was found in [69], that the coefficients of the expansion (3.0.2) have the same sign and grow factorially at higher orders. As a result, the $1/g$ expansion of $\Gamma_{\rm cusp}(g)$ is merely asymptotic (has zero-radius of convergence) and is not Borel summable. The Borel summation is a procedure that allows one to improve the convergency of certain series, and to construct a function effectively asymptotic to this series. When a series is not Borel summable, it means that the procedure is ambiguous. This ambiguity is usually associated with exponentially small contribution for $g \to \infty$. This suggests that the cusp anomalous dimension receives non-perturbative corrections at strong coupling

$$\Gamma_{\rm cusp}(g) = \sum_{k=-1}^{\infty} c_k/g^k - \frac{\sigma}{4\sqrt{2}}\, m_{\rm cusp}^2 + o(m_{\rm cusp}^2)\,. \tag{3.0.3}$$

The dependence of the non-perturbative scale $m_{\rm cusp}^2$ on the coupling constant $m_{\rm cusp} \propto g^{1/4}\, {\rm e}^{-\pi g}$ follows, through a standard analysis [117, 118, 119], from the large order behavior of the expansion coefficients, $c_k \sim \Gamma(k+\frac{1}{2})$ for $k \to \infty$ [69]. The value of the coefficient σ in (3.0.3) depends on the regularization of the Borel singularities in the perturbative $1/g$ expansion,[1] while the numerical prefactor was introduced for the later convenience. The relation (3.0.3) sheds light on the properties of $\Gamma_{\rm cusp}(g)$ in the transition region $g \sim 1$. Going from $g \gg 1$ to $g \sim 1$, we find that $m_{\rm cusp}^2$ increases and, as a consequence, non-perturbative $O(m_{\rm cusp}^2)$ corrections to $\Gamma_{\rm cusp}(g)$ become comparable with perturbative $O(1/g)$ corrections.

The discussion of the non-perturbative corrections to the cusp anomalous dimension is not purely academic. In the next chapter, we will show indeed that the non-perturbative scale $m_{\rm cusp}$ admits a string-theory interpretation. It will acquire the meaning of the mass gap m of the non-linear O(6) sigma model – embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ world-sheet σ-model [46, 69]. However, from the Bethe ansatz point of view, to understand the matching $m_{\rm cusp} = m$, we will need to consider a generalization of the BES equation, proposed by Freyhult, Rej and Staudacher [47]. The latter equation incorporates the subleading corrections to the logarithmic scaling, that are enhanced by powers of the twist L and resummed into the scaling function $\epsilon(g, j)$. We recall that the scaling function enters the formula for the minimal anomalous dimension as

$$\delta\Delta_{\rm min} - 2\Gamma_{\rm cusp}(g)\log N = \epsilon(g, j)\log N + \dots\,, \tag{3.0.4}$$

where the dots stand for contributions of order $\sim \log^0 N$, suppressed in the limit $N, L \to \infty$, keeping fixed the scaling variable $j = L/\log N$. Computing the scaling function $\epsilon(g, j)$ at strong coupling, we will uncover a non-perturbative regime controlled by the O(6) sigma model [46, 73]. It will then be quite easy to read the expression for the mass gap m off the behavior of the scaling function ($j \ll m$)

$$\epsilon(g, j) + j \sim mj + \dots\,, \tag{3.0.5}$$

and to conclude that $m_{\rm cusp} = m$ to any order in $1/g$ at strong coupling. At the end of this chapter, we will give a heuristic argument explaining the agreement between the two non-perturbative scales $m_{\rm cusp}$ and m, apparently unrelated.

[1] That is, it depends on the prescription used to sum the perturbative series.

The present chapter is organized as follows. First we formulate the BES equation, recast it in a more suitable form and briefly sketch the analysis of [69] (see also [70]) to outline the construction of the strong-coupling solution. Next following [71], we start the general analysis that allows us to incorporate the non-perturbative corrections. It consists of finding a convenient representation for the solution and to identify the source of non-perturbative contributions. We will construct, in parallel, the perturbative and (first) non-perturbative parts of the solution, and extract their corresponding contributions to the cusp anomalous dimension. We will argue that the non-perturbative corrections play a crucial role in the transition from the strong to weak coupling regime. To describe the transition, we will present a simplified model for the cusp anomalous dimension. This model correctly captures the properties of $\Gamma_{\text{cusp}}(g)$ at strong coupling and, most importantly, it allows us to obtain a closed expression for the cusp anomalous dimension which turns out to be remarkably close to the exact value of $\Gamma_{\text{cusp}}(g)$, throughout the entire range of the coupling constant. We will furthermore introduce, by hand, into our analysis the expression for the mass scale m. With its help we will derive one of the main results of this chapter, namely the identity $m_{\text{cusp}} = m$ between the scale of the leading non-perturbative correction to the cusp anomalous dimension and the mass gap of the O(6) model.

3.1 BES Equation

3.1.1 Original Formulation

The distribution of Bethe roots describing the minimal (or ground-state) trajectory of the spectrum of anomalous dimensions can be found in the large-spin (continuum) limit as a solution to a linear integral equation. This equation was derived in [44] directly from the all-loop asymptotic Bethe ansatz and is known as Beisert-Eden-Staudacher (BES) equation.[2] The BES equation is written in terms of a density distribution of fluctuations $\sigma(u)$ [63]. This latter is related to the density distribution of Bethe roots $\rho(u)$ as

$$\rho(u) = \rho_0(u) - 8\log(2c)\,\sigma(u)\,, \tag{3.1.1}$$

supported on the interval $u^2 \leqslant c^2$, with c large in the high-spin limit. Here, the first term in the right-hand side is the one-loop twist-two distribution, discussed before, while the second term describes higher-loop $O(g^2)$ contributions, enhanced by an overall logarithmic factor. When u is kept fixed and c is large, we found in the previous section that $\rho_0(u) = 2\log(2c)/\pi$, while when $u \gg 1$ it goes like $\rho_0(u) = -\log u^2/\pi + 2\log(2c)/\pi$ and matches the twist-independent density distribution $\rho_0(\bar{u})$ at small rescaled rapidities $\bar{u} = u/c \ll 1$. We stress that this matching is left unchanged at higher-loop because $\sigma(u) \sim 1/u^2$ for $u \gg 1$ [63, 44]. It means that the large-rapidity regime $\bar{u} = u/c = O(1)$ is unaffected, to leading order when $c \gg 1$, by the higher-loop contributions and so remains described there by $\rho_0(\bar{u})$. It follows that quantities controlled by the regime $\bar{u} = O(1)$ keep the same one-loop dependence at large c. In particular, the normalization condition

$$\int_{-c}^{c} du\,\rho(u) = N + \ldots\,, \tag{3.1.2}$$

[2]An alternative approach can be found in [91] whose starting point is the the all-loop asymptotic Baxter equation [90].

still relates c and N by the one-loop relation $2c = N + \ldots$, where dots stand for subleading higher-loop dependent corrections that scale as $\log N$. However, the anomalous dimension receives dominant contribution from u kept fixed, when c is large. Therefore, the cusp anomalous dimension, which controls the large spin limit of the minimal anomalous dimension $\delta\Delta(g) = 2\Gamma_{\rm cusp}(g)\log N$, will receive as expected radiative corrections coming from $\sigma(u)$. We recall also that all distributions in Eq. (3.1.1) are even functions of u, as required for the minimal anomalous dimension.

The BES equation, that determines the distribution of fluctuations $\sigma(u)$ to all-loop, reads

$$\widehat{\sigma}(t) = \frac{t}{{\rm e}^t - 1}\left[K(2gt, 0) - 4\int_0^\infty dt' K(2gt, 2gt')\widehat{\sigma}(t')\right], \tag{3.1.3}$$

where $\widehat{\sigma}(t)$ is related to the Fourier transform of $\sigma(u)$,

$$\widehat{\sigma}(t) = {\rm e}^{-t/2}\int_{-\infty}^{\infty} du\, {\rm e}^{iut}\,\sigma(u)\,, \tag{3.1.4}$$

for $t \geqslant 0$. The kernel $K(t, t')$, in Eq. (3.1.3), can be written in terms of Bessel functions $J_n(t)$ as

$$K(t, t') = g^2 K_+(t, t') + g^2 K_-(t, t') + 8g^4\int_0^\infty \frac{dt''\, t''}{{\rm e}^{t''} - 1} K_-(t, 2gt'')K_+(2gt'', t')\,, \tag{3.1.5}$$

with the parity even/odd kernels $K_\pm(-t, t') = K_\pm(t, -t') = \pm K_\pm(t, t')$ given by

$$\begin{aligned} K_+(t, t') &= \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geq 1}(2n-1)J_{2n-1}(t)J_{2n-1}(t')\,, \\ K_-(t, t') &= \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geq 1}(2n)J_{2n}(t)J_{2n}(t')\,. \end{aligned} \tag{3.1.6}$$

Finally, the solution to the BES equation determines the all-loop cusp anomalous dimension thanks to the relation

$$\Gamma_{\rm cusp}(g) = 8\widehat{\sigma}(0)\,. \tag{3.1.7}$$

In its original form, Eq. (3.1.3), the BES equation can be easily solved at weak coupling as an expansion in g^2. To this end, it is sufficient to observe that the integral term in Eq. (3.1.3) is subleading compared to the inhomogeneous term, permitting the solution to be constructed by iteration. That procedure provides a convergent expansion of the solution $\widehat{\sigma}(t)$ around $g = 0$, uniformely in the variable t. The first few terms of the weak coupling expansion of the cusp anomalous dimension can be easily obtained and read

$$\Gamma_{\rm cusp}(g) = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - \left(\frac{292\pi^6}{315} + 32\zeta_3^2\right)g^8 + \ldots\,. \tag{3.1.8}$$

The series above is convergent with a numerical estimation of the radius of convergence found to be $g_c^2 = 1/16$ [44].

3.1.2 Alternative Formulation

The strong coupling analysis of the equation (3.1.3) is more difficult, and it is convenient to first look for some simplifications. A possibility is to introduce two even/odd functions $\gamma_\pm(-t) = \pm\gamma_\pm(t)$ defined by

$$\widehat{\sigma}(t) = \frac{g}{2}\,\frac{\gamma_+(2gt) + \gamma_-(2gt)}{\mathrm{e}^t - 1}\,. \tag{3.1.9}$$

From the analytical property of the kernel and of the inhomogeneous term in (3.1.3), we conclude that both $\gamma_+(t)$ and $\gamma_-(t)$ extend to holomorphic functions in the full complex plane. Then, following [64, 65, 66], we may expand $\gamma_\pm(t)$ into Bessel-function Neumann series

$$\begin{aligned} \gamma_+(t) &= 2\textstyle\sum_{n\geq 1}(2n)\gamma_{2n}(g)J_{2n}(t)\,, \\ \gamma_-(t) &= 2\textstyle\sum_{n\geq 1}(2n-1)\gamma_{2n-1}(g)J_{2n-1}(t)\,. \end{aligned} \tag{3.1.10}$$

Here the dependence on the coupling constant is absorbed into the coefficients $\gamma_n(g)$. The expansions in (3.1.10) are convergent for any complex value of t, thanks to holomorphicity of $\gamma_\pm(t)$. Concerning the cusp anomalous dimension, it follows from Eqs. (3.1.7), (3.1.9), (3.1.10) and the asymptotics at small t of the Bessel functions, $J_n(t) \propto t^n$, that it can be found as[3]

$$\Gamma_{\mathrm{cusp}}(g) = 8g^2\gamma_1(g) = 8g^2 \lim_{t\to 0}\gamma_-(t)/t\,. \tag{3.1.12}$$

The coefficients $\gamma_n(g)$ are completely fixed by the BES equation. Indeed, after a bit of algebra, one can show that the equation (3.1.3) is equivalent to the infinite set of equations

$$\begin{aligned} 0 &= \gamma_{2n}(g) - \int_0^\infty \frac{dt}{t}\,\frac{\gamma_-(t) - \gamma_+(t)}{\mathrm{e}^{t/2g} - 1}J_{2n}(t)\,, \\ \frac{\delta_{n,1}}{2} &= \gamma_{2n-1}(g) + \int_0^\infty \frac{dt}{t}\,\frac{\gamma_-(t) + \gamma_+(t)}{\mathrm{e}^{t/2g} - 1}J_{2n-1}(t)\,, \end{aligned} \tag{3.1.13}$$

with $n \geq 1$. Replacing, above, the functions $\gamma_\pm(t)$ by their Neumann series (3.1.10) provides a closed system of equations, determining uniquely the coefficients $\gamma_n(g)$. The system of equations (3.1.13) can be solved easily at weak coupling and it perfectly reproduces the expansion (3.1.8) of the cusp anomalous dimension. We will see that it can be solved also in the strong coupling limit, and even beyond after a last simplification. In the form of Eq. (3.1.13), the BES equation was solved numerically in [64], over a large range of values for g, after truncating the sums over the Bessel functions in (3.1.10). It was found, in this way, that the cusp anomalous dimension is a smooth function of the coupling constant, scaling at strong coupling as[4]

$$\Gamma_{\mathrm{cusp}}(g) = 2g + \ldots\,, \tag{3.1.14}$$

[3]More generally, it is interesting to note that the spin-chain higher conserved charges $q_{2n}(g)$ (evaluated for the large N ground state) are directly related to the coefficients $\gamma_{2n-1}(g)$ as

$$q_{2n}(g) = 8(2ig)^{2-2n}\gamma_{2n-1}(g)\log N\,. \tag{3.1.11}$$

Equation (3.1.12) is the particular case $\delta\Delta(g) \equiv 2g^2 q_2(g) = 2\Gamma_{\mathrm{cusp}}(g)\log N = 16g^2\gamma_1(g)\log N$. However, since odd higher conserved charges $q_{2n+1}(g)$ all vanish for the ground state, there seems to be no simple spin-chain interpretation for the coefficients $\gamma_{2n}(g)$ in (3.1.10).

[4]Subleading corrections were also estimated in [64] and found to agree with string theory results.

in agreement with the string theory semiclassical result [14]. Analytically, the leading-order strong-coupling solution was constructed in [66], starting from the limit $g \to \infty$ of Eq. (3.1.13) given by

$$0 = 4g \int_0^\infty \frac{dt}{t^2} \Big[\gamma_-(t) - \gamma_+(t) \Big] J_{2n}(t) \,, \qquad (3.1.15)$$
$$\delta_{n,1} = 4g \int_0^\infty \frac{dt}{t^2} \Big[\gamma_-(t) + \gamma_+(t) \Big] J_{2n-1}(t) \,.$$

The strong-coupling solution of [66] can be written in closed form as

$$\gamma_+(t) + i\gamma_-(t) = \frac{it}{2\sqrt{2}\pi g} \int_{-1}^{1} du \left(\frac{1+u}{1-u} \right)^{1/4} \mathrm{e}^{-iut} + \dots \,. \qquad (3.1.16)$$

Here, it is assumed that t is kept fixed when $g \to \infty$. The solution (3.1.16) agrees with the solution of [67, 68], obtained by other means, and it predicts correctly the strong coupling value of the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) = 2g + \dots \,. \qquad (3.1.17)$$

There is however a subtlety, which becomes more and more problematic when the question of the subleading corrections is addressed. Namely, the solution (3.1.16) is not uniquely specified by the system of equation (3.1.15). There are indeed zero-mode solutions that can be added to Eq. (3.1.16) with arbitrary coefficients. One may nevertheless rely on a comparison with the numerics to rule out the contribution of the zero-mode solutions, in the strict strong coupling limit, supporting the validity of both Eqs. (3.1.16) and (3.1.17). At higher loops the expansion of the kernel in Eq. (3.1.13) becomes more and more singular and the question of the zero-mode ambiguity is enhanced. Indeed, it turns out [69, 70] that the number of 'zero-mode' contributions[5] increases with the powers of $1/g$. The strong coupling solution will therefore takes the form of an expansion over an infinite number of functions with unknown coefficients. Once the basis of functions has been correctly identified, the arbitrary coefficients was shown in [69, 70] to be fixed, order by order in $1/g$, by resumming the solution in the double scaling limit $t, g \to \infty$, t/g fixed, and by imposing that it satisfies correct analyticity conditions. This strategy was developed in [69], with some help from numerics to guess the correct procedure, and was proved and improved analytically in [70] (see also [71] for incorporation of non-perturbative contributions). These analysis determine the strong coupling expansion of the cusp anomalous dimension, iteratively order by order in $1/g$. Up to two-loop, it reads

$$\Gamma_{\text{cusp}}(g) = 2g - \frac{3\log 2}{2\pi} - \frac{\mathrm{K}}{8\pi^2 g} - \dots \,, \qquad (3.1.18)$$

and it agrees with the string theory result [14, 15, 60].

To realize the program outlined above and to circumvent the difficulty of the direct strong coupling expansion of the BES equation, it was proposed in [69] to first recast the system of

[5]Strictly speaking, the zero modes of [69] are not exactly solutions to the homogeneous equation, because they are too singular at large t to be integrable when the kernel is expanded in powers of $1/g$. However, they provide a correct basis of expansion for the solution, with coefficients to be fixed in a next step.

equations (3.1.15) in the form

$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(t)}{1-\mathrm{e}^{-t/2g}} - \frac{\gamma_-(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n}(t) = 0\,, \qquad (3.1.19)$$
$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\gamma_+(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n-1}(t) = \frac{1}{2}\delta_{n,1}\,,$$

with $n \geq 1$. To prove that this system is equivalent to (3.1.15), it is sufficient to make use of the orthogonality property of the Bessel functions as ($n \geqslant 1$)

$$\int_0^\infty \frac{dt}{t}\, J_{2n-1}(t) J_{2m-1}(t) = \frac{\delta_{n,m}}{2(2n-1)}\,, \qquad \int_0^\infty \frac{dt}{t}\, J_{2n}(t) J_{2m}(t) = \frac{\delta_{n,m}}{2(2n)}\,. \qquad (3.1.20)$$

Then, assuming that the sums over the Bessel functions can be interchanged with the integration, we may write

$$\gamma_{2n-1}(g) = \int_0^\infty \frac{dt}{t}\,\gamma_-(t) J_{2n-1}(t)\,, \qquad \gamma_{2n}(g) = \int_0^\infty \frac{dt}{t}\,\gamma_+(t) J_{2n}(t)\,, \qquad (3.1.21)$$

converting Eq. (3.1.13) into Eq. (3.1.19). The possibility to interchange summation and integration relies on the large-order behavior of the coefficients $\gamma_n(g)$ and the corresponding large-t behavior of the partial sums over the Bessel functions. The BES equation in the form (3.1.13) predicts that the coefficients $\gamma_n(g)$ are suppressed at large n, permitting the use of Eq. (3.1.21). Note, however, that the condition of holomorphicity for $\gamma_\pm(t)$ is not sufficient to prove Eq. (3.1.21) and, therefore, the system of equations (3.1.19) has more solutions than the BES equation. But, a holomorphic solution to Eq. (3.1.19) respecting Eq. (3.1.21) should be equivalent to the BES one.

The system of equations (3.1.19) has a remarkable property that considerably simplifies its analysis. Introducing two even/odd functions $\Gamma_\pm(t)$ by

$$\Gamma_\pm(t) = \gamma_\pm(t) \mp \coth\left(\frac{t}{4g}\right)\gamma_\mp(t)\,, \qquad (3.1.22)$$

or conversely

$$2\gamma_\pm(t) = \left(1 - \frac{1}{\cosh\left(t/2g\right)}\right)\Gamma_\pm(t) \pm \tanh\left(\frac{t}{2g}\right)\Gamma_\mp(t)\,, \qquad (3.1.23)$$

the system of equations (3.1.19) takes the g-independent form ($n \geqslant 1$)

$$\int_0^\infty \frac{dt}{t}\left[\Gamma_-(t) + (-1)^n \Gamma_+(t)\right] J_n(t) = \delta_{n,1}\,. \qquad (3.1.24)$$

This system has a large family of solutions. This freedom allows us to split the problem into two steps. First, find a general solution with a 'minimal' dependence on the coupling constant [6] and parameterized by an infinite number of arbitrary coefficients (zero-modes ambiguity). Next, impose to the general solution the analyticity conditions that single out the BES solution. These conditions can be cast into a set of equations for the unknown coefficients, to which we refer

[6] Here by 'minimal' we mean that the general solution takes into account some of the analyticity properties of the BES solution, as the fact that $\Gamma_\pm(t)$ has poles at some fixed positions depending on g. That permits to restrict a bit the generality of the solution and avoid to deal with unneccessary degrees of freedom.

as quantization conditions. In the next sections, we will develop this program, starting from the system of equations above. We will solve the quantization conditions in a double expansion in both the perturbative coupling $1/g$ and a non-perturbative scale $\Lambda^2 \propto g^{1/2}\exp(-2\pi g)$, whose existence is tied to the non-Borel summability of the perturbative expansion. Our analysis strictly follows the work [71].

3.2 General Solution

3.2.1 Integral Equation

We define, for later convenience, the function $\gamma(t)$ as

$$\gamma(t) = \gamma_+(t) + i\gamma_-(t)\,, \tag{3.2.1}$$

where $\gamma_\pm(t)$ are the real functions of t, with definite parity $\gamma_\pm(\pm t) = \pm\gamma_\pm(t)$, introduced before. For arbitrary coupling, the functions $\gamma_\pm(t)$ satisfy the (infinite) system of integral equations (3.1.19). These relations are equivalent to the BES equation provided that $\gamma_\pm(t)$ verify certain analyticity conditions, that we will discuss further below. The equations (3.1.19) can be significantly simplified with the help of the transformation[7] $\gamma(t) \to \Gamma(t)$

$$\Gamma(t) = \left(1 + i\coth\frac{t}{4g}\right)\gamma(t) \equiv \Gamma_+(t) + i\Gamma_-(t)\,. \tag{3.2.2}$$

We find from (3.1.12) and (3.2.2) the following representation for the cusp anomalous dimension

$$\Gamma_{\rm cusp}(g) = -2g\Gamma(0)\,. \tag{3.2.3}$$

It follows from (3.2.1) and (3.1.19) that $\Gamma_\pm(t)$ are real functions with a definite parity, $\Gamma_\pm(-t) = \pm\Gamma_\pm(t)$, satisfying the system of integral equations (3.1.24) or equivalently

$$\begin{aligned}\int_0^\infty dt\,\cos(ut)\Big[\Gamma_-(t) - \Gamma_+(t)\Big] &= 2\,,\\ \int_0^\infty dt\,\sin(ut)\Big[\Gamma_-(t) + \Gamma_+(t)\Big] &= 0\,,\end{aligned} \tag{3.2.4}$$

with u being an arbitrary real parameter such that $-1 \le u \le 1$. Since $\Gamma_\pm(t)$ take real values, we can rewrite these relations in a more compact form

$$\int_0^\infty dt\,\Big[\mathrm{e}^{iut}\,\Gamma_-(t) - \mathrm{e}^{-iut}\,\Gamma_+(t)\Big] = 2\,. \tag{3.2.5}$$

To recover (3.1.24), we replace in (3.2.4) trigonometric functions by their Bessel series expansions

$$\begin{aligned}\cos(ut) &= 2\sum_{n\ge1}(2n-1)\frac{\cos((2n-1)\varphi)}{\cos\varphi}\frac{J_{2n-1}(t)}{t}\,,\\ \sin(ut) &= 2\sum_{n\ge1}(2n)\frac{\sin(2n\varphi)}{\cos\varphi}\frac{J_{2n}(t)}{t}\,,\end{aligned} \tag{3.2.6}$$

[7] With a slight abuse of notations, we use here the same notation as for Euler gamma-function.

with $u = \sin\varphi$, and finally compare coefficients in front of $\cos((2n-1)\varphi)/\cos\varphi$ and $\sin(2n\varphi)/\cos\varphi$ in both sides of (3.2.4). It is important to stress that, doing this calculation, we interchanged the sum over n with the integral over t. This is only justified for φ real and, therefore, the relation (3.2.4) only holds for $-1 \le u \le 1$.

Comparing (3.2.5) and (3.1.19) we observe that the transformation $\gamma_\pm \to \Gamma_\pm$ eliminates the dependence of the integral kernel in the left-hand side of (3.2.5) on the coupling constant. One may then wonder where does the dependence of the functions $\Gamma_\pm(t)$ on the coupling constant come from? We will show in the next subsection that it is dictated by additional conditions imposed on analytical properties of solutions to the equation (3.2.5).

3.2.2 Analyticity Conditions

The integral equations (3.2.5) and (3.1.19) determine $\Gamma_\pm(t)$ and $\gamma_\pm(t)$, or equivalently the functions $\Gamma(t)$ and $\gamma(t)$, up to a contribution of zero modes. The latter satisfy the same integral equations (3.2.5) and (3.1.19) but without inhomogeneous term in the right-hand side. To fix the zero modes, we have to impose additional conditions on solutions to (3.2.5) and (3.1.19). The first requirement is that $\gamma(it)$ is an entire function of t. After rewriting the relation (3.2.2) as

$$\Gamma(it) = \gamma(it)\frac{\sin(\frac{t}{4g}+\frac{\pi}{4})}{\sin(\frac{t}{4g})\sin(\frac{\pi}{4})} = \gamma(it)\sqrt{2}\prod_{k=-\infty}^{\infty}\frac{t-4\pi g\big(k-\frac{1}{4}\big)}{t-4\pi g k}\,, \tag{3.2.7}$$

we conclude that $\Gamma(it)$ has an infinite number of zeros, $\Gamma(it_{\rm zeros}) = 0$, and poles, $\Gamma(it) \sim 1/(t - t_{\rm poles})$, on real $t-$axis located at

$$t_{\rm zeros} = 4\pi g\big(\ell - \tfrac{1}{4}\big)\,, \qquad\qquad t_{\rm poles} = 4\pi g \ell'\,, \tag{3.2.8}$$

where $\ell, \ell' \in \mathbb{Z}$ and $\ell' \neq 0$ so that $\Gamma(it)$ is regular at the origin (see Eq. (3.1.12)). Notice that $\Gamma(it)$ has an additional (infinite) set of zeros coming from the function $\gamma(it)$ but, in distinction with (3.2.8), their position is not fixed.

As said before, the analyticity property of $\gamma(it)$ is not sufficient to specify uniquely the solution. Indeed, to achieve the determination of $\gamma(it)$, one has to make sure that the coefficients $\gamma_n(g)$, appearing in the Neumann series (3.1.10), do satisfy the integral representation (3.1.21). That condition, which stems from the commutation between summation over the Bessel functions and integration over t, will be implemented by imposing that the Fourier transform of $\gamma(it)$ has support on the interval $[-1, 1]$. Taking into account the consequence of the latter property for the function $\Gamma(it)$, we will construct in the following the solution to the integral equation (3.2.4) which satisfies the relations (3.2.8).

3.2.3 Toy Model

To understand the relationship between analytical properties of $\Gamma(it)$ and properties of the cusp anomalous dimension, it is instructive to slightly simplify the problem and consider a 'toy' model in which the function $\Gamma(it)$ is replaced with $\Gamma^{(\rm toy)}(it)$.

We require that $\Gamma^{(\rm toy)}(it)$ satisfies the same integral equation (3.2.4) and define, following (3.2.3), the cusp anomalous dimension in the toy model as

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = -2g\Gamma^{(\rm toy)}(0)\,. \tag{3.2.9}$$

The only difference compared to $\Gamma(it)$ is that $\Gamma^{(\rm toy)}(it)$ has different analytical properties dictated by the relation

$$\Gamma^{(\rm toy)}(it) = \gamma^{(\rm toy)}(it)\frac{t+\pi g}{t}\,, \tag{3.2.10}$$

while $\gamma^{(\rm toy)}(it)$ has the same analytical properties as the function $\gamma(it)$.[8] This relation can be considered as a simplified version of (3.2.7). Indeed, it can be obtained from (3.2.7) if we retained in the product only one term with $k = 0$. As compared with (3.2.8), the function $\Gamma^{(\rm toy)}(it)$ does not have poles and it vanishes for $t = -\pi g$.

The main advantage of the toy model is that the expression for $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ can be found in a closed form for arbitrary value of the coupling constant (see Eq. (3.2.47) below). We will then compare it with the exact expression for $\Gamma_{\rm cusp}(g)$ and identify the difference between the two functions.

3.2.4 Exact Bounds and Uniqueness

Before we turn to finding the solution to (3.2.4), or equivalently to (3.1.19), let us demonstrate that this integral equation leads to non-trivial constraints for the cusp anomalous dimension valid for arbitrary coupling g.

Let us multiply both sides of the two relations in (3.1.19) by $2(2n-1)\gamma_{2n-1}(g)$ and $2(2n)\gamma_{2n}(g)$, respectively, and perform summation over $n \geq 1$. Then, we convert the sums into the functions $\gamma_\pm(t)$ using (3.1.10) and add the second relation to the first one to obtain

$$\gamma_1(g) = \int_0^\infty \frac{dt}{t}\frac{(\gamma_+(t))^2 + (\gamma_-(t))^2}{1-{\rm e}^{-t/(2g)}}\,. \tag{3.2.11}$$

Since $\gamma_\pm(t)$ are real functions of t and the denominator is positively definite for $0 \leq t < \infty$, this relation leads to the following inequality

$$\gamma_1(g) \geq \int_0^\infty \frac{dt}{t}(\gamma_-(t))^2 \geq 2\gamma_1^2(g) \geq 0\,. \tag{3.2.12}$$

Here we replaced the function $\gamma_-(t)$ by its Bessel series (3.1.10) and made use of the orthogonality condition for the Bessel functions with odd indices (3.1.20). We deduce from (3.2.12) that

$$0 \leq \gamma_1(g) \leq \frac{1}{2} \tag{3.2.13}$$

and, then, apply (3.1.12) to translate this inequality into the following relation for the cusp anomalous dimension

$$0 \leq \Gamma_{\rm cusp}(g) \leq 4g^2\,. \tag{3.2.14}$$

This relation should hold in planar $\mathcal{N} = 4$ SYM theory for arbitrary coupling g. Notice that the lower bound on the cusp anomalous dimension, $\Gamma_{\rm cusp}(g) \geq 0$, holds in any gauge theory [49]. It is the upper bound $\Gamma_{\rm cusp}(g) \leq 4g^2$ that is a distinguished feature of $\mathcal{N} = 4$ theory.

[8] Notice that the function $\gamma^{(\rm toy)}(t)$ does not satisfy the integral equation (3.1.19) anymore. Substitution of (3.2.10) into (3.2.5) yields an integral equation for $\gamma^{(\rm toy)}(t)$ which can be obtained from (3.1.19) by replacing $1/\left(1-{\rm e}^{-t/(2g)}\right) \to \frac{\pi g}{2t} + \frac{1}{2}$ and $1/\left({\rm e}^{t/(2g)}-1\right) \to \frac{\pi g}{2t} - \frac{1}{2}$ in the kernel in the left-hand side of (3.1.19).

Let us verify the validity of (3.2.14). At weak coupling $\Gamma_{\rm cusp}(g)$ admits a perturbative expansion in powers of g^2

$$\Gamma_{\rm cusp}(g) = 4g^2\left[1 - \frac{1}{3}\pi^2 g^2 + \frac{11}{45}\pi^4 g^4 - 2\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right)g^6 + \ldots\right], \qquad (3.2.15)$$

while at strong coupling it has the form

$$\Gamma_{\rm cusp}(g) = 2g\left[1 - \frac{3\log 2}{4\pi}g^{-1} - \frac{\rm K}{16\pi^2}g^{-2} - \left(\frac{3{\rm K}\log 2}{64\pi^3} + \frac{27\zeta_3}{2048\pi^3}\right)g^{-3} + \ldots\right], \qquad (3.2.16)$$

with K being the Catalan's constant. It is easy to see that the relations (3.2.15) and (3.2.16) are in an agreement with (3.2.14). For arbitrary g, we can verify the relation (3.2.14) by using the results for the cusp anomalous dimension obtained from the numerical solution of the BES equation [64, 69, 120]. The comparison is shown in Figure 3.1. We observe that the upper bound condition $\Gamma_{\rm cusp}(g)/(2g) \le 2g$ is indeed satisfied for arbitrary $g > 0$.

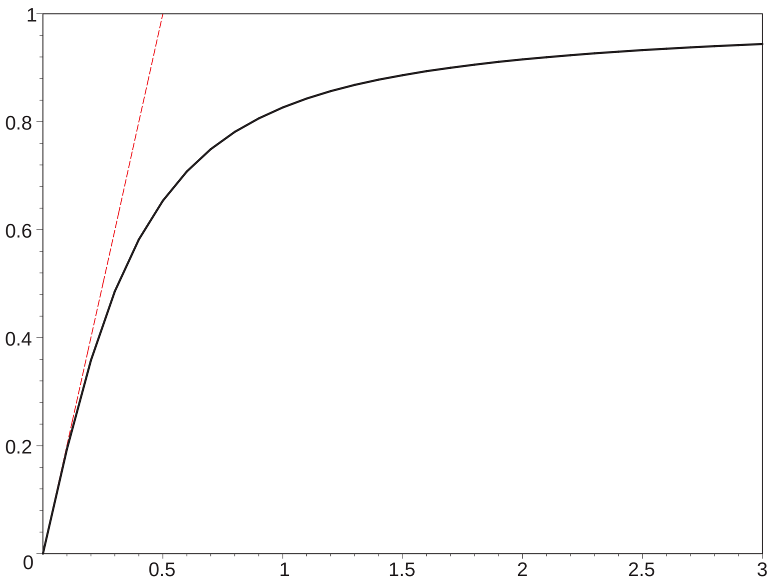

Figure 3.1: Dependence of the cusp anomalous dimension $\Gamma_{\rm cusp}(g)/(2g)$ on the coupling constant. Dashed line denotes the upper bound $2g$.

A similar argument proves the uniqueness of the solution to (3.1.19). As was already mentioned, solutions to (3.1.19) are defined modulo contribution of zero modes, $\gamma(t) \to \gamma(t) + \delta\gamma^{(\rm zero)}(t)$, with $\delta\gamma^{(\rm zero)}(t)$ being solution to the homogeneous equations. Going through the same steps that led us to (3.2.11), we obtain

$$0 = \int_0^\infty \frac{dt}{t}\frac{(\delta\gamma_+^{(\rm zero)}(t))^2 + (\delta\gamma_-^{(\rm zero)}(t))^2}{1 - {\rm e}^{-t/(2g)}}\,, \qquad (3.2.17)$$

where zero on the left-hand side is due to absence of the inhomogeneous term. Since the integrand is a positively definite function, we immediately deduce that $\delta\gamma^{(\text{zero})}(t) = 0$ and, therefore, the solution for $\gamma(t)$ is unique.[9]

3.2.5 Riemann-Hilbert Problem

Let us now construct the exact solution to the integral equations (3.2.5) and (3.1.19). To this end, it is convenient to Fourier transform the functions (3.2.1) and (3.2.2)

$$\widetilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \,\mathrm{e}^{ikt}\,\Gamma(t)\,, \qquad \widetilde{\gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \,\mathrm{e}^{ikt}\,\gamma(t)\,. \tag{3.2.18}$$

According to (3.2.1) and (3.1.10), the function $\gamma(t)$ is given by the Neumann series over Bessel functions. Then, we perform the Fourier transform on both sides of (3.1.10) and use the well-known fact that the Fourier transform of the Bessel function $J_n(t)$ vanishes for $k^2 > 1$ to deduce that the same is true for $\gamma(t)$ leading to[10]

$$\widetilde{\gamma}(k) = 0\,, \qquad \text{for } k^2 > 1. \tag{3.2.19}$$

This implies that the Fourier integral for $\gamma(t)$ only involves modes with $-1 \le k \le 1$ and, therefore, the function $\gamma(t)$ behaves at large (complex) t as

$$\gamma(t) \sim \mathrm{e}^{|t|}\,, \qquad \text{for } |t| \to \infty. \tag{3.2.20}$$

Let us now examine the function $\widetilde{\Gamma}(k)$. We find from (3.2.18) and (3.2.7) that $\widetilde{\Gamma}(k)$ admits the following representation

$$\widetilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \,\mathrm{e}^{ikt}\, \frac{\sinh(\frac{t}{4g} + i\frac{\pi}{4})}{\sinh(\frac{t}{4g})\sin(\frac{\pi}{4})}\gamma(t)\,. \tag{3.2.21}$$

Here the integrand has poles along the imaginary axis at $t = 4\pi i g n$ (with $n = \pm1, \pm2, \ldots$).[11]

It is suggestive to evaluate the integral (3.2.21) by deforming the integration contour to infinity and by picking up residues at the poles. However, taking into account the relation (3.2.20), we find that the contribution to (3.2.21) at infinity can be neglected for $k^2 > 1$ only. In this case, closing the integration contour into the upper (or lower) half-plane for $k > 1$ (or $k < -1$) we find

$$\widetilde{\Gamma}(k) \overset{k^2 \ge 1}{=} \theta(k-1) \sum_{n\ge1} c_+(n,g)\,\mathrm{e}^{-4\pi n g(k-1)} + \theta(-k-1) \sum_{n\ge1} c_-(n,g)\,\mathrm{e}^{-4\pi n g(-k-1)}\,. \tag{3.2.22}$$

Here the notation was introduced for $k-$independent expansion coefficients

$$c_\pm(n,g) = \mp 4g\gamma(\pm 4\pi i g n)\,\mathrm{e}^{-4\pi n g}\,, \tag{3.2.23}$$

[9]Note that this proof of uniqueness relies on the assumption that $\gamma(t) \sim \sum_n \gamma_n J_n(t)$ is holomorphic, and that summation over the Bessel functions and integration over t can be interchanged when deducing (3.2.11) and (3.2.17) from (3.1.19). Indeed, it is possible to show that non-vanishing homogeneous solutions to (3.1.19) exist, if one at least of the latter conditions is relaxed.

[10]Here, we tacitly assume that the sum over the Bessel functions can be commuted with the integration over t. This assumption is justified by our previous remark about the large-order behavior, $n \sim \infty$, of the coefficients γ_n.

[11]We recall that $\gamma(t) = O(t)$ and, therefore, the integrand is regular at $t = 0$.

where the factor $e^{-4\pi ng}$ is inserted to compensate exponential growth of $\gamma(\pm 4\pi i g n) \sim e^{4\pi ng}$ at large n (see Eq. (3.2.20)). For $k^2 \le 1$, we are not allowed to neglect the contribution to (3.2.21) at infinity and the relation (3.2.22) does not hold anymore. As we will see in a moment, for $k^2 \le 1$ the function $\widetilde{\Gamma}(k)$ can be found from (3.2.5).

Comparing the relations (3.2.19) and (3.2.22), we conclude that, in distinction with $\widetilde{\gamma}(k)$, the function $\widetilde{\Gamma}(k)$ does not vanish for $k^2 > 1$. Moreover, each term in the right-hand side of (3.2.22) is exponentially small at strong coupling and the function scales at large k as $\widetilde{\Gamma}(k) \sim e^{-4\pi g(|k|-1)}$. This implies that nonzero values of $\widetilde{\Gamma}(k)$ for $k^2 > 1$ are of non-perturbative origin. Indeed, in the perturbative approach of [69], the function $\Gamma(t)$ is given by a Bessel-function series analogous to (3.1.10) and, similarly to (3.2.19), the function $\widetilde{\Gamma}(k)$ vanishes for $k^2 > 1$ to any order in the $1/g$ expansion.[12]

We note that the sum in the right-hand side of (3.2.22) runs over poles of the function $\Gamma(it)$ specified in (3.2.8). We recall that in the toy model (3.2.10), $\Gamma^{(\text{toy})}(it)$ and $\gamma^{(\text{toy})}(it)$ are entire functions of t. At large t they have the same asymptotic behavior as the Bessel functions, $\Gamma^{(\text{toy})}(it) \sim \gamma^{(\text{toy})}(it) \sim e^{\pm it}$. Performing their Fourier transformation (3.2.18), we find

$$\widetilde{\gamma}^{(\text{toy})}(k) = \widetilde{\Gamma}^{(\text{toy})}(k) = 0\,, \qquad \text{for } k^2 > 1\,, \tag{3.2.24}$$

in a close analogy with (3.2.19). Comparison with (3.2.22) shows that the coefficients (3.2.23) vanish in the toy model for arbitrary n and g

$$c_+^{(\text{toy})}(n,g) = c_-^{(\text{toy})}(n,g) = 0\,. \tag{3.2.25}$$

The relation (3.2.22) defines the function $\widetilde{\Gamma}(k)$ for $k^2 > 1$ but it involves the coefficients $c_\pm(n,g)$ that need to be determined. In addition, we have to construct the same function for $k^2 \le 1$. To achieve both goals, let us return to the integral equations (3.2.4) and replace $\Gamma_\pm(t)$ by Fourier integrals (see Eqs. (3.2.18) and (3.2.2))

$$\Gamma_+(t) = \int_{-\infty}^{\infty} dk\, \cos(kt)\, \widetilde{\Gamma}(k)\,, \tag{3.2.26}$$

$$\Gamma_-(t) = -\int_{-\infty}^{\infty} dk\, \sin(kt)\, \widetilde{\Gamma}(k)\,.$$

In this way, we obtain from (3.2.4) the following remarkably simple integral equation for $\widetilde{\Gamma}(k)$

$$⨍_{-\infty}^{\infty} \frac{dk\, \widetilde{\Gamma}(k)}{k-u} + \pi \widetilde{\Gamma}(u) = -2\,, \qquad (-1 \le u \le 1)\,, \tag{3.2.27}$$

where the integral is defined using the principal value prescription. This relation is equivalent to the functional equation obtained in [70] (see Eq. (55) there).

Let us split the integral in (3.2.27) into $k^2 \le 1$ and $k^2 > 1$ and rewrite (3.2.27) in the form of a singular integral equation for the function $\widetilde{\Gamma}(k)$ on the interval $-1 \le k \le 1$

$$\widetilde{\Gamma}(u) + \frac{1}{\pi} ⨍_{-1}^{1} \frac{dk\, \widetilde{\Gamma}(k)}{k-u} = \phi(u)\,, \qquad (-1 \le u \le 1)\,, \tag{3.2.28}$$

[12] There is a subtlety however. The function $\Gamma(t)$ has poles in the complex t-plane. It implies that its expansion over the Bessel functions holds around the origin, but is divergent for large enough value of t. Therefore, strictly speaking, it is not allowed to commute the sum over the Bessel functions with the integration over t. Nevertheless, within the accuracy of the strong coupling expansion, these formal manipulations may be given some meaning.

where the inhomogeneous term is given by

$$\phi(u) = -\frac{1}{\pi}\left(2 + \int_{-\infty}^{-1} \frac{dk\,\widetilde{\Gamma}(k)}{k-u} + \int_{1}^{\infty} \frac{dk\,\widetilde{\Gamma}(k)}{k-u}\right). \tag{3.2.29}$$

Since the integration in (3.2.29) goes over $k^2 > 1$, the function $\widetilde{\Gamma}(k)$ can be replaced in the right-hand side of (3.2.29) by its expression (3.2.22) in terms of the coefficients $c_\pm(n,g)$.

The integral equation (3.2.28) can be solved by standard methods [121]. A general solution for $\widetilde{\Gamma}(k)$ reads (for $-1 \le k \le 1$)

$$\widetilde{\Gamma}(k) = \frac{1}{2}\phi(k) - \frac{1}{2\pi}\left(\frac{1+k}{1-k}\right)^{1/4} \fint_{-1}^{1} \frac{du\,\phi(u)}{u-k}\left(\frac{1-u}{1+u}\right)^{1/4} - \frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\frac{c}{1+k}\,, \tag{3.2.30}$$

where the last term describes the zero mode contribution with c being an arbitrary function of the coupling. We replace $\phi(u)$ by its expression (3.2.29), interchange the order of integration and find after some algebra

$$\widetilde{\Gamma}(k) \overset{k^2 \le 1}{=} -\frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1 + \frac{c}{1+k} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{dp\,\widetilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\theta(p^2-1)\right]. \tag{3.2.31}$$

Notice that the integral in the right-hand side of (3.2.31) goes along the real axis except the interval $[-1,1]$ and, therefore, $\widetilde{\Gamma}(p)$ can be replaced by its expression (3.2.22).

Being combined together, the relations (3.2.22) and (3.2.31) define the function $\widetilde{\Gamma}(k)$ for $-\infty < k < \infty$ in terms of (an infinite) set of yet unknown coefficients $c_\pm(n,g)$ and $c(g)$. To fix these coefficients we will first perform the Fourier transform of $\widetilde{\Gamma}(k)$ to obtain the function $\Gamma(t)$ and, then, require that $\Gamma(t)$ should have correct analytical properties (3.2.8).

3.2.6 General Solution

We are now ready to write down a general expression for the function $\Gamma(t)$. According to (3.2.18), it is related to $\widetilde{\Gamma}(k)$ through the inverse Fourier transformation

$$\Gamma(t) = \int_{-1}^{1} dk\,\mathrm{e}^{-ikt}\,\widetilde{\Gamma}(k) + \int_{-\infty}^{-1} dk\,\mathrm{e}^{-ikt}\,\widetilde{\Gamma}(k) + \int_{1}^{\infty} dk\,\mathrm{e}^{-ikt}\,\widetilde{\Gamma}(k)\,, \tag{3.2.32}$$

where we split the integral into three terms since $\widetilde{\Gamma}(k)$ has a different form for $k < -1$, $-1 \le k \le 1$ and $k > 1$. Then, we use the obtained expressions for $\widetilde{\Gamma}(k)$, Eqs. (3.2.22) and (3.2.31), to find after some algebra the following relation (see Appendix A.1 for details)

$$\Gamma(it) = f_0(t)V_0(t) + f_1(t)V_1(t)\,. \tag{3.2.33}$$

Here the notation was introduced for

$$f_0(t) = -1 + \sum_{n\ge1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{4\pi ng + t}\right], \tag{3.2.34}$$

$$f_1(t) = -c(g) + \sum_{n\ge1} 4\pi ng\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{4\pi ng + t}\right].$$

Also, V_r and $U_r^{\pm}$ (with $r = 0, 1$) stand for the integrals

$$V_r(x) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} dk\, (1+k)^{1/4-r}(1-k)^{-1/4}\, \mathrm{e}^{kx} \,, \qquad (3.2.35)$$
$$U_r^{\pm}(x) = \frac{1}{2} \int_{1}^{\infty} dk\, (k \pm 1)^{-1/4}(k \mp 1)^{1/4-r}\, \mathrm{e}^{-(k-1)x} \,,$$

which can be expressed in terms of Whittaker functions of 1st and 2nd kind [122] (see Appendix A.2).

The sum over the coefficients $c_{\pm}(n, g)$ and $c(g)$ in (3.2.34) is a sum over homogeneous solutions to the integral equation (3.2.5). The unknowns $c_{\pm}(n, g)$ and $c(g)$ have therefore the meaning of zero-mode coefficients. They are needed in order for the function $\Gamma(it)$ to respect the correct analyticity property of the BES solution. Putting all of them to zero, Eqs. (3.2.33) and (3.2.34) reduce to

$$\Gamma(it) = -V_0(t) \,, \qquad (3.2.36)$$

which is a particular solution to the integral equation (3.2.5). Mapping Γ into γ with

$$\Gamma(it) = \gamma(it) \frac{\sin(\frac{t}{4g} + \frac{\pi}{4})}{\sin(\frac{t}{4g}) \sin(\frac{\pi}{4})} \,, \qquad (3.2.37)$$

and expanding at $g = \infty$ with t fixed, we get

$$\gamma(it) = -\frac{t}{4g} V_0(t) \,, \qquad (3.2.38)$$

reproducing the leading-order solution of the BES equation, found in [64, 67, 68]. From the strong coupling point of view, the zero-mode coefficients $c_{\pm}(n, g)$ and $c(g)$ take into account higher-loop corrections. But, we emphasize that the solution (3.2.33) is exact for arbitrary coupling $g > 0$, with only undetermined ingredients being the expansion coefficients $c_{\pm}(n, g)$ and $c(g)$.

In the special case of the toy model (3.2.25), due to the absence of poles most of the coefficients vanishes, $c_{\pm}^{(\mathrm{toy})}(n, g) = 0$. The relation (3.2.34) takes a simpler form given by

$$f_0^{(\mathrm{toy})}(t) = -1 \,, \qquad f_1^{(\mathrm{toy})}(t) = -c^{(\mathrm{toy})}(g) \,. \qquad (3.2.39)$$

Substituting these expressions into (3.2.33), we obtain a general solution to the integral equation (3.2.5) in the toy model

$$\Gamma^{(\mathrm{toy})}(it) = -V_0(t) - c^{(\mathrm{toy})}(g) V_1(t) \,. \qquad (3.2.40)$$

It involves an arbitrary $g-$dependent constant $c^{(\mathrm{toy})}(g)$ which will be determined later.

3.2.7 Quantization Conditions

The relation (3.2.33) defines a general solution to the integral equation (3.2.5). It still depends on the coefficients $c_{\pm}(n, g)$ and $c(g)$ that need to be determined. We recall that $\Gamma(it)$ should have poles and zeros specified in (3.2.8).

Let us first examine poles in the right-hand side of (3.2.33). It follows from (3.2.35) that $V_0(t)$ and $V_1(t)$ are entire functions of t and, therefore, poles can only come from the functions $f_0(t)$ and $f_1(t)$. Indeed, the sums entering (3.2.34) produce an infinite sequence of poles located at $t = \pm 4\pi n$ (with $n \geq 1$) and, as a result, the solution (3.2.33) has the correct pole structure (3.2.8). Let us now require that $\Gamma(it)$ should vanish for $t = t_{\text{zero}}$ specified in (3.2.8). This leads to an infinite set of relations

$$\Gamma\left(4\pi i g\big(\ell - \tfrac{1}{4}\big)\right) = 0\,, \qquad\qquad \ell \in \mathbb{Z}\,. \tag{3.2.41}$$

Replacing $\Gamma(it)$ by its expression (3.2.33), we rewrite these relations in equivalent form

$$f_0\left(t_\ell\right) V_0\left(t_\ell\right) + f_1(t_\ell) V_1(t_\ell) = 0\,, \qquad\qquad t_\ell = 4\pi g\big(\ell - \tfrac{1}{4}\big)\,. \tag{3.2.42}$$

The relations (3.2.41) and (3.2.42) provide the quantization conditions for the coefficients $c(g)$ and $c_\pm(n, g)$ that we will analyse in Sect. 3.3.

Let us substitute (3.2.33) into the expression (3.2.3) for the cusp anomalous dimension. The result involves the functions $V_k(t)$ and $f_k(t)$ (with $k = 0, 1$) evaluated at $t = 0$. It is easy to see from (3.2.35) that $V_0(0) = 1$ and $V_1(0) = 2$. In addition, we obtain from (3.2.34) that $f_0(0) = -1$ for arbitrary coupling leading to

$$\Gamma_{\text{cusp}}(g) = 2g\big(1 - 2f_1(0)\big)\,. \tag{3.2.43}$$

Replacing $f_1(0)$ by its expression (3.2.34) we find the following relation for the cusp anomalous dimension in terms of the coefficients c and $c_\pm$

$$\Gamma_{\text{cusp}}(g) = 2g\left\{1 + 2c(g) - 2\sum_{n\geq 1}\Big[c_-(n,g)U_0^-(4\pi n g) + c_+(n,g)U_0^+(4\pi n g)\Big]\right\}\,. \tag{3.2.44}$$

The relations (3.2.43) and (3.2.44) are exact and hold for arbitrary coupling $g > 0$. This implies that, at weak coupling, they should reproduce the known expansion of $\Gamma_{\text{cusp}}(g)$ in *positive integer* powers of g^2 [58]. Similarly, at strong coupling, it should reproduce the known $1/g$ expansion [69, 70] and describe non-perturbative, exponentially suppressed corrections to $\Gamma_{\text{cusp}}(g)$.

3.2.8 Cusp Anomalous Dimension in the Toy Model

As before, the situation simplifies for the toy model (3.2.40). In this case, we have only one quantization condition $\Gamma^{(\text{toy})}(-i\pi g) = 0$ which follows from (3.2.10). Together with (3.2.40) it allows us to fix the coefficient $c^{(\text{toy})}(g)$ as

$$c^{(\text{toy})}(g) = -\frac{V_0(-\pi g)}{V_1(-\pi g)}\,. \tag{3.2.45}$$

Then, we substitute the relations (3.2.45) and (3.2.25) into (3.2.44) and obtain

$$\Gamma^{(\text{toy})}_{\text{cusp}}(g) = 2g\big(1 + 2c^{(\text{toy})}(g)\big) = 2g\left[1 - 2\frac{V_0(-\pi g)}{V_1(-\pi g)}\right]\,. \tag{3.2.46}$$

Replacing $V_0(-\pi g)$ and $V_1(-\pi g)$ by their expressions in terms of Whittaker function of the first kind (see Eq. (A.2.3)), we find the following relation

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = 2g\left[1-(2\pi g)^{-1/2}\frac{M_{1/4,1/2}(2\pi g)}{M_{-1/4,\,0}(2\pi g)}\right], \qquad (3.2.47)$$

which defines the cusp anomalous dimension in the toy model for arbitrary coupling $g > 0$.

Using (3.2.47) it is straightforward to compute $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ for arbitrary positive g. By construction, $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ should be different from $\Gamma_{\rm cusp}(g)$. Nevertheless, evaluating (3.2.47) for $0 \le g \le 3$, we found that the numerical values of $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ are very close to the exact values of the cusp anomalous dimension shown by the solid line in Figure 3.1. Also, as we will show in a moment, the two functions have similar properties at strong coupling. To compare these functions, it is instructive to examine the asymptotic behavior of $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ at weak and at strong coupling.

Weak Coupling

At weak coupling, we find from (3.2.47)

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = \frac{3}{2}\pi\, g^2 - \frac{1}{2}\pi^2 g^3 - \frac{1}{64}\pi^3 g^4 + \frac{5}{64}\pi^4 g^5 - \frac{11}{512}\pi^5 g^6 - \frac{3}{512}\pi^6 g^7 + O(g^8)\,. \qquad (3.2.48)$$

Comparison with (3.2.15) shows that this expansion is quite different from the weak coupling expansion of the cusp anomalous dimension. In distinction with $\Gamma_{\rm cusp}(g)$, the expansion in (3.2.48) runs both in even and odd powers of the coupling. In addition, the coefficient in front of g^n in the right-hand side of (3.2.48) has transcendentality $(n-1)$ while for $\Gamma_{\rm cusp}(g)$ it equals $(n-2)$ (with n taking even values only). Despite of this and similarly to the weak coupling expansion of the cusp anomalous dimension [44], the series (3.2.48) has a finite radius of convergence $|g_0| = 0.796$. It is determined by the position of the zero of the Whittaker function closest to the origin, $M_{-1/4,0}(2\pi g_0) = 0$ for $g_0 = -0.297 \pm i\,0.739$.

Strong coupling

At strong coupling, we can replace the Whittaker functions in (3.2.47) by their asymptotic expansion for $g \gg 1$. It is convenient however to apply (3.2.46) and replace the functions $V_0(-\pi g)$ and $V_1(-\pi g)$ by their expressions given in Appendix B.2.[13] In particular, we have

$$V_0(-\pi g) = {\rm e}^{1/(2\alpha)}\,\frac{\alpha^{5/4}}{\Gamma(\frac34)}\left[F\big(\tfrac14,\tfrac54|\alpha+i0\big)+\sigma\Lambda^2 F\big(-\tfrac14,\tfrac34|-\alpha\big)\right], \qquad \alpha = 1/(2\pi g)\,, \qquad (3.2.49)$$

where the parameter Λ^2 is defined as

$$\Lambda^2 = \alpha^{-1/2}\,{\rm e}^{-1/\alpha}\,\frac{\Gamma(\frac34)}{\Gamma(\frac54)}\,, \qquad \sigma = {\rm e}^{-3i\pi/4}\,. \qquad (3.2.50)$$

[13] See Eqs. (A.2.17) and (A.2.19) there.

Here, the functions $F(a,b|-\alpha)$ are related to the functions $U_0^{\pm}(\pi g)$ (introduced before) as[14]

$$F\left(\tfrac{1}{4},\tfrac{5}{4}|-\alpha\right) = \alpha^{-5/4}U_0^{+}\left(1/(2\alpha)\right)/\Gamma\left(\tfrac{5}{4}\right), \tag{3.2.51}$$
$$F\left(-\tfrac{1}{4},\tfrac{3}{4}|-\alpha\right) = \alpha^{-3/4}U_0^{-}\left(1/(2\alpha)\right)/\Gamma\left(\tfrac{3}{4}\right).$$

The function $F(a,b|-\alpha)$ defined in this way is an analytical function of α with a cut along the negative semi-axis and an essential singularity at the origin. We stress that the relation (3.2.49), here defined for $\alpha > 0$, holds true in the whole upper-half plane, $\mathrm{Im}\,\alpha \geqslant 0$. A similar formula applies for $V_1(-\pi g)$ (see Appendix B.2). We refer to the first term in square brackets, on the right-hand side of (3.2.49), as the perturbative contribution, and to the second one as the non-perturbative correction. Indeed, the latter is exponentially suppressed as compared to the former, when α is small and positive.[15]

Let us first analyse each term in Eq. (3.2.49) separately. It will help us to motivate the rationale for the decomposition of the V-function into F(or U)-functions. So, for positive $\alpha = 1/(2\pi g)$, the function $F\left(-\frac{1}{4},\frac{3}{4}|-\alpha\right)$ entering (3.2.49) is defined away from the cut and its large g expansion is given by a Borel summable asymptotic series (for $a = -\frac{1}{4}$ and $b = \frac{3}{4}$)

$$F(a,b|-\alpha) = \sum_{k\geq 0}\frac{(-\alpha)^k}{k!}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)} = 1 - \alpha ab + O(\alpha^2), \tag{3.2.52}$$

with the expansion coefficients growing factorially to higher orders in α. This series can be immediately resummed by means of the Borel resummation method. Namely, replacing $\Gamma(a+k)$ by its integral representation and performing the sum over k we find for $\mathrm{Re}\,\alpha > 0$

$$F(a,b|-\alpha) = \frac{\alpha^{-a}}{\Gamma(a)}\int_0^\infty ds\, s^{a-1}(1+s)^{-b}\,\mathrm{e}^{-s/\alpha}, \tag{3.2.53}$$

in agreement with (3.2.51) and with the integral representation for $U_0^{\pm}(\pi g)$, Eq. (3.2.35). The relation (3.2.52) holds in fact for arbitrary complex α and the functions $F(a,b|\alpha \pm i0)$, defined for $\alpha > 0$ above and below the cut, respectively, are given by the same asymptotic expansion (3.2.52) with α replaced by $-\alpha$. An important difference is that now the series (3.2.52) is not Borel summable anymore. Namely, if one attempted to resum this series using the Borel summation method, one would immediately find a branch point singularity along the integration contour at $s = 1$

$$F(a,b|\alpha \pm i0) = \frac{\alpha^{-a}}{\Gamma(a)}\int_0^\infty ds\, s^{a-1}(1-s\mp i0)^{-b}\,\mathrm{e}^{-s/\alpha}. \tag{3.2.54}$$

The ambiguity related to the choice of the prescription to integrate over the singularity is known as Borel ambiguity. In particular, deforming the $s-$integration contour above or below the cut, one obtains two different functions $F(a,b|\alpha \pm i0)$. They define analytical continuation of the same function $F(a,b|-\alpha)$ from $\mathrm{Re}\,\alpha > 0$ to the upper and lower edge of the cut running along the negative semi-axis. Its discontinuity across the cut, $F(a,b|\alpha+i0)-F(a,b|\alpha-i0)$, is exponentially

[14] They can be expressed in terms of the confluent hypergeometric function of the second kind [122]. See also Appendix B.2, Eqs. (A.2.14) and (A.2.9).

[15] If α is small but negative, their roles are exchanged of course. Hence, as we shall see, the V-function has different asymptotic behaviors depending on the sign of its argument.

suppressed at small $\alpha > 0$ and is proportional to the non-perturbative scale Λ^2 (see Eq. (A.2.20) in Appendix B.2). In the following, we will implicitely assume the analytic continuation of our formulae in the upper-half plane, $\text{Im}\,\alpha \geqslant 0$, corresponding to the choice of $F(a,b|\alpha+i0)$. Had we decided to work in the lower-half plane, $\text{Im}\,\alpha \leqslant 0$, associated to the choice $F(a,b|\alpha-i0)$, we would have to adjust (3.2.49) accordingly. It would simply amount to the replacement of σ in Eq. (3.2.50) by its complex conjugate. Of course, both choices are equivalent since the function $V_0(-\pi g)$, that we are representing as a sum of a perturbative part, $F(a,b|\alpha \pm i0)$, with a non-perturbative part, $\sigma\Lambda^2 F(a,b|-\alpha)$, is an entire function of $\alpha = 1/(2\pi g)$ (except at $\alpha = 0$).

We can now elucidate the reason for decomposing the entire V_0-function in (3.2.49) into the sum of two $F-$functions. In spite of the fact that analytical properties of the former function are simpler compared to the latter functions, its asymptotic behavior at large g is more complicated. Indeed, the $F-$functions admit asymptotic expansions valid in the whole complex $g-$plane and they can be unambiguously defined through the Borel resummation once their analytical properties are specified. In distinction with this, the entire function $V_0(-\pi g)$ admits different asymptotic behaviors at large g, depending[16] on $\text{Re}\,g > 0$ or $\text{Re}\,g < 0$. This situation is called the Stokes phenomenon, and the two domains, $\text{Re}\,g > 0$ and $\text{Re}\,g < 0$, are separated by the Stokes line, $\text{Re}\,g = 0$. The transition from the domain $\text{Re}\,g > 0$ to the other, going across the Stokes line, is driven by the non-perturbative contribution. The latter correction also impacts on the transition from large to small g, where it ceases to be suppressed as compared to the perturbative contribution. Therefore, resumming the perturbative series by the function $F(a,b|\alpha+i0)$, we need to keep track of the non-perturbative contribution $\sim \sigma\Lambda^2 F(a,b|-\alpha)$ in order to correctly describe the function $V_0(-\pi g)$.[17] A similar conclusion applies for $V_1(-\pi g)$.

We are now in position to discuss the strong coupling expansion of the cusp anomalous dimension in the toy model, including into our consideration both perturbative and non-perturbative contributions. Substituting (3.2.49) and similar relation for $V_1(-\pi g)$ (see Eq. (A.2.19)) into (3.2.46) we find (for $\alpha^+ \equiv \alpha + i0$ and $\alpha = 1/(2\pi g)$)

$$\Gamma^{(\text{toy})}_{\text{cusp}}(g)/(2g) = 1 - \alpha \frac{F\left(\frac{1}{4},\frac{5}{4}|\alpha^+\right) + \sigma\Lambda^2 F\left(-\frac{1}{4},\frac{3}{4}|-\alpha\right)}{F\left(\frac{1}{4},\frac{1}{4}|\alpha^+\right) + \sigma\Lambda^2 \frac{\alpha}{4} F\left(\frac{3}{4},\frac{3}{4}|-\alpha\right)} \,. \tag{3.2.55}$$

Because the parameter Λ^2 is exponentially suppressed at strong coupling, Eq. (3.2.50), and, at the same time, the $F-$functions are all of the same order, it makes sense to expand the right-hand side of (3.2.55) in powers of Λ^2 and, then, study separately each coefficient function. In this way, we identify the leading, Λ^2-independent term as perturbative contribution to $\Gamma^{(\text{toy})}_{\text{cusp}}(g)$ and the $O(\Lambda^2)$ term as the leading non-perturbative correction. More precisely, expanding the right-hand side of (3.2.55) in powers of Λ^2 we obtain

$$\Gamma^{(\text{toy})}_{\text{cusp}}(g)/(2g) = C_0(\alpha) - \alpha\sigma\Lambda^2 C_2(\alpha) + \frac{1}{4}\alpha^2\sigma^2\Lambda^4 C_4(\alpha) + O(\Lambda^6) \,. \tag{3.2.56}$$

[16]The marginal cases ($\text{Re}\,g = 0$) corresponding to $\text{Im}\,g > 0$ or $\text{Im}\,g < 0$ may also require different asymptotic expansions.

[17]We stress that the function $F(a,b|\alpha+i0)$ does not show any transition at small $\alpha = 1/(2\pi g)$, when α evolves from $\text{Re}\,\alpha > 0$ to $\text{Re}\,\alpha < 0$ in the upper-half plane, while $V_0(-\pi g)$ does, due to the Stokes phenomenon. In this transition, the non-perturbative contribution $\sigma\Lambda^2 F(a,b|-\alpha)$, with $\Lambda^2 \sim \exp(-1/\alpha)$, is more and more important, and finally becomes dominant when $\text{Re}\,\alpha < 0$. Roughly speaking, the fact that the perturbative and non-perturbative contributions exchange their role, when $\text{Re}\,\alpha$ changes its sign, requires two F-functions to completely describe the V-function.

Here the expansion runs in even powers of Λ and the coefficient functions $C_k(\alpha)$ are given by algebraic combinations of $F-$functions

$$C_0 = 1 - \alpha \frac{F\left(\frac{1}{4}, \frac{5}{4}|\alpha^+\right)}{F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)}, \qquad C_2 = \frac{1}{\left[F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)\right]^2}, \qquad C_4 = \frac{F\left(\frac{3}{4}, \frac{3}{4}|-\alpha\right)}{\left[F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)\right]^3}, \tag{3.2.57}$$

where we applied (A.2.11) and (A.2.14) to simplify the last two relations. Since the coefficient functions $C_k(\alpha)$ are expressed in terms of the functions $F(a, b|\alpha^+)$ and $F(a, b|-\alpha)$, with a cut along the positive and negative semi-axis respectively, they are analytical functions of α in the upper-half plane, with an essential singularity at the origin. Notice that the functions $C_k(\alpha)$ take complex values for $\alpha > 0$, despite the fact that their asymptotic expansions at $\alpha \sim 0$ have real expansion coefficients, see Eqs. (3.2.58) below. Their imaginary parts, indeed, are exponentially small for $\alpha \sim 0$, of order Λ^2, and such that their sum in (3.2.56) vanishes.

Let us now examine the strong coupling expansion of the coefficient functions (3.2.57). Replacing $F-$functions in (3.2.57) by their asymptotic series representation (3.2.52) we get

$$\begin{aligned} C_0 &= 1 - \alpha - \frac{1}{4}\alpha^2 - \frac{3}{8}\alpha^3 - \frac{61}{64}\alpha^4 - \frac{433}{128}\alpha^5 + O\left(\alpha^6\right), \\ C_2 &= 1 - \frac{1}{8}\alpha - \frac{11}{128}\alpha^2 - \frac{151}{1024}\alpha^3 - \frac{13085}{32768}\alpha^4 + O\left(\alpha^5\right), \\ C_4 &= 1 - \frac{3}{4}\alpha - \frac{27}{32}\alpha^2 - \frac{317}{128}\alpha^3 + O\left(\alpha^4\right). \end{aligned} \tag{3.2.58}$$

Not surprisingly, these expressions inherit the properties of the $F-$functions – the series (3.2.58) are asymptotic and non-Borel summable. If one simply substituted the relations (3.2.58) into the right-hand side of (3.2.56), one would then worry about the meaning of the non-perturbative $O(\Lambda^2)$ corrections, given the fact that the perturbative $O(\Lambda^0)$ contribution $C_0(\alpha)$ is not unambiguously determined by its *asymptotic* strong coupling expansion. However, we stress that the functions $C_0(\alpha), C_2(\alpha), \ldots$, are all expressible in terms of F-functions, as in Eq. (3.2.57). As was already mentioned, the $F-$functions do not suffer from the Stokes phenomenon and, as a consequence, their asymptotic expansions, supplemented with additional analyticity conditions, permit to reconstruct them through the Borel transformation, Eqs. (3.2.53) and (3.2.54). We may expect, therefore, that the same should be true for the $C-$functions. Indeed, it follows from a general theorem discussed in [117, 119], that the functions $C_0(\alpha)$, $C_2(\alpha)$, $C_4(\alpha)$, ... are uniquely determined by their series representations (3.2.58) as soon as the latter are understood as asymptotic expansions valid in the upper-half plane $\mathrm{Im}\,\alpha \geq 0$.[18] This implies that the exact expressions

[18] More precisely, according to [117, 119], the asymptotic series for $C_0(\alpha), \ldots$, assumed to be valid on the domain $S = \{\alpha \text{ small enough with } \mathrm{Im}(\alpha) \geqslant 0\}$ and exhibiting a factorial growth, will fix the functions $C_0(\alpha), \ldots$, if the bound

$$\left|C_0(\alpha) - \sum_{n\geqslant 0}^{N} C_0^{(n)}\alpha^n\right| \leqslant D_0^{(N+1)}|\alpha|^{N+1},$$

applies on S with $D_0^{(N+1)} \sim C_0^{(N+1)}$, for any N. In our case, this bound means that the best possible estimate of $C_0(\alpha)$, obtained by partially resumming the asymptotic series, is of order $\exp(-1/|\alpha|)$. It prevents the addition of terms, as $\exp(-1/\sqrt{-i\alpha})$ for instance, that would not change the asymptotic series on S and would spoil uniqueness. From a physical point of view, it means that the typical divergence of the asymptotic series is a good indicator of the for the functions (3.2.57) can be unambiguously constructed starting from their asymptotic series (3.2.58), by means of the Borel resummation for instance. It would be interesting to verify it explicitely.

Since the expression (3.2.55) is exact for arbitrary coupling g we may address the question: how does the transition from the strong to the weak coupling regime occur? We recall that, in the toy model, $\Gamma^{(\rm toy)}_{\rm cusp}(g)/(2g)$ is given for $g \ll 1$ and $g \gg 1$ by the relations (3.2.48) and (3.2.56), respectively. Let us choose some sufficiently small value of the coupling constant, say $g = 1/4$, and compute $\Gamma^{(\rm toy)}_{\rm cusp}(g)/(2g)$ using three different representations. Firstly, we substitute $g = 0.25$ into (3.2.55) and find the exact value as 0.4424(3). Then, we use the weak coupling expansion (3.2.48) and obtain a close value 0.4420(2). Finally, we use the strong coupling expansion (3.2.56) and evaluate the first few terms in the right-hand side of (3.2.56) for $g = 0.25$ to get

$$\begin{aligned}\text{Eq. (3.2.56)} &= (0.2902 - 0.1434\, i) + (0.1517 + 0.1345\, i) \\ &\quad + (0.0008 + 0.0086\, i) - (0.0002 - 0.0003\, i) + \ldots = 0.4425 + \ldots \end{aligned} \tag{3.2.59}$$

Here the four expressions inside the round brackets correspond to contributions proportional to Λ^0, Λ^2, Λ^4 and Λ^6, respectively, with $\Lambda^2(g = 0.25) = 0.3522$ being the non-perturbative scale (3.2.50).

We observe that each term in (3.2.59) takes complex values. This is due to our prescription to sum the asymptotic series (3.2.58). The total sum however does not suffer from this ambiguity. Indeed, we verify in Eq.(3.2.59) that the imaginary part vanishes and that the real part is remarkably close to the exact value. In addition, the leading $O(\Lambda^2)$ non-perturbative correction (the second term) is comparable with the perturbative correction (the first term). Moreover, the former term starts to dominate over the latter one as we go to smaller values of the coupling constant. Thus, the transition from the strong to weak coupling regime is driven by non-perturbative corrections parameterized by the scale Λ^2. The numerical analysis indicates that the expansion of $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ in powers of Λ^2 is convergent for $\operatorname{Re} g > 0$.

From the Toy Model to the Exact Solution

The relation (3.2.56) is remarkably similar to the expected strong coupling expansion of the cusp anomalous dimension (3.0.3) with the function $C_0(\alpha)$ providing perturbative contribution and Λ^2 defining the leading non-perturbative contribution. Let us compare $C_0(\alpha)$ with the known perturbative expansion (3.2.16) of $\Gamma_{\rm cusp}(g)$. In terms of the coupling $\alpha = 1/(2\pi g)$, the first few terms of this expansion look as

$$\Gamma_{\rm cusp}(g)/(2g) = 1 - \frac{3\log 2}{2}\alpha - \frac{\rm K}{4}\alpha^2 - \left(\frac{3{\rm K}\log 2}{8} + \frac{27\zeta_3}{256}\right)\alpha^3 + \ldots\,, \tag{3.2.60}$$

where ellipses denote both higher order corrections in α and non-perturbative corrections in Λ^2. Comparing (3.2.60) and the first term, $C_0(\alpha)$, in the right-hand side of (3.2.56), we observe that both expressions approach the same value 1 as $\alpha \to 0$.

size of the non-perturbative corrections. We believe, in the case of the toy-model, that this bounds can be proved exactly by a more detailed analysis. For the BES case, we have verified that it is consistent with the numerical solution.

As was already mentioned, the expansion coefficients of the two series have different transcendentality – they are rational for the toy model, Eq. (3.2.58), and have maximal transcendentality[19] for the cusp anomalous dimension, Eq. (3.2.60). Notice that the two series would coincide if one formally replaced the transcendental numbers in (3.2.60) by appropriate rational constants. In particular, replacing

$$\frac{3\log 2}{2} \to 1\,, \qquad \frac{\mathrm{K}}{2} \to \frac{1}{2}\,, \qquad \frac{9\zeta_3}{32} \to \frac{1}{3}\,, \qquad \dots\,, \tag{3.2.61}$$

one obtains from (3.2.60) the first few terms of perturbative expansion (3.2.58) of the function C_0 in the toy model. This rule can be generalized to all loops as follows. Introducing an auxiliary parameter τ, we define the generating function for the transcendental numbers in (3.2.61) and rewrite (3.2.61) as

$$\exp\left[\frac{3\log 2}{2}\,\tau - \frac{\mathrm{K}}{2}\,\tau^2 + \frac{9\zeta_3}{32}\,\tau^3 + \dots\right] \;\to\; \exp\left[\tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} + \dots\right]. \tag{3.2.62}$$

Going to higher loops, we have to add higher order terms in τ to both exponents. In the right-hand side, these terms are resummed into $\exp(\log(1+\tau)) = 1+\tau$, while in the left-hand side they produce a ratio of Euler gamma-functions leading to

$$\frac{\Gamma(\frac{1}{4})\Gamma(1+\frac{\tau}{4})\Gamma(\frac{3}{4}-\frac{\tau}{4})}{\Gamma(\frac{3}{4})\Gamma(1-\frac{\tau}{4})\Gamma(\frac{1}{4}+\frac{\tau}{4})} \;\to\; (1+\tau)\,. \tag{3.2.63}$$

Taking logarithms in both sides of this relation and subsequently expanding them in powers of τ, we obtain the subtitution rules which generalize (3.2.61) to the complete family of transcendental numbers entering into the strong coupling expansion (3.2.60). The relation (3.2.63) can be thought of as an empirical rule, which allows us to map the strong coupling expansion of the cusp anomalous dimension (3.2.60) into that in the toy model, Eq. (3.2.58). To some extent, it takes into account the difference between the toy model and the BES case, encoded into the position of poles and zeros. It originates from the structure of the strong coupling solution that we will discuss later. It would be interesting to understand if this substitution rule has some counterpart on the string theory side, at a Feynman-graph level for instance.

In spite of the fact that the numbers entering both sides of (3.2.61) have different transcendentality, we may compare their numerical values. Given that $3\log 2/2 = 1.0397(2)$, $\mathrm{K}/2 = 0.4579(8)$ and $9\zeta_3/32 = 0.3380(7)$ we observe that the relation (3.2.61) defines a meaningful approximation to the transcendental numbers. Moreover, examining the coefficients in front of τ^n in both sides of (3.2.62) at large n, we find that the accuracy of the approximation increases as $n \to \infty$. This

[19] To any of the numbers $(\log 2)^j$, $\zeta(2k+1)$ or $\beta(2l)$, we associate a degree of transcendentality j, $2k+1$ or $2l$, respectively. Here, $\zeta(z)$ and $\beta(z)$ denote the Riemann and Dirichlet zeta-functions, respectively. It was observed in [69], up to some high order, that the contribution $\sim \alpha^n$ to the cusp anomalous dimension only involves products of these special numbers with a sum of their degrees of transcendentality equal to n (maximal transcendentality). Note that $\beta(2) \sim \mathrm{K}$. This situation is reminiscent of what happens at weak coupling, for which a maximal transcendentality principle has been argued in [123, 56] to restrict the form of radiative corrections to anomalous dimensions of Wilson operators. In this context, the transcendentality is maximal in $\mathcal{N}=4$ SYM as compared to QCD where less-transcendental contributions appear. Moreover, the maximal transcendentality contribution computed in QCD, in MS-like schemes, was proposed to be identical to the full $\mathcal{N}=4$ SYM result [123, 56].

is in agreement with the observation made before that the cusp anomalous dimension in the toy model is close numerically to the exact (BES) prediction. In addition, the same property suggests that the coefficients in the strong coupling expansion of $\Gamma_{\text{cusp}}^{(\text{toy})}(g)$ and $\Gamma_{\text{cusp}}(g)$ should have the same large order behavior. It was found in [69] that the expansion coefficients in the right-hand side of (3.2.60) grow factorially at higher orders

$$\Gamma_{\text{cusp}}(g) \sim \sum_k \Gamma\left(k+\frac{1}{2}\right)\alpha^k\,. \tag{3.2.64}$$

It is straightforward to verify using (3.2.57) and (3.2.52) that the expansion coefficients of $C_0(\alpha)$ in the toy model have the same behavior. This suggests that the non-perturbative corrections to $\Gamma_{\text{cusp}}(g)$ are also parameterized by the scale $\Lambda^2 \propto g^{1/2}\exp(-2\pi g)$. Indeed, we will show this in the next section by explicit calculation.

In this section, we identified the origin of non-perturbative corrections in the toy-model. They arise due to the Stokes phenomenon for the functions $V_{0,1}(-\pi g)$, that requires the introduction at strong coupling of the non-perturbative scale $\Lambda^2 \propto g^{1/2}\exp(-2\pi g)$. We will see in the next section that a similar conclusion applies for the cusp anomalous dimension, as predicted by the BES equation.

3.3 Solving the Quantization Conditions

Let us now solve the quantization conditions for the cusp anomalous dimension. We recall that they read

$$f_0\,(t_\ell)\,V_0\,(t_\ell) + f_1(t_\ell)V_1(t_\ell) = 0\,, \qquad t_\ell = 4\pi g\big(\ell-\tfrac{1}{4}\big)\,, \tag{3.3.1}$$

with $\ell \in \mathbb{Z}$ and where

$$\begin{aligned} f_0(t) &= -1 + \sum_{n\geq 1} t\left[c_+(n,g)\frac{U_1^+(4\pi n g)}{4\pi n g - t} + c_-(n,g)\frac{U_1^-(4\pi n g)}{4\pi n g + t}\right]\,, \\ f_1(t) &= -c(g) + \sum_{n\geq 1} 4\pi n g\left[c_+(n,g)\frac{U_0^+(4\pi n g)}{4\pi n g - t} + c_-(n,g)\frac{U_0^-(4\pi n g)}{4\pi n g + t}\right]\,. \end{aligned} \tag{3.3.2}$$

The quantization conditions (3.3.1) form an infinite set of equations for the unknown coefficients $c_\pm(n,g)$ and $c(g)$. Once solved, we can compute the cusp anomalous dimension for arbitrary coupling with the help of

$$\Gamma_{\text{cusp}}(g) = 2g\big(1-2f_1(0)\big)\,. \tag{3.3.3}$$

Examining (3.3.2) we observe that the dependence on the coupling resides both in the expansion coefficients and in the functions $U_{0,1}^\pm(4\pi n g)$. The latter functions are related to the F-functions of the previous section, see Eq. (3.2.51), and as such they do not suffer from the Stokes phenomenon. Therefore, non-perturbative corrections to the cusp anomalous dimension (3.3.3) could only come from the coefficients $c_\pm(n,g)$ and $c(g)$. The quantization conditions (3.3.1) indicate moreover that the non-perturbative contributions to the expansion coefficients originate from the functions $V_0(t)$ and $V_1(t)$, evaluated at $t = (4\ell-1)\pi g$. Since these functions depend, at strong coupling, on the hidden non-perturbative scale $\Lambda^2 \propto g^{1/2}\exp(-2\pi g)$, we will look for a solution to the quantization conditions as a series in powers of Λ^2. We will construct both the leading $O(\Lambda^0)$ and subleading $O(\Lambda^2)$ parts of the solution, in the form of an expansion in powers of $1/g$. We expect, as we argued to be the case for the toy model, that each of these parts are uniquely determined by their asymptotic series in $1/g$, assumed to be valid in the lower-half plane $\operatorname{Im} g \leqslant 0$. We believe that the latter assumption is reasonable, recalling that the toy model provides a meaningful approximation to the BES solution, at least as long as the cusp anomalous dimension is concerned.

3.3.1 Quantization Conditions

Let us replace $f_0(t)$ and $f_1(t)$ in (3.3.1) by their explicit expressions (3.3.2) and rewrite the quantization conditions (3.3.1) as

$$V_0(4\pi g x_\ell) + c(g) V_1(4\pi g x_\ell) = \sum_{n\geq 1}\Big[c_+(n,g)A_+(n,x_\ell) + c_-(n,g)A_-(n,x_\ell)\Big]\,, \tag{3.3.4}$$

where $x_\ell = \ell - \frac{1}{4}$ (with $\ell = 0, \pm 1, \pm 2, \ldots$) and the notation was introduced for

$$A_\pm(n,x_\ell) = \frac{nV_1(4\pi g x_\ell)U_0^\pm(4\pi n g) + x_\ell V_0(4\pi g x_\ell)U_1^\pm(4\pi n g)}{n \mp x_\ell}\,. \tag{3.3.5}$$

The relation (3.3.4) provides an infinite system of linear equations for $c_\pm(g,n)$ and $c(g)$. The coefficients in this system depend on $V_{0,1}(4\pi g x_\ell)$ and $U_{0,1}^\pm(4\pi n g)$ which are known functions.

Let us show that the quantization conditions (3.3.4) lead to $c(g) = 0$ for arbitrary coupling. To this end, we examine (3.3.4) for $|x_\ell| \gg 1$. In this limit, for $g =$ fixed we are allowed to replace the functions $V_0(4\pi g x_\ell)$ and $V_1(4\pi g x_\ell)$ in both sides of (3.3.4) by their asymptotic behaviors at infinity. Making use of (A.2.12) and (A.2.14), we find for $|x_\ell| \gg 1$

$$r(x_\ell) \equiv \frac{V_1(4\pi g x_\ell)}{V_0(4\pi g x_\ell)} = \begin{cases} -16\pi g x_\ell + \ldots\,, & (x_\ell < 0) \\ \frac{1}{2} + \ldots\,, & (x_\ell > 0) \end{cases} \tag{3.3.6}$$

where ellipses denote terms suppressed by powers of $1/(g x_\ell)$ and $\mathrm{e}^{-8\pi g |x_\ell|}$. We divide both sides of (3.3.4) by $V_1(4\pi g x_\ell)$ and observe that for $x_\ell \to -\infty$ the first term in the left-hand side of (3.3.4) is subleading and can be safely neglected. In the similar manner, one has $A_\pm(n,x_\ell)/V_1(4\pi g x_\ell) = O(1/x_\ell)$ for fixed n in the right-hand side of (3.3.4). Therefore, going to the limit $x_\ell \to -\infty$ in both sides of (3.3.4) we get

$$c(g) = 0 \tag{3.3.7}$$

for arbitrary g.

Arriving at (3.3.7), we tacitly assumed that the sum over n in (3.3.4) remains finite in the limit $x_\ell \to -\infty$. Taking into account the large n behaviors

$$U_0^+(4\pi n g) \sim n^{-5/4}\,, \qquad U_0^-(4\pi n g) \sim U_1^-(4\pi n g) \sim n^{-3/4}\,, \qquad U_1^+(4\pi n g) \sim n^{-1/4}\,, \tag{3.3.8}$$

we obtain that this condition translates into ($n \gg 1$)

$$c_+(n,g) = o(n^{1/4})\,, \qquad c_-(n,g) = o(n^{-1/4})\,. \tag{3.3.9}$$

These conditions also ensure that the sum in the expression for the cusp anomalous dimension

$$\Gamma_{\rm cusp}(g) = 2g\left\{1+2c(g)-2\sum_{n\geq 1}\left[c_-(n,g)U_0^-(4\pi ng)+c_+(n,g)U_0^+(4\pi ng)\right]\right\}, \qquad (3.3.10)$$

is convergent.

3.3.2 Strong Coupling Solution

Let us divide both sides of (3.3.4) by $V_0(4\pi g x_\ell)$ and use (3.3.7) to get (for $x_\ell = \ell - \frac{1}{4}$ and $\ell \in \mathbb{Z}$)

$$\begin{aligned}1 = &\sum_{n\geq 1} c_+(n,g)\left[\frac{nU_0^+(4\pi ng)r(x_\ell)+U_1^+(4\pi ng)x_\ell}{n-x_\ell}\right] \\ &+\sum_{n\geq 1} c_-(n,g)\left[\frac{nU_0^-(4\pi ng)r(x_\ell)+U_1^-(4\pi ng)x_\ell}{n+x_\ell}\right],\end{aligned} \qquad (3.3.11)$$

where the function $r(x_\ell)$ was defined in (3.3.6).

Let us now examine the large g asymptotics of the functions multiplying $c_\pm(n,g)$ in the right-hand side of (3.3.11). The functions $U_0^\pm(4\pi ng)$ and $U_1^\pm(4\pi ng)$ admit an asymptotic expansion in $1/g$ given by (A.2.14). For the function $r(x_\ell)$ the situation is different. As follows from its definition, Eqs. (3.3.6) and (A.2.12), large g expansion of $r(x_\ell)$ runs in two parameters: perturbative $1/g$ and non-perturbative exponentially small parameter $\Lambda^2 \propto g^{1/2}\,\mathrm{e}^{-2\pi g}$ which we already encountered in the toy model, Eq. (3.2.50). Moreover, we deduce from (3.3.6) and (A.2.12) that the leading non-perturbative correction to $r(x_\ell)$ scales as

$$\delta r(x_\ell) = O\big(\Lambda^{|8\ell-2|}\big)\,, \qquad (x_\ell = \ell - \tfrac{1}{4}\,,\ \ell\in\mathbb{Z})\,, \qquad (3.3.12)$$

so that the power of Λ grows with ℓ. We observe that $O(\Lambda^2)$ corrections are only present in $r(x_\ell)$ for $\ell = 0$. Therefore, as far as the leading $O(\Lambda^2)$ correction to the solutions of (3.3.11) are concerned, we are allowed to neglect non-perturbative (Λ^2-dependent) corrections to $r(x_\ell)$ in the right-hand side of (3.3.11) for $\ell \neq 0$ and retain them for $\ell = 0$ only.

Since the coefficient functions in the linear equations (3.3.11) admit a double series expansion in powers of $1/g$ and Λ^2, we expect that the same should be true for their solutions $c_\pm(n,g)$. Let us determine the first few terms of this expansion using the following ansatz:

$$c_\pm(n,g) = (8\pi g n)^{\pm 1/4}\left\{\left[a_\pm(n)+\frac{b_\pm(n)}{4\pi g}+\ldots\right]+\sigma\Lambda^2\left[\alpha_\pm(n)+\frac{\beta_\pm(n)}{4\pi g}+\ldots\right]+O(\Lambda^4)\right\}, \qquad (3.3.13)$$

where ellipses denote terms suppressed by powers of $1/g$ and with Λ^2 the non-perturbative parameter defined in (3.2.50).

Here the functions $a_\pm(n), b_\pm(n), \ldots$ are assumed to be $g-$independent. We recall that the functions $c_\pm(n,g)$ have to verify the relation (3.3.9). This implies that the functions $a_\pm(n), b_\pm(n), \ldots$ should vanish as $n\to\infty$. To determine them we substitute (3.3.13) into (3.3.11) and compare the coefficients in front of powers of $1/g$ and Λ^2 in both sides of (3.3.11).

Perturbative Corrections

Let us start with 'perturbative', Λ^2-independent part of (3.3.13) and compute the functions $a_\pm(n)$ and $b_\pm(n)$.

To determine $a_\pm(n)$, we substitute (3.3.13) into (3.3.11), replace the functions $U^\pm_{0,1}(4\pi gn)$ and $r(x_\ell)$ by their large g asymptotic expansion, Eqs. (A.2.14) and (3.3.6), respectively, neglect corrections in Λ^2 and compare the leading $O(g^0)$ terms in both sides of (3.3.11). In this way, we obtain from (3.3.11) the following relations for $a_\pm(n)$ (with $x_\ell = \ell - \frac{1}{4}$)

$$2x_\ell\,\Gamma(\tfrac{5}{4})\sum_{n\geq 1}\frac{a_+(n)}{n-x_\ell} = 1\,, \qquad (\ell\geq 1) \tag{3.3.14}$$

$$-2x_\ell\,\Gamma(\tfrac{3}{4})\sum_{n\geq 1}\frac{a_-(n)}{n+x_\ell} = 1\,, \qquad (\ell\leq 0)$$

One can verify that the solutions to this system satisfying $a_\pm(n)\to 0$ for $n\to\infty$ have the form

$$a_+(n) = \frac{2\Gamma(n+\frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})}\,, \tag{3.3.15}$$

$$a_-(n) = \frac{\Gamma(n+\frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})}\,.$$

In the similar manner, we compare the subleading $O(1/g)$ terms in both sides of (3.3.11) and find that the functions $b_\pm(n)$ satisfy the following relations (with $x_\ell = \ell - \frac{1}{4}$)

$$2x_\ell\,\Gamma(\tfrac{5}{4})\sum_{n\geq 1}\frac{b_+(n)}{n-x_\ell} = -\frac{3}{32x_\ell} - \frac{3\pi}{64} - \frac{15}{32}\log 2\,, \qquad (\ell\geq 1) \tag{3.3.16}$$

$$-2x_\ell\,\Gamma(\tfrac{3}{4})\sum_{n\geq 1}\frac{b_-(n)}{n+x_\ell} = -\frac{5}{32x_\ell} - \frac{5\pi}{64} + \frac{9}{32}\log 2\,, \qquad (\ell\leq 0)$$

where in the right-hand side we made use of (3.3.15). Solutions to these relations are

$$b_+(n) = -a_+(n)\left(\frac{3\log 2}{4} + \frac{3}{32n}\right), \tag{3.3.17}$$

$$b_-(n) = \quad a_-(n)\left(\frac{3\log 2}{4} + \frac{5}{32n}\right).$$

It is straightforward to extend analysis to subleading perturbative corrections to $c_\pm(n,g)$.[20]

Let us substitute (3.3.13) into expression (3.3.10) for the cusp anomalous dimension. Taking into account the identities (A.2.14) we find the 'perturbative' contribution to $\Gamma_{\rm cusp}(g)$ as

$$\Gamma_{\rm cusp}(g) = 2g - \sum_{n\geq 1}(2\pi n)^{-1}\Big[\Gamma(\tfrac{5}{4})\Big(a_+(n) + \frac{b_+(n)}{4\pi g} + \ldots\Big)\Big(1 - \frac{5}{128\pi ng} + \ldots\Big) \tag{3.3.18}$$

$$+\Gamma(\tfrac{3}{4})\Big(a_-(n) + \frac{b_-(n)}{4\pi g} + \ldots\Big)\Big(1 + \frac{3}{128\pi ng} + \ldots\Big)\Big] + O(\Lambda^2)\,.$$

[20]For instance, the k-th corrections can be found in the form $a_\pm(n)\,P^{(k)}_\pm(1/n)$, where $P^{(k)}_\pm(1/n)$ is a polynomial of degree k in the variable $1/n$.

Replacing $a_\pm(n)$ and $b_\pm(n)$ by their expressions (3.3.15) and (3.3.17), we find after some algebra

$$\Gamma_{\rm cusp}(g) = 2g\left[1-\frac{3\log 2}{4\pi g}-\frac{\rm K}{16\pi^2 g^2}+O(1/g^3)\right]+O(\Lambda^2)\,, \tag{3.3.19}$$

where K is the Catalan's constant.

Non-Perturbative Corrections

Let us now compute the leading $O(\Lambda^2)$ non-perturbative correction to the coefficients $c_\pm(n,g)$. According to (3.3.13), it is described by the functions $\alpha_\pm(n)$ and $\beta_\pm(n)$. To determine them from (3.3.11), we have to retain in $r(x_\ell)$ corrections proportional to Λ^2. As was already explained, they only appear for $\ell = 0$. Combining together the relations (3.3.6), (A.2.12) and (A.2.14) we find after some algebra

$$\delta r(x_\ell) = -\delta_{\ell,0}\sigma\Lambda^2\left[4\pi g-\frac{5}{4}+O(g^{-1})\right]+O(\Lambda^4)\,. \tag{3.3.20}$$

Let us substitute this relation into (3.3.11) and equate to zero the coefficient in front of Λ^2 in the right-hand side of (3.3.11). This coefficient is given by a series in $1/g$ and, examining the first two terms, we obtain the defining relations for the functions $\alpha_\pm(n)$ and $\beta_\pm(n)$.

In this way, we find that the leading functions $\alpha_\pm(n)$ satisfy the relations (with $x_\ell = \ell - \frac{1}{4}$)

$$\begin{aligned} 2x_\ell\,\Gamma(\tfrac{5}{4})\sum_{n\ge1}\frac{\alpha_+(n)}{n-x_\ell} &= 0\,, && (\ell\ge1)\\ -2x_\ell\,\Gamma(\tfrac{3}{4})\sum_{n\ge1}\frac{\alpha_-(n)}{n+x_\ell} &= \frac{\pi}{2\sqrt{2}}\,\delta_{\ell,0}\,, && (\ell\le0) \end{aligned} \tag{3.3.21}$$

where in the right-hand side we applied (3.3.15). Solution to (3.3.21) satisfying $\alpha_\pm(n)\to0$ as $n\to\infty$ reads

$$\begin{aligned} \alpha_+(n) &= 0\,,\\ \alpha_-(n) &= a_-(n-1)\,. \end{aligned} \tag{3.3.22}$$

with $a_-(n)$ defined in (3.3.15). For subleading functions $\beta_\pm(n)$ we have similar relations

$$\begin{aligned} 2x_\ell\,\Gamma(\tfrac{5}{4})\sum_{n\ge1}\frac{\beta_+(n)}{n-x_\ell} &= -\frac{1}{2}\,, && (\ell\ge1)\\ -2x_\ell\,\Gamma(\tfrac{3}{4})\sum_{n\ge1}\frac{\beta_-(n)}{n+x_\ell} &= -\frac{1}{8}+\frac{3\pi}{16\sqrt{2}}(1-2\log 2)\delta_{\ell,0}\,, && (\ell\le0) \end{aligned} \tag{3.3.23}$$

In a close analogy with (3.3.17), the solutions to these relations can be written in terms of the leading-order functions $a_\pm(n)$ defined in (3.3.15)

$$\begin{aligned} \beta_+(n) &= -\frac{1}{2}\,a_+(n)\,,\\ \beta_-(n) &= a_-(n-1)\left(\frac{1}{4}-\frac{3\log 2}{4}+\frac{1}{32n}\right)\,. \end{aligned} \tag{3.3.24}$$

We are now ready to compute non-perturbative correction to the cusp anomalous dimension (3.3.10). Substituting (3.3.13) into (3.3.10) we obtain

$$\delta\Gamma_{\text{cusp}}(g) = -\sigma\Lambda^2 \sum_{n\geq 1}(2\pi n)^{-1}\Big[\Gamma(\tfrac{5}{4})\Big(\alpha_+(n) + \frac{\beta_+(n)}{4\pi g} + \dots\Big)\Big(1 - \frac{5}{128\pi g n} + \dots\Big) \tag{3.3.25}$$
$$+\Gamma(\tfrac{3}{4})\Big(\alpha_-(n) + \frac{\beta_-(n)}{4\pi g} + \dots\Big)\Big(1 + \frac{3}{128\pi g n} + \dots\Big)\Big] + O(\Lambda^4)\,.$$

We replace $\alpha_\pm(n)$ and $\beta_\pm(n)$ by their explicit expressions (3.3.22) and (3.3.24), evaluate the sums and find

$$\delta\Gamma_{\text{cusp}}(g) = -\frac{\sigma\Lambda^2}{\pi}\left[1 + \frac{3 - 6\log 2}{16\pi g} + O(1/g^2)\right] + O(\Lambda^4)\,, \tag{3.3.26}$$

with Λ^2 defined in (3.2.50).

We summarize our findings by the formula

$$\Gamma_{\text{cusp}}(g) = \left[2g - \frac{3\log 2}{2\pi} - \frac{\mathrm{K}}{8\pi^2 g} + O(1/g^2)\right] - \frac{\sigma}{4\sqrt{2}} m_{\text{cusp}}^2 + O(m_{\text{cusp}}^4)\,, \tag{3.3.27}$$

where we introduced a non-perturbative parameter m_{cusp} whose strong coupling expansion starts as

$$m_{\text{cusp}} = \frac{\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)}(2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\left[1 + \frac{3 - 6\log 2}{32\pi g} + O(1/g^2)\right]. \tag{3.3.28}$$

3.4 Mass Scale

In this section, we will analyse a new non-perturbative scale m given in terms of the BES solution $\Gamma(t) \equiv \Gamma_+(t) + i\Gamma_-(t)$ as

$$m \equiv \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^\infty \frac{dt}{t + i\pi g}\,\mathrm{e}^{it - i\pi/4}\Gamma(t)\right]. \tag{3.4.1}$$

The original motivation to introduce this quantity is that it controls the small j scaling function $\epsilon(g, j)$ at strong coupling, as we shall see in the next chapter. In this context, we will prove that m has a nice interpretation since it can be given the meaning of the mass gap of the non-linear O(6) sigma model. In this section, we will show that the mass scale m is also relevant to our discussion of the leading non-perturbative contribution to the cusp anomalous dimension. The principal result that we will obtain is the identity $m_{\text{cusp}} = m$, valid to any order in $1/g$.

3.4.1 General Expression

The expression for the mass gap m can be written as

$$m = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^{-i\infty} \frac{dt}{t + \frac{1}{4}}\,\mathrm{e}^{-4\pi g t - i\pi/4}\,\Gamma(4\pi g i t)\right], \tag{3.4.2}$$

where integration goes along the imaginary axis. Replacing $\Gamma(4\pi g i t)$ in (3.4.2) by its expression

$$\Gamma(4\pi g i t) = f_0(4\pi g t)V_0(4\pi g t) + f_1(4\pi g t)V_1(4\pi g t)\,, \tag{3.4.3}$$

and evaluating the $t-$integral, we find after some algebra (see Appendix A.3 for details) [76]

$$m = -\frac{16g\sqrt{2}}{\pi}\, \mathrm{e}^{-\pi g}\left[f_0(-\pi g)U_0^-(\pi g) + f_1(-\pi g)U_1^-(\pi g)\right] . \tag{3.4.4}$$

This relation can be further simplified with the help of the quantization conditions (3.3.1). For $\ell = 0$, we obtain from (3.3.1) that $f_0(-\pi g)V_0(-\pi g) + f_1(-\pi g)V_1(-\pi g) = 0$. Together with the Wronskian relation for the Whittaker functions (A.2.10) this leads to the following relation for the mass gap

$$m = \frac{16\sqrt{2}}{\pi^2}\frac{f_1(-\pi g)}{V_0(-\pi g)} . \tag{3.4.5}$$

It is instructive to compare this relation with the similar relation (3.3.3) for the cusp anomalous dimension. We observe that both quantities involve the same function $f_1(4\pi gt)$ but evaluated for different values of its argument, that is $t = -1/4$ for the mass gap and $t = 0$ for the cusp anomalous dimension. As a consequence, there are no reasons to expect that the two functions $m = m(g)$ and $\Gamma_{\text{cusp}}(g)$ could be related to each other in a simple way. Nevertheless, we will demonstrate that m^2 determines the leading non-perturbative correction to $\Gamma_{\text{cusp}}(g)$ at strong coupling.

3.4.2 Strong Coupling Expression

We will first determine the strong coupling expression of the functions $f_0(4\pi gt)$ and $f_1(4\pi gt)$. We recall that they depend on the coefficients $c_\pm(n,g)$ and the functions $U_{0,1}^\pm(4\pi ng)$ as follows,

$$\begin{aligned} f_0(4\pi gt) &= \sum_{n\geq 1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{n+t}\right] - 1 , \\ f_1(4\pi gt) &= \sum_{n\geq 1} n\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{n+t}\right] . \end{aligned} \tag{3.4.6}$$

The coefficients $c_\pm(n,g)$ admit an expansion in integer powers of $\Lambda^2 \propto g^{1/2}\exp(-2\pi g)$, while the U-functions do not[21]. As a consequence, the functions $f_0(4\pi gt)$ and $f_1(4\pi gt)$ have the form

$$f_k(4\pi gt) = f_k^{(\text{PT})}(4\pi gt) + \delta f_k(4\pi gt) , \qquad (k = 0, 1) . \tag{3.4.7}$$

Here $f_k^{(\text{PT})}$ is the perturbative $O(\Lambda^0)$ contribution, given at strong coupling by a (non-Borel summable) asymptotic series in $1/g$. The remainder δf_k takes into account non-perturbative corrections running in integer powers of Λ^2. In the following, we will only be interested in the first non-perturbative contribution, $\delta f_k = O(\Lambda^2)$, but subleading terms can be incorporated in a similar way.

To compute the functions $f_k^{(\text{PT})}$ at strong coupling, we replace the coefficients $c_\pm(n,g)$ in (3.4.6) by their $O(\Lambda^0)$ contributions, see Eq. (3.3.13), and take into account the obtained results for the

[21] The U-functions are directly related to the F-functions, which we know do not suffer from the Stokes phenomenon. The latter property implies that the U-functions should be completely determined by their asymptotic series. Therefore, they do not depend on the non-perturbative scale Λ^2.

functions $a_\pm, b_\pm, \dots$, Eqs. (3.3.15), (3.3.17). In addition, we replace in (3.4.6) the U-functions by their strong coupling expansion, Eqs. (A.2.14). We proceed similarly for δf_k, using the explicit expressions for the functions $\alpha_\pm, \beta_\pm, \dots$, Eqs. (3.3.22), (3.3.24). Doing so, we find that $f_0(4\pi g t)$ and $f_1(4\pi g t)$ can be expressed in terms of two sums involving the functions $a_\pm(n)$ defined in (3.3.15)

$$2\Gamma(\tfrac{5}{4})\sum_{n\geq 1}\frac{a_+(n)}{t-n} = \frac{1}{t}\left[\frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{\Gamma(\frac{3}{4}-t)} - 1\right],$$
$$2\Gamma(\tfrac{3}{4})\sum_{n\geq 1}\frac{a_-(n)}{t+n} = \frac{1}{t}\left[\frac{\Gamma(\frac{1}{4})\Gamma(1+t)}{\Gamma(\frac{1}{4}+t)} - 1\right]. \quad (3.4.8)$$

Going through calculation of (3.4.6), we find after some algebra that the perturbative corrections to $f_0(4\pi g t)$ and $f_1(4\pi g t)$ are given by linear combinations of the ratios of Euler gamma-functions

$$\begin{aligned} f_0^{(\mathrm{PT})}(4\pi g t) &= -\frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{\Gamma(\frac{3}{4}-t)} \\ &+ \frac{1}{4\pi g}\left[\left(\frac{3\log 2}{4} + \frac{1}{8t}\right)\frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{\Gamma(\frac{3}{4}-t)} - \frac{\Gamma(\frac{1}{4})\Gamma(1+t)}{8t\,\Gamma(\frac{1}{4}+t)}\right] + O(g^{-2}), \\ f_1^{(\mathrm{PT})}(4\pi g t) &= \frac{1}{4\pi g}\left[\frac{\Gamma(\frac{1}{4})\Gamma(1+t)}{4t\,\Gamma(\frac{1}{4}+t)} - \frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{4t\,\Gamma(\frac{3}{4}-t)}\right] \\ &- \frac{1}{(4\pi g)^2}\left[\frac{\Gamma(\frac{1}{4})\Gamma(1+t)}{4t\,\Gamma(\frac{1}{4}+t)}\left(\frac{1}{4t} - \frac{3\log 2}{4}\right) - \frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{4t\,\Gamma(\frac{3}{4}-t)}\left(\frac{1}{4t} + \frac{3\log 2}{4}\right)\right] + O(g^{-3}). \end{aligned} \quad (3.4.9)$$

In a similar manner, we compute the non-perturbative corrections to (3.4.7)

$$\begin{aligned} \delta f_0(4\pi g t) &= \sigma\Lambda^2\left\{\frac{1}{4\pi g}\left[\frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{2\,\Gamma(\frac{3}{4}-t)} - \frac{\Gamma(\frac{5}{4})\Gamma(1+t)}{2\,\Gamma(\frac{5}{4}+t)}\right] + O(g^{-2})\right\} + \dots, \\ \delta f_1(4\pi g t) &= \sigma\Lambda^2\left\{\frac{1}{4\pi g}\frac{\Gamma(\frac{5}{4})\Gamma(1+t)}{\Gamma(\frac{5}{4}+t)}\right. \\ &\left.+ \frac{1}{(4\pi g)^2}\left[\frac{\Gamma(\frac{3}{4})\Gamma(1-t)}{8t\Gamma(\frac{3}{4}-t)} - \frac{\Gamma(\frac{5}{4})\Gamma(1+t)}{\Gamma(\frac{5}{4}+t)}\left(\frac{1}{8t} + \frac{3}{4}\log 2 - \frac{1}{4}\right)\right] + O(g^{-3})\right\} + \dots, \end{aligned} \quad (3.4.10)$$

where ellipses denote $O(\Lambda^4)$ terms. Notice that the strong coupling expansion of both $f_{0,1}^{(\mathrm{PT})}(4\pi g t)$ and $\delta f_{0,1}(4\pi g t)$ is taken at a fixed value of t.

To verify the obtained expressions, we apply (3.3.3) to calculate the cusp anomalous dimension

$$\Gamma_{\mathrm{cusp}}(g) = 2g - 4g f_1^{(\mathrm{PT})}(0) - 4g\,\delta f_1(0). \quad (3.4.11)$$

Replacing $f_1^{(\mathrm{PT})}(0)$ and $\delta f_1(0)$ by their expressions, Eqs. (3.4.9) and (3.4.10), we obtain

$$\Gamma_{\mathrm{cusp}}(g) = 2g\left[1 - \frac{3\log 2}{4\pi g} - \frac{\mathrm{K}}{(4\pi g)^2} + \dots\right] - \frac{\sigma\Lambda^2}{\pi}\left[1 + \frac{3 - 6\log 2}{16\pi g} + \dots\right] + O(\Lambda^4), \quad (3.4.12)$$

in agreement with our previous findings (3.3.19) and (3.3.26), respectively.

To obtain the strong coupling expression for the mass scale (3.4.5), we first recall that $V_0(-\pi g)$ depends on the non-perturbative scale Λ^2. Therefore, we decompose it in two F-functions, see Eq. (A.2.17), and then replace each term by its asymptotic series, Eq. (A.2.14). Taking into account (3.4.9) and (3.4.10) we get[22]

$$m = \frac{\sqrt{2}}{\Gamma(\frac{5}{4})}(2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\left[1 + \frac{3-6\log 2}{32\pi g} + O\left(1/g^2\right)\right]. \qquad (3.4.13)$$

The comparison with the parameter $m_{\rm cusp}$, given in Eq. (3.3.28), shows that the identity $m = m_{\rm cusp}$ is satisified, at least up to $O(1/g)$. We will show in the next subsection that it holds at strong coupling to all orders in $1/g$.

3.4.3 Cusp Anomalous Dimension and Mass Gap

We demonstrated that the strong coupling expansion of the cusp anomalous dimension has the form

$$\Gamma_{\rm cusp}(g) = \left[2g - \frac{3\log 2}{2\pi} - \frac{\rm K}{8\pi^2 g} + O(1/g^2)\right] - \frac{\sigma}{4\sqrt{2}} m_{\rm cusp}^2 + O(m_{\rm cusp}^4)\,, \qquad (3.4.14)$$

with the leading non-perturbative correction given to the first few orders in the $1/g$ expansion by the mass scale m, that is $m_{\rm cusp} = m$. Let us show that this relation is exact to all orders in the strong coupling expansion.

According to (3.4.11), the non-perturbative corrections to the cusp anomalous dimension are given by

$$\delta\Gamma_{\rm cusp} = -4g\,\delta f_1(0)\,, \qquad (3.4.15)$$

with $\delta f_1(0)$ denoting $O(\Lambda^2)$ correction to the function $f_1(t=0)$, Eq. (3.4.7). The latter function verifies the quantization conditions (3.3.1). As was already explained, the $O(\Lambda^2)$ corrections to solutions of (3.3.1) originate from subleading, exponentially suppressed terms in the strong coupling expansion of the functions $V_0(-\pi g)$ and $V_1(-\pi g)$, that we shall denote as $\delta V_0(-\pi g)$ and $\delta V_1(-\pi g)$, respectively. Using the identities (A.2.17) and (A.2.19), we find

$$\begin{aligned} \delta V_0(-\pi g) &= \sigma\frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}\,U_0^-(\pi g)\,, \\ \delta V_1(-\pi g) &= \sigma\frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}\,U_1^-(\pi g)\,, \end{aligned} \qquad (3.4.16)$$

where $\sigma = \exp(-3i\pi/4)$. Then, we split the functions $f_0(t)$ and $f_1(t)$ entering the quantization conditions (3.3.1) into perturbative and non-perturbative parts according to (3.4.7) and compare exponentially small terms in both sides of (3.3.1) to get

$$\delta f_0(t_\ell)V_0(t_\ell) + \delta f_1(t_\ell)V_1(t_\ell) = -\xi\delta_{\ell,0}\,, \qquad (3.4.17)$$

[22]Note that the mass scale m also receives subleading non-perturbative corrections.

where $t_\ell = 4\pi g\big(\ell - \frac{1}{4}\big)$ and the notation was introduced for[23]

$$\xi = f_0(-\pi g)\delta V_0(-\pi g) + f_1(-\pi g)\delta V_1(-\pi g)\,. \tag{3.4.18}$$

Taking into account the relations (3.4.16) and comparing the resulting expression for ξ with (3.4.4) we find that

$$\xi = -\frac{\sigma}{8g}m\,. \tag{3.4.19}$$

Now, to compute non-perturbative $O(\Lambda^2)$ correction to the cusp anomalous dimension, we have to solve the system of relations (3.4.17), determine the function $\delta f_1(t)$ and, then, apply (3.4.15). We will show in this subsection that the result reads

$$\delta f_1(0) = -\frac{\sqrt{2}}{4}\xi m = \sigma\frac{\sqrt{2}}{32g}\,m^2\,, \tag{3.4.20}$$

Together with (3.4.15) this leads to the desired expression $m = m_{\rm cusp}$ for the leading non-perturbative correction to the cusp anomalous dimension (3.4.14). Note that we will prove that the relation (3.4.20) is an exact consequence of (3.4.17) and (3.4.19), valid at arbitrary coupling $g > 0$. However, given the fact that the equations (3.4.17) and (3.4.19) only describe the functions $\delta f_{0,1}$ up to subleading $O(m_{\rm cusp}^4)$ non-perturbative corrections, the result $m = m_{\rm cusp}$ will be valid only to all orders in the $1/g$ expansion. Nevertheless, given this perturbative agreement, it is always possible to define the parameter $m_{\rm cusp}$ such that $m = m_{\rm cusp}$ exactly.

To begin with, let us introduce the function

$$\delta\Gamma(it) = \delta f_0(t)V_0(t) + \delta f_1(t)V_1(t)\,. \tag{3.4.21}$$

Here $\delta f_0(t)$ and $\delta f_1(t)$ are given by

$$\begin{aligned}\delta f_0(t) &= \sum_{n\geq 1} t\left[\delta c_+(n,g)\frac{U_1^+(4\pi ng)}{4\pi ng - t} + \delta c_-(n,g)\frac{U_1^-(4\pi ng)}{4\pi ng + t}\right]\,,\\ \delta f_1(t) &= \sum_{n\geq 1} 4\pi ng\left[\delta c_+(n,g)\frac{U_0^+(4\pi ng)}{4\pi ng - t} + \delta c_-(n,g)\frac{U_0^-(4\pi ng)}{4\pi ng + t}\right]\,,\end{aligned} \tag{3.4.22}$$

where the coefficients $\delta c_\pm(n,g) = O(\Lambda^2)$ are the leading non-perturbative contributions to $c_\pm(n,g)$. We find for $t = 0$ that $\delta f_0(0) = 0$ for arbitrary coupling, leading to

$$\delta\Gamma(0) = 2\delta f_1(0) \tag{3.4.23}$$

We recall that, for arbitrary $\delta c_\pm(n,g)$, the function (3.4.21) satisfies the homogeneous integral equation (3.2.5), i.e.

$$\int_0^\infty dt\left[{\rm e}^{itu}\,\delta\Gamma_-(t) - {\rm e}^{-itu}\,\delta\Gamma_+(t)\right] = 0\,, \qquad (-1\leq u\leq 1)\,, \tag{3.4.24}$$

[23]Note that we replaced $f_{0,1}^{(\rm PT)}(-\pi g)$ on the right-hand side of (3.4.18) by the total functions $f_{0,1}(-\pi g)$, and, similarly, we replaced $V_{0,1}^{(\rm PT)}(t_\ell)$ on the left-hand side of (3.4.17) by their complete expressions $V_{0,1}(t_\ell) = V_{0,1}^{(\rm PT)}(t_\ell) + \delta V_{0,1}(t_\ell)$. The discrepancy between the two set of functions induces a subleading $O(\Lambda^4)$ effect, that we will not consider.

where $\delta\Gamma(t) = \delta\Gamma_+(t) + i\delta\Gamma_-(t)$ and $\delta\Gamma_\pm(-t) = \pm\delta\Gamma_\pm(t)$. In other words, the function $\delta\Gamma(t)$ is a sum of zero-mode solutions, each of them parameterized by the coefficient $\delta c_+(n,g)$ or $\delta c_-(n,g)$.

As before, in order to construct the solution to (3.4.24), we have to specify additional conditions for $\delta\Gamma(it)$. The function $\delta\Gamma(it)$ shares with $\Gamma(it)$ an infinite set of simple poles located at the same position

$$\delta\Gamma(it) \sim \frac{1}{t - 4\pi g\ell}\,, \qquad (\ell \in \mathbb{Z}^*)\,. \tag{3.4.25}$$

In addition, we deduce from (3.4.17) that it also satisfies the relation (with $x_\ell = \ell - \frac{1}{4}$)

$$\delta\Gamma(4\pi i g x_\ell) = -\xi\delta_{\ell,0}\,, \qquad (\ell \in \mathbb{Z})\,, \tag{3.4.26}$$

and, therefore, has an infinite number of zeros. An important difference with $\Gamma(it)$ is that $\delta\Gamma(it)$ does not vanish at $t = -\pi g$ and its value is fixed by the parameter ξ defined in (3.4.19).

Having in mind the similarities between the functions $\Gamma(it)$ and $\delta\Gamma(it)$ we define a new function

$$\delta\gamma(it) = \frac{\sin(t/4g)\sin(\pi/4)}{\sin(t/4g + \pi/4)}\delta\Gamma(it)\,. \tag{3.4.27}$$

As before, the poles and zeros of $\delta\Gamma(it)$ are compensated by the ratio of sinus functions. However, in distinction with $\gamma(it)$ and in virtue of $\delta\Gamma(-i\pi g) = -\xi$, the function $\delta\gamma(it)$ has a single pole at $t = -\pi g$ with the residue equal to $2g\xi$. For $t \to 0$ we find from (3.4.27) that $\delta\gamma(it)$ vanishes as

$$\delta\gamma(it) = \frac{t}{4g}\delta\Gamma(0) + O(t^2) = \frac{t}{2g}\delta f_1(0) + O(t^2)\,, \tag{3.4.28}$$

where in the second relation we applied (3.4.23). It is convenient to split the function $\delta\gamma(t)$ into the sum of two terms of a definite parity, $\delta\gamma(t) = \delta\gamma_+(t) + i\delta\gamma_-(t)$ with $\delta\gamma_\pm(-t) = \pm\delta\gamma_\pm(t)$. Then, combining together (3.4.24) and (3.4.27) we obtain that the functions $\delta\gamma_\pm(t)$ satisfy the infinite system of homogeneous equations (for $n \geqslant 1$)

$$\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_-(t)}{1 - \mathrm{e}^{-t/2g}} + \frac{\delta\gamma_+(t)}{\mathrm{e}^{t/2g} - 1}\right] J_{2n-1}(t) = 0\,,\\
&\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_+(t)}{1 - \mathrm{e}^{-t/2g}} - \frac{\delta\gamma_-(t)}{\mathrm{e}^{t/2g} - 1}\right] J_{2n}(t) = 0\,.
\end{aligned} \tag{3.4.29}$$

By construction, the solution to this system $\delta\gamma(t)$ should vanish at $t = 0$ and have a simple pole at $t = -i\pi g$. We recognize in (3.4.29), up to an inhomogeneous term in the right-hand side, the system determining the solution to the BES equation $\gamma(t)$. As we show in Appendix A.4, this fact permits to derive a Wronskian-like relation between the functions $\delta\gamma(t)$ and $\gamma(t)$. This relation turn out to be powerful enough to determine the small t asymptotics of the function $\delta\gamma(t)$ in terms of $\gamma(t)$, or equivalently $\Gamma(t)$. In this way we obtain

$$\delta\gamma(it) = -\xi t\left[\frac{2}{\pi^2 g}\,\mathrm{e}^{-\pi g} - \frac{\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\int_0^\infty \frac{dt'}{t' + i\pi g}\,\mathrm{e}^{i(t'-\pi/4)}\,\Gamma(t')\right] + O(t^2)\,. \tag{3.4.30}$$

Comparing the expression inside the square brackets with the definition of the mass scale m, Eq. (3.4.1), we find that

$$\delta\gamma(it) = -\xi m\frac{t\sqrt{2}}{8g} + O(t^2)\,. \tag{3.4.31}$$

Matching this relation into (3.4.28), we obtain the desired expression for $\delta f_1(0)$, Eq. (3.4.20). Then, we substitute it into (3.4.15) and compute the leading non-perturbative correction to the cusp anomalous dimension leading to

$$m_{\rm cusp} = m\,. \tag{3.4.32}$$

Thus, we demonstrated in this section that the leading non-perturbative, exponentially small correction to the cusp anomalous dimensions at strong coupling are determined to all orders in $1/g$ by the mass scale m.

Chapter 4

Scaling Function and O(6) Sigma Model

In this chapter, we will explore further the scaling behavior of the minimal anomalous dimension in the limit when both the spin of the Wilson operators N and their twist L are large. Namely, we will allow L to grow logarithmically with N and determine the large N expansion in the limit $N,\ L \to \infty$ for fixed scaling variable $j = L/\log N$. In this double scaling limit, the minimal anomalous still scales logarithmically with N but the coefficient in front of $\log N$ receives besides the cusp anomalous dimension an additional contribution which depends explicitely on j [45]. The minimal anomalous dimension thus takes the following form for $N, L \to \infty$ with $j = L/\log N$ fixed [45, 59, 46, 47]

$$\delta\Delta_{\min} = f(g,j)\log N + \dots\,, \tag{4.0.1}$$

where dots stand for $O(\log^0 N)$ corrections. For $j = 0$, the scaling function $f(g,j)$ reduces to the cusp anomalous dimension

$$f(g,j=0) = 2\Gamma_{\text{cusp}}(g)\,, \tag{4.0.2}$$

and the non-trivial dependence on j is encoded in the function $\epsilon(g,j)$ defined as

$$\epsilon(g,j) = f(g,j) - f(g,0)\,. \tag{4.0.3}$$

This function can be computed in planar $\mathcal{N}=4$ SYM theory thanks to integrability [45, 47]. In a previous chapter, we illustrated the power of integrability on the example of one-loop Bethe ansatz equations in the $\mathfrak{sl}(2)$ sector. Applying the all-loop Bethe ansatz equations [41, 43, 44], it is possible to derive an integral equation [47] whose solution determines the scaling function $\epsilon(g,j)$ exactly for any value of j and g. At strong coupling, this function can be investigated with the help of AdS/CFT correspondence [45, 15, 59, 61] by computing the semiclassical energy of a string rotating in AdS_3 with a large spin N and boosted along a big circle of S^5 with a large momentum L [14, 15].

For general values of g and j, the scaling function $\epsilon(g,j)$ has a quite complicated form. To get an insight into the properties of this function, it is instructive to examine its asymptotic behavior in different parts of the parameter space. We will study in detail the regime of small j and strong coupling $g \gg 1$. In this regime, the scaling function can be computed exactly from string theory and, according to the proposal put forward in [46], it should be related to the energy density in the ground state of non-linear O(6) sigma model with the particle density $\rho \equiv j/2 \sim m$ and $m \sim \mathrm{e}^{-\pi g}$ being the mass gap of the O(6) sigma model. In this chapter, we will verify this proposal on both sides of the AdS/CFT correspondence [46, 61, 74, 73, 75, 138, 76] and, then, compare the two predictions for the mass gap, coming from gauge [74, 73, 76] and string [46, 76] theory considerations, respectively.

4.1 Scaling Function in String Theory

In this section we study the scaling function $\epsilon(g, j)$ at strong coupling from string theory. We first analyse it with the help of the semiclassical string expansion and then by means of the O(6) sigma model. By comparing the two approaches, we will extract the expression of the mass gap $m = m(g)$ up to two loops at strong coupling.

4.1.1 Semiclassical String Expansion

We start with few remarks about the string theory dual to planar $\mathcal{N} = 4$ SYM theory and the spinning string solution dual to spin N, twist L Wilson operator. We refer the reader to the literature mentioned below for comprehensive discussions.

General Remarks

The AdS/CFT correspondence predicts that the spectrum of scaling dimensions of single-trace operators in planar $\mathcal{N} = 4$ SYM theory coincides with the energy spectrum of closed strings propagating on $\mathrm{AdS}_5 \times \mathrm{S}^5$. To find the spectrum of string excitations, one needs to solve the string sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. The latter is given by a two-dimensional field theory which is uniquely fixed by its symmetries [12] and only depends on the string tension $\sqrt{\lambda}/2\pi = 2g$ with g^2 being the 't Hooft coupling constant in gauge theory. The inverse string tension $\sim 1/\sqrt{\lambda}$ plays the role of a coupling constant in the σ-model and it controls the size of the quantum fluctuations of the string. Then when the tension is large $\sqrt{\lambda} \gg 1$, the σ-model is weakly coupled while the dual gauge theory is strongly coupled $g \gg 1$.

To compute the energy spectrum of the string, one has to quantize the σ-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. This is a very complicated problem that still awaits it solution. It turned out that for higher excited states in the energy spectrum of the string, the problem can be circumvented by applying semiclassical methods [14, 15]. In this approach, the starting point is an identification of the solitonic-like solution to the classical string equations of motion with the relevant global conserved charges fixed by quantum numbers of Wilson operators in dual gauge theory. Expanding and quantizing the σ-model around this background solution, one builds up the string spectrum above the corresponding excited state, which here plays the role of a ground state for the two-dimensional theory. Note that one usually starts with a classical solution with minimal energy for given global conserved charges to ensure the stability of the vacuum. In general, if the background solution is not too involved, one may perform a perturbative quantization of the theory at $\sqrt{\lambda} \gg 1$. It allows in particular to find the ground-state energy in the form of an expansion in inverse powers of $\sqrt{\lambda} \propto g \gg 1$. The leading order contribution to this semiclassical or strong coupling expansion is simply given by the classical energy of the soliton, while the subleading terms are induced by the quantum fluctuations. In this way one can work out the energy of a string dual to a Wilson operator carrying a large Lorentz spin N and/or a large twist L [14, 15].

Before considering it, let us finally say few words about the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [11]. The metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ decomposes into the direct sum of the metrics of two subspaces. By construction, the curvature radius of AdS_5 and S^5 are equal to each other and we normalize their value to be 1. The Euclidean metric of S^5 takes the well-known form while the Minkowskian metric of AdS_5 in global coordinates reads

$$ds^2 = \cosh^2 r\, dt^2 - dr^2 - \sinh^2 r\, d\Omega_3^2 \,. \tag{4.1.1}$$

Here $t \in (-\infty, \infty)$ and $r \in (0, \infty)$ are respectively the time and radial coordinates, while $d\Omega_3^2$ is the metric on the unit sphere S^3. The boundary of AdS_5 is located at $r = \infty$ and it is conformally equivalent to $\mathbb{R} \times \mathrm{S}^3$ with metric $ds^2 = dt^2 - d\Omega_3^2$. The dual gauge theory "lives" on the boundary of AdS_5 and the AdS/CFT correspondence establishes the correspondence between the Wilson operators in this theory and quantum states of the strings propagating in $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the following, we will be dealing with strings rotating in $\mathrm{AdS}_3 \subset \mathrm{AdS}_5$. In that case the sphere S^3 in (4.1.1) is reduced to a big circle S^1 with angular coordinate $\varphi \in (0, 2\pi)$ and the metric takes the form

$$ds^2 = \cosh^2 r\, dt^2 - dr^2 - \sinh^2 r\, d\varphi^2 \,. \tag{4.1.2}$$

The boundary $r = \infty$ of AdS_3 space is then conformally equivalent to a cylinder $\mathbb{R} \times \mathrm{S}^1$ with metric $ds^2 = dt^2 - d\varphi^2$. The Lorentz spin N of a Wilson operator in the dual gauge theory is translated into angular momentum of the string, that is the charge conjugated to shift in φ. In the similar manner, the twist L of the Wilson operator in consideration is conjugated to a rotation in S^5.

Spinning String

The string state that is dual to a Wilson operator carrying a Lorentz spin N and a twist L has two angular momenta N and L corresponding to rotations on $\mathrm{AdS}_3 \subset \mathrm{AdS}_5$ and on $\mathrm{S}^1 \subset \mathrm{S}^5$, respectively [11]. Applying the AdS/CFT dictionary [8] its energy E is equal to the scaling dimension Δ of the dual Wilson operator

$$E = \Delta = N + L + \delta\Delta \,, \tag{4.1.3}$$

where $\delta\Delta = O(g^2)$ is the anomalous dimension in $\mathcal{N} = 4$ SYM theory. Since we are interested in conformal operators with minimal anomalous dimension, we will focus on a string state with minimal energy for given N and L. In flat space-time, a string state with minimal energy for a given spin N lies on leading Regge trajectory. In the limit of a large number of excitations $N \gg 1$ the state becomes semiclassical and it can be described by a stretched (folded) classical string rotating around its center-of-mass [5, 14]. As proposed in [14], the same physical picture should also apply in AdS and the corresponding classical configuration is a folded string spinning in the curved AdS_3 background with the angular momentum N. In addition, for large values of twist L, the string gets boosted along a big circle of S^5 with the angular momentum L [15].

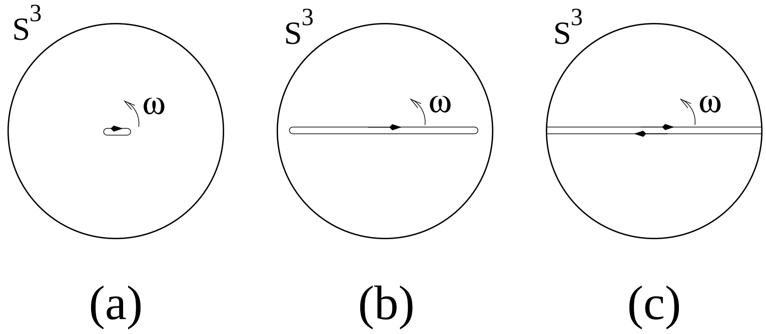

Figure 4.1: Transition from the short folded rotating string to the infinitely long string limit [14, 15]: (a) short string $\mathcal{S} \ll 1$ with length $\sim 4\,(\mathcal{S}/2\pi)^{1/2}$, (b) long string $\mathcal{S} \gg 1$ with length $\sim 2\log\mathcal{S}$ and (c) infinitely long string $\mathcal{S} = \infty$ stretched up to the boundary of AdS_5. Here the interior of the disk represents a section of AdS_5 at a given time t and the circle its boundary S^3. The frequency ω interpolates between (a) $\omega \sim (2\pi/\mathcal{S})^{1/2}$ at $\mathcal{S} \sim 0$ and (c) $\omega = 1$ at $\mathcal{S} = \infty$. The momentum $\mathcal{J}$ is assumed all the way much smaller than the spin $\mathcal{S}$.

To obtain the strong coupling expansion of the string energy E at large N and L, one has to quantize semiclassically the σ-model around the classical string solution mentioned above [14, 15]. The coupling constant does not play any role in the classical string dynamics except to set the scale of the Noether charges. It is then convenient to introduce[1]

$$\mathcal{E} \equiv E/g\,, \qquad \mathcal{S} \equiv N/g\,, \qquad \mathcal{J} \equiv L/g\,. \tag{4.1.4}$$

A classical solution is parameterized by $\mathcal{S}$ and $\mathcal{J}$, that are kept fixed in the semiclassical string expansion $g \gg 1$. The energy $\mathcal{E}$ is then found as a series in inverse powers of the coupling constant [14, 15, 59, 61]

$$E = g\,\mathcal{E}_0(\mathcal{S},\mathcal{J}) + \mathcal{E}_1(\mathcal{S},\mathcal{J}) + O(1/g)\,, \tag{4.1.5}$$

with $\mathcal{E}_0, \mathcal{E}_1, \ldots$ being functions of $\mathcal{S}$ and $\mathcal{J}$, but independent of the coupling constant. Here, the leading contribution $\mathcal{E}_0$ is the classical string energy while $\mathcal{E}_1$ is the one-loop correction to the energy induced by the quadratic fluctuations around the classical solution. The semiclassical expansion (4.1.5) being defined for large g and fixed $(\mathcal{S},\mathcal{J})$, it yields predictions for anomalous dimension at strong coupling of Wilson operators carrying large spin N and twist L.

Due to the curvature of the AdS_3 background, the leading classical contribution $\sim \mathcal{E}_0(\mathcal{S},\mathcal{J})$ to the energy of the string (4.1.5) is given by a complicated function of $\mathcal{S}$ and $\mathcal{J}$ that interpolates between different regimes [14, 15].[2] For instance, for small angular momentum $\mathcal{S} \ll 1$ the string is short. If, in addition, the rotation of the center-of-mass is slow, $\mathcal{J} \ll 1$, the energy $\mathcal{E}_0(\mathcal{S},\mathcal{J})$ is small. We expect that in this limit the string does not feel the curvature of the background. Indeed, according to [14, 15] the energy for $\mathcal{S},\mathcal{J} \ll 1$ reads

$$\mathcal{E}_0(\mathcal{S},\mathcal{J}) = \sqrt{\mathcal{J}^2 + 8\pi\mathcal{S}} + \ldots\,, \tag{4.1.6}$$

[1] Note that the notations here are similar to those of [15, 61] but differ by an overall factor 4π due to the relation $\sqrt{\lambda} = 4\pi g$.

[2] An exact parametric representation valid for any $(\mathcal{S},\mathcal{J})$ can be found in [15]. Here, we will not directly make use of it since we will focus on the large spin regime $\mathcal{S} \gg 1$ for which the explicit expressions are known [14, 15, 45].

and, making use of (4.1.4) and (4.1.5), one finds that it corresponds to the familiar relativistic dispersion relation of a classical string in a flat space-time

$$E \simeq E_{\text{flat-space}} = \sqrt{L^2 + 8\pi g N}\,. \tag{4.1.7}$$

Here L is the momentum of the center-of-mass of the string while $M^2 = 8\pi g N = 2N/\alpha'$ is the squared mass of the folded rotating string with angular momentum N and Regge slope $\alpha' = 1/4\pi g$.[3]

The physics is however quite different in the limit $\mathcal{S} \gg 1$, assuming for a while that $\mathcal{J}$ is kept fixed. In that case it is no longer possible to neglect curvature effects since the string is appreciably stretched out and its energy becomes large, see Figure 4.1. Indeed, according to [14, 15] the string length scales as $\sim 2\log\mathcal{S} \gg 1$ and the energy $\mathcal{E}_0(\mathcal{S},\mathcal{J})$ behaves as

$$\mathcal{E}_0(\mathcal{S},\mathcal{J}) = \mathcal{S} + 4\log\mathcal{S} + \dots\,. \tag{4.1.8}$$

Substituting this relation into (4.1.5), we find for $N \gg g \gg 1$

$$E = N + 4g\log N + \dots\,. \tag{4.1.9}$$

We immediately verify that this expression for the energy E of a long rotating string (4.1.9) is in perfect agreement with the all-loop gauge theory prediction for scaling dimension of Wilson operators [49]

$$\Delta_{\text{min}} = N + 2\Gamma_{\text{cusp}}(g)\log N + \dots\ , \tag{4.1.10}$$

with the cusp anomalous dimension $\Gamma_{\text{cusp}}(g)$ being a function of the coupling constant independent of the twist L. The AdS/CFT correspondence implies that $E = \Delta_{\text{min}}$, thus leading to the strong coupling expression of the cusp anomalous dimension [14]

$$\Gamma_{\text{cusp}}(g) = 2g + O(1)\,. \tag{4.1.11}$$

In order to put this matching on firm grounds, it is important to verify that the radiative corrections to the classical energy of a long rotating string (4.1.9) do not scale stronger than the first power of $\log\mathcal{S} \simeq \log N$. As proposed in [14] and argued in [15, 46] this is indeed the case and the energy of a long rotating string scales to all orders in the strong coupling expansion as

$$E = N + 2\mathcal{F}(g)\log N + \dots\,. \tag{4.1.12}$$

Here $\mathcal{F}(g) = 2g + O(1)$ is a function of the coupling constant only. It has been computed explicitely to two loops in [14, 15, 60]. Complete agreement with the gauge theory requires that $\mathcal{F}(g) = \Gamma_{\text{cusp}}(g)$. By definition, the cusp anomalous dimension governs the scale dependence of the vev of a cusped lightlike Wilson loop in $\mathcal{N} = 4$ SYM theory. At strong coupling, in the dual stringy picture, the Wilson loop is determined by the area of the minimal surface ending at the boundary of AdS_5 on the contour drawn by the Wilson loop [124, 125]. This offers a way

[3] One-loop quantum correction $\mathcal{E}_1(\mathcal{S},\mathcal{J})$ to the energy of a short spinning string has been computed in [126] for $\mathcal{J} = 0$.

to compute the cusp anomalous dimension at strong coupling. To two loop order this was done in [113, 114, 115, 116] leading to

$$\Gamma_{\rm cusp}(g) = \mathcal{F}(g) = 2g - \frac{3\log 2}{2\pi} - \frac{\rm K}{8\pi^2 g} + O(1/g^2)\,, \tag{4.1.13}$$

with K the Catalan's constant. More generally the relation $\mathcal{F}(g) = \Gamma_{\rm cusp}(g)$ was shown in [115] to be exact to all orders in the strong coupling expansion.

4.1.2 String Scaling Function

As was already mentioned, the energy of a long rotating string (4.1.12) gets corrected when its angular momentum $\mathcal{J}$ corresponding to rotation of the center-of-mass of the string along $\mathrm{S}^1 \subset \mathrm{S}^5$ becomes large simultaneously with $\mathcal{S}$. Based on gauge theory considerations, we expect that the appropriate scaling variable in the limit when both the Lorent spin N and the twist L take large values is given by $j = L/\log N$. For N, $L \to \infty$ and j fixed, the minimal scaling dimension has the asymptotic behavior [45, 46, 47]

$$\Delta_{\rm min} = N + L + f(g,j)\log N + \dots\,. \tag{4.1.14}$$

where the function $f(g,j)$ split into the sum of two terms

$$f(g,j) = 2\Gamma_{\rm cusp}(g) + \epsilon(g,j)\,. \tag{4.1.15}$$

By construction, the scaling function $\epsilon(g,j)$ vanishes at $j = 0$, and for small j it admits an expansion in integer powers of $j = L/\log N$ [45, 47].

At strong coupling, the scaling function $f(g,j)$ should match the prediction of the string theory. It is convenient to rewrite the relation (4.1.14) as

$$E = N + 2\mathcal{F}(g,j)\log N + \dots\,, \tag{4.1.16}$$

with

$$\mathcal{F}(g,j) = \frac{f(g,j)+j}{2} = \Gamma_{\rm cusp}(g) + \frac{\epsilon(g,j)+j}{2}\,. \tag{4.1.17}$$

We expect that in the semiclassical stringy approach the scaling function $\mathcal{F}(g,j)$ is given by an expansion in $1/g$ with coefficients being functions of the variables $\mathcal{S} = N/g$ and $\mathcal{J} = L/g$. Namely, introducing a new variable $\ell \equiv \mathcal{J}/4\log\mathcal{S}$ and using the relation $j \equiv L/\log N \simeq L/\log\mathcal{S} = 4g\ell$, we expect that the scaling function $\mathcal{F}(g,j) = \mathcal{F}(g,4g\ell)$ can be found semiclassically as [45, 15, 59, 61]

$$\mathcal{F}(g,j) = \mathcal{F}(g,4g\ell) = g\mathcal{F}_0(\ell) + \mathcal{F}_1(\ell) + \dots\,, \tag{4.1.18}$$

where the strong coupling expansion is performed for a fixed value of the scaling variable ℓ. For $\ell = j = 0$ this expansion takes the form

$$\mathcal{F}(g,0) = \Gamma_{\rm cusp}(g) = g\mathcal{F}_0(0) + \mathcal{F}_1(0) + \dots\,, \tag{4.1.19}$$

In order to understand better the origin of this generalized scaling in the string theory, we will first comment a bit about some remarkable features of the long rotating string $\mathcal{S} \gg 1$ [14, 15, 115, 46]. After this few remarks, we will find quite easily the classical string scaling function $\mathcal{F}_0(\ell)$ and then discuss what happens at the quantum level.

Classical String Scaling Function

To begin with, let us examine the properties of a long rotating string with $\mathcal{S} \gg 1$ first assuming that $\mathcal{J} = 0$. Then, in order to isolate the 'anomalous' $\sim \log N \simeq \log \mathcal{S}$ contribution to its energy, let us define the energy in the rotating frame of the string. This frame rotates with a frequency ω in the global coordinates of AdS_3 and the charge conjugate to shift in time is given by $E - \omega N$. The frequency ω is function of the spin N and it can be found as

$$\omega = \frac{dE}{dN} \,. \tag{4.1.20}$$

This is so because the string has minimal energy E for given spin N. In other words, $F(\omega) = E - \omega N$ defines the free energy of the theory at chemical potential ω, the latter being determined by the minimization condition $\delta F = \delta E - \omega \delta N = 0$, or equivalently (4.1.20). Plugging the long string classical energy $E = N + 4g \log N + \ldots$ into (4.1.20) we obtain that $\omega = 1 + 4g/N + \ldots = 1 + 4/\mathcal{S} + \ldots$ [14]. Thus the energy in the rotating frame is given by $E - N$, up to subleading $o(\log \mathcal{S})$ corrections at large $\mathcal{S}$.

Now, let us work out the induced metric on the worldsheet of the classical string. It can be easily obtained from (4.1.2) by using that $\varphi = \omega t$ and one finds

$$ds^2 = (\cosh^2 r - \omega^2 \sinh^2 r)\, dt^2 - dr^2 \,. \tag{4.1.21}$$

Here the radial coordinate r is assumed to take negative values $-r_0 \leq r \leq r_0$ which correspond to the string stretched along the segment $(-r_0, r_0)$ with r_0 the distance between the turning points of the string and the origin $r = 0$. Note that since the string is folded its length R is given by $R = 4r_0$. Then, taking into account the redshift factor in (4.1.21), we find that the energy $E - \omega N$ is given by [14]

$$E - \omega N = 8g \int_0^{r_0} dr \sqrt{\cosh^2 r - \omega^2 \sinh^2 r} \,, \tag{4.1.22}$$

where the string tension $2g$ has been multiplied by 4 to account for the four equivalent segments of the folded string. We can now observe a remarkable simplification of (4.1.22) when $\omega = 1$. Namely, the energy at rest is uniformely distributed along the string and thus is proportional to the length R

$$E - N = 2gR \,. \tag{4.1.23}$$

As was already explained, for $\mathcal{S} \gg 1$ the length of the string is given by $R = 2 \log \mathcal{S}$. Indeed, it follows from the expression of the induced metric (4.1.21) that the extension of the string along the radial coordinate r_0 and the frequency ω are related to each other as [14]

$$\omega = \coth r_0 \,. \tag{4.1.24}$$

This is so because the string fold points propagate at the speed of light. Using this relation we verify that $r_0 \to \infty$ corresponds to $\omega \to 1$ and that $\omega = 1 + 4/\mathcal{S} + \ldots$ implies $R = 4r_0 = 2 \log \mathcal{S} + \ldots$. Substituting this relation into (4.1.23), we find that the cusp anomalous dimension is equal at strong coupling to the string tension $2g$, or equivalently to the energy density along the string.

Let us understand what happens when the string is boosted with a momentum L along a big circle of S^5. The string that was at rest now effectuates a rigid rotation on a cylinder (r, ψ) in a flat space-time

$$ds^2 = dt^2 - dr^2 - d\psi^2\,, \tag{4.1.25}$$

where ψ is the angular variable on the big circle of S^5. Here we used (4.1.21) evaluated at $\omega = 1$ for the relevant contribution coming from AdS_3. It immediately follows that the energy density $(E-N)/R$ is given by the relativistic law

$$\frac{E-N}{R} = \sqrt{\left(\frac{M}{R}\right)^2 + \left(\frac{L}{R}\right)^2}\,, \tag{4.1.26}$$

where M/R and L/R are respectively the density of mass and of momentum along the string. We found previously that $M/R = 2g$ and $R = 2\log\mathcal{S}$ at large $\mathcal{S}$. Then factoring out the dependence on the coupling constant and introducing the semiclassical variable $\ell = L/M = L/4g\log\mathcal{S} = \mathcal{J}/4\log\mathcal{S}$, we get

$$E - N = 2g\mathcal{F}_0(\ell)\log\mathcal{S}\,, \tag{4.1.27}$$

with

$$\mathcal{F}_0(\ell) = 2\sqrt{1+\ell^2}\,. \tag{4.1.28}$$

This is the correct classical string scaling function [45, 15], which has been successfully reproduced with help of the Bethe ansatz technology in [137, 138, 139].

We thus verified, at the classical level, that the string energy admits a generalized logarithmic scaling at large $\mathcal{S}$, for any fixed value of $\ell = \mathcal{J}/4\log\mathcal{S}$. Expanding (4.1.27) at small ℓ we obtain [15, 45]

$$E - N = 4g\log\mathcal{S}\left(1 + \frac{\mathcal{J}^2}{32\log^2\mathcal{S}} + O(\mathcal{J}^4/\log^4\mathcal{S})\right)\,. \tag{4.1.29}$$

This relation illustrates how the string scaling function resums corrections at large $\mathcal{S}$ enhanced by a large momentum $\mathcal{J}$. Arriving at (4.1.29) we tacitly assumed that $\ell = \mathcal{J}/4\log\mathcal{S}$ is small. In the opposite large ℓ limit, or equivalently $\mathcal{J} \gg \log\mathcal{S}$, the classical energy (4.1.27) looks as [15]

$$E = N + 4g\left(\ell + \frac{1}{2\ell} + O(1/\ell^3)\right)\log\mathcal{S} = N + L + 8g\frac{\log^2\mathcal{S}}{\mathcal{J}} + O(\log^4\mathcal{S}/\mathcal{J}^3)\,. \tag{4.1.30}$$

Notice that this relation is valid at large $\mathcal{S}$ only for $\mathcal{S} \gg \mathcal{J} \gg \log\mathcal{S}$ whereas for $\mathcal{J} \gg \mathcal{S}$ the properties of E are different. A similar phenomenon also happens in the gauge theory [45] and has been discussed in Chapter 2. In string theory, one can easily understand the change of asymptotic behavior by noting that for $\mathcal{S} \sim \mathcal{J} \gg 1$ the string is not long anymore. Indeed, the string length R depends on the momentum $\mathcal{J}$, because it is fixed at the classical level by the lightlike condition imposed at the fold points [15]. The solution to this condition depends on whether the string is boosted along S^1 or not. At large $\mathcal{S}$ and finite ℓ we have $R = 2\log\mathcal{S}$ up to subleading $o(\log\mathcal{S})$ corrections. However, for $\ell \gg 1$ we find instead that $R = 2\log(\mathcal{S}/\mathcal{J}) = 2\log(N/L)$ [15] indicating that the string shortens when $\mathcal{J}$ becomes of order $\mathcal{S}$. Eventually, the string shrinks into a point when $\mathcal{J} \gg \mathcal{S}$ [14, 15] and one recovers the BMN scaling

$$E = N + L + 8g\pi^2\frac{\mathcal{S}}{\mathcal{J}^2} + \dots\,. \tag{4.1.31}$$

The study of this regime was done in [95] to leading order on both sides of the AdS/CFT correspondence.

Quantum String Scaling Function

The previous discussion of the classical string scaling function can be extended to the quantum level. Namely, by expanding the string σ-model around the classical solution for a long rotating string with $\mathcal{J} = 0$ and assuming a static gauge in which the radial and time coordinate do not fluctuate, one obtains a two-dimensional QFT for the transverse (and fermionic) fluctuations of the (super)string [15, 46]. The latter excitations can be viewed as propagating above the classical string background and carrying energy measured with the 2d Hamiltonian H = (global-time) energy − spin = scaling dimension − Lorentz spin.[4] Given that the energy of a long rotating string scales linearly with its length $R = 2\log\mathcal{S}$ to all loops, we expect the cusp anomalous dimension to be equal to the effective (quantum) tension of the string. In other words, the cusp anomalous dimension is equal to the vacuum energy density of the σ-model expanded around the classical string solution [15, 46].

Similar to the cusp anomalous dimension, the scaling function also admits a direct generalization at the quantum level. Namely, $\mathcal{F}(g,j) = (E-N)/R = (E-N)/2\log\mathcal{S}$ is the energy density of an excited string state with minimal energy for a given charge density $\rho = L/R = j/2$ [46]. Here we prefer to think of ρ as a charge density instead of a momentum density, because from the 2d σ-model point of view the target-space momentum L is an internal charge. It is indeed the Noether charge corresponding to rotation of the center of mass of the (super)string along a big circle on S^5. At zero density $\rho = j = 0$, we find the vacuum state (string at rest w.r.t. $S^1 \subset S^5$) with energy density $\mathcal{F}(g, j = 0) = \Gamma_{\rm cusp}(g)$. At finite density $\rho \neq 0$ we have a steady state, made out of transverse fluctuations at (zero-temperature) thermodynamic equilibrium, with energy density

$$\varepsilon(g,j) \equiv \mathcal{F}(g,j) - \mathcal{F}(g,j=0) = \mathcal{F}(g,j) - \Gamma_{\rm cusp}(g) = \frac{\epsilon(g,j)+j}{2}\,. \qquad (4.1.32)$$

Here we used the equation (4.1.17) to relate $\varepsilon(g,j)$ to the gauge scaling function $\epsilon(g,j)$. Given the interpretation above, the string theory should provide a definite prediction for the scaling function at strong coupling and arbitrary j.

Let us start with the semiclassical string scaling function. As was already said, it is found, at large g, as [45, 15, 59, 61]

$$\mathcal{F}(g,j) = \mathcal{F}(g,4g\ell) = g\mathcal{F}_0(\ell) + \mathcal{F}_1(\ell) + g^{-1}\mathcal{F}_2(\ell) + \dots\,, \qquad (4.1.33)$$

where $\mathcal{F}_0(\ell), \mathcal{F}_1(\ell), \dots$ are the classical, one-loop, ... contribution, respectively. We stress that the expansion (4.1.33) assumes $\ell \equiv j/4g \equiv \rho/2g$ to be fixed. Let us understand why. The starting point of the expansion (4.1.33) is the classical boosted string solution, that is characterized by the frequency ν of the rotation along S^1, namely $\psi = \nu t$. The frequency ν parameterizes the background solution and it has to be kept fixed when the perturbative quantization is done

[4] Here we assume the frequency $\omega = 1$ corresponding to $\mathcal{S} \sim \infty$.

around it. To relate the frequency ν to the conserved charges one can make use of the relation

$$\nu = \frac{dE}{dL}\,, \tag{4.1.34}$$

that is similar to the one used for the frequency ω, which was conjugate to the spin N. Then, classically and to leading order at large $\mathcal{S}$, we get [15]

$$\nu = \frac{\ell}{\sqrt{1+\ell^2}}\,. \tag{4.1.35}$$

Therefore, keeping the frequency ν fixed becomes equivalent to working at finite ℓ.[5] We note, moreover, that from the point of view of the string σ-model defined around the string at rest, that is $\nu = 0$, a state with finite ℓ is a highly-excited state with a large charge density $\rho \equiv 2g\ell$, since $g \gg 1$. It is thus not completely surprising to find that this regime of large density can be captured semiclassically by shifting the vacuum to the relevant soliton solution, which in the present case describes a macrosopic rotation at frequency ν.

The classical contribution, $\mathcal{F}_0(\ell)$, to the semiclassical string scaling function (4.1.33) was easy to compute, because the relevant metric was flat. Quantum mechanically the transverse fluctuations of the string feel the curvature of the space-time around the classical string and induce more involved corrections [15, 59, 61]. Remarkably enough the one-loop correction $\mathcal{F}_1(\ell)$ can be computed explicitely [59] and its expression has been reproduced by methods based on the Bethe ansatz equations [137, 138, 139]. Here we will not need this expression but will later give its leading behavior at small ℓ. The two-loop contribution $\mathcal{F}_2(\ell)$ has been obtained in [61] at small ℓ. The small ℓ or more exactly the small $j \equiv 4g\ell$ regime of the string scaling function $\mathcal{F}(g,j)$ is precisely the one we would like to investigate and to compare with the gauge theory. So let us examine the small ℓ expansion of the semiclassical string scaling function in more detail.

The prediction for the gauge scaling function at strong coupling goes as follows. To leading order, we have

$$\varepsilon(g,j) = g\left(\mathcal{F}_0(\ell) - \mathcal{F}_0(0)\right) = 2g\left(\sqrt{1+\ell^2} - 1\right)\,, \tag{4.1.36}$$

which after expanding at small $\ell \equiv j/4g$ and using Eq. (4.1.32) leads to

$$\epsilon(g,j) = -j + \frac{j^2}{8g} - \frac{j^4}{512g^3} + \dots\,. \tag{4.1.37}$$

Comparing Eq. (4.1.37) with the gauge theory small j expansion [45, 47]

$$\epsilon(g,j) = \epsilon_1(g)\,j + \epsilon_2(g)\,j^2 + \epsilon_3(g)\,j^3 + \dots\,, \tag{4.1.38}$$

with one-loop expressions

$$\epsilon_1(g) = -8\log 2\,g^2 + \dots\,, \qquad \epsilon_2(g) = 0 + \dots\,, \qquad \epsilon_3(g) = \frac{7}{12}\pi^2\zeta_3\,g^2 + \dots\,, \tag{4.1.39}$$

one would naively expect that $\epsilon_1(g)$ interpolates between $\epsilon_1(g) = -8\log 2\,g^2$ at weak coupling and $\epsilon_1(g) = -1$ at strong coupling, and so on for $\epsilon_2(g), \dots$. That sounds indeed reasonable since both

[5] Things remain the same at higher loops but the relation between ν and ℓ gets corrected by contributions suppressed by powers of $1/g$ [59, 61].

the gauge and string scaling functions appear to be analytic around $j = \ell = 0$. However, this is not what happens, except for the coefficient $\epsilon_1(g)$. For instance, as we shall see later, the all-loop asymptotic Bethe ansatz equations predicts that $\epsilon_2(g)$ is exactly zero [47], at any value of the coupling constant. It is thus not possible to match the would-be strong coupling string prediction $\epsilon_2(g) = 1/8g + \dots$ ($g \gg 1$). Moreover, the expansion of the scaling function in integer powers of ℓ does not hold anymore at higher-loop on the string theory side [59, 61]. Indeed, the first two terms of the small ℓ expansion of the two-loop string scaling function reads explicitely [59, 61]

$$\varepsilon(g,j) = g\ell^2 \left[1 + \frac{1}{\pi g}\left(\frac{3}{4} - \log\ell\right) + \frac{1}{4\pi^2 g^2}\left(\frac{q_{02}}{2} - 3\log\ell + 4\log^2\ell\right)\right] + O(\ell^4)\,, \qquad (4.1.40)$$

where q_{02} is a two-loop constant equal to [61]

$$q_{02} = -2\mathrm{K} - \frac{3}{2}\log 2 + \frac{7}{4}\,, \qquad (4.1.41)$$

where K is the Catalan's constant. Clearly, the result (4.1.40) with its logarithmic dependence on $\ell \equiv j/4g$ prevents any direct interpolation with the gauge theory expansion (4.1.38).

Similar logarithmic enhancements apply for corrections to (4.1.40) suppressed by higher powers of ℓ [59]. The situation becomes worse at higher loops since higher polynomials of $\log\ell$, whose degrees are given by the loop order, are generated [61]. That singular behavior of the expansion (4.1.40) with its accumulation of powers of $\log\ell$ seems to invalidate the semiclassical analysis for $j \sim 0$. A rough estimate of the 'convergency radius' of the (extrapolated) term in square brackets in (4.1.40) points toward a transition at $\log\ell \sim (-\pi g)$, and thus to the existence of a non-perturbative small j regime: $j \sim \mathrm{e}^{-\pi g}$ [46]. It would mean that a 'resummation' of the double expansion in powers of $1/g$ and $\log\ell$ is needed to correctly probe the scaling function around $j = 0$ at strong coupling, and possibly restore the agreement with the gauge theory expansion (4.1.38). However, without further information about the structure of the semiclassical expansion (4.1.40), this program looks quite impossible to tackle on analytical grounds.[6] Fortunately, the way out was proposed in [46] in which the conjecture was put forward that the small ℓ string scaling function is exactly governed by a 2d non-linear O(6) sigma model. This model appears as a low-energy effective theory for the string σ-model. Let us see how it solves the interpolation of the two expansions (4.1.38) and (4.1.40).

According to [46], the $O(\ell^2)$ contributions in the small ℓ expansion of the string scaling function (4.1.40) are controlled exactly by the O(6) sigma model. More precisely, when $\ell \equiv j/4g \equiv \rho/2g \ll 1$, we have the relation

$$\varepsilon(g,j) \equiv \frac{\epsilon(g,j) + j}{2} = \varepsilon_{\mathrm{O}(6)}(\rho) + \dots\,, \qquad (4.1.42)$$

where the dots stand for $O(\ell^4)$ corrections. Here, $\varepsilon_{\mathrm{O}(6)}(\rho)$ is the (thermodynamic) density of energy, measured above the vacuum, for an excited state of the O(6) model with minimal energy for a given charge density $\rho \equiv j/2$.[7] The O(6) sigma model has a non-trivial dynamics that

[6] A discussion of the resummation of the logarithms based on a world-sheet superstring approach can be found in [61].

[7] The charge is measured with respect to one of the Cartan generators of $\mathfrak{so}(6)$, the other ones annihilating the state. We may choose the generator of the rotation in the plane $(1,2)$ for definiteness.

splits into two different regimes, which affect the expression of $\varepsilon_{\mathrm{O}(6)}(\rho)$. Perturbatively, the O(6) model describes the interaction of an O(5) multiplet of massless Goldstone bosons. It has a running coupling constant and it is asymptotically free at short distances [127, 128]. In the infrared, however, the O(6) model develops a mass gap $m \sim \exp(-g/\beta_0)$, with one-loop beta-function coefficient $\beta_0 = 1/\pi$, and describes the interaction of an O(6) multiplet of massive (asymptotic) particles with mass m. The corresponding regimes for the energy density $\varepsilon_{\mathrm{O}(6)}(\rho)$ are then summarized as follows.

- In the non-perturbative regime $\rho \ll m$, the O(6) energy density admits an expansion in integer powers of ρ [130]

$$\varepsilon_{\mathrm{O}(6)}(\rho) = m^2 \left[\frac{\rho}{m} + \frac{\pi^2}{6} \left(\frac{\rho}{m} \right)^3 + O\left(\rho^4/m^4 \right) \right] . \tag{4.1.43}$$

  This regime is directly connected to the small $j \equiv 2\rho$ expansion of the gauge theory (4.1.38). We note, in particular, the absence of term proportional to $\rho^2 \sim j^2$ in (4.1.43). Comparison of (4.1.43) and (4.1.38) suggests that $\epsilon_1(g)$ interpolates bewteen $\epsilon_1(g) = -8 \log 2\, g^2 + \ldots$ at weak coupling ($g \ll 1$) and $\epsilon_1(g) = -1 + m + \ldots$ at strong coupling ($g \gg 1$), and so on for $\epsilon_3(g), \ldots$. We will see later that this is effectively the transition predicted by the all-loop Bethe ansatz equation for the scaling function in the gauge theory [47, 74, 73, 75]. Note that the dots in the relation $\epsilon_1(g) = -1 + m + \ldots$ stand for subleading contributions at strong coupling. The latter arise from resummation of terms $\sim \ell^4 \log^k \ell, \ell^6 \log^k \ell, \ldots$ in (4.1.40) to all orders in $1/g$. From the *effective* theory point of view, they correspond to irrelevant perturbations of the O(6) model associated to dimension $4, 6, \ldots$ operators. Thus it seems reasonable to believe that their contributions at a given order in the small j expansion of the scaling function will be suppressed by higher powers of the mass $m \sim \mathrm{e}^{-\pi g}$, e.g. $\epsilon_1(g) = -1 + m + O(\mathrm{e}^{-3\pi g})$. These sort of contributions are predicted [73, 75] by the all-loop Bethe ansatz equation for the scaling function [47].

- In the perturbative regime $\rho \gg m$, the O(6) energy density reads [130]

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \rho^2 \left[\frac{\pi}{4 \log(\rho/m)} + O\left(\frac{\log\log(\rho/m)}{\log^2(\rho/m)} \right) \right] . \tag{4.1.44}$$

  This regime is connected to the small $\ell \equiv \rho/2g$ expansion of the semiclassical string result (4.1.40). The expression (4.1.44) resums through renormalization group (an infinite number of) perturbative corrections in $1/g$ with coefficients proportional to $\ell^2 \sim \rho^2$ and enhanced by powers of $\log \ell$ [46, 61]. Using the leading-order (one-loop) strong coupling expression for the mass gap $m \propto \exp(-g/\beta_0) = \mathrm{e}^{-\pi g}$ and expanding (4.1.44) at strong coupling, one immediately recovers the classical string result $\varepsilon(g, j) = g\ell^2 + \ldots$ [46]. To match higher-loop corrections to (4.1.40) into (4.1.44), one needs to know the strong coupling expansion of the mass gap m. This expansion depends on the embedding of the O(6) model into the superstring σ-model, and it is not fixed by the O(6) model solely. Comparing the O(6) result (4.1.44) with the string result (4.1.40) order by order in $1/g$, one may find the strong coupling expression of the mass gap $m = m(g)$ [46, 76]. We will explain it in more details in the following subsection, but it is important to stress that the O(6) model predicts that the string expansion (4.1.40) has a renormalization-group structure [46, 61]. It implies for instance that the mapping of (4.1.40) into (4.1.44), with the help of the one-loop mass gap $m \propto \exp(-g/\beta_0)$, definitely determines the numerical coefficient $\sim (-\beta_0)^k$ multiplying the leading logarithm $\log^k \ell$ of (4.1.40) at k-loop order [46].

- In the intermediate regime ρ/m = fixed, the function $\varepsilon_{\mathrm{O}(6)}(\rho)$ does not admit a simple representation. However, thanks to the complete integrability of the O(6) sigma model [72, 127, 128], the energy density can be found for an arbitrary charge density as the solution to the (zero-temperature) thermodynamic Bethe ansatz (TBA) equations [130]. We shall discuss it later, since it is in this form that the O(6) model is embbeded [73] into the gauge-theory all-loop Bethe ansatz equation for the scaling function [47] (at strong coupling).

Thus the proposal of [46] solves the apparent disagreement between the small j expansion in the gauge theory at strong coupling (4.1.38) and the small ℓ expansion in the string theory (4.1.40) by establishing a non-perturbative transition between these two regimes. That transition is governed by the O(6) sigma model and takes place at $j \sim m \sim \mathrm{e}^{-\pi g}$ where m is the O(6) mass gap.

In the next subsection, we will verify explicitely that the $O(\ell^2)$ contribution to the small ℓ string scaling function (4.1.40) can be interpreted as the semiclassical expansion of the O(6) energy density, which holds in the perturbative regime $j \gg m$. Namely, we will check that the polynomials of $\log \ell$ in (4.1.40) have a renormalization-group structure that originates from the running of the effective coupling constant of the O(6) model. From this mapping, we will extract the two-loop expression of the O(6) mass gap $m = m(g)$. Finally, we will sketch the construction of the O(6) TBA equations [72, 130], in view of a later embedding into the gauge theory.

4.1.3 Non-Linear O(6) Sigma Model

According to [46], the effective dynamics of the gapless excitations of the string σ-model, expanded around the long rotating string at rest w.r.t. $\mathrm{S}^1 \subset \mathrm{S}^5$, is described by the bosonic non-linear O(6) sigma model. Let us give an argument supporting this proposal by looking at the spectrum of quadratic fluctuations. To one-loop order, the spectrum of masses of these fluctuations has been computed in [59]. It can be easily deduced from the following representation for the one-loop contribution to the cusp anomalous dimension [59, 46]

$$-\frac{3\log 2}{2\pi} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[5|p| + \sqrt{p^2+4} + 2\sqrt{p^2+2} - 8\sqrt{p^2+1}\right]. \tag{4.1.45}$$

Indeed, we recall that the cusp anomalous dimension has the meaning of the vacuum energy density for the 2d worldsheet theory: $\Gamma_{\mathrm{cusp}}(g) = E_{\mathrm{vac}}/R = E_{\mathrm{vac}}/2\log\mathcal{S} = 2g - 3\log 2/2\pi + \ldots$. As a consequence, $O(g^0)$ correction to the cusp anomalous dimension is given by the sum of the individual energies E_k of each of the string modes. Namely,

$$E_{\mathrm{vac}} = E_{\mathrm{classical}} + 1/2\sum_k E_k + O(1/g)\,, \tag{4.1.46}$$

where $E_{\mathrm{classical}} = 4g\log\mathcal{S}$ and $\sum_k \to 2\log\mathcal{S}\times\int_p$ with $p \sim 2\pi k/2\log\mathcal{S}$ in the long string limit $\mathcal{S} \gg 1$.

Clearly, the one-loop vacuum energy density (4.1.45) decomposes into the sum of bosonic and fermionic contributions with the latter entering with minus sign. The correct balance between the number of these modes and the values of the corresponding masses ensures the UV finiteness of the result. We see that the excitations have relativistic dispersion law and all of them except five are massive. The five gapless modes correspond to small fluctuations of the string in $S^5 \subset AdS_5 \times S^5$. They are massless by the Nambu-Goldstone mechanism. Indeed, the 2d Hamiltonian has an exact O(6) symmetry, reflecting the invariance with respect to rotations in S^5. But, at the perturbative level $g \gg 1$, the symmetry is spontaneously broken down to O(5) by the semiclassical vacuum or equivalently by the choice of the position of the string in S^5. In this way, one has an O(5) multiplet of Goldstone bosons. As we will discuss later, the embedding of the string into S^5 is governed at low-energy by a non-linear O(6) sigma model. We conclude therefore that the O(6) model can be considered as a low-energy effective theory with a UV cutoff of order $O(1)$ [46]. We stress that this is possible at strong coupling $g \gg 1$ only. Indeed, the strong infrared dynamics of the O(6) model generates a mass gap $m \sim e^{-\pi g}$, that has to be kept much smaller than the mass of the string massive modes for consistency. We require thus that $m \sim e^{-\pi g} \ll 1$.

Looking back at (4.1.45), we note that the cusp anomalous dimension does not receive at strong coupling a dominant contribution from the O(6) model. This is because the vacuum energy density of the O(6) model is a UV divergent quantity with no external scale to probe the low-energy physics. The situation is different for the scaling function $\varepsilon(g,j)$ that depends on the the charge density $\rho \equiv j/2$. Then, as long as the mean energy per charge (particle) $\varepsilon(g,j)/j$ stays smaller than the UV scale ~ 1, we expect the O(6) model to provide the dominant contribution to the energy density (= scaling function) [46]

$$\varepsilon(g,j) = \varepsilon_{O(6)}(\rho) + \dots\,, \tag{4.1.47}$$

where the dots stand for contributions induced by irrelevant interactions. Due to the renormalizability of the O(6) sigma model, the dependence of the energy density $\varepsilon_{O(6)}(\rho)$ on the physics at the UV scale ~ 1 can be absorbed into an effective coupling constant or more conveniently into the mass gap m. The latter observable is the natural low-energy parameter of the O(6) model. Therefore, $\varepsilon_{O(6)}(\rho)$ can be found as a function of ρ and m. Moreover, it assumes the form $\varepsilon_{O(6)}(\rho) = \rho^2 f_{O(6)}(\rho/m)$, as dictated by dimensional analysis. The function $f_{O(6)}(\rho/m)$ is scheme independent and it can be computed unambiguoulsy in the O(6) model. The information about the regularization or UV completion of the O(6) model by the string σ-model is encoded in the dependence of the mass gap m on the UV coupling constant: $m = m(g)$. In summary, the equation (4.1.47) reads

$$\varepsilon(g,j) \equiv \frac{\epsilon(g,j)+j}{2} = \rho^2 f_{O(6)}(\rho/m(g)) + \dots\,. \tag{4.1.48}$$

The function $f_{O(6)}$ interpolates between two different regimes that have been already presented. In the low-density regime $\rho \ll m$, one has a dilute gas of massive particles with energy density [130]

$$\varepsilon_{O(6)}(\rho) = m\rho + O(\rho^3)\,, \tag{4.1.49}$$

while in the regime $\rho \gg m$, the perturbative expansion of the O(6) model is reliable and the energy density reads [130]

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{\pi\rho^2}{4\log(\rho/m)} + O\left(\frac{\log\log(\rho/m)}{\log^2(\rho/m)}\right) . \tag{4.1.50}$$

The rough estimate of the validity of the O(6) description, given by $\varepsilon_{\mathrm{O}(6)}(\rho)/\rho \ll 1$, provides $m \ll 1$ in the non-perturbative regime and $\rho \ll g$, or equivalently $\ell \sim j/g \sim \rho/g \ll 1$, in the perturbative regime (making use of $m \sim \mathrm{e}^{-\pi g}$).

Perturbative Regime

In order to check the interpretation of the O(6) model as a low-energy effective theory, we will verify up to two-loop in the semiclassical expansion $g \gg 1$, $\ell \equiv \rho/2g$ fixed, that the string scaling function agrees when $\ell \ll 1$ with the perturbative expansion of the O(6) energy density. In parallel, we will make more precise the condition of validity of the O(6) model. Finally, we will compute the mass gap expression $m = m(g)$ up to two-loop at strong coupling $g \gg 1$.

Classical limit

To begin with, let us first consider the O(6) model at the classical level. The two-dimensional O(6) sigma model is a theory for a scalar field taking values on the unit five-sphere $\mathrm{S}^5 \subset \mathbb{R}^6$. More explicitely, introducing the field multiplet $X = X(r,t) = (X^1, \dots, X^6) \in \mathbb{R}^6$ satisfying the condition $X \cdot X = \sum_i X^i X^i = 1$, its action is given by

$$S = g \int dr dt\, \partial X \cdot \partial X . \tag{4.1.51}$$

Here g is the coupling constant of the O(6) model matching the string tension at the classical level. Note that it is not the conventional coupling of the O(6) model that reads $\bar{e}^2 = 1/2g$. At the quantum level, it follows that the O(6) model is weakly coupled when $g \gg 1$ corresponding to $\bar{e}^2 \ll 1$. Despite its simple action (4.1.51), the O(6) model is a non-linear theory. It can be easily seen by solving the constraint $X \cdot X = 1$ in a given system of local coordinates on the sphere S^5, revealing that the O(6) model is an interacting theory. For instance, in terms of the unconstrained (stereographic) fields $Y = Y(r,t) = (Y^1, \dots, Y^5) \in \mathbb{R}^5$ with $Y^i = X^i/(1+X^6)$, the action (4.1.51) assumes the more involved expression

$$S = 4g \int dr dt\, \frac{\partial Y \cdot \partial Y}{(1+Y^2)^2} . \tag{4.1.52}$$

This form of the action is suitable to analyse the O(6) model perturbatively, that is for $g \gg 1$, around the Goldstone vacuum $Y = 0 \longleftrightarrow X^6 = 1$, after rescaling the fields as $Y \to Y/\sqrt{g}$ and expanding in $1/g$. It makes explicit that the symmetry is broken down to O(5), due to the non-linear realization of the O(6) symmetry by the Y fields, and that the quantum fluctuations form an O(5) vector multiplet of interacting massless Goldstone bosons, at the perturbative level.[8]

[8] The O(6) symmetry prevents the fields to pick up a mass, but, as said before, the perturbative expansion breaks down at low-energy and the theory develops a mass gap non-perturbatively $m \sim \mathrm{e}^{-\pi g}$.

The masslessness of the fields reflects the fact the O(6) model is a conformal field theory at the classical level. At the quantum level however the coupling starts to run and a dynamical scale is generated, as we will discuss later.

After this brief presentation of the O(6) model, let us come back to our initial classical problem. We are interested in the configuration describing the rigid rotation of the string along a big circle in the plane $(1, 2)$. It corresponds to a solitonic solution of the equations of motion that is more conveniently analysed in terms of the field multiplet X. It is not difficult to find the corresponding solution as

$$X_1 = \cos(\nu t)\,, \qquad X_2 = \sin(\nu t)\,, \qquad X_{3,\ldots,6} = 0\,, \tag{4.1.53}$$

up to a shift of the time coordinate t. To verify that (4.1.53) solves the equations of motion, it is important to take into account the constraint $X \cdot X = 1$. That can be done by working in the Y coordinates, or more directly by relaxing the condition $X \cdot X = 1$ with the help of a Lagrange multiplier λ. Then the Lagrangian density of the O(6) model reads

$$\mathcal{L} = \partial X \cdot \partial X - \lambda(X \cdot X - 1)\,, \tag{4.1.54}$$

and the equations of motion are given by

$$\Box X + \lambda X = 0\,, \qquad X \cdot X = 1\,. \tag{4.1.55}$$

They are obviously satisfied by the fields (4.1.53) for $\lambda = \nu^2$. We see that the frequency ν is left arbitrary and defines the relevant energy scale of the problem. Since the O(6) model has no dimensionful parameter, we expect the energy density $\varepsilon_{\mathrm{O}(6)}$ and the charge density ρ to scale respectively as $\varepsilon_{\mathrm{O}(6)} \propto \nu^2$ and $\rho \propto \nu$. It is no difficult to verify it explicitely. The energy density reads

$$\varepsilon_{\mathrm{O}(6)} = g\left(\dot{X} \cdot \dot{X} + X' \cdot X'\right) = g\,\nu^2\,, \tag{4.1.56}$$

where dot and prime stand for derivative with respect to time and spacial coordinates, respectively. The Noether charge density associated to rotation, say in the plane $(1, 2)$, is given by

$$\rho = 2g\left(X_1\dot{X}_2 - X_2\dot{X}_1\right) = 2g\,\nu\,. \tag{4.1.57}$$

Resolving the parametric representation, we obtain the classical energy density as

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{\rho^2}{4g}\,. \tag{4.1.58}$$

Since $\rho \equiv 2g\ell$, we verify immediately the agreement with the $O(\ell^2)$ contribution to the classical string result [45]

$$\varepsilon(g, j) = \varepsilon_{\mathrm{O}(6)}(\rho) + \ldots = g\ell^2 + \ldots\,, \tag{4.1.59}$$

where ellipses denote subleading corrections, suppressed by powers of ℓ^2, which are not captured by the O(6) model. From the point of view of the O(6) model, they correspond to contributions of operators with dimensions $\geqslant 4$ and thus stand for irrelevant interactions. At the quantum level, we expect such contributions to appear after integrating out the massive modes of the string, but their origin at the classical level is different. The reason is that further non-linear interactions for the fields $X = (X^1, \ldots, X^6)$ are generated by the string dynamics at the classical level already.

To understand it, let us consider in more details the Nambu-Goto action for the classical string. The latter is given by the area of the string and reads explicitely as

$$\mathcal{S}_{\rm NG} = -2g \int drdt \sqrt{(1-\dot{X}\cdot\dot{X})(1+X'\cdot X')+(\dot{X}\cdot X')^2}\,. \qquad (4.1.60)$$

Here the fields $X=(X^1,\dots,X^6)$ stand for the embedding coordinates of the string in S^5 satisfying $X\cdot X=1$. Two of the embedding coordinates of the string in AdS_5, namely r and t, have been identified with the world-sheet coordinates. Their contributions produce '1' under square root in (4.1.60), because, as seen before, the relevant metric is flat in the long string limit. The remaining embedding coordinates of the string in AdS_5 take constant values over the world-sheet and thus do not participate to the dynamics at the classical level. A similar conclusion applies for the fermionic excitations of the string. The Nambu-Goto action (4.1.60) obviously predicts non-linear corrections to the action of the O(6) sigma model (4.1.51). Indeed, its Lagrangian density admits the expansion

$$\mathcal{L}_{NG} = -2g + g\,\partial X\cdot\partial X + \dots\,. \qquad (4.1.61)$$

Here the constant term gives the classical value of the vacuum energy density $\Gamma_{\rm cusp}(g) = -\mathcal{L}_{NG}(X=0)=2g$, the marginal operator $g\,\partial X\cdot\partial X$ matches the Lagrangian density of the O(6) model (4.1.51) and the dots stand for operators with dimensions $\geqslant 4$. Thus we verify explicitely at the classical that the O(6) model is the low-energy effective theory for the fluctuations of the string in S^5.

To be more explicit about the validity of this approximation, let us work out the classical string scaling function from the Nambu-Goto action (4.1.60). The relevant classical solution is still given by (4.1.53) but the charge density ρ and the energy density ε are both corrected. Namely,

$$\rho = \frac{2g\,\nu}{\sqrt{1-\nu^2}} = 2g\,\nu + \dots\,, \qquad (4.1.62)$$

and

$$\varepsilon = 2g\left(\frac{1}{\sqrt{1-\nu^2}}-1\right) = g\,\nu^2 + \dots\,. \qquad (4.1.63)$$

Eliminating ν we recover the classical string result of [45]

$$\varepsilon(g,j) = 2g\Big(\sqrt{1+\ell^2}-1\Big) = g\ell^2 + \dots\,. \qquad (4.1.64)$$

It is clear that in order to trust the O(6) model prediction for the string scaling function, $\varepsilon(g,j) = \varepsilon_{\rm O(6)}(\rho)+\dots$, we must require the frequency ν to be much smaller than 1 [46]. It translates into the condition $\rho/2g \equiv \ell \ll 1$.

Another characterization of the decoupling condition, which also applies at the quantum level, can be found with the help of the chemical potential $h\equiv d\varepsilon/d\rho$. It defines an intensive quantity which measures the typical individual energy of the relevant fluctuations. In the O(6) model, $h = d\varepsilon_{\rm O(6)}/d\rho$ is a function of the density ρ and of the mass gap m, or equivalenlty of the coupling g, since $m=m(g)\sim \mathrm{e}^{-\pi g}$. One finds $h=\ell\equiv\rho/2g$ at the classical level, while, quantum mechanically, $h=m$ at very small density $\rho\ll m$ [130]. More generally $h\geqslant m$ whatever ρ is ($\rho\geqslant 0$), since one cannot excite fluctuations with energy lower than m. The decoupling condition requires h to be much smaller than the UV cut-off, $h \ll 1$. In the following, we will make use of h as a renormalization scale for the perturbative expansion of the O(6) energy density. In that case, $h = \ell(1 + O(1/g))$ and it is kept fixed when $g \to \infty$. Note that this procedure corresponds to the semiclassical expansion of the string scaling function: ℓ kept fixed with $g \gg 1$, with in addition ℓ assumed to be smaller than the UV cut-off.

Semiclassical Expansion

We will now consider the radiative corrections to the classical result (4.1.58). They are given by a perturbative expansion in $1/g$ ($g \gg 1$) which suffers from UV divergences. Since the O(6) model is embedded into the UV finite string σ-model, these divergences are automatically regularized. It follows that the $1/g$ expansion of the O(6) energy density is already finite but renormalized in a way that depends on the details of the embedding. So what we will do is to check that the two-loop string theory result [59, 61] is compatible with a renormalized O(6) perturbative expansion [46, 61]. Namely, we will verify that the logarithmic terms $\sim \log^k \ell \sim \log^k h$ that appear in the two-loop string scaling function do fulfill the constraints imposed by the Gell-Mann–Low (renormalization group) equation of the O(6) model. After that, we will explain how to work out the two-loop expression for the mass gap $m(g)$ and with its help we will bring the O(6) contribution to the string scaling function into a scheme independent form [46, 76].

As already mentioned, at the semiclassical level, the O(6) model only captures the contributions $\sim \ell^2 \sim \rho^2$ in the small ℓ expansion of the string scaling function. Indeed, by dimensional analysis, the semiclassical O(6) energy density still scales as ρ^2 at higher loops. However, in response to the breakdown of the conformal invariance of the O(6) model at the quantum level, radiative corrections acquire a dependence on the renormalization scale $h = d\varepsilon_{O(6)}/d\rho = \ell(1+O(1/g)) \ll 1$. It arises as an enhancement of the coefficients in the $1/g$ expansion of $\varepsilon_{O(6)}(\rho)$ by integer powers of $\log h$. Since the energy density is a physical quantity, these logarithms can be eliminated by expanding $\varepsilon_{O(6)}(\rho)$ not in terms of the bare coupling g but instead in terms of an effective coupling constant $g(h)$. Therefore, the energy density reads

$$\varepsilon_{O(6)}(\rho) = \frac{\rho^2}{4g(h)} \left(1 + c_1/g(h) + c_2/g^2(h) + \ldots\right) , \tag{4.1.65}$$

where $c_1, c_2, \ldots$ are numerical coefficients. Since all the coefficients $c_1, c_2, \ldots$ can be absorbed into a redefinition of the coupling $g(h)$, we can put all of them equal to zero without loss of generality, leading to

$$\varepsilon_{O(6)}(\rho) = \frac{\rho^2}{4g(h)} \,. \tag{4.1.66}$$

The effective coupling constant $g(h)$ satisfies the Gell-Mann–Low (renormalization group) equation (for $h \ll 1$)

$$h\frac{dg(h)}{dh} = \beta(g(h)) = \beta_0 + \frac{\beta_1}{g(h)} + \frac{\beta_2}{g^2(h)} + \ldots \,, \tag{4.1.67}$$

and it admits the expansion (for $h/h_0 \sim 1$)

$$g(h) = g_0 + \beta_0 \log(h/h_0) + \frac{\beta_1}{g_0} \log(h/h_0) + O(1/g_0^2) \,, \tag{4.1.68}$$

with the initial condition $g_0 = g(h_0) \gg 1$. Here $\beta_0 = 1/\pi$ and $\beta_1 = 1/4\pi^2$ are respectively the one-loop and two-loop beta-function coefficients of the O(6) model. Higher-loop corrections $\beta_2, \ldots$ to the beta-function are not universal and depend on the renormalization scheme. Here we fixed the scheme by imposing the relation (4.1.66) between $\varepsilon(\rho)$ and $g(h)$ and by identifying the renormalization scale h with the chemical potential.[9] These two conditions determine the coefficients $\beta_2, \ldots$ unambiguously. In other words, the coefficients $\beta_2, \ldots$ are accessible by direct perturbative calculations in the O(6) model, independently of the embedding into the string σ-model. For example, from the two-loop O(6) result of [76] one can derive that $\beta_2 = -3/16\pi^3$. The details of the embedding into the string σ-model are absorbed into g_0 and its unknown relation with the bare coupling g. Classically $g_0 = g$ while perturbatively ($g \gg 1$) we assume that $g_0 = g + a_0 + a_1/g + \ldots$ for some given value of $h_0 \ll 1$ and some UV sensitive numerical coefficients $a_0, a_1, \ldots$. Then it follows from (4.1.68) that

$$g(h) = g + (\beta_0 \log h + \delta_0) + \frac{1}{g}(\beta_1 \log h + \delta_1) + O(1/g^2)\,, \tag{4.1.69}$$

with the numerical coefficients $\delta_0, \delta_1, \ldots$ in a one-to-one correspondence with $a_0, a_1, \ldots$. For instance, $a_0 = (\beta_0 \log h_0 + \delta_0)$, $a_1 = (\beta_1 \log h_0 + \delta_1)$ and the remaining relations can be found iteratively by solving the Gell-Mann–Low equation (4.1.67).

We are now in position to check if the two-loop string theory result [59, 61] is compatible with a renormalized O(6) perturbative expansion. Plugging the expression (4.1.69) for the effective coupling $g(h)$ into the formula (4.1.66) for the O(6) energy density $\varepsilon_{O(6)}(\rho)$, we get the double expansion of $\varepsilon_{O(6)}(\rho)/\rho^2$ in powers of $1/g$ and $\log h$. Then, eliminating h in favor of ρ with the help of $h = d\varepsilon_{O(6)}/d\rho = \rho/2g + \ldots$,[10] we expand the energy density in $1/g$ and $\log \ell$, with $\ell \equiv \rho/2g$, which is more suitable to compare with the string semiclassical expansion. Doing so, we get

$$\varepsilon_{O(6)}(\rho) = g\ell^2 \left[1 + \frac{1}{g}\Big(c_{01} + c_{11}\log\ell\Big) + \frac{1}{g^2}\Big(c_{02} + c_{12}\log\ell + c_{22}\log^2\ell\Big) + \ldots\right]\,, \tag{4.1.71}$$

with the numerical constants

$$c_{11} = -\beta_0\,, \qquad c_{01} = -\delta_0\,, \tag{4.1.72}$$

$$c_{22} = \beta_0^2\,, \qquad c_{12} = \beta_0^2 + 2\beta_0\delta_0 - \beta_1\,, \qquad c_{02} = \beta_0\delta_0 + \beta_0^2/2 + \delta_0^2 - \delta_1\,. \tag{4.1.73}$$

We see that the expansion (4.1.71) has the correct logarithmic structure to match the corresponding string semiclassical expansion. We recall that the two-loop small ℓ string scaling function [59, 61] assumes the form

$$\varepsilon(g,j) = g\ell^2\left[1 + \frac{1}{g}\left(\frac{3}{4\pi} - \frac{1}{\pi}\log\ell\right) + \frac{1}{g^2}\left(\frac{q_{02}}{8\pi^2} - \frac{3}{4\pi^2}\log\ell + \frac{1}{\pi^2}\log^2\ell\right)\right] + O(\ell^4)\,. \tag{4.1.74}$$

[9]Namely $h = d\varepsilon_{O(6)}/d\rho$ with the derivative taken at fixed renormalized coupling $g(h_0) = g_0$ when the energy density $\varepsilon_{O(6)}(\rho)$ is expanded in $1/g_0$.

[10]To the relevant order for the computation of the two-loop energy density, one finds

$$h = \ell\left(1 - \frac{1}{g}\left(\beta_0\log\ell + \left(\delta_0 + \frac{\beta_0}{2}\right)\right) + \ldots\right)\,, \tag{4.1.70}$$

where $\ell \equiv \rho/2g$.

Given the one-loop beta-function $\beta_0 = 1/\pi$, we immediately check the agreement between (4.1.71) and (4.1.74) at the level of the leading logarithms $\sim c_{kk}\,(\log \ell)^k/g^k$, namely $c_{11} = -1/\pi$ and $c_{22} = 1/\pi^2$ [46, 61]. More generally, all the leading logarithms originate from the one-loop running of the effective coupling constant (4.1.69), leading to the higher-loop predictions for the string scaling function: $c_{kk} = (-\beta_0)^k = (-1/\pi)^k$ [46]. The one-loop constant $c_{01} = -\delta_0$ in (4.1.71) is left undetermined by the O(6) model, since it has the meaning of an embedding parameter. Its value has to be fixed by a direct matching with the string result (4.1.74), namely $c_{01} = -\delta_0 = 3/4\pi$. It follows that the subleading-logarithm coefficient $c_{12} = \beta_0^2 + 2\beta_0\delta_0 - \beta_1$ is fixed, with the help of the two-loop beta-function coefficient $\beta_1 = 1/4\pi^2$. We find $c_{12} = -3/4\pi^2$ in agreement wih the string result (4.1.74) [46, 61]. One can show moreover that all the subleading logarithms $\sim c_{(k-1)k}\,(\log \ell)^{k-1}/g^k$ in (4.1.71) are definitely determined by δ_0 and $\beta_{0,1}$, which are further predictions for a higher-loop string computation [46]. Finally, we can express the unknown embedding parameter δ_1, appearing in c_{02}, in terms of the two-loop string constant q_{02} as

$$\delta_1 = \frac{5}{16\pi^2} - \frac{q_{02}}{8\pi^2}\,. \tag{4.1.75}$$

Its explicit expression follows from the two-loop string result of [61] (K is the Catalan's constant)

$$q_{02} = -2\mathrm{K} - \frac{3}{2}\log 2 + \frac{7}{4}\,. \tag{4.1.76}$$

We have thus verified up to two-loop that the small ℓ semiclassical string scaling function can be cast into the form of the O(6) energy density in the perturbative regime: $g \gg 1$ with $h \sim \ell \equiv \rho/2g$ kept fixed and assumed much smaller than the UV cut-off ~ 1. The two-loop matching above illustrates the constraints imposed on the string scaling function by the renormalization-group structure of the O(6) energy density. At an arbitrary order n in the loop expansion, we have the unknown constant c_{0n} or equivalently δ_{n-1} that has to be determined by a direct matching with an explicit string theory calculation. Once done, a perturbative computation of the $(n+1)$-th beta-function coefficient β_n in the O(6) model permits the determination of all the subleading logarithms $\sim c_{(k-n)k}(\log \ell)^{k-n}/g^k$. They are indeed consequences of the running of the coupling constant $g(h)$ whose dependence on $\log h$ is governed by the Gell-Mann–Low equation.

O(6) *Mass Gap*

Given the interpretation of the O(6) model as a low-energy effective theory, it should be possible to cast the O(6) contribution to the string scaling function into a form that is explicitely scheme independent. As said before, that can be done by absorbing the bare coupling g into the natural low-energy parameter of the O(6) model, namely its mass gap $m = m(g)$. To find that relation at strong coupling $g \gg 1$, let us go back to the Gell-Mann–Low equation (4.1.67). We observe that it does not depend explicitely on h and thus can be immediately integrated as

$$h_0 = h \exp\left[-\int_{g(h_0)}^{g(h)} \frac{dg}{\beta(g)}\right]. \tag{4.1.77}$$

For $g(h_0), g(h) \gg 1$, the integral in square brackets can be performed perturbatively after expanding the beta function at large coupling, $\beta(g) = \beta_0 + \beta_1/g + \dots$, giving

$$\int_{g(h_0)}^{g(h)} \frac{dg}{\beta(g)} = \frac{1}{\beta_0}(g(h) - g(h_0)) - \frac{\beta_1}{\beta_0^2}(\log g(h) - \log g(h_0)) + \dots \,, \tag{4.1.78}$$

where the dots stand for corrections suppressed by powers of $1/g(h)$ or $1/g(h_0)$. The expression on the right-hand side of (4.1.78) can be written as

$$\int_{g(h_0)}^{g(h)} \frac{dg}{\beta(g)} = \int^{g(h)} \frac{dg}{\beta(g)} - \int^{g(h_0)} \frac{dg}{\beta(g)} \,, \tag{4.1.79}$$

where the lower bound of the integral $\int^{g(h)} dg/\beta(g)$ is implicitely determined by imposing the following asymptotic behavior at $g(h) \gg 1$

$$\int^{g(h)} \frac{dg}{\beta(g)} = \frac{g(h)}{\beta_0} - \frac{\beta_1}{\beta_0^2} \log g(h) + O(1/g(h)) \,, \tag{4.1.80}$$

Note that the condition (4.1.80) can be thought alternatively as fixing the freedom of an arbitrary constant of integration. Combining (4.1.77) and (4.1.79), we can define a mass scale $\Lambda_{\mathrm{O}(6)}$ as

$$\Lambda_{\mathrm{O}(6)} \equiv h \exp\left[-\int^{g(h)} \frac{dg}{\beta(g)}\right] = h_0 \exp\left[-\int^{g(h_0)} \frac{dg}{\beta(g)}\right] , \tag{4.1.81}$$

with the property to be invariant along a renormalization-group trajectory.[11] The scale $\Lambda_{\mathrm{O}(6)}$ defined in this way originates from the running of the effective coupling constant $g(h)$ and is referred to as the dynamical scale. Its value is fixed by the intial condition $g_0 = g(h_0)$ and it can be found perturbatively as ($g_0 \gg 1$)

$$\Lambda_{\mathrm{O}(6)} = h_0\, g_0^{\beta_1/\beta_0^2} \exp\left(-g_0/\beta_0\right) \left(1 + \frac{1}{g_0}\left(\frac{\beta_1^2}{\beta_0^3} - \frac{\beta_2}{\beta_0^2}\right) + O(1/g_0^2)\right) . \tag{4.1.82}$$

From the point of view of the string theory, it is more natural to express the mass scale $\Lambda_{\mathrm{O}(6)}$ in terms of the coupling constant g. That can be easily done with the help of the relation $g_0 = g + (\beta_0 \log h_0 + \delta_0) + (\beta_1 \log h_0 + \delta_1)/g + \dots$ ($g \gg 1$) giving

$$\Lambda_{\mathrm{O}(6)} = g^{\beta_1/\beta_0^2} \exp\left(-(g+\delta_0)/\beta_0\right) \left(1 + \frac{1}{g}\left(\frac{\beta_1^2}{\beta_0^3} - \frac{\beta_2}{\beta_0^2} + \frac{\beta_1\delta_0}{\beta_0^2} - \frac{\delta_1}{\beta_0}\right) + O(1/g^2)\right) . \tag{4.1.83}$$

We note the disappearance of the renormalization scale h_0 in the relation (4.1.83). This reflects the fact that g can be thought as an effective coupling constant defined at the UV scale ~ 1. Finally, plugging the numerical values of $\beta_{0,1,2}$ and $\delta_{0,1}$ into (4.1.83) provides

$$\Lambda_{\mathrm{O}(6)} = g^{1/4}\, \mathrm{e}^{-\pi g + 3/4} \left(1 + \frac{1}{\pi g}\left(\frac{q_{02}}{8} - \frac{1}{4}\right) + O(1/g^2)\right) . \tag{4.1.84}$$

[11] Note that the mass scale $\Lambda_{\mathrm{O}(6)}$ is defined in such a way that

$$\int_{g(\Lambda)}^{g(h)} \frac{dg}{\beta(g)} = \int^{g(h)} \frac{dg}{\beta(g)} = \frac{g(h)}{\beta_0} - \frac{\beta_1}{\beta_0^2} \log g(h) + O(1/g(h)) \,,$$

when $g(h) \gg 1$.

The advantage of introducing the dynamical scale $\Lambda_{O(6)}$ is that it absorbs the dependence of the effective coupling constant $g(h)$ on both the bare coupling g and the embedding parameters $\delta_0, \delta_1, \ldots$, see Eq. (4.1.83). Indeed, the effective coupling constant $g(h)$ can be found by means of (4.1.81) as a function of the dimensionless ratio $h/\Lambda_{O(6)}$. Then, since the equation (4.1.81) only involves the beta function that can be computed directly in the O(6) model, the function f in the expression $g(h) = f(h/\Lambda_{O(6)})$ no longer depends on the details of the UV (string) regularization.

Moreover, the scale $\Lambda_{O(6)}$ has a direct physical meaning in the O(6) model as it should stand for the mass gap m, which is the only mass scale in this theory. More exactly, $\Lambda_{O(6)}$ and m should be proportional to each other

$$m = c\,\Lambda_{O(6)}\,, \tag{4.1.85}$$

where the *numerical* factor c is needed to restore the scheme independence of the mass gap m. A strategy to determine c is to calculate the O(6) energy density $\varepsilon_{O(6)}(\rho)$ in two different ways: from the thermodynamic Bethe ansatz, that is scheme independent and expressed in terms of m, and from a direct perturbative calculation, that involves the dynamical scale $\Lambda_{O(6)}$ in a given scheme [130]. The comparison of the two results fixes c. This was done for instance in [130] for the $\overline{\text{MS}}$ scheme and it was found that

$$c_{\overline{\text{MS}}} = \left(\frac{8}{e}\right)^{1/4} \frac{1}{\Gamma(5/4)}\,. \tag{4.1.86}$$

This result has been checked by using Monte Carlo results and also by comparing finite-volume mass-gap values computed in perturbation theory [134] to those obtained from a (non-linear) TBA integral equation [135]. It is not difficult to convert it to the current scheme by relating the effective coupling $g(h)$, defined in (4.1.66), to the effective coupling of the $\overline{\text{MS}}$ scheme. This can be done with the help of formulae in [76] and one finds after some algebra

$$c = \pi^{1/4}\left(\frac{2}{e}\right)^{3/4} \frac{1}{\Gamma(5/4)}\,. \tag{4.1.87}$$

Combining the expression for $\Lambda_{O(6)}$, Eq. (4.1.84), its relation to the mass gap (4.1.85) and the value of the constant c, we arrive at the desired expression of the mass gap in terms of g. Namely

$$m = k\, g^{1/4}\, \mathrm{e}^{-\pi g}\left[1 + \frac{m_1}{\pi g} + O(1/g^2)\right]\,, \tag{4.1.88}$$

with the prefactor k and the two-loop constant m_1 given respectively by

$$k = 2^{3/4}\pi^{1/4}/\Gamma(\tfrac{5}{4})\,, \qquad m_1 = \frac{q_{02}}{8} - \frac{1}{4} = -\frac{\mathrm{K}}{4} - \frac{3\log 2}{16} - \frac{1}{32}\,, \tag{4.1.89}$$

in agreement with the findings of [46, 76].

Finally, to complete the picture, let us cast the O(6) contribution to the semiclassical string scaling function into a scheme independent form. To do that, we need to solve the relation (4.1.81) for $g(h) = f(h/\Lambda_{O(6)})$, replace the dynamical scale $\Lambda_{O(6)}$ by the mass gap $m = c\,\Lambda_{O(6)}$, plug the obtained expression into

$$\varepsilon_{O(6)}(\rho) = \frac{\rho^2}{4g(h)}\,, \tag{4.1.90}$$

and finally solve the dependence of h on the density ρ with the help of $h = d\varepsilon_{\text{O}(6)}/d\rho$. To begin with, we recall that in the perturbative regime the coupling is large $g \gg 1$ and the renormalization scale h is fixed. Since $\Lambda_{\text{O}(6)} \sim \text{e}^{-\pi g}$ at strong coupling $g \gg 1$, it follows that $h \gg \Lambda_{\text{O}(6)}$, or equivalently $h \gg m$, in agreement with the asymptotic freedom. Thus we can rely on the perturbative expansion of the beta function to solve the relation (4.1.81). Its solution reads

$$g(h) = \alpha(h) + \frac{\beta_1}{\beta_0}\log\alpha(h) + \frac{\beta_1^2}{\beta_0^2}\frac{\log\alpha(h)}{\alpha(h)} + \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0}\right)\frac{1}{\alpha(h)} + O\left(\frac{\log^2\alpha(h)}{\alpha(h)^2}\right), \tag{4.1.91}$$

with the one-loop running coupling $\alpha(h) \equiv \beta_0 \log\left(h/\Lambda_{\text{O}(6)}\right) \gg 1$. Then, after some algebra, one finds the chemical potential h as

$$h = \frac{\pi\rho}{2\log(\rho/m)}\left[1 + \frac{3}{4}\frac{\log(\kappa\log(\rho/m)) - \frac{1}{6}}{\log(\rho/m)} + \dots\right], \tag{4.1.92}$$

where $\log\kappa = \frac{1}{2} - \frac{5}{3}\log 2 - \frac{4}{3}\log\Gamma\left(\frac{3}{4}\right)$. From the equation above, we verify that the condition $h \gg m$ requires the density ρ to be large as compared to the mass scale m, namely $\rho \gg 2m\log(\rho/m)/\pi$. Finally, one obtains easily the (renormalization-group improved) two-loop expression for the O(6) energy density in the perturbative regime $\rho \gg m$ as [130, 76]

$$\varepsilon_{\text{O}(6)}(\rho) = \frac{\pi\rho^2}{4\log(\rho/m)}\left[1 + \frac{3}{4}\frac{\log(\kappa\log(\rho/m)) + \frac{1}{2}}{\log(\rho/m)} + \frac{9}{16}\frac{\log^2(\kappa\log(\rho/m)) + \frac{7}{36}}{\log^2(\rho/m)} + \dots\right]. \tag{4.1.93}$$

Replacing the mass scale in the scheme independent formula (4.1.93) by its expression (4.1.88), (4.1.89) and re-expanding the right-hand side of (4.1.93) in powers of $1/g$ at fixed $\ell \equiv \rho/2g$, one can reproduce the two-loop string semiclassical expression (4.1.74).

Equations (4.1.93), (4.1.88) and (4.1.89) achieve our discussion of the string scaling function in the O(6) perturbative regime $m \ll h \ll 1$. In the following, we will address the problem of the determination of the O(6) energy density in the non-perturbative regime $h \sim m$, corresponding to a low density $\rho \ll m$.

Non-Perturbative Regime

The previous perturbative analysis of the energy density in the O(6) model assumed that the density was much greater than the mass scale $\rho \gg m$, and it breaks down when ρ becomes comparable with m. The low-density regime $\rho \ll m$ is of a special interest since it allows us to understand the transition from the gauge to the string theory. Indeed, the low-density regime of the O(6) energy density is nothing else than the small j regime of the scaling function at strong coupling [46]. According to the the all-loop Bethe ansatz equations of the gauge theory [45, 47], in this regime the scaling function admits an expansion in integer powers of j with expansion coefficients being some functions of the coupling constant

$$\epsilon(g, j) = \epsilon_1(g)\, j + \epsilon_3(g)\, j^3 + O(j^4)\,. \tag{4.1.94}$$

The AdS/CFT correspondence suggests that this relation should coincide with the expansion of the O(6) energy density in powers of $\rho \equiv j/2$ at low-density $\rho \ll m$.[12] To verify it, we will now

[12] More precisely, we expect an expansion in integer powers of ρ/m since the O(6) energy density is a function of this dimensionless ratio, up to the overall factor of ρ^2 or, equivalently, of m^2.

develop a convenient approach of this non-perturbative regime of the O(6) model. We will be able to find an exact determination of the O(6) energy density, which is valid for any value of the density, thanks to the integrability of the quantum O(6) sigma model [72, 127, 128].

In the following, we will outline the construction of a linear integral equation that controls exactly, in the thermodynamic limit, the ground-state energy density $\varepsilon_{O(6)}(\rho)$ associated with an arbitrary charge density ρ. The procedure does not depend essentially on the restriction to $O(n = 6)$, so we will consider the more generic case $n \geqslant 3$. The three main steps are the construction of the exact (infinite volume) S-matrix for the non-linear $O(n)$ sigma model, the approximative diagonalization of the (finite volume) Hamiltonian by means of an asymptotic Bethe ansatz, and, finally, the derivation of an integral equation in the thermodynamic limit. The first step was achieved in [72] by exploiting one of the most remarkable property of an integrable system, namely the factorization of the scattering. The material for the second and last steps can be found in [87, 130].

Exact S-Matrix

As already mentioned, the exact spectrum of the two-dimensional non-linear $O(n)$ sigma model includes a vector multiplet of massive particles with mass m. Hence, a particle carries both an isospin index $i = 1, \ldots, n$, and a rapidity θ, related to energy and momentum through $p^0 \pm p^1 = m \exp(\pm\theta)$. We will denote an incoming/outgoing state of M asymptotic particles $(\theta_1, i), \ldots, (\theta_M, j)$, by

$$\psi^{i\ldots j}_{\text{in/out}}(\theta_1, \ldots, \theta_M)\,. \tag{4.1.95}$$

Without loss of generality, it would be sufficient to consider the case of decreasingly ordered rapidities, $\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_M$, that provides a complete basis of asymptotic states. Restricting ourselves to the two-body problem, we could also argue that the $2 \to 2$ scattering amplitude depends on one Mandelstam variable only, say $s = (p_1 + p_2)^2 = 2m^2(1 + \cosh\theta_{12})$, whose physical range, $s > 4m^2$, is completely exhausted by restricting to $\theta_{12} = \theta_1 - \theta_2 > 0$. Nevertheless, when dealing with the construction of the exact S-matrix, it is more than convenient to consider the analytic continuation of the scattering amplitude for complex values of the kinematical invariant s or equivalently θ_{12}. For the relevant case of two-dimensional integrable relativistic quantum field theories, these analytical properties are examined in detail in [86]. In the following, we will implicitely assume analytic continuation to complex rapidities. Then consequences of crossing-symmetry, unitarity and integrability will be valid in the full complex plane of $\theta_{12} = \theta_1 - \theta_2$. We recall that, in addition to the conservation of all individual momenta and the absence of particle production, the property of complete integrability, implemented in the theory through an infinite set of conservation laws, is believed to impose the factorization of the multiparticle scattering. A general discussion of the relation between the absence of particle production, the factorizability of the scattering and the existence of local, higher-spin and Abelian conserved charges, can be found in [86]. An explicit construction of nonabelian and non-local conserved charges for the quantum non-linear $O(n)$ sigma model was done in [129], and used there to check the aforementioned remarkable feature of the scattering amplitudes.

Let us now construct the exact S-matrix along the lines of [72]. The $2 \to 2$ S-matrix is defined by

$$\psi_{\rm in}^{kl}(\theta_1,\theta_2) = S_{ij}^{kl}(\theta_{12})\,\psi_{\rm out}^{ij}(\theta_1,\theta_2)\,, \tag{4.1.96}$$

where a sum over isotopic indices, $i,j = 1,\ldots,n$, is assumed. By Lorentz invariance, the two-body S-matrix, Eq. (4.1.96), only depends on the difference of rapidities $\theta_{12} = \theta_1 - \theta_2$, and, due to O($n$)-symmetry, it admits the following tensor decomposition

$$S_{ij}^{kl}(\theta) = \sigma_1(\theta)\delta_{ij}\delta^{kl} + \sigma_2(\theta)\delta_i^k\delta_j^l + \sigma_3(\theta)\delta_i^l\delta_j^k\,. \tag{4.1.97}$$

The crossing transformation interchanges particles (i,k) and permutes the two Mandelstam variables $s = 2m^2(1+\cosh\theta)$ and $t = 2m^2(1-\cosh\theta)$, which is equivalent to the substitution $\theta \to i\pi - \theta$. Therefore, by crossing symmetry, the s-channel and t-channel amplitudes, $\sigma_a(\theta)$ and $\sigma_b(i\pi-\theta)$ respectively, are related to one another by

$$\sigma_2(\theta) = \sigma_2(i\pi-\theta)\,, \qquad \sigma_3(\theta) = \sigma_1(i\pi-\theta)\,. \tag{4.1.98}$$

Furthermore, the absence of particle production implies the unitarity of the $2\to 2$ S-matrix at any value of θ, that is

$$S_{ij}^{mn}(\theta)S_{nm}^{lk}(-\theta) = \delta_i^k\delta_j^l\,. \tag{4.1.99}$$

Expanding Eq.(4.1.99), with help of the decomposition (4.1.97) for the two-body S-matrix, provides a set of quadratic relations between the amplitudes $\sigma_a(\theta)$ and $\sigma_b(-\theta)$, whose explicit expressions can be found in [72].

Altogether, Eqs. (4.1.98) and (4.1.99) are not strong enough to fix the exact S-matrix for the non-linear O(n) sigma model. Fortunately, an additionnal set of constraints arises from the assumption of factorized scattering. It states that a general M-body scattering can be decomposed as a product of $M(M-1)/2$ consecutive two-body scatterings. This procedure explicitely introduces an ordering of the $2\to 2$ scattering events. The latter sequence has no counterpart in the definition of the S-matrix that already takes care of all possible intermediate processes with same initial and final boundary conditions. Imposing that the *factorized* M-body scattering does not depend on the sequence of events provides the so-called factorization equations for the two-body S-matrix. All these constraints are reducible to the consistency equation for the factorized three-body S-matrix $S_{123}(\theta_{12},\theta_{23})$ that reads

$$S_{123}(\theta_{12},\theta_{23}) = S_{23}(\theta_{23})S_{13}(\theta_{13})S_{12}(\theta_{12}) = S_{12}(\theta_{12})S_{13}(\theta_{13})S_{23}(\theta_{23})\,. \tag{4.1.100}$$

Here, the first equality implements the assumption of factorized scattering and the last one its consistency condition. Incidentally, we recover the same algebraic relation that was already at work for the integrable Heisenberg spin chain, that is the Yang-Baxter equation. The equation above holds as a matrix identity on the space of three-particle states $\psi_{\rm out}^{ijk}(\theta_{12},\theta_{23})$, and it could have been equivalently written as

$$S_{bc}^{mn}(\theta_{23})S_{ak}^{lc}(\theta_{12}+\theta_{23})S_{ij}^{ab}(\theta_{12}) = S_{ab}^{lm}(\theta_{12})S_{ic}^{an}(\theta_{12}+\theta_{23})S_{jk}^{bc}(\theta_{23})\,, \tag{4.1.101}$$

where (i,j,k) and (l,m,n) are free integers while an implicit summation is assumed for (a,b,c).

For the O(n)-symmetric two-body S-matrix (4.1.97), the factorization equation (4.1.100), or (4.1.101), provides a set of cubic relations between $\sigma_a(\theta), \sigma_b(\theta+\theta')$ and $\sigma_c(\theta')$. The solution to this equation, compatible with crossing-symmetry and unitarity, was constructed in [72]. It expresses $\sigma_1(\theta)$ and $\sigma_3(\theta)$ in terms of $\sigma_2(\theta)$ and reads

$$\sigma_1(\theta) = -\frac{\Delta}{x+1/2}\sigma_2(\theta)\,, \qquad \sigma_3(\theta) = \frac{\Delta}{x}\sigma_2(\theta)\,, \tag{4.1.102}$$

where $x = i\theta/2\pi$ and $\Delta = 1/(n-2)$. The component $\sigma_2(\theta)$ cannot be fixed in this way since it may be considered as an overall (scalar) factor, always left undetermined by the Yang-Baxter equation. It does not behave, however, as a global factor for the crossing-symmetry and unitarity equations, which combined with the solution (4.1.102) translate into [72]

$$\sigma_2(i\pi-\theta) = \sigma_2(\theta), \qquad \sigma_2(\theta)\sigma_2(-\theta) = \frac{x^2}{x^2-\Delta^2}\,. \tag{4.1.103}$$

A 'minimal' solution to Eq. (4.1.103) was proposed in [72] to describe the scattering of the O(n) sigma model.[13] It reads

$$\sigma_2(\theta) = -\frac{x}{\Delta+x}\frac{\Gamma(1+x)\Gamma(\frac{1}{2}-x)\Gamma(\Delta+\frac{1}{2}+x)\Gamma(\Delta-x)}{\Gamma(1-x)\Gamma(\frac{1}{2}+x)\Gamma(\Delta+\frac{1}{2}-x)\Gamma(\Delta+x)}\,, \tag{4.1.104}$$

and it completes the construction of the exact S-matrix. In Appendix B.1., we propose an iterative procedure to derive the solution (4.1.104) from the unitarity and crossing-symmetry equations (4.1.103).

The procedure above illustrates perfectly the general attitude to adopt with a factorizable scattering. The outline consists of three steps. First, identify the exact spectrum of asymptotic states and the global symmetry algebra. Then, resolve the isotopic structure of the two-body S-matrix, up to a scalar factor, with help of the Yang-Baxter (factorization) equation. Finally, fix the overall factor thanks to crossing-symmetry and unitarity equations. This procedure has been applied successfully to various integrable relativistic 2D QFT, like the sine-Gordon or the Gross-Neveu models [72], for instance. It works as well to fix the exact S-matrix of the AdS/CFT correspondence [42, 104, 43, 107, 108]. In the latter case, however, the lack of Lorentz invariance has required some guessworks and adjustments [104].

Before to end this subsection, let us consider the scattering of particles all charged only with respect to one of the Cartan generators of the $\mathfrak{so}(n)$ algebra, say the generator of the rotation in the plane $(1,2)$. The conservation of the angular momentum, combined with the absence of particle-antiparticle creation, prevent the mixing of such a polarized beam of particles with other states and lead to a diagonal scattering. The projection of the exact S-matrix, Eqs. (4.1.97), (4.1.102) and (4.1.105), onto this Abelian subspace is given by [130]

$$S(\theta) = \sigma_2(\theta)+\sigma_3(\theta) = -\frac{\Gamma(1+x)\Gamma(\frac{1}{2}-x)\Gamma(\Delta+\frac{1}{2}+x)\Gamma(\Delta-x)}{\Gamma(1-x)\Gamma(\frac{1}{2}+x)\Gamma(\Delta+\frac{1}{2}-x)\Gamma(\Delta+x)}\,, \tag{4.1.105}$$

[13] Physical solution to unitarity and crossing-symmetry equations should fulfill general analyticity conditions [86]. In particular, the scattering amplitude $\sigma_2(\theta)$ should define a meromorphic function of the complex rapidity θ. The minimal solution of [72] satisfies all these general requirements and assumes, as a further input, the absence of bound-state singularities in the physical strip, $0 \leqslant \mathrm{Im}(\theta) \leqslant \pi$. The absence of bound-states in the O(n) sigma model is confirmed both by semiclassical and large-n analysis [132, 131], and it can be thought of as a consequence of the repulsive interaction between particles of the vector multiplet.

with $x = i\theta/2\pi$ and $\Delta = 1/(n-2)$.

Asymptotic Bethe Ansatz Equations

We will now proceed to an approximative quantization of the energy spectrum for a system of total charge L, confined on a circle of radius R. The infinite-volume property of conservation of the number of asymptotic particles, in any scattering event, suggests a quantum-mechanical approach to the spectral problem. Henceforth, we will follow this way and discuss later its range of validity.

Let us specify first the asymptotic ($R \gg 1$) Hilbert space. We note that we can always add to a system an arbitrary number of pairs of particle and antiparticle, without changing the total charge L. However, for the minimal-energy state, we can certainly restrict the consideration to states build out of only L asymptotic particles, each one carrying one unit of charge. Asymptotically ($R \gg 1$) this Hilbert space is stable, thanks to integrability, and, as we found in the last subsection, the scattering is diagonal there. Any of these states can be described by an asymptotic wave-function ψ, that depends, in coordinate representation, on the positions $x_1 \ll \ldots \ll x_L \ll x_1 + R$ of the L asymptotic particles, assumed to be well separated from each other. For simplicity, let us proceed with the particular case of a two-body wave-function for a system of two particles with rapidity θ_1 and θ_2, respectively. Then we have that the asymptotic wave-function is simply given by the superposition of an incident wave with a scattered one. It reads explicitely as

$$\psi(x_1, x_2) = \mathrm{e}^{ip_1x_1 + ip_2x_2} + S(\theta_1 - \theta_2)\, \mathrm{e}^{ip_2x_1 + ip_1x_2}\,, \tag{4.1.106}$$

where $p_{1,2} = m \sinh \theta_{1,2}$. In the infinite volume limit, all values of p_1 and p_2 are possible. But on an cylinder of length R, we require the wave-function to be periodic

$$\psi(x_1, x_2) = \psi(x_2, x_1 + R)\,, \tag{4.1.107}$$

wich translates into the equations

$$\mathrm{e}^{ip_1R} = S(\theta_2 - \theta_1)\,, \qquad \mathrm{e}^{ip_2R} = S(\theta_1 - \theta_2)\,. \tag{4.1.108}$$

Note that these equations have to be solved for $p_1 \neq p_2$ since otherwise the asymptotic wave-function vanishes due to $S(0) = -1$, see Eq. (4.1.105). The generalization to the case of L asymptotic particles is straightforward and, thanks to the factorizability of the scattering, one finds that the periodicity conditions turns into a set of Bethe ansatz equations

$$\mathrm{e}^{ip_kR} = \prod_{j\neq k}^{L} S(\theta_j - \theta_k)\,. \tag{4.1.109}$$

Solutions satisfying the exclusion principle $p_1 \neq \ldots \neq p_L$ lead to the asymptotic spectrum of L particles states with energy given by the sum of the individual asymptotic energies.

The Bethe ansatz equations (4.1.109) are only asymptotic and correctly describe the spectrum of the theory when defined on a cylinder with a sufficiently "large" radius R. Here we mean large as compared to the Compton wave-length of a particle of mass m. On physical grounds, if not sufficient, the latter condition is at least necessary. Indeed, over a distance smaller than the Compton wave-length, it looks suspicious to ascribe a wave-function to our system since the (asymptotic) particles completely loose their individuality and are impossible to enumerate. Then, off-shell processes can no longer be neglected and a full-fledged field-theory treatment is required. In conclusion, we enforce the condition $mR \gg 1$ to trust the validity of the Eqs. (4.1.109). This condition is obtained in the thermodynamic limit that we will consider, but we note that in terms of the string theory variables it translates into a condition for the spin at strong coupling to be surprisingly large, $\log \mathcal{S} \gg \exp(\pi g)$. It seems to indicate that the Bethe equations are reliable only if the infinite volume limit is understood, which may not be completely wrong given the quasi-masslessness of the particles at strong coupling.

Thermodynamic Limit

The next step is to take the thermodynamic limit of the asymptotic Bethe ansatz equations (4.1.109), for the (ground) state with minimal energy for a fixed number L of particles. It consists of taking $R \to \infty$, simultaneously with $L \to \infty$, while keeping fixed the density $\rho = L/R$. Similarly to the large spin limit of the minimal anomalous dimension, we will find that it is possible to solve the algebraic problem by means of a linear integral equation. The analysis below is standard and follows the one of the non-relativistic gas of bosons interacting via a repulsive $\delta-$function potential [87]. We will therefore only sketch the different steps and let the reader adapts the lines of [87] to the present case, for a more rigorous treatment.

We start by rewriting Eq. (4.1.109) as

$$e^{imR\sinh\theta_k} = (-1)^{L-1}\prod_{j=1}^{L}\Big[-S(\theta_j-\theta_k)\Big]\,, \tag{4.1.110}$$

in order for the term in the product to be normalized to one at zero-momentum exchange. Then, on both sides of Eq. (4.1.110), we take the logarithm, choosing for convenience the principal branch, and get

$$2\pi n_k = mR\sinh\theta_k + i\sum_{j=1}^{L}\log\Big[-S(\theta_j-\theta_k)\Big]\,. \tag{4.1.111}$$

Here, the mode numbers n_k belong to $\mathbb{Z}$ or $(2\mathbb{Z}+1)/2$, for L odd or even, respectively. They enumerate the possible states with L particles, including the ground state, and they are allowed to take arbitrary large values, corresponding to highly excited states. We look for real solutions to Eq. (4.1.111), assuming that the true ground state belongs to this set. We note that for θ real, each term of the sum in Eq. (4.1.111) is purely imaginary and it is consistent to consider real solutions to Eq. (4.1.111). Moreover, it seems reasonable to believe that complex solutions are ruled out by the absence of bound states in the infinite-volume limit. We don't know, however, a general proof supporting the argument.

Before we take the thermodynamic scaling of the Bethe ansatz equations (4.1.111), it is instructive to consider the non-relativistic limit, first. It helps to understand how to correctly characterize the ground-state distribution of rapidities. In the non-relativistic limit, all the rapidities are small, $\theta_k \sim 1/R$, and we can safely neglect the scattering term in (4.1.111), as long as the number of particles L is small enough. Doing this way, we end up with the equations

$$\theta_k = \frac{2\pi}{mR} n_k \,, \tag{4.1.112}$$

which are the free-particle quantization conditions for the momenta $p_k = m \sinh(\theta_k) \simeq m\theta_k$. We recall that by construction the rapidities are ordered decreasingly and do not coincide, $\theta_1 > \ldots > \theta_L$. Consequently, all the mode numbers n_k in Eq. (4.1.112) are strictly different, in agreement with the exclusion principle. The ground-state is then given by the Fermi sea of a system of L non-interacting particles, described by the (symmetric-)distribution

$$n_k \in \mathfrak{S} = \left\{ \frac{L-1}{2}, \frac{L-3}{2}, \ldots, -\frac{L-3}{2}, -\frac{L-1}{2} \right\}. \tag{4.1.113}$$

Given the set $\mathfrak{S}$ and observing that the distribution of rapidities (4.1.112) is dense at large volume $mR \gg 1$, it is straightforward to compute the density of energy ε in the thermodynamic limit. It turns into a function of the density ρ and reads

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{m}{R} \sum_{k=1}^{L} \cosh \theta_k = \frac{m}{R} \sum_{k=1}^{L} \left(1 + \frac{\theta_k^2}{2} + \ldots \right) = m\rho + \frac{\pi^2}{6m}\rho^3 + \ldots \,, \tag{4.1.114}$$

where dots stand both for corrections to the thermodynamic limit, suppressed by powers of $1/R$, and for relativistic contributions, suppressed by higher powers of the density. To compute properly the latter contributions, one needs to take into account scattering effects. Indeed, one can easily verify, by evaluating the scattering term in (4.1.111) with the help of Eqs. (4.1.112) and (4.1.113), that the size of the interaction grows with the density. Therefore, in the termodynamic limit, the non-relativistic approximation coincides with the low-density regime $\rho = L/R \ll m$.

We now determine the exact ground-state energy density $\varepsilon_{\mathrm{O}(6)}(\rho)$ at arbitrary value of ρ. We assume that the distribution of mode numbers n_k parameterizing the ground state is still given by (4.1.113). In the thermodynamic limit, the distribution is dense in rapidity space and the sum in the right-hand side of (4.1.111) can be approximated by an integral. As for the large spin limit of the $\mathfrak{sl}(2)$ Bethe ansatz equations, the continuum limit is facilitated by introducing a smooth function $\chi(\theta)$ which interpolates the distibution of rapidities. It is defined as

$$\int_0^{\theta} d\theta' \, \chi(\theta') = m \sinh \theta + \frac{i}{R} \sum_{j=1}^{L} \log \Big[- S(\theta_j - \theta) \Big] \,, \tag{4.1.115}$$

and it is positive and symmetric (for the ground-state) $\chi(\theta) = \chi(-\theta)$. Due to the Bethe ansatz equations (4.1.111), it satisfies

$$\int_0^{\theta_k} d\theta' \, \chi(\theta') = \frac{2\pi}{R} n_k \,, \tag{4.1.116}$$

where the mode numbers n_k fill the set $\mathfrak{S}$, Eq. (4.1.113). In particular, denoting $B = \theta_1$ the Fermi rapidity (maximal rapidity), we have the normalization condition ($\rho = L/R$)

$$\int_{-B}^{B} d\theta \, \chi(\theta) = 2\pi \frac{L-1}{R} = 2\pi\rho + O(1/R) \,. \tag{4.1.117}$$

Applying the Euler-Maclaurin summation formula to the right-hand side of (4.1.115), we obtain

$$\int_0^\theta d\theta'\,\chi(\theta') = m\sinh\theta + \frac{i}{2\pi}\int_{-B}^{B} d\theta'\,\chi(\theta')\log\Big[-S(\theta'-\theta)\Big] + \ldots\,, \tag{4.1.118}$$

where ellipses stand for contributions suppressed by $1/R$. Differentiating both sides of the equation above with respect to θ, we obtain an integral equation determining the rapidity density distribution $\chi(\theta)$ in the thermodynamic limit $R, L\to\infty$ with $\rho = L/R$ fixed. It reads [130]

$$\chi(\theta) = m\cosh\theta + \int_{-B}^{B} d\theta'\,K(\theta-\theta')\chi(\theta')\,, \tag{4.1.119}$$

with the (real and symmetric) kernel expressed in terms of the logarithmic derivative of the S-matrix (4.1.105) as $K(\theta) = (\log S(\theta))'/(2\pi i)$, i.e.

$$\begin{aligned} K(\theta) = \frac{1}{4\pi^2}\Big[&\psi\left(1+\frac{i\theta}{2\pi}\right)+\psi\left(1-\frac{i\theta}{2\pi}\right)-\psi\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right)-\psi\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right) \\ &+\psi\left(\Delta+\frac{1}{2}+\frac{i\theta}{2\pi}\right)+\psi\left(\Delta+\frac{1}{2}-\frac{i\theta}{2\pi}\right)-\psi\left(\Delta+\frac{i\theta}{2\pi}\right)-\psi\left(\Delta-\frac{i\theta}{2\pi}\right)\Big]\,, \end{aligned} \tag{4.1.120}$$

where ψ is the Euler psi-function and $\Delta = 1/(n-2)$. Finally, one finds that the density of energy is given in the thermodynamic limit as

$$\varepsilon_{\mathrm{O}(6)} = \frac{m}{2\pi}\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\theta\,, \tag{4.1.121}$$

up to subleading corrections suppressed by $1/R$.

The solution to the equation (4.1.119) fully determines the density of energy $\varepsilon_{\mathrm{O}(6)}$ as a function of the density ρ, after eliminating B with the help of (4.1.117) and (4.1.121). In the low-density regime $\rho \ll m$, corresponding to $B \ll 1$, the equation (4.1.119) can be solved by iteration of the inhomogeneous term as

$$\chi(\theta) = m\sum_{n\geqslant 0}\left(K^n * \cosh\right)(\theta) = m\cosh\theta + m\int_{-B}^{B} d\theta'\,K(\theta-\theta')\cosh\theta' + \ldots\,. \tag{4.1.122}$$

It follows that both $\varepsilon_{\mathrm{O}(6)}$ and ρ can be found to admit an expansion in integer powers of B. Reciprocally, B admits an expansion in integer powers of the dimensionless ratio ρ/m, and so does $\varepsilon_{\mathrm{O}(6)}(\rho)$ up to an overall factor of m^2. For instance, at leading order one gets $B = \pi\rho/m + O(\rho^2/m^2)$ and

$$\varepsilon_{\mathrm{O}(6)}(\rho) = m^2\left[\left(\frac{\rho}{m}\right)+\frac{\pi^2}{6}\left(\frac{\rho}{m}\right)^3 + O(\rho^4/m^4)\right]\,, \tag{4.1.123}$$

in agreement with (4.1.114). Subleading corrections can be obtained iteratively. They have been considered in [75] for the O(6) model, in relation with the small j expansion of the scaling function of $\mathcal{N}=4$ SYM theory at strong coupling.

The analysis of the large-density regime $\rho \gg m$, which is relevant for the matching with the semiclassical expansion of the string scaling function, turns out to be more difficult. It corresponds to the regime $B \gg 1$ of the equation (4.1.119), whose study was performed at leading order in [130]. A systematic analysis including arbitrary subleading corrections has been recently developed in [136]. Restricting to the case of the $\mathrm{O}(n=6)$ model, it was found in [130, 136] that the solution for $B \gg 1$ to the TBA equation (4.1.119) yields

$$\rho = m\left[\frac{\sqrt{2}\Gamma(5/4)}{\pi}\sqrt{B}\ \mathrm{e}^{B-1/4} + \dots\right], \qquad \varepsilon_{\mathrm{O}(6)} = m^2\left[\frac{\Gamma(5/4)^2}{2\pi}\ \mathrm{e}^{2B-1/2} + \dots\right]. \tag{4.1.124}$$

Solving the parametric dependence, the energy density reads for $\rho \gg m$ as

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{\pi\rho^2}{4\log(\rho/m)} + \dots\,, \tag{4.1.125}$$

in agreement with the leading (renormalization-group improved) perturbative result [130]. Subleading corrections [130, 136] were found to match the O(6) perturbative result [130, 76] up to two loops.

Equipped with the TBA equations for the O(6) model, Eqs. (4.1.119), (4.1.121), (4.1.117), we are now in position to examine the all-loop integral equation for the scaling function of the gauge theory [47]. This is the subject of the next section.

4.2 Scaling Function in $\mathcal{N}=4$ SYM Theory

In the gauge theory, the derivation of the scaling function $\epsilon(g,j)$ relies on the integrability of the (planar) dilatation operator in the $\mathfrak{sl}(2)$ sector [45, 47]. We exemplified it at the one-loop level in Chapter 2 starting from the Bethe ansatz equations of the $\mathrm{XXX}_{-1/2}$ Heisenberg spin chain. The Bethe ansatz solution is conveniently characterized in this limit by two sets of parameters, the Bethe roots and the roots of the (auxiliary) transfer matrix, which describe magnon and hole excitations, respectively. In the generalized scaling limit, both sets of parameters form a dense distribution on the real axis with the holes confined to the interval $[-a,a]$ and the magnons to the union of two intervals $[-\infty,-a]\cup[a,\infty]$. The scaling function $\epsilon(g,j)$ is uniquely determined by the corresponding distributions of holes and magnons arising as solutions to an integral equation. In this section, we will analyse this equation proposed in [47] by Freyhult, Rej and Staudacher, and known as FRS equation.

We will start reformulating the FRS equation as a system of coupled equations for the hole and magnon density distributions [73]. We will see that the magnon equation takes essentially the same form as the BES equation up to an inhomogeneous term. This term acts as a source and accounts for the influence of a given static distribution of hole rapidities on the distribution of Bethe roots. The analogy with the BES equation will be very helpful to analyse the small j regime of the scaling function $\epsilon(g,j)$ on which we will focus. After a quick look at the solution at weak coupling, we will solve the magnon equation at strong coupling and derive an effective equation for the hole dynamics. We will find that the effective hole equation exactly coincides with the thermodynamic Bethe ansatz equation for the O(6) sigma model, as expected from the string theory analysis of [46]. The holes are then naturally identified with the massive interacting particles of the O(6) model. This mapping implies, in particular, that the FRS equation predicts the dependence on the coupling constant of the mass gap m of the O(6) model. We will see that m is determined uniquely by the BES solution. Moreover, it perfectly matches the scale m_{cusp}

that controls the leading non-perturbative correction to the cusp anomalous dimension. We found in Chapter 3, see Eq. (3.3.28), that to leading-order at strong coupling $m_{\rm cusp}$ is given by

$$m = m_{\rm cusp} = k\, g^{1/4}\, {\rm e}^{-\pi g}\,(1 + O(1/g))\,, \qquad\qquad k = 2^{3/4}\pi^{1/4}/\Gamma(\tfrac{5}{4})\,. \tag{4.2.1}$$

We immediately verify the agreement with the string theory prediction [46], see Eqs. (4.1.88) and (4.1.89). However, the comparison of the subleading, $1/g$ suppressed contribution in Eq. (3.3.28) and Eq. (4.1.88), shows that the gauge and string expressions for the O(6) mass gap m disagree at the two-loop level. We will come back to this mismatch at the end of this chapter.

4.2.1 FRS Equation

The FRS equation, as given in [47], is an integral equation for the (Fourier-like transform) of the density distribution of fluctuations $\widehat{\sigma}(t)$. It is very similar to the BES equation [44] and reads

$$\widehat{\sigma}(t) = \frac{t}{{\rm e}^t - 1}\left(\widehat{\mathcal{K}}(t,0) - 4\int_0^\infty dt'\, \widehat{\mathcal{K}}(t,t')\, \widehat{\sigma}(t')\right). \tag{4.2.2}$$

Here $\widehat{\mathcal{K}}(t,t')$ is a complicated kernel depending on both the coupling constant g and the parameter a. In the particular limit $a \to 0$, i.e. $j \to 0$, the kernel reduces to the kernel of the BES equation and therefore we have that $\widehat{\sigma}(t; j = 0) = \widehat{\sigma}^{\rm BES}(t)$. The explicit expression for the kernel $\widehat{\mathcal{K}}(t,t')$ can be found in [47].

Given the solution to the FRS equation $\widehat{\sigma}(t)$, one can compute the all-loop density distribution $\rho(u)$ in the generalized scaling limit. The precise relation between the two is

$$\rho(u) = \rho_0(u) - 8\sigma(u)\log N\,, \tag{4.2.3}$$

where the density distribution of fluctuations $\sigma(u)$ is given by

$$\sigma(u) = \frac{1}{\pi}\int_0^\infty dt\, \cos(ut)\, {\rm e}^{t/2}\, \widehat{\sigma}(t)\,. \tag{4.2.4}$$

The function $\sigma(u)$ encodes both higher-loop and j-dependent contributions to the density distribution $\rho(u)$, while $\rho_0(u) = 2\log N/\pi$ is the twist-two one-loop density distribution in the limit $N \to \infty$ at fixed rapidity u. On the interval $u^2 > a^2$ the function $\rho(u)$ describes the density distribution of Bethe roots, while on the interval $u^2 < a^2$ it can be used to compute the density distribution of holes rapidities

$$\rho_{\rm h}(u) \equiv \rho(u)/\log N = \frac{2}{\pi} - 8\sigma(u)\,, \tag{4.2.5}$$

normalized such that

$$\int_{-a}^{a} du\, \rho_{\rm h}(u) = j\,. \tag{4.2.6}$$

For later convenience, we introduce the Fourier transform of the hole distribution as

$$\gamma_{\rm h}(t) = \frac{1}{8}\int_{-a}^{a} du\, \cos(ut)\rho_{\rm h}(u)\,. \tag{4.2.7}$$

It is an even entire function of t satisfying $\gamma_{\rm h}(0) = j/8$. Using the relations (4.2.4), (4.2.5) and (4.2.7), we find that it can expressed in terms of the function $\widehat{\sigma}(t)$ as

$$\gamma_{\rm h}(t) = K_{\rm h}(t,0) - 4\int_0^\infty dt'\, K_{\rm h}(t,t')\, {\rm e}^{t'/2}\, \widehat{\sigma}(t')\,, \tag{4.2.8}$$

where the hole kernel $K_{\rm h}(t,t')$ is given by

$$K_{\rm h}(t,t') = \frac{1}{4\pi}\left[\frac{\sin(a(t-t'))}{t-t'} + \frac{\sin(a(t+t'))}{t+t'}\right]. \tag{4.2.9}$$

In particular, we obtain from $\gamma_{\rm h}(0) = j/8$ that the identity

$$j = \frac{4a}{\pi} - \frac{16}{\pi}\int_0^\infty \frac{dt}{t}\,\sin(at)\,{\rm e}^{t/2}\,\widehat{\sigma}(t) \tag{4.2.10}$$

holds true. This relation can be used, given the solution $\widehat{\sigma}(t) = \widehat{\sigma}(t;a)$ to the FRS equation [47], to eliminate the parameter a in favor of the scaling variable j.

Last but not least, it can be shown that the FRS solution $\widehat{\sigma}(t)$ predicts the all-loop scaling function $f(g,j)$ by means of [47]

$$f(g,j) \equiv 2\Gamma_{\rm cusp}(g) + \epsilon(g,j) = j + 16\widehat{\sigma}(0)\,. \tag{4.2.11}$$

Magnon and Hole Equations

As said before, given the solution $\widehat{\sigma}(t)$ to the FRS equation, one can compute the all-loop density distributions of both hole and magnon rapidities [47]. However, at strong coupling, the FRS equation (4.2.2) is difficult to solve directly in terms of the function $\widehat{\sigma}(t)$. Hence, similarly to our previous treatment of the BES equation, we will introduce more suitable functions $\gamma_\pm(t)$ and rewrite the FRS equation as a system of integral equations for them. They arise from the splitting of $\widehat{\sigma}(t)$ into the sum

$$\widehat{\sigma}(t) = \frac{1}{{\rm e}^t - 1}\left[\frac{g}{2}\Big(\gamma_+(2gt) + \gamma_-(2gt)\Big) + {\rm e}^{-t/2}\,\gamma_{\rm h}(t) - \frac{j}{8}J_0(2gt)\right]. \tag{4.2.12}$$

where J_0 is an (even) Bessel function. In the limit $j \to 0$, i.e. $a \to 0$, we have $\gamma_{\rm h}(t) \to 0$ and the relation (4.2.12) reduces to

$$\widehat{\sigma}(t; j=0) = \widehat{\sigma}^{\rm BES}(t) = \frac{g}{2}\frac{\gamma_+^{(j=0)}(2gt) + \gamma_-^{(j=0)}(2gt)}{{\rm e}^t - 1}\,. \tag{4.2.13}$$

It coincides with the formula (3.1.9) used in Chapter 3 to simplify the analysis of the BES equation. If instead, we keep j fixed but assume $g \to 0$, the equation (4.2.12) becomes

$$\widehat{\sigma}(t; g=0) = \frac{1}{{\rm e}^t - 1}\left[{\rm e}^{-t/2}\,\gamma_{\rm h}(t; g=0) - \frac{j}{8}\right]. \tag{4.2.14}$$

Using Eqs. (4.2.3), (4.2.4) and (4.2.7), it translates into

$$\begin{aligned}\frac{2\pi}{\log N}\rho(u; g=0) =& 4 - j\Big[\psi\left(\tfrac{1}{2}+iu\right) + \psi\left(\tfrac{1}{2}-iu\right)\Big] \\ &+ \int_{-a}^{a} dv\,\rho_{\rm h}(v; g=0)\Big[\psi\left(1+i(u-v)\right) + \psi\left(1-i(u-v)\right)\Big],\end{aligned} \tag{4.2.15}$$

which coincides with the one-loop formula (2.3.49) derived in Chapter 2. Thus we verified that the relation (4.2.12) is consistent with our previous analysis.

The even/odd functions $\gamma_\pm(-t) = \pm\gamma_\pm(t)$ have exactly the same analytic properties as in Chapter 3. Namely, they are entire functions of t, satisfying $\gamma_-(t) = O(t)$ and $\gamma_+(t) = O(t^2)$ at small t, and their Fourier transforms have support on the interval $[-1, 1]$. The function $\gamma_-(t)$ can be used to compute the scaling function $f(g, j)$ with the help of

$$f(g,j) = 16g^2 \lim_{t\to 0} \frac{\gamma_-(2gt)}{2gt}\,, \tag{4.2.16}$$

where we applied Eqs. (4.2.11) and (4.2.12) and used that $\gamma_{\rm h}(0) = j/8$.[14]

The functions $\gamma_\pm(t)$ are subject to an (infinite) system of equations, to which we shall refer as magnon equation. It was derived in [73] from the FRS equation (4.2.2) and reads ($n \geqslant 1$)

$$\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(2gt)}{1-{\rm e}^{-t}} + \frac{\gamma_+(2gt)}{{\rm e}^t-1}\right] J_{2n-1}(2gt) = \frac{1}{2}\,\delta_{n,1} + h_{2n-1}(g,j)\,, \\
&\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(2gt)}{1-{\rm e}^{-t}} - \frac{\gamma_-(2gt)}{{\rm e}^t-1}\right] J_{2n}(2gt) = h_{2n}(g,j)\,.
\end{aligned} \tag{4.2.17}$$

Here the notation was introduced for the coefficients $h_n(g, j)$ that depend on the hole distribution $\gamma_{\rm h}(t)$ as

$$h_n(g,j) = -\frac{2}{g}\int_0^\infty \frac{dt}{t}\,\frac{J_n(2gt)}{{\rm e}^t-1}\left[{\rm e}^{t/2}\,\gamma_{\rm h}(t) - \frac{j}{8}J_0(2gt)\right]. \tag{4.2.18}$$

We note that the system of equations (4.2.17) depends on j only through the inhomogeneous terms $h_n(g, j)$. Moreover, when $j \to 0$, we have that $\gamma_{\rm h}(t; j = 0) = h_n(g, j = 0) = 0$ and the system (4.2.17) becomes identical to the BES one. We denote its solution, analysed in Chapter 3, as

$$\gamma_\pm^{(j=0)}(t) \equiv \gamma_\pm^{(0)}(t)\,. \tag{4.2.19}$$

For non-vanishing value of j and for given 'static' hole distribution $\gamma_{\rm h}(t)$, we can solve the magnon equation (4.2.17) for $\gamma_\pm(2gt)$. Doing so, the function $\gamma_{\rm h}(t)$, which acts as a source for the system (4.2.17), can be considered as being arbitrary. But it should satisfy $\gamma_{\rm h}(0) = j/8$ in order for the integral in $h_1(g, j)$ to be well-defined.[15] If, furthermore, the functions $\gamma_\pm(t)$ are required to have the correct analytic properties, one can show that the solution to the magnon equation is uniquely determined.[16]

When the solution to the magnon equation is known, one can compute the distribution of fluctuations $\widehat{\sigma}(t)$ via Eq. (4.2.12) and, then, the density distribution $\rho(u)$ with the help of (4.2.3) and (4.2.4). Eventually, one needs to impose the identity $\rho_{\rm h}(u) = \rho(u)/\log N$ when $u^2 < a^2$ in order to determine the hole distribution. It is equivalent to requiring for the relation (4.2.8) between $\gamma_{\rm h}(t)$ and $\widehat{\sigma}(t)$ to be satisfied. In other words, after assuming $\gamma_{\rm h}(t)$ to be fixed and extracting $\gamma_\pm(t)$ from the magnon equation, the relation (4.2.8) becomes an equation to be solved

[14] Note that, similarly to what happens in the BES limit $j \to 0$, the function $\gamma_-(t)$ generates, after expansion over the Bessel functions, the infinite set of (even) higher conserved charges. The function $\gamma_+(t)$ produces an infinite sequence of charges, whose spin-chain interpretation remains obscure.

[15] Moreover, we will always assume that the Fourier transform of $\gamma_{\rm h}(t)$ is supported on the interval $[-a, a]$.

[16] The proof goes along the same lines as the ones given in Chapter 3 in the BES $j \to 0$ limit.

for the function $\gamma_{\rm h}(t)$. We shall refer, therefore, to the relation (4.2.8) as the hole equation. We note, finally, that these two steps in the construction of the complete solution are identical to those followed at one-loop order in Chapter 2. The only difference is that it turns out to be more difficult to completely eliminate the magnons from consideration, namely to find the two functions $\gamma_\pm(t)$ for a given $\gamma_{\rm h}(t)$, and then obtain a closed 'effective' equation for the holes. Nevertheless, we will find a regime of interest in which that can be done explicitely.

Hole Energy Formula

As in one-loop case, discussed in Chapter 2, it is possible to find a representation of the scaling function in terms of the hole function $\gamma_{\rm h}(t)$. It can be found by using the fact that the FRS and BES solutions satisfy a Wronskian-like relation, derived in Appendix B.2. It leads to the following expression for the scaling function

$$\epsilon(g,j) = 32g\int_0^\infty \frac{dt}{t}\,\frac{\gamma_+^{(0)}(2gt)-\gamma_-^{(0)}(2gt)}{{\rm e}^t-1}\left[{\rm e}^{t/2}\,\gamma_{\rm h}(t)-\frac{j}{8}\right] - 4gj\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)\,. \qquad (4.2.20)$$

This identity generalizes to all-loop the one-loop hole energy formula found in Chapter 2, see Eq. (2.3.53). To verify it, we substitute in (4.2.20) the weak coupling expression of the BES solution given by

$$\gamma_-^{(0)}(t) = J_1(t) + O(g^2)\,, \qquad\qquad \gamma_+^{(0)}(t) = O(g^3)\,, \qquad (4.2.21)$$

and expand the resulting integral to leading order at weak coupling. We find that

$$\epsilon(g,j) = -32g^2\int_0^\infty \frac{dt}{{\rm e}^t-1}\left[{\rm e}^{t/2}\,\gamma_{\rm h}(t)-\frac{j}{8}\right] + O(g^4)\,. \qquad (4.2.22)$$

Then, introducing the hole density distribution as in (4.2.7) and performing the integral (4.2.22), we obtain

$$\epsilon(g,j) = 2g^2\int_{-a}^{a} dv\,\rho_{\rm h}(v)\left[\psi\big(\tfrac12+iv\big)+\psi\big(\tfrac12-iv\big)-2\psi(1)\right] + O(g^4)\,, \qquad (4.2.23)$$

which agrees with the one-loop formula (2.3.53). It is straightforward to compute higher-loop corrections to (4.2.23) by solving iteratively the BES equation at weak coupling. We shall see, moreover, that a useful representation for the scaling function at small j can be obtained from the relation (4.2.20) at both weak and strong coupling.

4.2.2 From Weak to Strong Coupling

To find the scaling function $\epsilon(g,j)$, we have to solve a complicated system of coupled integral equations for the functions $\gamma_\pm(t)$ and $\gamma_{\rm h}(t)$. In this subsection we shall evaluate the scaling function $\epsilon(g,j)$ at small j. As we will see in a moment, the first few terms of the small j expansion of $\epsilon(g,j)$ can be expressed directly in terms of the BES solution without solving the integral equations for the functions $\gamma_\pm(t)$. Thanks to this simplification, we will be able to uncover a transition from the *perturbative* weak-coupling regime, with a running in powers of g^2, to a non-perturbative strong-coupling regime, controlled by the scale $m \propto g^{1/4}\,{\rm e}^{-\pi g}$. The latter parameter will be given the meaning of the mass gap of the O(6) sigma model in the next subsection.

Small j Expansion

To begin with let us consider the integral equation (4.2.8) for the function $\gamma_{\rm h}(t)$. It involves the kernel $K_{\rm h}(t,t')$ defined in (4.2.9). According to its definition, $K_{\rm h}(t,t')$ depends on the parameter a which depends on its turn on j. For a small value of j, we have $a \sim j$ and we may expand the kernel $K_{\rm h}(t,t')$ in powers of a. We obtain from (4.2.9)

$$K_{\rm h}(t,t') = \frac{1}{2\pi}\left[a - \frac{a^3}{6}\left(t^2+t'^2\right) + O\left(a^5\right)\right] = \frac{\sin(at')}{2\pi t'}\left[1-\frac{1}{6}(at)^2\right] + O\left(a^5\right). \qquad (4.2.24)$$

Substituting this relation into the right-hand side of the hole equation (4.2.8) and using (4.2.10), we find that

$$\gamma_{\rm h}(t) = \frac{j}{8}\left[1-\frac{1}{6}(at)^2 + O(a^4)\right]. \qquad (4.2.25)$$

Here, a is related to j as

$$a = \frac{j\pi}{2\kappa} + O(j^2), \qquad (4.2.26)$$

where the $g-$dependence resides in the normalization factor

$$\kappa = 2 - 2g\int_0^\infty dt\,\frac{\gamma_+^{(0)}(2gt)+\gamma_-^{(0)}(2gt)}{\sinh(t/2)}. \qquad (4.2.27)$$

To derive the last identity we used (4.2.10), (4.2.25), (4.2.12) and $\gamma_\pm(2gt) = \gamma_\pm^{(0)}(2gt) + O(j)$.

According to the representation (4.2.20), the $j-$dependence of the scaling function $\epsilon(g,j)$ is controlled by the function $\gamma_{\rm h}(t)$. Replacing $\gamma_{\rm h}(t)$ in (4.2.20) by its small j expansion (4.2.25), we get

$$\epsilon(g,j) = \epsilon_1(g)\,j + \epsilon_3(g)\,j^3 + O(j^4), \qquad (4.2.28)$$

where the coefficient in front of j^2 equals zero for any g [47] and the coefficient functions $\epsilon_1(g)$ and $\epsilon_3(g)$ are given by

$$\begin{aligned}\epsilon_1(g) &= -4g\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+^{(0)}(2gt)}{{\rm e}^{-t/2}+1} + \frac{\gamma_-^{(0)}(2gt)}{{\rm e}^{t/2}+1}\right], \\ \epsilon_3(g) &= -\frac{\pi^2 g}{12\kappa^2}\int_0^\infty dt\,t\,\frac{\gamma_+^{(0)}(2gt)-\gamma_-^{(0)}(2gt)}{\sinh(t/2)}.\end{aligned} \qquad (4.2.29)$$

These relations hold for arbitrary coupling g and thus can be investigated at weak and/or strong coupling given the BES solution $\gamma_\pm^{(0)}(2gt)$.

Weak Coupling

At weak coupling, the iterative solution to the BES equation [44, 90] leads to

$$\gamma_-^{(0)}(t) = \left(1-\frac{\pi^2 g^2}{3}\right) J_1(t) + O\left(g^4\right), \qquad \gamma_+^{(0)}(t) = 4\zeta_3\, g^3 J_2(t) + O\left(g^5\right). \qquad (4.2.30)$$

Plugging these expressions into (4.2.29) and expanding $\epsilon_1(g)$ and $\epsilon_3(g)$ at small g^2 we get the expected perturbative expansion in g^2 of the gauge theory

$$\begin{aligned}\epsilon_1(g) &= -8\log 2\, g^2 + \left(\frac{8}{3}\log 2\,\pi^2 + 16\zeta_3\right) g^4 + O(g^6)\,, \\ \epsilon_3(g) &= \frac{7}{12}\zeta_3\pi^2 g^2 + \left(\frac{35}{36}\zeta_3\pi^4 - \frac{31}{2}\zeta_5\pi^2\right) g^4 + O(g^6)\,.\end{aligned} \tag{4.2.31}$$

The leading contribution $\sim g^2$ agrees with our previous findings in Chapter 2 while subleading correction was obtained in [47].

Strong Coupling

At strong coupling, the evaluation of $\epsilon_1(g)$ leads to (see details in Appendix B.3)

$$\epsilon_1(g) = -1 + m + O\left(\mathrm{e}^{-3g\pi}\right)\,, \tag{4.2.32}$$

where the parameter $m = m(g)$ is defined as

$$m = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^\infty \frac{dt\ \mathrm{e}^{i(t-\pi/4)}}{t+i\pi g}\left(\Gamma_+^{(0)}(t) + i\Gamma_-^{(0)}(t)\right)\right]. \tag{4.2.33}$$

According to (4.2.32), the function $\epsilon_1(g)$ does not receive perturbative corrections in $1/g$ and the leading non-trivial correction is given by m which is exponentially small in g. We immediately recognize that the parameter m exactly coincides with the non-perturbative scale m_{cusp} introduced in Chapter 3. There, it was shown that the scale $m = m_{\mathrm{cusp}}$ controls the leading non-perturbative correction to the cusp anomalous dimension. To leading-order at strong coupling, we found that, see Eq. (3.3.28),

$$m = m_{\mathrm{cusp}} = \frac{\sqrt{2}}{\Gamma(\frac{5}{4})}(2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\,. \tag{4.2.34}$$

The expression (4.2.34) is consistent with the numerical solution to the FRS equation constructed in [74]. Moreover, it is in perfect agreement with the expression for the O(6) mass gap that we found by matching the string theory prediction for the scaling function against its O(6) interpretation [46], see Eqs. (4.1.88) and (4.1.89). To uncover if m has the same meaning in the gauge-theory Bethe ansatz approach, let us consider the subleading $O(j^3)$ contribution $\epsilon_3(g)$ at strong coupling.

The evaluation of the integrals entering the expressions for $\epsilon_3(g)$ and κ, Eqs. (4.2.29) and (4.2.27) respectively, can be found in [73]. It is shown that the leading contributions to $\epsilon_3(g)$ and κ as $g \to \infty$ are proportional to the scale m and read

$$\epsilon_3(g) = \frac{\pi^4}{96\kappa^2}m + O\left(\mathrm{e}^{-3\pi g}\right)\,, \qquad \kappa = \frac{\pi}{2}m + O\left(\mathrm{e}^{-3\pi g}\right)\,, \tag{4.2.35}$$

so that the relation (4.2.26) takes the form $a = j/m + \ldots$. Then combining together the relations (4.2.28), (4.2.32) and (4.2.35), we obtain the small j expansion of the scaling function at strong coupling as

$$\epsilon(g,j) + j = m^2\left[\frac{j}{m} + \frac{\pi^2}{24}\left(\frac{j}{m}\right)^3 + \ldots\right]\,, \tag{4.2.36}$$

with m given by (4.2.34). Introducing notation for a density of particles $\rho \equiv j/2$ and a density of energy $\varepsilon(\rho) = (\epsilon(g,j)+j)/2$, the result (4.2.36) can be written as

$$\varepsilon(\rho) = m\rho + \frac{\pi^2}{6m}\rho^3 + \dots \,. \tag{4.2.37}$$

We already encountered this expression and found that it has the meaning of the energy density of a dilute, nonrelativistic Fermi gas of massive particles of mass m [130]. This suggests, as expected from the string theory side consideration, that the non-perturbative parameter m has indeed the meaning of a mass scale. To understand whether it is the mass gap of the O(6) sigma model, we have to include subleading corrections suppressed by higher powers of j. We expect that the expansion of the scaling function will run in the parameter j/m and that it will match the low-density expansion of the O(6) model. Indeed, we will show in the next section that the hole equation, in the (non-perturbative) regime $j \sim m$, perfectly coincides with the O(6) model TBA equation with m being the mass gap.

4.2.3 Non-Linear O(6) Sigma Model

For $j \sim m \sim g^{1/4}\,\mathrm{e}^{-\pi g}$, the small j expansion employed in the previous section is not applicable. In this section, we will show that for $g \to \infty$ and $j/m =$ fixed, the scaling function $\epsilon(g,j)$ coincides with the energy density of the ground state of the two-dimensional O(6) sigma model.

O(6) TBA Equation

We start by recalling the exact solution for the ground state energy in the two-dimensional O(6) sigma model [130], constructed before. It can be summarized as follows. The energy density $\varepsilon_{\mathrm{O}(6)}(\rho)$ in the ground state and the particle density ρ are given by

$$\varepsilon_{\mathrm{O}(6)}(\rho) = \frac{m}{2\pi}\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\theta\,, \qquad \rho = \frac{1}{2\pi}\int_{-B}^{B} d\theta\,\chi(\theta)\,, \tag{4.2.38}$$

where the density distribution $\chi(\theta)$ has support on the interval $[-B,B]$ and satisfies the TBA integral equation

$$\chi(\theta) = \int_{-B}^{B} d\theta'\,K(\theta-\theta')\chi(\theta') + m\cosh\theta\,. \tag{4.2.39}$$

Here, m is the mass of the O(6)-multiplet of asymptotic states, B is the Fermi rapidity and the kernel $K(\theta) = (\log S(\theta))'/(2\pi i)$ is related to the logarithmic derivative of the exact $S-$matrix of the O(6) model [72]

$$K(\theta) = \frac{1}{4\pi^2}\left[\psi\left(1+\frac{i\theta}{2\pi}\right) + \psi\left(1-\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right) + \frac{2\pi}{\cosh\theta}\right], \tag{4.2.40}$$

where $\psi(x) = (\log\Gamma(x))'$ is the Euler psi-function. Later in this subsection we will encounter its Fourier transform

$$K(\theta) = \frac{2}{\pi^2}\int_0^\infty dt\,\cos\left(2\theta t/\pi\right)\frac{\mathrm{e}^t+1}{\mathrm{e}^{2t}+1}\,. \tag{4.2.41}$$

We will see that the mapping with the scaling variable j and the scaling function $\epsilon(g,j)$ is given by

$$\varepsilon(\rho) \equiv \frac{\epsilon(g,j)+j}{2} = \varepsilon_{\mathrm{O}(6)}(\rho) + \dots\,, \qquad \rho = \frac{j}{2}\,. \tag{4.2.42}$$

Here the dots in the first relation stand for corrections that are subleading at large g when $j \sim m$. Moreover, we will demonstrate that the hole equation can be written in the form of (4.2.39) upon the identification of holes with O(6) particles,

$$\chi(\theta) = \frac{8}{\pi}\int_{-\infty}^{\infty} dt\, \cos(2\theta t/\pi)\, \gamma_{\mathrm{h}}(t)\,, \tag{4.2.43}$$

or conversely

$$\gamma_{\mathrm{h}}(t) = \frac{1}{8\pi}\int_{-B}^{B} d\theta\, \cos(2\theta t/\pi)\, \chi(\theta)\,, \tag{4.2.44}$$

with $B = a\pi/2$. Note that we have introduced the hole density distribution in a different way than in (4.2.7). The two distributions are related by $\chi(\theta) = 2\rho_{\mathrm{h}}(2\theta/\pi)$.

Rapidity Distribution

Let us first demonstrate that the Fourier transform of the function $\gamma_{\mathrm{h}}(t)$ fulfills the same integral equation (4.2.39) as the rapidity density distribution for the O(6) model.

By construction, the Fourier transform of $\gamma_{\mathrm{h}}(t)$ satisfies the hole equation (4.2.8). In rapidity space, it can be written as

$$\chi(\theta) \equiv 2\rho_{\mathrm{h}}(u) = \frac{4}{\pi} + I(\theta) - \frac{16}{\pi}\int_0^{\infty} dt\, \frac{\cos(ut)}{\mathrm{e}^t - 1}\left(\gamma_{\mathrm{h}}(t) - \frac{j}{8}\,\mathrm{e}^{t/2}\, J_0(2gt)\right)\,, \tag{4.2.45}$$

where $\theta = u\pi/2$ belongs to the interval $[-B,B]$, with $B = a\pi/2$, and the notation was introduced for

$$I(\theta) = -\frac{4g}{\pi}\int_0^{\infty} dt\, \frac{\cos(ut)}{\sinh(t/2)}\Big(\gamma_+(2gt) + \gamma_-(2gt)\Big)\,. \tag{4.2.46}$$

The derivation of Eq. (4.2.45) makes use of the relations (4.2.5), (4.2.4) and (4.2.12).

In order to find a closed equation for the density $\chi(\theta)$, we need to evaluate the integral $I(\theta)$. To this end, we make use of the magnon solution $\gamma_\pm(t)$ constructed in Appendix B.4. There it is shown that the solution can be decomposed as

$$\gamma_\pm(2gt) = \gamma_\pm^{(0)}(2gt) + \delta\gamma_\pm^{\mathrm{p}}(2gt) + \delta\gamma_\pm^{\mathrm{np}}(2gt)\,. \tag{4.2.47}$$

Here the first term in the right-hand side is the BES solution. It is independent on j and on the function $\gamma_{\mathrm{h}}(t)$. The second term is a particular solution of the j-dependent part of the magnon equation (4.2.17). It is given explicitely by

$$\delta\gamma_+^{\mathrm{p}}(2gt) = -\frac{2}{g}\left(\frac{\cosh(t/2)}{\cosh t}\gamma_{\mathrm{h}}(t) - \frac{j}{8}J_0(2gt)\right)\,, \qquad \delta\gamma_-^{\mathrm{p}}(2gt) = -\frac{2}{g}\frac{\sinh(t/2)}{\cosh t}\gamma_{\mathrm{h}}(t)\,. \tag{4.2.48}$$

To leading order at small j, $\gamma_{\mathrm{h}}(t) = j/8 + \dots$, the particular solution, above, agrees with the findings of [74]. Finally, the last term in the right-hand side of (4.2.47) is a homogeneous solution to the magnon equation, or, equivalently, a homogeneous solution to the BES equation. It is introduced in order to restore the correct analytic properties of $\gamma_\pm(2gt)$, see Appendix B.4.

The properties of the homogeneous solution $\delta\gamma_\pm^{\rm np}(t)$ are discussed in Appendix B.4. There, it is shown that the construction of this solution goes along the same lines as the one to the BES equation provided that the condition $B < \pi g$ is fulfilled. So, before to continue, let us be more precise about the limit we are considering. We found previously that the Fermi rapidity is expressed as $a \equiv 2B/\pi = j/m + \dots$ at low density $j \ll m$ and at strong coupling. Of course, j/m may not be small, in which case we expect a more complicated relation between B and j. Nevetheless, the previous equality suggests that keeping j/m fixed is equivalent for B to be fixed, as it is the case for the O(6) TBA equation. We will assume therefore that the strong coupling limit is taken at a given, but arbitrary, value of B. It follows, in particular, that the condition $B < \pi g$ is necessarily satisfied. We can therefore apply the results obtained in Appendix B.4 and observe that the homogeneous solution is exponentially suppressed in g. We may verify it at the level of the scaling function. Namely, plugging $\gamma_-(2gt)$, as given in (4.2.47), into Eq. (4.2.19), we find that

$$f(g,j) = 16g^2 \lim_{t\to 0} \frac{\gamma_-(2gt)}{2gt} = 2\Gamma_{\rm cusp}(g) - j + 16g^2 \lim_{t\to 0} \frac{\delta\gamma_-^{\rm np}(2gt)}{2gt}\,, \tag{4.2.49}$$

or equivalently

$$\epsilon(g,j) = -j + 16g^2 \lim_{t\to 0} \frac{\delta\gamma_-^{\rm np}(2gt)}{2gt}\,. \tag{4.2.50}$$

Here the term $(-j)$ in the right-hand side of (4.2.50) is the contribution to the scaling function originating from the particular solution (4.2.48). Comparing Eq. (4.2.50) with our previous result (4.2.36), we conclude that

$$16g^2 \lim_{t\to 0} \frac{\delta\gamma_-^{\rm np}(2gt)}{2gt} = mj + \dots\,. \tag{4.2.51}$$

Now, since $m \sim {\rm e}^{-\pi g}$, we check that $\delta\gamma_-^{\rm np}(t\sim 0)$ is exponentially suppressed as compared to the two first terms in the right-hand side of the last equality in (4.2.49).

We are now in position to compute the integral $I(\theta)$, Eq. (4.2.46), entering the right-hand side of the hole equation (4.2.45). Plugging the decomposition (4.2.47) into (4.2.46) and making use of (4.2.48), we get that ($\theta = u\pi/2$)

$$I(\theta) = I^{(0)}(\theta) + \delta I^{\rm np}(\theta) + \frac{16}{\pi}\int_0^\infty dt\, \frac{\cos(ut)}{{\rm e}^t - 1}\left(\frac{2\,{\rm e}^{2t}}{{\rm e}^{2t}+1}\gamma_{\rm h}(t) - \frac{j}{8}\,{\rm e}^{t/2}\, J_0(2gt)\right), \tag{4.2.52}$$

where $I^{(0)}(\theta)$ and $\delta I^{\rm np}(\theta)$ are obtained from the integral (4.2.46) by the substitutions $\gamma_\pm(t) \to \gamma_\pm^{(0)}(t)$ and $\gamma_\pm(t) \to \delta\gamma_\pm^{\rm np}(t)$, respectively. Now, evaluating the right-hand side of the hole equation (4.2.45) with the help of (4.2.52), we find

$$\chi(\theta) = \frac{16}{\pi}\int_0^\infty dt\, \cos(ut)\frac{{\rm e}^t + 1}{{\rm e}^{2t}+1}\gamma_{\rm h}(t) + \frac{4}{\pi} + I^{(0)}(\theta) + \delta I^{\rm np}(\theta)\,, \tag{4.2.53}$$

or equivalently, after using (4.2.44), (4.2.41) and $\theta = u\pi/2$,

$$\chi(\theta) = \int_{-B}^{B} d\theta'\, K(\theta-\theta')\chi(\theta') + \frac{4}{\pi} + I^{(0)}(\theta) + \delta I^{\rm np}(\theta)\,. \tag{4.2.54}$$

Here $K(\theta)$ is the kernel of the O(6) model given in Eq. (4.2.40). We have thus already obtained half of the O(6) TBA equation simply from the contribution to $I(\theta)$ associated with the particular solution $\delta\gamma_\pm^{\rm p}(t)$. It remains to check that the inhomogeneous term $4/\pi + I^{(0)}(\theta) + \delta I^{\rm np}(\theta)$ of the hole equation (4.2.54) takes the correct form.

The integral $I^{(0)}(\theta)$ was computed in [73] at strong coupling and at fixed value of θ, assuming that $B < \pi g$. It reads explicitely as

$$I^{(0)}(\theta) = -\frac{4}{\pi} + m\cosh\theta + \widehat{m}\cosh(3\theta) + \dots\,, \tag{4.2.55}$$

where $\widehat{m} = O(\mathrm{e}^{-3\pi g})$. The integral $\delta I^{\rm np}(\theta)$ can be found from a similar analysis and it reads

$$\delta I^{\rm np}(\theta) = \delta m\cosh\theta \int_{-B}^{B} d\theta'\,\chi(\theta')\cosh\theta' + \dots\,, \tag{4.2.56}$$

where $\delta m = \pi m^2/4g = O(\mathrm{e}^{-2\pi g})$ to leading order at strong coupling. We see that the integral $\delta I^{\rm np}(\theta)$ and the contribution $\widehat{m}\cosh(3\theta)$ in (4.2.55) generate subleading corrections, that are exponentially suppressed in g, for a fixed value of θ, as compared to $m\cosh\theta$ in (4.2.55). We can therefore neglect the former in a first approximation. Then, substituting (4.2.55) into (4.2.54) yields

$$\chi(\theta) = \int_{-B}^{B} d\theta'\,K(\theta-\theta')\chi(\theta') + m\cosh\theta\,, \tag{4.2.57}$$

which is precisely the O(6) TBA equation (4.2.39).

Finally, it follows from (4.2.38) and (4.2.44) that the density of particles in the O(6) model is related to the scaling parameter j as

$$\rho = \frac{1}{2\pi}\int_{-B}^{B} d\theta\,\chi(\theta) = 4\gamma_{\rm h}(0) = \frac{j}{2}\,, \tag{4.2.58}$$

this without any approximation.

We can verify that if j/m is kept fixed at strong coupling, then $\chi(\theta)/m$ and B are also fixed, as a consequence of the O(6) equations (4.2.57) and (4.2.58). Then the correction $\widehat{m}\cosh(3\theta)$ in (4.2.55) and the contribution $\delta I^{\rm np}(\theta)$ are both of order $O(\mathrm{e}^{-3\pi g})$, and are therefore suppressed at strong coupling. Note, however, that if B takes a large value, these contributions get typically enhanced by a factor e^{3B}, while the solution $\chi(\theta)$ to the O(6) equation scales as $m\,\mathrm{e}^{B}$ [130, 136]. This suggests that the parameter controlling the validity of the O(6) approximation is $z^2 = \mathrm{e}^{2(B-\pi g)}$. The O(6) equations, Eqs. (4.2.57) and (4.2.58), predict that $B \sim \log\rho/m$ in the perturbative regime $\rho \gg m$ corresponding to a large value of B, see Eq. (4.1.124). It leads to $z^2 \sim \rho^2$, suggesting that the corrections suppressed by powers of z^2 are presumably associated to the irrelevant deformation of the O(6) sigma model, due to operators of dimensions $4, 6, \dots$ in the string σ-model.

Energy of the Ground State

It remains to show that the scaling function $\epsilon(g, j)$ is related to the energy of the ground state of the O(6) model (4.2.38) through relation (4.2.42).

As follows from (4.2.20), the scaling function admits the representation

$$\epsilon(g,j) = 16g \int_0^\infty \frac{dt}{t} \frac{\gamma_+^{(0)}(2gt) - \gamma_-^{(0)}(2gt)}{\sinh(t/2)} (\gamma_{\rm h}(t) - \gamma_{\rm h}(0)) + j\epsilon_1(g)\,, \tag{4.2.59}$$

where we separated terms linear in j into the function $\epsilon_1(g)$ given by (4.2.29). The relation (4.2.59) can be written as

$$\epsilon(g,j) = \frac{2g}{\pi} \int_{-B}^{B} d\theta\, \chi(\theta)\, E(\theta) + j\,\epsilon_1(g)\,, \tag{4.2.60}$$

where the explicit expression for $E(\theta)$ can be found in [73]. Given the solution $\gamma_\pm^{(0)}(t)$ to the BES equation, the function $E(\theta)$ can be evaluated at large g, and for $\theta \leqslant B$ fixed, as

$$E(\theta) = \frac{m}{2g}(\cosh\theta - 1) + \dots\,, \tag{4.2.61}$$

where dots stand for subleading corrections of order $O(\mathrm{e}^{-3\pi g})$ at a given value of θ. Plugging this expression into (4.2.60) and making use of (4.2.32), we get the scaling function as

$$\epsilon(g,j) = \frac{m}{\pi} \int_{-B}^{B} d\theta\, \chi(\theta)\, (\cosh\theta - 1) + j(-1+m) = \frac{m}{\pi} \int_{-B}^{B} d\theta\, \chi(\theta)\, \cosh\theta - j\,, \tag{4.2.62}$$

in a perfect agreement with (4.2.42) and (4.2.38).

Thus, we demonstrated that the scaling function $\epsilon(g,j)$ is related to the energy density in the ground state of the two-dimensional O(6) sigma model (4.2.42) and that it can be found from the exact solution to the O(6) equations, Eqs. (4.2.38) and (4.2.39).

4.2.4 Concluding Remarks

On the gauge theory side, we found that the expression for the O(6) mass gap $m^{(\mathrm{g})} \equiv m\,\big|_{\text{gauge-theory}}$ was identical to the scale $m_{\rm cusp}$ governing the leading non-perturbative correction to the cusp anomalous dimension

$$\Gamma_{\rm cusp}(g) = \Gamma_{\rm cusp}^{\rm pert}(g) - \frac{\sigma}{4\sqrt{2}} m_{\rm cusp}^2 + O(m_{\rm cusp}^4)\,. \tag{4.2.63}$$

Here, $\Gamma_{\rm cusp}^{\rm pert}(g)$ is the perturbative contribution, given at large g by a non-Borel summable series in $1/g$, and

$$m_{\rm cusp} = m^{(\mathrm{g})} = k\, g^{1/4}\, \mathrm{e}^{-\pi g} \left[1 + \frac{m_1^{(\mathrm{g})}}{\pi g} + O(1/g^2)\right]\,, \tag{4.2.64}$$

with k and m_1 given as, see Eq. (3.3.28),

$$k = 2^{3/4} \pi^{1/4} / \Gamma\left(\tfrac{5}{4}\right)\,, \qquad m_1^{(\mathrm{g})} = \frac{3}{32} - \frac{3\log 2}{16}\,. \tag{4.2.65}$$

The coefficient σ in Eq. (4.2.63) depends on the prescription used to separate the perturbative and non-perturbative contribution.

The result $m^{(\mathrm{g})} = m_{\rm cusp}$ agrees with the proposal in [46] that, in string theory, the leading non-perturbative correction to the cusp anomalous dimension coincides with the one to the vacuum energy density of the two-dimensional bosonic O(6) model, embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string σ-model. The O(6) model only describes the effective dynamics of massless modes of the string and the mass of the massive excitations $\mu \sim 1$ defines an ultraviolet cut-off for this model. The vacuum energy density in the O(6) model, and more generally in the O(n) model, is an ultraviolet divergent quantity, and it admits the following form

$$\varepsilon_{\text{vac}} = \mu^2 \varepsilon_{\text{pert}}(g(\mu)) + \kappa\, m^2_{\mathrm{O}(n)} + O(m^4_{\mathrm{O}(n)}/\mu^2)\,. \tag{4.2.66}$$

Here, μ^2 is the UV cut-off, $\varepsilon_{\text{pert}}(g(\mu))$ stands for the perturbative contribution (with expansion in $1/g(\mu)$ at large $g(\mu)$), κ is a coefficient and the mass gap $m_{\mathrm{O}(n)}$ is

$$m_{\mathrm{O}(n)} = c\,\mu\, \mathrm{e}^{-g(\mu)/\beta_0}\, g(\mu)^{\beta_1/\beta_0^2} \left[1 + O(1/g(\mu))\right]\,, \tag{4.2.67}$$

where β_0 and β_1 are the beta-function coefficients for the O(n) model and the normalization factor c ensures the independence of $m_{\mathrm{O}(n)}$ on the renormalization scheme. The two terms in the right-hand side of (4.2.66) describe perturbative and non-perturbative corrections to ε_{vac}. For $n \to \infty$, each of them is well-defined separately and can be computed exactly [140, 141]. For n finite, including $n = 6$, the $1/g(\mu)$ expansion of $\varepsilon_{\text{pert}}(g(\mu))$ is non-Borel summable, in a generic renormalization scheme, and, thus, it does not define uniquely $\varepsilon_{\text{pert}}(g(\mu))$. In a close analogy with (4.2.63), the coefficient κ in front of $m^2_{\mathrm{O}(n)}$, in the right-hand side of (4.2.66), depends on the prescription used to separate the perturbative and non-perturbative contribution to ε_{vac}.

For $n = 6$, we have $\beta_0 = 1/\pi$ and $\beta_1 = 1/4\pi^2$, and the relations (4.2.67) and (4.2.66) should be compared with (4.2.64) and (4.2.63), respectively, assuming $g(\mu) = g + \ldots$ for $\mu \sim 1$. It is therefore rewarding to verify from the gauge theory Bethe ansatz approach that the scale m_{cusp} and the O(6) mass gap $m^{(\mathrm{g})}$ do coincide.

On the string theory side, we found that the two-loop small $\ell \equiv j/4g$ semiclassical string scaling function [45, 15, 59, 61], given by

$$\varepsilon(g,j) = 2\ell^2 \left[g + \frac{1}{\pi}\left(\frac{3}{4} - \log\ell\right) + \frac{1}{4\pi^2 g}\left(\frac{q_{02}^{(\mathrm{s})}}{2} - 3\log\ell + 4(\log\ell)^2\right) + O\left(1/g^2\right)\right] + O(\ell^4)\,, \tag{4.2.68}$$

can be cast into the form of the O(6) energy density in the perturbative regime. From this, we extracted the expression for the O(6) mass gap $m^{(\mathrm{s})} \equiv m\,|_{\text{string-theory}}$ as

$$m^{(\mathrm{s})} = k\, g^{1/4}\, \mathrm{e}^{-\pi g} \left[1 + \frac{m_1^{(\mathrm{s})}}{\pi g} + O(1/g^2)\right]\,, \tag{4.2.69}$$

with k and $m_1^{(\mathrm{s})}$ introduced for, see Eqs. (4.1.88) and (4.1.89),

$$k = 2^{3/4}\pi^{1/4}/\Gamma\left(\tfrac{5}{4}\right)\,, \qquad m_1^{(\mathrm{s})} = \frac{q_{02}^{(\mathrm{s})}}{8} - \frac{1}{4}\,. \tag{4.2.70}$$

The explicit value for the constant $q_{02}^{(\mathrm{s})}$ was obtained in [61] by a direct two-loop world-sheet computation and reads

$$q_{02}^{(\mathrm{s})} = -2\mathrm{K} - \frac{3\log 2}{2} + \frac{7}{4}\,. \tag{4.2.71}$$

Plugging this value of $q_{02}^{(\mathrm{s})}$ into the expression for $m_1^{(\mathrm{s})}$, we get

$$m_1^{(\mathrm{s})} = -\frac{\mathrm{K}}{4} - \frac{3\log 2}{16} - \frac{1}{32}\,. \tag{4.2.72}$$

Comparing $m^{(\mathrm{g})}$ and $m^{(\mathrm{s})}$, Eqs. (4.2.64) and (4.2.69), we find a remarkable agreement between gauge and string predictions [46, 74, 73] for the leading contribution $\sim k\, g^{1/4}\, \mathrm{e}^{-\pi g}$. However, the matching does not extend to the subleading corrections associated with $m_1^{(\mathrm{g})}$ and $m_1^{(\mathrm{s})}$, see Eq. (4.2.65) and (4.2.72), $m_1^{(\mathrm{g})} \neq m_1^{(\mathrm{s})}$. We observe, indeed, that the two results agree with each other in the term $\sim \log 2$ but disagree in the rest [76].

Note that, instead of comparing the predictions for the mass gap, we could equivalently compare predictions for the scaling function in the small ℓ semiclassical regime directly. It requires assuming first that $j \sim m$ and taking then $j \gg m$ to fall in the perturbative regime of the O(6) model. As was previously explained, the O(6) energy density in this limit can be cast into the form of a expansion in $1/g$ with coefficients all being proportional to $\ell^2 \equiv j/4g$ up to polynomials in $\log \ell$. More precisely, the O(6) prediction for the gauge theory scaling function assumes the form (4.2.68) with the two-loop constant

$$q_{02}^{(\mathrm{g})} = 8m_1^{(\mathrm{g})} + 2 = -\frac{3\log 2}{2} + \frac{11}{4}\,, \tag{4.2.73}$$

instead of $q_{02}^{(\mathrm{s})}$ in Eq. (4.2.68). Of course, the value for $q_{02}^{(\mathrm{g})}$, above, disagrees with $q_{02}^{(\mathrm{s})}$, see Eq. (4.2.71), for the obvious reason that $m_1^{(\mathrm{g})} \neq m_1^{(\mathrm{s})}$. But, remarkably, the expression (4.2.73) reproduces the result of [138] derived from the Bethe ansatz equations, within approach alternative to the FRS equation (at strong coupling).

Finally, one could be interested in computing the scaling function in the semiclassical regime, that is as an expansion in $1/g$ with $\ell \equiv j/4g$ kept fixed and arbitrary, directly from the FRS equation. This analysis was performed in [139] and agreement with the classical and one-loop string scaling function of [45, 59] was obtained. But, at the two-loop level, a singular contribution at small ℓ was found. It is not clear how this contribution can be interpreted from the O(6) model point of view. Nevertheless, it seems to indicate that the 'irrelevant' corrections to the O(6) TBA equations, predicted by the FRS equation, could play a more prominent role in the perturbative regime $j \gg m$ than naively expected. Finally, we note that the regular piece of the two-loop scaling function of [139] matches the result of [138], and, in particular, it agrees with the prediction based on the (gauge theory) O(6) model approach, as far as the $O(\ell^2)$ contribution is concerned.

All these remarks indicate that the two-loop matching of gauge and string theory is subtle. In particular, the reason for the discrepancy between $m_1^{(\mathrm{g})}$ and $m_1^{(\mathrm{s})}$ remains unclear.

Chapter 5

Conclusion

In this thesis, we used the (conjectured) all-loop integrability of the dilatation operator of the planar $\mathcal{N}=4$ Super-Yang-Mills theory [41] to unravel the strong coupling regime of distinguished observables, the cusp anomalous dimension and the scaling function. These functions are accessible by considering the scaling dimensions of Wilson operators carrying large quantum numbers, both Lorentz spin and twist [62, 45, 63, 44, 47]. For them we can rely on the all-loop asymptotic Bethe ansatz equations [41, 43, 44] to tackle the strong coupling analysis in the gauge theory.

Thanks to integrability, the cusp anomalous dimension and the scaling function can be found as solution to integral equations, the Beisert-Eden-Staudacher [44, 63] and Freyhult-Rej-Staudacher [47] equation, respectively. At weak coupling, the solution to the BES equation yields the cusp anomalous dimension as an expansion in powers of g^2 [63, 44] matching explicit four-loop perturbative calculation in the gauge theory [51, 49, 55, 56, 57, 58]. Solving the BES equation at strong coupling, we found the first few terms of the strong coupling expansion of the cusp anomalous dimension, extending the findings of [65, 66, 67, 68] and confirming the numerical estimate of [64]. Up to two-loop, they read [69, 70]

$$\Gamma_{\text{cusp}}(g) = 2g - \frac{3\log 2}{2\pi} - \frac{\text{K}}{8\pi^2 g} + \dots\ , \tag{5.0.1}$$

where K is the Catalan's constant. This result was independently derived from analysis based on the (quantum string) Bethe ansatz [137, 138] or on the asymptotic Baxter equation [91]. Moreover, it is in a remarkable agreement with the string theory prediction obtained either from the semiclassical quantization of the energy of a folded spinning string [14, 15, 60] or from the area scaling of a cusped minimal suface [113, 114, 115, 116]. We see that the cusp anomalous dimension, obtained by solving the BES equation, interpolates, with high accuracy, between explicit gauge and string theory results. This is a rather non-trivial dynamical test of the AdS/CFT correspondence.

Note that the cusp anomalous dimension, and the scaling function, are not the only observables that can be determined with the help of the all-loop Bethe ansatz equations and compared with string theory predictions. For instance, the so-called virtual scaling function, which is related to subleading $O(\log^0 N)$ corrections to the logarithmic scaling of the finite twist, large spin N, minimal anomalous dimension, has been analyzed in [142, 143] and compared successfully with the string theory result of [144].

Concerning the scaling function $\epsilon(g, j)$, we verified the proposal of [46] that it can be found exactly, at strong coupling $g \gg 1$ and for $j \sim m$, as solution to the TBA equations for the O(6) sigma model. Thanks to this result, we extracted from the FRS equation the strong coupling expression for the mass gap m of the O(6) model and we made comparison with the prediction from the string theory [46, 61, 76]. The two results match remarkably well up to one-loop order at strong coupling, but a discrepancy is found at the two-loop level [61, 138, 76, 139]. We stress that this disagreement does not invalidate the relation between the scaling function and the O(6) model, either on the gauge or string theory side, since it only affects the dependence on the coupling constant of the mass gap m. It is important however to understand its origin and its consequence for the AdS/CFT correspondence. In order to clarify this point, it would be interesting to consider in more detail the subleading corrections to the O(6) TBA equations, that are predicted by the FRS equation.

In thesis, we have also analyzed the nature of the strong coupling expansion for the cusp anomalous dimension and uncovered its asymptotic nature. The series tuns out to be divergent and non-Borel summable, the latter property indicating that the cusp anomalous dimension should receive non-perturbative corrections at strong coupling. We explained how to compute the leading non-perturbative contribution to the cusp anomalous dimension from the BES equation. We found that it is controlled to all orders at strong coupling by the scale m^2, with m the mass gap of the O(6) model.

It would be interesting to understand if the divergence of the strong coupling expansion is a generic phenomenon. A first element of response is that the dressing phase, appearing in the all-loop asymptotic Bethe ansatz equations, has expansion coefficients that admit divergent expansion at strong coupling [43, 44]. It suggests that anomalous dimensions of operators carrying large charges (long operators) will have, generically, divergent series description at strong coupling. But, what about anomalous dimension of short operators with finite quantum numbers?

For short operators, the all loop asymptotic Bethe ansatz equations are not reliable because of the wrapping effects [39, 112, 41]. A famous example is the anomalous dimension of the Konishi operator, which can be represented as $\mathrm{tr}[[\mathcal{Z}(0), \mathcal{Y}(0)][\mathcal{Z}(0), \mathcal{Y}(0)]]$, where $\mathcal{Z}(0)$ and $\mathcal{Y}(0)$ are two complex scalar fields. The all-loop asymptotic Bethe ansatz equations predict the anomalous dimension of the Konishi operator up to three-loop at weak coupling [31, 21], and the result is in agreement with the explicit perturbative gauge theory computation [145](see also references therein). At four loops, wrapping effects occur and the all-loop Bethe ansatz equations cannot be trusted [146]. Recently, it has been demonstrated how these finite-size corrections can be computed [147, 148] and, impressively, the obtained four-loop anomalous dimension for the Konishi operator was found to match perfectly the explicit four-loop perturbative calculation [149] in the gauge theory. The equations proposed in [148, 150] (see also references therein) were argued to capture all type of finite-size corrections and thus should describe the exact spectrum of anomalous dimensions of the planar gauge theory. With their help, one should be able to find the strong coupling expansion of the anomalous dimension of the Konishi operator and make comparison with the string theory result of [151]. It has been done recently by numerical means in [152](to be compared with the result of [153] obtained with the all-loop asymptotic Bethe ansatz equations). This numerical estimate seems to be consistent with the string theory computation of [151] up to next-to-leading order.

It would be very interesting to see whether the methods developed to deal with the strong coupling regime of the BES and FRS equations [70, 73, 71, 108] can be applied to solving the equations of [148] for the anomalous dimension of the Konishi operator at strong coupling. It would perhaps shed light on the nature of the strong coupling expansion in string theory.

Appendix A

BES Equation

A.1 General Solution

In this appendix, we give further details on the construction of the general solution $\Gamma(t) = \Gamma_+(t) + i\Gamma_-(t)$ to the BES equation. By construction, the function $\Gamma(t)$ is given by the Fourier integral

$$\Gamma(t) = \int_{-\infty}^{\infty} dk\, \mathrm{e}^{-ikt}\, \widetilde{\Gamma}(k)\,, \tag{A.1.1}$$

with the function $\widetilde{\Gamma}(k)$ having different form for $k^2 \leq 1$ and $k^2 > 1$:

- For $-\infty < k < -1$:
$$\widetilde{\Gamma}(k) = \sum_{n\geq 1} c_-(n,g)\, \mathrm{e}^{-4\pi n g(-k-1)}\,, \tag{A.1.2}$$
- For $1 < k < \infty$:
$$\widetilde{\Gamma}(k) = \sum_{n\geq 1} c_+(n,g)\, \mathrm{e}^{-4\pi n g(k-1)}\,, \tag{A.1.3}$$
- For $-1 \leq k \leq 1$:
$$\widetilde{\Gamma}(k) = -\frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1 + \frac{c(g)}{1+k} + \frac{1}{2}\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)\frac{dp\,\widetilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\right], \tag{A.1.4}$$

  where $\widetilde{\Gamma}(p)$ inside the integral is replaced by (A.1.2) and (A.1.3).

We recall that the expression of $\widetilde{\Gamma}(k)$ for $k^2 > 1$ is dictated by the analyticity properties, while for $k^2 < 1$ it follows from solving the BES equation. Integration over $k^2 > 1$ in the integral (A.1.1) can be done immediately, while the integral over $-1 \leq k \leq 1$ can be expressed in terms of special functions. Namely, we find

$$\begin{aligned}\Gamma(t) = &\sum_{n\geq 1} c_+(n,g)\left[\frac{\mathrm{e}^{-it}}{4\pi n g + it} - V_+(-it, 4\pi n g)\right] \\ &+ \sum_{n\geq 1} c_-(n,g)\left[\frac{\mathrm{e}^{it}}{4\pi n g - it} + V_-(it, 4\pi n g)\right] - V_0(-it) - c(g)V_1(-it)\,,\end{aligned} \tag{A.1.5}$$

where the notation was introduced for the functions (with $n = 0, 1$)

$$V_\pm(x,y) = \frac{1}{\sqrt{2}\pi}\int_{-1}^{1} dk\,\mathrm{e}^{\pm xk}\int_1^\infty \frac{dp\ \mathrm{e}^{-y(p-1)}}{p-k}\left(\frac{1+k}{1-k}\frac{p-1}{p+1}\right)^{\pm 1/4},$$
$$V_n(x) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dk\ \mathrm{e}^{xk}}{(k+1)^n}\left(\frac{1+k}{1-k}\right)^{1/4},$$
$$U_n^\pm(y) = \frac{1}{2}\int_1^\infty \frac{dp\ \mathrm{e}^{-y(p-1)}}{(p\mp 1)^n}\left(\frac{p+1}{p-1}\right)^{\mp 1/4}. \qquad \text{(A.1.6)}$$

The reason why we also introduced $U_n^\pm(y)$ is that the functions $V_\pm(x,y)$ can be further simplified with the help of master identities (we shall return to them in a moment)

$$(x+y)V_-(x,y) = xV_0(x)U_1^-(y) + yV_1(x)U_0^-(y) - \mathrm{e}^{-x},$$
$$(x-y)V_+(x,y) = xV_0(x)U_1^+(y) + yV_1(x)U_0^+(y) - \mathrm{e}^{x}. \qquad \text{(A.1.7)}$$

Combining together (A.1.7) and (A.1.5) we arrive at the following expression for the function $\Gamma(it)$

$$\Gamma(it) = -V_0(t) - c(g)V_1(t)$$
$$+\sum_{n\geq 1} c_+(n,g)\left[\frac{4\pi ngV_1(t)U_0^+(4\pi ng) + tV_0(t)U_1^+(4\pi ng)}{4\pi ng - t}\right]$$
$$+\sum_{n\geq 1} c_-(n,g)\left[\frac{4\pi ngV_1(t)U_0^-(4\pi ng) + tV_0(t)U_1^-(4\pi ng)}{4\pi ng + t}\right], \qquad \text{(A.1.8)}$$

which is the result stated in Chapter 3. As follows from their integral representation (A.1.6), $V_0(t)$ and $V_1(t)$ are holomorphic functions of t. As a result, $\Gamma(it)$ is a meromorphic function of t with (an infinite) set of poles located at $t = \pm 4\pi ng$ with n positive integer.

Finally, let us prove the master identities (A.1.7). We start with the second relation in (A.1.7) and make use of (A.1.6) to rewrite the expression in the left-hand side of (A.1.7) as

$$(x-y)V_+(x,y)\,\mathrm{e}^{-y} = (x-y)\int_0^\infty ds\, V_0(x+s)U_0^+(y+s)\,\mathrm{e}^{-y-s}. \qquad \text{(A.1.9)}$$

Let us introduce two auxiliary functions

$$z_1(x) = V_1(x), \qquad z_1(x) + z_1'(x) = V_0(x),$$
$$z_2(x) = \mathrm{e}^{-x}\,U_1^+(x), \qquad z_2(x) + z_2'(x) = -\,\mathrm{e}^{-x}\,U_0^+(x), \qquad \text{(A.1.10)}$$

with $V_n(x)$ and $U_n^+(x)$ given by (A.1.6) and where $z_{1,2}'(x)$ stand for the derivatives of $z_{1,2}(x)$. They satisfy the second-order differential equation

$$\frac{d}{dx}\big(xz_i'(x)\big) = \big(x - \tfrac{1}{2}\big)\, z_i(x). \qquad \text{(A.1.11)}$$

Applying this relation it is straightforward to verify the following identity

$$-(x-y)\big[z_1(x+s)+z_1'(x+s)\big]\big[z_2(y+s)+z_2'(y+s)\big]$$
$$=\frac{d}{ds}\big\{(y+s)[z_2(y+s)+z_2'(y+s)]z_1(x+s)\big\}$$
$$-\frac{d}{ds}\big\{(x+s)[z_1(x+s)+z_1'(x+s)]z_2(y+s)\big\}. \tag{A.1.12}$$

It is easy to see that the expression in the left-hand side coincides with the integrand in (A.1.9). Therefore, integrating both sides of (A.1.12) over $0\le s<\infty$, we obtain

$$(x-y)V_+(x,y) = -\,\mathrm{e}^{-s}\left[(x+s)V_0(x+s)U_1^+(y+s)+(y+s)V_1(x+s)U_0^+(y+s)\right]\Big|_{s=0}^{s=\infty}$$
$$= -\mathrm{e}^{x}+xV_0(x)U_1^+(y)+yV_1(x)U_0^+(y)\,, \tag{A.1.13}$$

where in the second relation we took into account the asymptotic behavior of the functions (A.1.6) (see Eqs. (A.2.12) and (A.2.14)), $V_n(s)\sim \mathrm{e}^s\, s^{-3/4}$ and $U_n^+(s)\sim s^{n-5/4}$ as $s\to\infty$.

The derivation of the first relation in (A.1.7) goes along the same lines.

A.2 Relation to Whittaker Functions

In this appendix we summarize properties of special functions encountered in the analysis of the BES solution.

Integral Representations

Let us first consider the functions $V_n(x)$ (with $n=0,1$) introduced as

$$V_n(x)=\frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dk\,\mathrm{e}^{xk}}{(k+1)^n}\left(\frac{1+k}{1-k}\right)^{1/4}. \tag{A.2.1}$$

As follows from their integral representation, $V_0(x)$ and $V_1(x)$ are entire function on a complex $x-$plane. Changing the integration variable in (A.2.1) as $k=1-2t$ and $k=2t-1$ we obtain two equivalent representations

$$V_n(x)=\frac{1}{\pi}2^{3/2-n}\,\mathrm{e}^{x}\int_0^1 dt\,t^{-1/4}(1-t)^{1/4-n}\,\mathrm{e}^{-2tx}\,,$$
$$=\frac{1}{\pi}2^{3/2-n}\,\mathrm{e}^{-x}\int_0^1 dt\,t^{1/4-n}(1-t)^{-1/4}\,\mathrm{e}^{2tx}\,, \tag{A.2.2}$$

which give rise to the following expressions for $V_n(x)$ (with $n=0,1$) in terms of Whittaker functions of the first kind

$$V_n(x)=2^{-n}\frac{\Gamma(\frac54-n)}{\Gamma(\frac54)\Gamma(2-n)}(2x)^{n/2-1}M_{n/2-1/4,1/2-n/2}(2x)\,,$$
$$=2^{-n}\frac{\Gamma(\frac54-n)}{\Gamma(\frac54)\Gamma(2-n)}(-2x)^{n/2-1}M_{1/4-n/2,1/2-n/2}(-2x)\,. \tag{A.2.3}$$

In distinction with $V_n(x)$, the Whittaker function $M_{n/2-1/4,1/2-n/2}(2x)$ is an analytical function of x on the complex plane with the cut along negative semi-axis. The same is true for the factor $(2x)^{n/2-1}$ so that the product of two functions in the right-hand side of (A.2.3) is a single-valued analytical function in the whole complex plane. The two representations (A.2.3) are equivalent in virtue of the relation

$$M_{n/2-1/4,1/2-n/2}(2x) = \mathrm{e}^{\pm i\pi(1-n/2)}\, M_{1/4-n/2,1/2-n/2}(-2x) \qquad (\text{for } \operatorname{Im} x \gtrless 0)\,, \tag{A.2.4}$$

where the upper and lower signs in the exponent correspond to $\operatorname{Im} x > 0$ and $\operatorname{Im} x < 0$, respectively.

Let us know consider the functions $U_0^\pm(x)$ and $U_1^\pm(x)$ defined for real positive x by

$$U_n^\pm(x) = \frac{1}{2}\int_1^\infty \frac{dp\ \mathrm{e}^{-x(p-1)}}{(p\mp 1)^n}\left(\frac{p+1}{p-1}\right)^{\mp 1/4}\,. \tag{A.2.5}$$

The four different integrals in (A.2.5) can be found as special cases of the following generic integral

$$U_{kl}(x) = \frac{1}{2}\int_1^\infty dp\ \mathrm{e}^{-x(p-1)}(p+1)^{k+l-1/2}(p-1)^{k-l-1/2}\,, \tag{A.2.6}$$

defined for $x > 0$. Changing the integration variable as $p = t/x + 1$ we obtain

$$U_{kl}(x) = 2^{k+l-3/2}x^{k-l-1/2}\int_0^\infty dt\ \mathrm{e}^{-t}\, t^{k-l-1/2}\left(1+\frac{t}{2x}\right)^{k+l-1/2}\,. \tag{A.2.7}$$

The integral entering this relation can be expressed in terms of Whittaker functions of second kind or equivalently confluent hypergeometric function of the second kind

$$\begin{aligned} U_{kl}(x) &= 2^{l-3/2}\Gamma(\tfrac{1}{2}-k+l)x^{-l-1/2}\,\mathrm{e}^{x}\,W_{kl}(2x)\,, \\ &= \tfrac{1}{2}\Gamma(\tfrac{1}{2}-k+l)\mathrm{U}\big(\tfrac{1}{2}-k+l, 1+2l; 2x\big)\,. \end{aligned} \tag{A.2.8}$$

This relation can be used to analytically continue $U_{kl}(x)$ from $x > 0$ to the whole complex $x-$plane with the cut along negative semi-axis. Matching (A.2.6) into (A.2.5) we obtain the following relations for the functions $U_0^\pm(x)$ and $U_1^\pm(x)$

$$\begin{aligned} U_0^+(x) &= \tfrac{1}{2}\Gamma(\tfrac{5}{4})x^{-1}\,\mathrm{e}^x\,W_{-1/4,1/2}(2x)\,, \qquad & U_1^+(x) &= \tfrac{1}{2}\Gamma(\tfrac{1}{4})(2x)^{-1/2}\,\mathrm{e}^x\,W_{1/4,0}(2x)\,, \\ U_0^-(x) &= \tfrac{1}{2}\Gamma(\tfrac{3}{4})x^{-1}\,\mathrm{e}^x\,W_{1/4,1/2}(2x)\,, \qquad & U_1^-(x) &= \tfrac{1}{2}\Gamma(\tfrac{3}{4})(2x)^{-1/2}\,\mathrm{e}^x\,W_{-1/4,0}(2x)\,. \end{aligned} \tag{A.2.9}$$

The functions $V_1(\pm x), U_1^\pm(x)$ and $V_0(\pm x), U_0^\pm(x)$ satisfy the same Whittaker differential equation and, as a consequence, they satisfy Wronskian relations

$$V_1(-x)U_0^-(x) - V_0(-x)U_1^-(x) = V_1(x)U_0^+(x) + V_0(x)U_1^+(x) = \frac{\mathrm{e}^x}{x}\,. \tag{A.2.10}$$

The same relations also follow from (A.1.7) for $x = \pm y$. In addition,

$$U_0^+(x)U_1^-(-x) + U_1^+(x)U_0^-(-x) = \frac{\pi}{2\sqrt{2}x}\,\mathrm{e}^{\pm\frac{3i\pi}{4}}\,, \qquad (\text{for } \operatorname{Im} x \gtrless 0)\,. \tag{A.2.11}$$

Combining together (A.2.10) and (A.2.11) we obtain the following relations between the functions

$$V_0(x) = \frac{2\sqrt{2}}{\pi}\, \mathrm{e}^{\mp\frac{3i\pi}{4}} \left[\mathrm{e}^{x}\, U_0^-(-x) + \mathrm{e}^{-x}\, U_0^+(x)\right],$$
$$V_1(x) = \frac{2\sqrt{2}}{\pi}\, \mathrm{e}^{\mp\frac{3i\pi}{4}} \left[\mathrm{e}^{x}\, U_1^-(-x) - \mathrm{e}^{-x}\, U_1^+(x)\right], \qquad \text{(A.2.12)}$$

where the upper and lower signs correspond to $\mathrm{Im}\, x > 0$ and $\mathrm{Im}\, x < 0$, respectively.

At first sight, the relations (A.2.12) look surprising since $V_0(x)$ and $V_1(x)$ are entire functions in the complex $x-$plane, while $U_0^\pm(x)$ and $U_1^\pm(x)$ are single-valued functions in the same plane but with the cut along the negative semi-axis. Indeed, one can use the relations (A.2.10) and (A.2.11) to compute the discontinuity of the these functions across the cut as

$$\Delta U_0^\pm(-x) = \pm\frac{\pi}{4}\, \mathrm{e}^{-x}\, V_0(\mp x)\, \theta(x),$$
$$\Delta U_1^\pm(-x) = -\frac{\pi}{4}\, \mathrm{e}^{-x}\, V_1(\mp x)\, \theta(x), \qquad \text{(A.2.13)}$$

where $\Delta U(-x) \equiv \lim_{\epsilon\to 0}[U(-x+i\epsilon) - U(-x-i\epsilon)]/(2i)$ and $\theta(x)$ is a step function. Then, one verifies with the help of these identities that the linear combinations of $U-$functions in the right-hand side of (A.2.12) have zero discontinuity across the cut and, therefore, they are well-defined in the whole complex plane.

Asymptotic Expansions

For our purposes, we need asymptotic expansion of functions $V_n(x)$ and $U_n^\pm(x)$ at large real x. Let us start with the latter functions and consider a generic integral (A.2.8).

To find asymptotic expansion of the function $U_{kl}(x)$ at large x, it suffices to replace the last factor in the integrand (A.2.7) in powers of $t/(2x)$ and integrate term by term. In this way, we find that

$$U_0^+(x) = (2x)^{-5/4}\Gamma(\tfrac{5}{4})F\big(\tfrac{1}{4},\tfrac{5}{4}| -\tfrac{1}{2x}\big) = (2x)^{-5/4}\Gamma(\tfrac{5}{4})\left[1 - \frac{5}{32x} + \dots\right],$$
$$U_0^-(x) = (2x)^{-3/4}\Gamma(\tfrac{3}{4})F\big(-\tfrac{1}{4},\tfrac{3}{4}| -\tfrac{1}{2x}\big) = (2x)^{-3/4}\Gamma(\tfrac{3}{4})\left[1 + \frac{3}{32x} + \dots\right],$$
$$U_1^+(x) = (2x)^{-1/4}\tfrac{1}{2}\Gamma(\tfrac{1}{4})F\big(\tfrac{1}{4},\tfrac{1}{4}| -\tfrac{1}{2x}\big) = (2x)^{-1/4}\tfrac{1}{2}\Gamma(\tfrac{1}{4})\left[1 - \frac{1}{32x} + \dots\right],$$
$$U_1^-(x) = (2x)^{-3/4}\tfrac{1}{2}\Gamma(\tfrac{3}{4})F\big(\tfrac{3}{4},\tfrac{3}{4}| -\tfrac{1}{2x}\big) = (2x)^{-3/4}\tfrac{1}{2}\Gamma(\tfrac{3}{4})\left[1 - \frac{9}{32x} + \dots\right], \qquad \text{(A.2.14)}$$

Here we introduced the function $F(a,b| -\frac{1}{2x})$ defined as

$$F\big(a,b| -\tfrac{1}{2x}\big) = \frac{(2x)^a}{\Gamma(a)}\int_0^\infty ds\, s^{a-1}(1+s)^{-b}\,\mathrm{e}^{-2xs}, \qquad \text{(A.2.15)}$$

with the property that $F(a,b| -\frac{1}{2x}) = 1 + O(1/x)$ at large x.

Notice that the expansion coefficients in (A.2.14) grow factorially to higher orders but the series are Borel summable for $x > 0$. For $x < 0$ one has to distinguish the functions $U_n^\pm(x+i\epsilon)$

and $U_n^{\pm}(x - i\epsilon)$ (with $\epsilon \to 0$) which define analytical continuation of the function $U_n^{\pm}(x)$ to the upper and lower edges of the cut, respectively. In contrast with this, the functions $V_n(x)$ are well-defined on the whole real axis. Still, to make use of the relations (A.2.12) we have to specify the $U-$functions on the cut. As an example, let us consider $V_0(-\pi g)$ in the limit $g \to \infty$ and apply (A.2.12)

$$V_0(-\pi g) = \frac{2\sqrt{2}}{\pi} \, \mathrm{e}^{-\frac{3i\pi}{4}} \, \mathrm{e}^{\pi g} \left[U_0^+(-\pi g + i\epsilon) + \mathrm{e}^{-2\pi g} \, U_0^-(\pi g) \right], \tag{A.2.16}$$

where $\epsilon \to 0$ and we have chosen to define the $U-$functions on the upper edge of the cut. Written in this form, both terms inside the square brackets are well-defined separately. Replacing $U_0^{\pm}$ functions in (A.2.16) by their expressions (A.2.14) in terms of F-functions we find

$$V_0(-\pi g) = \frac{(2\pi g)^{-5/4} \, \mathrm{e}^{\pi g}}{\Gamma(\frac{3}{4})} \left[F\left(\tfrac{1}{4}, \tfrac{5}{4} | \tfrac{1}{2\pi g} + i\epsilon\right) + \sigma \Lambda^2 F\left(-\tfrac{1}{4}, \tfrac{3}{4} | -\tfrac{1}{2\pi g}\right) \right], \tag{A.2.17}$$

with Λ^2 given by

$$\Lambda^2 = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} \, \mathrm{e}^{-2\pi g} (2\pi g)^{1/2}, \qquad \sigma = \mathrm{e}^{-\frac{3i\pi}{4}} . \tag{A.2.18}$$

Since the second term in the right-hand side of (A.2.17) is exponentially suppressed at large g we may treat it as a non-perturbative correction. Repeating the same analysis for $V_1(-\pi g)$, we obtain from (A.2.12) and (A.2.14)

$$V_1(-\pi g) = \frac{(2\pi g)^{-5/4} \, \mathrm{e}^{\pi g}}{2\Gamma(\frac{3}{4})} \left[8\pi g F\left(\tfrac{1}{4}, \tfrac{1}{4} | \tfrac{1}{2\pi g} + i\epsilon\right) + \sigma \Lambda^2 F\left(\tfrac{3}{4}, \tfrac{3}{4} | -\tfrac{1}{2\pi g}\right) \right], \tag{A.2.19}$$

We would like to stress that the '$+i\epsilon$' prescription in the first term in (A.2.17) and the phase factor $\sigma = \mathrm{e}^{-\frac{3i\pi}{4}}$ in (A.2.18) follow unambiguously from (A.2.16). Had we defined the $U-$functions on the lower edge of the cut, we would get the expression for $V_0(-\pi g)$ with '$-i\epsilon$' prescription and the phase factor $\mathrm{e}^{\frac{3i\pi}{4}}$. The two expressions are however equivalent since discontinuity of the first term in (A.2.17) compensates the change of the phase factor in front of the second term

$$F\left(\tfrac{1}{4}, \tfrac{5}{4} | \tfrac{1}{2\pi g} + i\epsilon\right) - F\left(\tfrac{1}{4}, \tfrac{5}{4} | \tfrac{1}{2\pi g} - i\epsilon\right) = i\sqrt{2}\Lambda^2 F\left(-\tfrac{1}{4}, \tfrac{3}{4} | -\tfrac{1}{2\pi g}\right). \tag{A.2.20}$$

If one neglected '$+i\epsilon$' prescription in (A.2.16) and formally expanded the first term in (A.2.17) in powers of $1/g$, this would lead to non-Borel summable series. This series suffers from Borel ambiguity which are exponentially small for large g and produce the contribution of the same order as the second term in the right-hand side of (A.2.17). The relation (A.2.17) suggests how to give a meaning to this series. Namely, one should first resum the series for negative g where it is Borel summable and, then, analytically continue it to the upper edge of the cut at positive g.

A.3 Expression for the Mass Scale

In this appendix we derive the expression for the mass scale given by

$$m = -\frac{16\sqrt{2}}{\pi} g \, \mathrm{e}^{-\pi g} \left[f_0(-\pi g) U_0^-(\pi g) + f_1(-\pi g) U_1^-(\pi g) \right] . \tag{A.3.1}$$

We recall that the functions $f_{0,1}(4\pi gt)$ read

$$f_0(4\pi gt) = \sum_{n\ge1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{n+t}\right] - 1\,, \tag{A.3.2}$$
$$f_1(4\pi gt) = \sum_{n\ge1} n\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{n+t}\right]\,,$$

and parameterize the BES solution $\Gamma(4\pi it)$ as

$$\Gamma(4\pi git) = f_0(4\pi gt)V_0(4\pi gt) + f_1(4\pi gt)V_1(4\pi gt)\,. \tag{A.3.3}$$

To obtain the representation (A.3.1), we replace $\Gamma(4\pi git)$ in the definition of m,

$$m \equiv \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^{-i\infty} dt\;\mathrm{e}^{-4\pi gt - i\pi/4}\frac{\Gamma(4\pi git)}{t+\frac14}\right]\,, \tag{A.3.4}$$

by its expression (A.3.3) and perform integration over t in the right-hand side of (A.3.4). It is convenient to decompose $\Gamma(4\pi git)/(t+\frac14)$ into a sum of simple poles as

$$\frac{\Gamma(4\pi git)}{t+\frac14} = \sum_{k=0,1} f_k(-\pi g)\frac{V_k(4\pi gt)}{t+\frac14} + \sum_{k=0,1}\frac{f_k(4\pi gt) - f_k(-\pi g)}{t+\frac14}V_k(4\pi gt)\,, \tag{A.3.5}$$

where the second term is regular at $t=-1/4$. Substituting this relation into (A.3.4) and replacing $f_k(4\pi gt)$ by their expressions (A.3.2), we encounter the following integral

$$R_k(4\pi gs) \equiv \mathrm{Re}\left[\int_0^{-i\infty} dt\;\mathrm{e}^{-4\pi gt-i\pi/4}\frac{V_k(4\pi gt)}{t-s}\right] = \mathrm{Re}\left[\int_0^{-i\infty} dt\;\mathrm{e}^{-t-i\pi/4}\frac{V_k(t)}{t-4\pi gs}\right]\,. \tag{A.3.6}$$

Then, the integral in (A.3.4) can be expressed in terms of $R-$function as

$$\begin{aligned}\mathrm{Re}\left[\int_0^{-i\infty} dt\;\mathrm{e}^{-4\pi gt-i\pi/4}\frac{\Gamma(4\pi git)}{t+\frac14}\right] &= f_0(-\pi g)R_0(-\pi g) + f_1(-\pi g)R_1(-\pi g)\\ &- \sum_{n\ge1}\frac{nc_+(n,g)}{n+\frac14}\left[U_1^+(4\pi ng)R_0(4\pi gn) + U_0^+(4\pi ng)R_1(4\pi gn)\right]\\ &+ \sum_{n\ge1}\frac{nc_-(n,g)}{n-\frac14}\left[U_1^-(4\pi ng)R_0(-4\pi gn) - U_0^-(4\pi ng)R_1(-4\pi gn)\right]\,,\end{aligned} \tag{A.3.7}$$

where the last two lines correspond to the second sum in the right-hand side of (A.3.5) and we took into account that the coefficients $c_\pm(n,g)$ are real.

Let us evaluate the integral (A.3.6) and choose for simplicity $R_0(s)$. We have to distinguish two cases: $s>0$ and $s<0$. For $s>0$ we have

$$\begin{aligned}R_0(s) &= -\,\mathrm{Re}\left[\mathrm{e}^{-i\pi/4}\int_{-\infty}^1 dv\;\mathrm{e}^{-(1-v)s}\int_0^{-i\infty} dt\;\mathrm{e}^{-vt}\,V_0(t)\right]\\ &= \frac{\sqrt{2}}{\pi}\,\mathrm{Re}\left[\mathrm{e}^{-i\pi/4}\int_{-\infty}^1 dv\;\mathrm{e}^{-(1-v)s}\int_{-1}^1 du\,\frac{(1+u)^{1/4}(1-u)^{-1/4}}{u-v-i\epsilon}\right]\,,\end{aligned} \tag{A.3.8}$$

where in the second relation we replaced $V_0(t)$ by its integral representation (A.2.1). Integration over u can be carried out with the help of identity

$$\frac{1}{\sqrt{2}\pi}\int_{-1}^{1} du\, \frac{(1+u)^{1/4-k}(1-u)^{-1/4}}{u-v-i\epsilon} = \delta_{k,0} - (v+1)^{-k} \times \begin{cases} \left(\frac{v+1}{v-1}\right)^{1/4}, & v^2 > 1 \\ \mathrm{e}^{-i\pi/4}\left(\frac{1+v}{1-v}\right)^{1/4}, & v^2 < 1 \end{cases} \tag{A.3.9}$$

In this way, we obtain from (A.3.8)

$$R_0(s) \stackrel{s\geq 0}{=} \sqrt{2}\left[\frac{1}{s} - \int_{-\infty}^{-1} dv\, \mathrm{e}^{-(1-v)s}\left(\frac{v+1}{v-1}\right)^{1/4}\right] = \sqrt{2}\left[\frac{1}{s} - 2\,\mathrm{e}^{-2s}\, U_0^+(s)\right], \tag{A.3.10}$$

with the function $U_0^+(s)$ defined in the previous appendix. In the similar manner, for $s < 0$ we get

$$R_0(s) \stackrel{s\leq 0}{=} \sqrt{2}\left[\frac{1}{s} + 2U_0^-(-s)\right], \tag{A.3.11}$$

together with

$$R_1(s) = 2\sqrt{2}\left[\theta(-s)U_1^-(-s) + \theta(s)\,\mathrm{e}^{-2s}\, U_1^+(s)\right]. \tag{A.3.12}$$

Then, we substitute the relations (A.3.10), (A.3.11) and (A.3.12) into (A.3.7) and find

$$\begin{aligned} &\mathrm{Re}\left[\int_0^{-i\infty} dt\, \mathrm{e}^{-4\pi g t - i\pi/4}\frac{\Gamma(4\pi g i t)}{t+\frac{1}{4}}\right] \\ &= 2\sqrt{2}f_0(-\pi g)\left[U_0^-(\pi g) - \frac{1}{2\pi g}\right] + 2\sqrt{2}f_1(-\pi g)U_1^-(\pi g) + \frac{\sqrt{2}}{\pi g}\left[f_0(-\pi g)+1\right], \end{aligned} \tag{A.3.13}$$

where the last term in the right-hand side corresponds to the last two lines in (A.3.7). Substitution of (A.3.13) into (A.3.4) yields the expression for the mass scale (A.3.1).

A.4 Wronskian-like Relation

In this appendix, we consider the homogeneous solution to the BES equation $\delta\Gamma(t)$ that satisfies the quantization conditions

$$\delta\Gamma(it_m) = -\xi\delta_{m,0}, \tag{A.4.1}$$

where $t_m = 4\pi g\big(m - \frac{1}{4}\big)$ with $m \in \mathbb{Z}$. More precisely, we will establish that the idendity

$$\delta\Gamma(0) = -\frac{\xi m}{\sqrt{2}}, \tag{A.4.2}$$

holds true at any value of the coupling constant $g > 0$, where m is the mass scale defined in terms of the BES solution $\Gamma(t)$ as

$$m = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\int_0^{\infty}\frac{dt}{t+i\pi g}\;\mathrm{e}^{i(t-\pi/4)}\,\Gamma(t). \tag{A.4.3}$$

The derivation of the identity (A.4.2) follows from the use of a Wronskian-like relation between the two functions $\delta\Gamma(t)$ and $\Gamma(t)$. This relation is most easily obtained by working with the $\gamma-$functions introduced as

$$\delta\gamma(it) = \frac{\sin(t/4g)\sin(\pi/4)}{\sin(t/4g+\pi/4)}\delta\Gamma(it)\,, \tag{A.4.4}$$

and similarly for the pair $(\gamma(t), \Gamma(it))$. The equation (A.4.2) then translates into the small t behavior

$$\delta\gamma(it) = -\frac{\xi m}{4\sqrt{2}g}\,t + O(t^2)\,. \tag{A.4.5}$$

We recall that the homogeneous solution $\delta\Gamma(t)$ satisfying the quantization conditions (A.4.1) is relevant to the study of the first non-perturbative correction $\sim \exp(-2\pi g)$ to the BES solution at strong coupling (see Section 3.4.3). In this context, the relation (A.4.2), or equivalently (A.4.5), permits to demonstrate that the leading non-perturbative correction to the cusp anomalous dimension is controlled by the mass scale m. The solution $\delta\Gamma(t)$ plays a similar role at the level of the FRS equation and another application of the formula (A.4.2) can be found in Appendix B.4.

By construction, $\delta\gamma(t)$ is a homogeneous solution to the BES equation. It means that it satisfies the system of equations ($n \geqslant 1$)

$$\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_-(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\delta\gamma_+(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n-1}(t) = 0\,,\\
&\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_+(t)}{1-\mathrm{e}^{-t/2g}} - \frac{\delta\gamma_-(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n}(t) = 0\,,
\end{aligned} \tag{A.4.6}$$

where $\delta\gamma(t) \equiv \delta\gamma_+(t) + i\delta\gamma_-(t)$ with $\delta\gamma_\pm(-t) = \pm\delta\gamma_\pm(t)$. The solution we are interested in has special analytic properties discussed in Section 3.4.3. In particular, $\delta\gamma(t)$ should vanish at $t=0$ and have a simple pole at $t=-i\pi g$ with residue $2ig\xi$. To fulfill these requirements, we split $\delta\gamma(it)$ into the sum of two functions

$$\delta\gamma(it) = \rho(it) - \frac{2\xi}{\pi}\frac{t}{t+\pi g}\,, \tag{A.4.7}$$

where $\rho(it)$ is an entire function of t. Moreover, the function $\rho(it)$ so-defined should have a Fourier transform supported on the interval $[-1,1]$. We remark that these analytic conditions on $\rho(it)$ are identical to those imposed on the BES solution $\gamma(it)$. In order to find a deeper relation between the two functions, it is advantageous to rewrite the system of equations (A.4.6) for the unknown $\rho(it)$. Proceeding to the parity decomposition of $\rho(t)$ as $\rho(t) = \rho_+(t) + i\rho_-(t)$, we obtain from (A.4.7) that

$$\begin{aligned}
\delta\gamma_+(t) &= \rho_+(t) - \frac{2\xi}{\pi}\frac{t^2}{t^2+(\pi g)^2}\,,\\
\delta\gamma_-(t) &= \rho_-(t) + \frac{2g\xi t}{t^2+\pi^2 g^2}\,.
\end{aligned} \tag{A.4.8}$$

Then, substituting these relations into (A.4.6), we derive a system of inhomogeneous integral equations for the functions $\rho_\pm(t)$ given by ($n \geqslant 1$)

$$\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left[\frac{\rho_-(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\rho_+(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n-1}(t) = h_{2n-1}(g)\,,\\
&\int_0^\infty \frac{dt}{t}\left[\frac{\rho_+(t)}{1-\mathrm{e}^{-t/2g}} - \frac{\rho_-(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n}(t) = h_{2n}(g)\,,
\end{aligned} \tag{A.4.9}$$

with the inhomogeneous terms

$$
\begin{aligned}
h_{2n-1}(g) &= \frac{2\xi}{\pi}\int_0^\infty \frac{dt\, J_{2n-1}(t)}{t^2+(\pi g)^2}\left[\frac{t}{\mathrm{e}^{t/2g}-1}-\frac{\pi g}{1-\mathrm{e}^{-t/2g}}\right], \\
h_{2n}(g) &= \frac{2\xi}{\pi}\int_0^\infty \frac{dt\, J_{2n}(t)}{t^2+(\pi g)^2}\left[\frac{\pi g}{\mathrm{e}^{t/2g}-1}+\frac{t}{1-\mathrm{e}^{-t/2g}}\right].
\end{aligned} \tag{A.4.10}
$$

These relations differ from the BES ones for $\gamma(t) = \gamma_+(t)+i\gamma_-(t)$ by the form of the inhomogeneous terms, and they can be obtained one from another through the substitution

$$
\rho_\pm(t) \to \gamma_\pm(t)\,, \qquad h_{2n-1}(g) \to \tfrac{1}{2}\delta_{n,1}\,, \qquad h_{2n}(g) \to 0\,. \tag{A.4.11}
$$

As we did for $\gamma(t)$ in Chapter 3, we may look for a solution to (A.4.9) in the form of Bessel series

$$
\begin{aligned}
\rho_-(t) &= 2\sum_{n\geqslant 1}(2n-1)J_{2n-1}(t)\rho_{2n-1}(g)\,, \\
\rho_+(t) &= 2\sum_{n\geqslant 1}(2n)\;J_{2n}(t)\rho_{2n}(g)\,,
\end{aligned} \tag{A.4.12}
$$

which lead at small t to

$$
\delta\gamma(it) = i\rho_-(it) - \frac{2\xi}{\pi^2 g}t + O(t^2) = -t\left(\rho_1(g)+\frac{2\xi}{\pi^2 g}\right) + O(t^2)\,. \tag{A.4.13}
$$

The similarities between $\gamma(t)$ and $\rho(t)$ permit to derive a Wronskian-like relation between them, whose immediate consequence is the desired result (A.4.5). To see this, let us multiply both sides of the first relation in (A.4.9) by $(2n-1)\gamma_{2n-1}(g)$ and sum both sides over $n \geq 1$ in order to form the function $\gamma_-(t)$. In parallel, we multiply the second relation in (A.4.9) by $(2n)\gamma_{2n}(g)$, sum over $n \geqslant 1$ and form $\gamma_+(t)$. Then, we subtract the second relation from the first one and obtain

$$
\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t)\rho_-(t)-\gamma_+(t)\rho_+(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\gamma_-(t)\rho_+(t)+\gamma_+(t)\rho_-(t)}{\mathrm{e}^{t/2g}-1}\right] \\
&\qquad = 2\sum_{n\geq 1}\Big[(2n-1)\gamma_{2n-1}(g)h_{2n-1}(g) - (2n)\gamma_{2n}(g)h_{2n}(g)\Big].
\end{aligned} \tag{A.4.14}
$$

We notice that the expression in the left-hand side of this relation is invariant under the exchange $\gamma_\pm(t) \leftrightarrow \rho_\pm(t)$. Therefore, the right-hand side should be also invariant under (A.4.11) leading to

$$
\rho_1(g) = 2\sum_{n\geq 1}\left[(2n-1)\gamma_{2n-1}(g)h_{2n-1}(g) - (2n)\gamma_{2n}(g)h_{2n}(g)\right]. \tag{A.4.15}
$$

After replacing $h_{2n-1}(g)$ and $h_{2n}(g)$ by their expressions (A.4.10) and summing over the Bessel functions, we obtain that $\rho_1(g)$ is given by an integral involving the functions $\gamma_\pm(t)$. It takes much simpler form when expressed in terms of the functions $\Gamma_\pm(t)$ (with $\Gamma(t) = \Gamma_+(t) + i\Gamma_-(t)$) as

$$
\rho_1(g) = -\frac{\xi}{\pi}\int_0^\infty dt\left[\frac{\pi g}{t^2+\pi^2 g^2}\left(\Gamma_-(t)-\Gamma_+(t)\right) + \frac{t}{t^2+\pi^2 g^2}\left(\Gamma_-(t)+\Gamma_+(t)\right)\right]. \tag{A.4.16}
$$

Making use of the identities

$$\frac{\pi g}{t^2+\pi^2 g^2} = \int_0^\infty du\ \mathrm{e}^{-\pi g u}\cos(ut)\,, \qquad \text{(A.4.17)}$$
$$\frac{t}{t^2+\pi^2 g^2} = \int_0^\infty du\ \mathrm{e}^{-\pi g u}\sin(ut)\,,$$

we rewrite $\rho_1(g)$ as

$$\rho_1(g) = -\frac{\xi}{\pi}\int_0^\infty du\ \mathrm{e}^{-\pi g u}\left[\int_0^\infty dt\ \cos(ut)\,(\Gamma_-(t)-\Gamma_+(t)) + \int_0^\infty dt\ \sin(ut)\,(\Gamma_-(t)+\Gamma_+(t))\right]. \qquad \text{(A.4.18)}$$

Now, let us split the $u-$integral into $0 \le u \le 1$ and $u > 1$. We observe that for $u^2 \le 1$ the $t-$integrals in this relation can be done exactly thanks to the BES equation solved by the functions $\Gamma_\pm(t)$, see Eq. (3.2.4). Then, performing the integration over $u \ge 1$, we find after some algebra

$$\rho_1(g) = -\frac{2\xi}{\pi^2 g}\big(1-\mathrm{e}^{-\pi g}\big) - \frac{\sqrt{2}\xi}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^\infty \frac{dt}{t+i\pi g}\ \mathrm{e}^{i(t-\pi/4)}\,\Gamma(t)\right]. \qquad \text{(A.4.19)}$$

Substituting this relation into (A.4.13) and using the expression (A.4.3) for the mass scale, we finally arrive at (A.4.5).

Appendix B

FRS Equation and O(6) Sigma Model

B.1 Scalar Factor of the O(n) Sigma Model S-Matrix

In this appendix we solve iteratively the unitarity and crossing-symmetry equations for the scalar factor $\sigma(\theta) \equiv \sigma_2(\theta)$ of the O(n) sigma model S-matrix. The strategy we use follows the steps recommended in [133] (see also references therein) to find particular and homogeneous solutions to this sort of functional equations.

Iterative Solution

The unitarity and crossing-symmetry equations for the scalar factor $\sigma(\theta)$ are respectively given by

$$\sigma(x)\sigma(-x) = \frac{x^2}{x^2 - \Delta^2}\,, \tag{B.1.1}$$

and

$$\sigma(-1/2 - x) = \sigma(x)\,, \tag{B.1.2}$$

where $x = i\theta/2\pi$ and $\Delta = 1/(n-2)$. Here we look for a solution of both equations represented by a meromorphic function in the complex θ-plane (or equivalently in the complex x-plane) with singularities lying along the imaginary θ axis (or real x axis) only. As explained in [86], these conditions are general analytic properties (in the rapidity θ-plane) for two-body amplitudes in a relativistic *integrable* two-dimensional QFT. The condition of meromorphicity is however not sufficient to single out one of the solutions of the equations (B.1.1) and (B.1.2). Indeed the equations (B.1.1) and (B.1.2) are not specific to the non-linear O(n) sigma model. They rely on relativistic invariance, factorizability of the scattering, gobal O(n) symmetry and hold for a (non-degenerate) vector multiplet of massive asymptotic particles. But all these features can be satisfied by other two-dimensional QFTs, as the Gross-Neveu model for instance [72]. According to [72], the solution that describes the scattering in the non-linear O(n) sigma model is minimal. It means that it has a minimal set of singularities along the real x axis and moreover none in the physical strip Re(x) $\in (-1/2, 0)$, reflecting the absence of bound states in the spectrum of the O(n) sigma model [72, 132, 131].[1] In the following we will look for this solution. We will first

[1] For a bound state of mass M, we expect to find two poles respectively at $s = M^2 \leqslant 4m^2$ (s-channel) and at $s = (4m^2 - M^2) \geqslant 0$ (t-channel) in the physical sheet of the two-body scattering amplitude, which is here continued

construct iteratively a particular solution, observing all the way the absence of singularities in the physical strip and introducing least possible singularities in general. We will verify that the procedure converges to the minimal solution of [72]. Later we will discuss its uniqueness after constructing a general homogeneous (meromorphic) solution.

We start with an obviously minimal meromorphic solution of the unitarity condition (B.1.1) given by

$$\sigma_0(x) = \frac{x}{\Delta - x}\,, \tag{B.1.3}$$

with no pole in the physical strip. That solution does not satisfy the crossing-symmetry equation (B.1.2) or equivalently it is not symmetric with respect to $x = -1/4$. To symmetrize it, we introduce a pole at $x = -(1/2 + \Delta)$ and a zero at $x = -1/2$ by defining

$$\sigma_1(x) = \sigma_0(x)\sigma_0(-1/2 - x)\,. \tag{B.1.4}$$

Unfortunately the new solution no longer respects unitarity. To restore it, we need to add a pole at $x = 1/2$ and a zero at $x = (1/2 + \Delta)$ leading to

$$\sigma_2(x) = \frac{\sigma_1(x)}{\sigma_0(-1/2 + x)} = \sigma_0(x)\frac{\sigma_0(-1/2 - x)}{\sigma_0(-1/2 + x)}\,. \tag{B.1.5}$$

Doing so we broke the crossing symmetry of $\sigma_1(x)$ and to restore it we introduce a pole at $x = -1$ and a zero at $x = -(1 + \Delta)$ as

$$\sigma_3(x) = \frac{\sigma_2(x)}{\sigma_0(-1 - x)} = \frac{\sigma_1(x)}{\sigma_0(-1/2 + x)\sigma_0(-1 - x)}\,. \tag{B.1.6}$$

Now to cure the lack of unitarity of $\sigma_3(x)$ we add a pole at $x = (1 + \Delta)$ and a zero at $x = 1$ by means of

$$\sigma_4(x) = \sigma_3(x)\sigma_0(-1 + x) = \sigma_0(x)\frac{\sigma_0(-1/2 - x)\sigma_0(-1 + x)}{\sigma_0(-1/2 + x)\sigma_0(-1 - x)}\,. \tag{B.1.7}$$

It is not difficult to figure out what will happen next. One pole and one zero are generated at each step, farther and farther from the physical strip, and after $4k$ steps one finds

$$\sigma_{4k}(x) = \sigma_0(x)\prod_{j=1}^{k}\frac{\sigma_0(1/2 - j - x)\sigma_0(-j + x)}{\sigma_0(1/2 - j + x)\sigma_0(-j - x)}\,, \tag{B.1.8}$$

that is

$$\sigma_{4k}(x) = \frac{x}{\Delta - x}\prod_{j=1}^{k}\frac{j - x}{j + x}\,\frac{j - 1/2 + x}{j - 1/2 - x}\,\frac{j + \Delta + x}{j + \Delta - x}\,\frac{j - 1/2 + \Delta - x}{j - 1/2 + \Delta + x}\,. \tag{B.1.9}$$

into a meromorphic function of the Mandelstam variable $s = 2m^2(1 + \cosh\theta)$ with two cuts along $s \geqslant 4m^2$ and $s \leqslant 0$ [86]. Thus bound states would manifest as poles in the physical strip $\mathrm{Im}(\theta) \in (0, i\pi)$ (i.e. $\mathrm{Re}(x) \in (-1/2, 0)$) which is the image of the physical sheet. Finally, to motivate the choice of a minimal solution, we may observe with [129] that the O(n) sigma model is the O(n) symmetric bosonic QFT with least possible number of degrees of freedom per site. It suggests to look for the simplest possible solution which, being meromorphic (in x-plane), should have minimal set of singularities.

Of course $\sigma_{4k}(x)$ is not solution of both equations (B.1.1) and (B.1.2) but $\sigma_\infty(x)$ will do the job if the limit $k \to \infty$ exists. And indeed, the fixed-point limit $k \to \infty$ can be taken with help of the identity

$$\prod_{j=1}^{\infty} \frac{n+a+x}{n+a-x} = \frac{\Gamma(1+a-x)}{\Gamma(1+a+x)}\,, \tag{B.1.10}$$

and one can easily verify that

$$\sigma_\infty(x) = \frac{x}{\Delta - x}\frac{\Gamma(1+x)\Gamma(1/2-x)\Gamma(1+\Delta-x)\Gamma(1/2+\Delta+x)}{\Gamma(1-x)\Gamma(1/2+x)\Gamma(1+\Delta+x)\Gamma(1/2+\Delta-x)}\,, \tag{B.1.11}$$

is solution to (B.1.1) and (B.1.2). It may be equivalently written as

$$\sigma_\infty(x) = \frac{x}{\Delta + x}\frac{\Gamma(1+x)\Gamma(1/2-x)\Gamma(\Delta-x)\Gamma(1/2+\Delta+x)}{\Gamma(1-x)\Gamma(1/2+x)\Gamma(\Delta+x)\Gamma(1/2+\Delta-x)}\,, \tag{B.1.12}$$

which is up to a sign the minimal solution of [72]. The multiplication by a factor -1 is a $\mathbb{Z}_2$ symmetry of the equations (B.1.1) and (B.1.2), and thus it has to be fixed by normalizing correctly the minimal solution. Here we impose the boundary condition $\sigma(i\infty) = 1$ at large rapidity $\theta \sim \infty$ because of the asymptotic freedom of the bosonic O(n) sigma model. Since $\sigma_\infty(i\infty) = -1$, the correct minimal solution is [72]

$$\sigma(x) = -\frac{x}{\Delta + x}\frac{\Gamma(1+x)\Gamma(1/2-x)\Gamma(\Delta-x)\Gamma(1/2+\Delta+x)}{\Gamma(1-x)\Gamma(1/2+x)\Gamma(\Delta+x)\Gamma(1/2+\Delta-x)}\,. \tag{B.1.13}$$

uniqueness and CDD Factor

We will discuss now the uniqueness of the previous solution. We remark that, necessarily, two solutions of the equations (B.1.1) and (B.1.2) differ multiplicatively by a function $f(x)$ satisfying the homogeneous equations

$$f(x)f(-x) = 1\,, \tag{B.1.14}$$

and

$$f(-1/2-x) = f(x)\,. \tag{B.1.15}$$

Thus to decide about the uniqueness of the minimal solution we have to consider a general homogeneous solution $f(x)$.

Let us start with holomorphic homogeneous solutions. The general solution reads $f(x) = \pm \exp g(x)$ where $g(x)$ is holomorphic and can be Fourier decomposed as

$$g(x) = \sum_{n\geqslant 1} a_n \sin\left[2\pi(2n+1)x\right]\,, \tag{B.1.16}$$

with $a_n \in \mathbb{C}$. We note that $g(x)$, and so $f(x)$, is periodic, and thus bounded, along any axis parallel to the real x axis. Since $f(x)$ is holomorphic it cannot be bounded all over the complex plane. It implies that $|f(x)|$ can be found unbounded when x becomes large with a large imaginary part. At the same time, we would like to keep fixed the asymptotic behavior of the minimal solution $\sigma(x) \sim 1$ when $\mathrm{Im}(x) \sim \infty$. It follows immediately that $f(x)$ has to be constant, which, according to (B.1.14), requires $f(x) = \pm 1$. Hence, holomorphic solutions are reduced to the $\mathbb{Z}_2$ ambiguity that has been already fixed. We are thus led to consider the case of a meromorphic homogeneous solution, with singularities lying along the real x-axis. As follows from the equations (B.1.14) and (B.1.15), the presence of one pole at say $x = -\alpha$ will automatically call for a zero at $x = \alpha$, that both will be repeated infinitely along the real x-axis. Running through the same iterative procedure as before, starting with a solution of the equation (B.1.14) given by

$$f_0(x;\alpha) = \frac{\alpha - x}{\alpha + x}\,, \tag{B.1.17}$$

we find in the fixed-point limit that

$$f_\infty(x;\alpha) = \prod_{j=-\infty}^{\infty} \frac{j-1/2+\alpha+x}{j-1/2+n-x}\,\frac{j+n-x}{j+n+x} = \frac{\cos\left(\pi(\alpha+x)\right)\sin\left(\pi(\alpha-x)\right)}{\cos\left(\pi(\alpha-x)\right)\sin\left(\pi(\alpha+x)\right)}\,. \tag{B.1.18}$$

Then the general (meromorphic) solution to the unitary and crossing-symmetry equations reads

$$\tilde{\sigma}(x) = \sigma(x)\prod_k f_\infty(x;\alpha_k)\,, \tag{B.1.19}$$

where $\sigma(x)$ is the minimal solution (B.1.13) while the product is called CDD factor [72]. Now, to verify that $\sigma(x)$ is the unic minimal solution, we have to say if it is possible or not to kill or shift poles of $\sigma(x)$ with help of a CDD factor, without adding singularities in the physical strip. It is sufficient for that to consider the particular case of one factor $f_\infty(x;\alpha)$ and to restrict attention to $\alpha \in (-1/4, 1/4)$. For $\alpha \in (0, 1/4)$, the factor $f_\infty(x;\alpha)$ introduces two poles in the physical strip and thus does not produce a minimal solution. For $\alpha \in (-1/4, 0)$, the situation is better since $f_\infty(x;\alpha)$ introduces two zeros and no pole in the physical strip. Nevertheless, by comparing the positions of the zeros of $f_\infty(x;\alpha)$ with those of the poles of $\sigma(x)$, one can easily see that it is not possible to reduce or displace the set of singularities of the minimal solution, and their number necessarily increases.

We conclude that the minimal solution $\sigma(x)$, describing scattering amplitude in the non-linear O(n) sigma model, is unic and given explicitely by (B.1.13).

B.2 Hole Energy Formula

This appendix contains a derivation of the (all-loop) hole energy formula

$$\epsilon(g,j) = 32g\int_0^\infty \frac{dt}{t}\,\frac{\gamma_+^{(0)}(2gt)-\gamma_-^{(0)}(2gt)}{\mathrm{e}^t-1}\left[\mathrm{e}^{t/2}\,\gamma_{\mathrm{h}}(t;j) - \frac{j}{8}\right] - 4gj\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)\,, \tag{B.2.1}$$

where $\gamma_\pm^{(0)}(t) \equiv \gamma_\pm(t; j=0)$ stands for the BES solution. The proof of (B.2.1) relies on the use of the Wronskian-like relation, applied previously in Appendix A.4 in another context. We repeat here the argument.

The scaling function $\epsilon(g,j)$ is given by

$$\epsilon(g,j) = f(g,j) - f(g,0) = 16g^2\Big[\gamma_1(g,j) - \gamma_1(g,0)\Big]\,, \tag{B.2.2}$$

where $\gamma_1(g,j)$ enters into the Bessel series expansion of the function $\gamma_-(t;j)$,

$$\gamma_-(t;j) = 2\sum_{n\geqslant 1}(2n-1)\gamma_{2n-1}(g,j)J_{2n-1}(t)\,, \qquad \gamma_+(t;j) = 2\sum_{n\geqslant 1}(2n-1)\gamma_{2n}(g,j)J_{2n}(t)\,. \quad \text{(B.2.3)}$$

The functions $\gamma_\pm(t;j)$ are solution to the integral equations (4.2.17) and satisfies a Wronskian-like relation. To derive this relation we choose some reference j', multiply both sides of the two relations in (4.2.17) by the coefficients $(2n-1)\gamma_{2n-1}(g,j')$ and $(2n)\gamma_{2n}(g,j')$, respectively, and sum over $n \geq 1$. Then, we convert the sums into the functions $\gamma_\pm(t;j')$ using the definition (B.2.3) and subtract the second relation from the first one to obtain (after rescaling $t \to t/2g$)

$$\begin{aligned}\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t;j)\gamma_-(t;j') - \gamma_+(t;j)\gamma_+(t;j')}{1-\mathrm{e}^{-t/2g}} + \frac{\gamma_-(t;j)\gamma_+(t;j') + \gamma_+(t;j)\gamma_-(t;j')}{\mathrm{e}^{t/2g}-1}\right] \\ = \gamma_1(g,j') + 2\sum_{n\geq 1}\left[(2n-1)h_{2n-1}(g,j)\gamma_{2n-1}(g,j') - (2n)h_{2n}(g,j)\gamma_{2n}(g,j')\right].\end{aligned} \quad \text{(B.2.4)}$$

The expression on the left-hand side is invariant under exchange $j \leftrightarrow j'$ and the same should be true on the right-hand side. Then, we replace $h_n(g,j)$ by their definition (4.2.18) and get

$$\gamma_1(g,j) - \gamma_1(g,j') = \frac{2}{g}\int_0^\infty \frac{dt}{t}\,\frac{\gamma_+(2gt;j') - \gamma_-(2gt;j')}{\mathrm{e}^t-1}\left[\mathrm{e}^{t/2}\,\gamma_{\mathrm{h}}(t;j) - \frac{j}{8}J_0(2gt)\right] - \left(j \leftrightarrow j'\right), \quad \text{(B.2.5)}$$

where we indicated explicitly the dependence of the $\gamma-$functions on the scaling parameters. We note that for $j'=0$ the relation (B.2.5) can be used to evaluate the scaling function (B.2.2). We take into account that $\gamma_{\mathrm{h}}(t;j'=0) = h_n(g,j'=0) = 0$ and $\gamma_\pm(t;j'=0) \equiv \gamma_\pm^{(0)}(t)$ to obtain

$$\epsilon(g,j) = 32g\int_0^\infty \frac{dt}{t}\,\frac{\gamma_+^{(0)}(2gt) - \gamma_-^{(0)}(2gt)}{\mathrm{e}^t-1}\left[\left(\mathrm{e}^{t/2}\,\gamma_{\mathrm{h}}(t;j) - \frac{j}{8}\right) + \frac{j}{8}\left(1 - J_0(2gt)\right)\right]. \quad \text{(B.2.6)}$$

Here the integral involving the Bessel function can be further simplified by making use of the identity $1 - J_0(z) = 2\sum_{n\geq 1} J_{2n}(z)$ leading to

$$\begin{aligned}8gj\sum_{n\geq 1}\int_0^\infty \frac{dt}{t}\frac{\gamma_+^{(0)}(2gt) - \gamma_-^{(0)}(2gt)}{\mathrm{e}^t-1}J_{2n}(2gt) \\ = -4gj\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)\left(1 - J_0(2gt)\right) = -4gj\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)\,,\end{aligned} \quad \text{(B.2.7)}$$

where in the first relation we applied the integral equations (4.2.17) for $j=0$ and in the second relation used Bessel series representation (B.2.3) together with the orthogonality condition (3.1.20). Combining relations (B.2.6) and (B.2.7) we arrive at (B.2.1).

B.3 Small j Scaling Function

In this appendix we present some details of the strong coupling expansion of the (small j) scaling function $\epsilon(g,j) = \epsilon_1(g)j + \epsilon_3(g)j^3 + \dots$. We only calculate $\epsilon_1(g)$ and we prove its relation to the mass scale m

$$\epsilon_1(g) = -1 + m + O\left(\mathrm{e}^{-3g\pi}\right)\,. \quad \text{(B.3.1)}$$

The analysis of $\epsilon_3(g)$ is essentially the same and can be found in [73].

We recall that m is given in terms of the BES solution $\Gamma^{(0)}(t) = \Gamma_+^{(0)}(t) + i\Gamma_-^{(0)}(t)$ as

$$m \equiv \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^\infty \frac{dt}{t+i\pi g}\;\mathrm{e}^{it-i\pi/4}\,\Gamma^{(0)}(t)\right], \tag{B.3.2}$$

and that the functions $\Gamma_\pm^{(0)}(t)$ are solutions to

$$\begin{aligned}\int_0^\infty dt\,\sin(ut)\left[\Gamma_-^{(0)}(t) + \Gamma_+^{(0)}(t)\right] &= 0\,,\\ \int_0^\infty dt\,\cos(ut)\left[\Gamma_-^{(0)}(t) - \Gamma_+^{(0)}(t)\right] &= 2\,,\end{aligned} \tag{B.3.3}$$

for $u^2 < 1$.

The starting point is the representation (4.2.29) for $\epsilon_1(g)$, or equivalently

$$\epsilon_1(g) = -2g\int_0^\infty \frac{dt}{t}\left[\left(1 - \frac{\cosh(t/4g)}{\cosh(t/2g)}\right)\left(\Gamma_-^{(0)}(t) + \Gamma_+^{(0)}(t)\right) + \frac{\sinh(t/4g)}{\cosh(t/2g)}\left(\Gamma_-^{(0)}(t) - \Gamma_+^{(0)}(t)\right)\right]. \tag{B.3.4}$$

Observing the similarity between the integrals in (B.3.4) and those in (B.3.3), it is suggestive to replace ratios of hyperbolic functions by their Fourier integrals. That can be done with the help of the identities

$$\begin{aligned}\frac{\cosh(t/4g)}{\cosh(t/2g)} &= \sqrt{2}g\int_{-\infty}^\infty du\,\cos(ut)\frac{\cosh(g\pi u)}{\cosh(2g\pi u)}\,,\\ \frac{\sinh(t/4g)}{\cosh(t/2g)} &= \sqrt{2}g\int_{-\infty}^\infty du\,\sin(ut)\frac{\sinh(g\pi u)}{\cosh(2g\pi u)}\,,\end{aligned} \tag{B.3.5}$$

valid for arbitrary real t and $g > 0$. Applying (B.3.5), we find from (B.3.4)

$$\begin{aligned}\epsilon_1(g) = -2\sqrt{2}g^2\Big\{&\int_{-\infty}^\infty du\,\frac{\cosh(g\pi u)}{\cosh(2g\pi u)}\int_0^\infty \frac{dt}{t}\,\Big(1-\cos(ut)\Big)\left[\Gamma_-^{(0)}(t) + \Gamma_+^{(0)}(t)\right]\\ &+\int_{-\infty}^\infty du\,\frac{\sinh(g\pi u)}{\cosh(2g\pi u)}\int_0^\infty \frac{dt}{t}\,\sin(ut)\,\left[\Gamma_-^{(0)}(t) - \Gamma_+^{(0)}(t)\right]\Big\}.\end{aligned} \tag{B.3.6}$$

Now, let us split the $u-$integrals into sum of two terms corresponding to $u^2 \le 1$ and $u^2 > 1$. In the first one, we replace $(1-\cos(ut))/t = \int_0^u dv\,\sin(vt)$ and $\sin(ut)/t = \int_0^u dv\,\cos(vt)$ and, then, evaluate the $t-$integral with the help of the equation (B.3.3). The resulting expression for $\epsilon_1(g)$ is given by

$$\begin{aligned}\epsilon_1(g) = -4\sqrt{2}g^2\Big\{&\int_{-1}^1 du\;u\,\frac{\sinh(g\pi u)}{\cosh(2g\pi u)}\\ &+\int_1^\infty du\,\frac{\cosh(g\pi u)}{\cosh(2g\pi u)}\int_0^\infty \frac{dt}{t}\,\Big(1-\cos(ut)\Big)\left[\Gamma_-^{(0)}(t) + \Gamma_+^{(0)}(t)\right]\\ &+\int_1^\infty du\,\frac{\sinh(g\pi u)}{\cosh(2g\pi u)}\int_0^\infty \frac{dt}{t}\,\sin(ut)\,\left[\Gamma_-^{(0)}(t) - \Gamma_+^{(0)}(t)\right]\Big\}.\end{aligned} \tag{B.3.7}$$

To find the large g expansion of the first term on the right-hand side of (B.3.7) we just do the opposite, that is rewrite the integral over $-1 \le u \le 1$ as a difference of two integrals over $-\infty < u < \infty$ and $u^2 > 1$

$$\int_{-1}^{1} du\, u \frac{\sinh(g\pi u)}{\cosh(2g\pi u)} = \int_{-\infty}^{\infty} du\, u \frac{\sinh(g\pi u)}{\cosh(2g\pi u)} - 2\int_{1}^{\infty} du\, u \frac{\sinh(g\pi u)}{\cosh(2g\pi u)}$$
$$= \frac{1}{4g^2\sqrt{2}} - 2\frac{1+\pi g}{(\pi g)^2}\, \mathrm{e}^{-g\pi} + O(\mathrm{e}^{-3g\pi})\,. \tag{B.3.8}$$

In the remaining two terms in (B.3.7), we replace hyperbolic functions by their leading large g asymptotics, integrate over u by parts, take into account the equation (B.3.3) to evaluate the boundary term and arrive at

$$\epsilon_1(g) = -1 + \delta + O\left(\mathrm{e}^{-3g\pi}\right)\,, \tag{B.3.9}$$

with $\delta \sim \mathrm{e}^{-\pi g}$ given by

$$\delta \equiv \frac{8\sqrt{2}g}{\pi}\Bigg\{\frac{\mathrm{e}^{-\pi g}}{\pi g} - \frac{1}{2}\int_{1}^{\infty} du\ \mathrm{e}^{-\pi g u}\int_{0}^{\infty} dt \tag{B.3.10}$$
$$\times\left[\cos(ut)\Big(\Gamma_{-}^{(0)}(t) - \Gamma_{+}^{(0)}(t)\Big) + \sin(ut)\Big(\Gamma_{-}^{(0)}(t) + \Gamma_{+}^{(0)}(t)\Big)\right]\Bigg\}.$$

Integration over u leads to the expression in the right-hand side of equation (B.3.2) and thus to $\delta = m$, as promised.

B.4 Magnon Solution

In this appendix we discuss the solution to the magnon equation at strong coupling.

The solution to the magnon equation can be decomposed as

$$\gamma_{\pm}(t) = \gamma_{\pm}^{(0)}(t) + \delta\gamma_{\pm}(t)\,, \tag{B.4.1}$$

where $\gamma_{\pm}^{(0)}(t) \equiv \gamma_{\pm}(t; j=0)$ is the solution to the BES equation corresponding to the $j \to 0$ limit of the FRS equation, while $\delta\gamma_{\pm}(t) \equiv \gamma_{\pm}(t;j) - \gamma_{\pm}(t; j=0)$ captures all corrections depending on j. In particular, the function $\delta\gamma_{-}(t)$ completely determines the scaling function $\epsilon(g,j)$ as

$$\epsilon(g,j) = 16g^2 \lim_{t\to 0} \delta\gamma_{-}(t)/t\,. \tag{B.4.2}$$

By construction, the functions $\delta\gamma_{+/-}(t)$ are even/odd entire functions of t with Fourier transforms supported on the interval $[-1,1]$ and they satisfy the system of equations ($n \geqslant 1$)

$$\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_{-}(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\delta\gamma_{+}(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n-1}(t) = -\frac{2}{g}\int_0^\infty \frac{dt}{t}\frac{J_{2n-1}(t)}{\mathrm{e}^{t/2g}-1}\left[\mathrm{e}^{t/4g}\,\gamma_{\mathrm{h}}(t/2g) - \frac{j}{8}J_0(t)\right],$$
$$\int_0^\infty \frac{dt}{t}\left[\frac{\delta\gamma_{+}(t)}{1-\mathrm{e}^{-t/2g}} - \frac{\delta\gamma_{-}(t)}{\mathrm{e}^{t/2g}-1}\right] J_{2n}(t) = -\frac{2}{g}\int_0^\infty \frac{dt}{t}\frac{J_{2n}(t)}{\mathrm{e}^{t/2g}-1}\left[\mathrm{e}^{t/4g}\,\gamma_{\mathrm{h}}(t/2g) - \frac{j}{8}J_0(t)\right]. \tag{B.4.3}$$

We recall that the function $\gamma_{\rm h}(t)$, in the right-hand side of (B.4.3), is the Fourier transform of the density $\chi(\theta)$,

$$\gamma_{\rm h}(t) = \frac{1}{8\pi}\int_{-B}^{B} d\theta\, \chi(\theta)\cos\left(2\theta t/\pi\right), \tag{B.4.4}$$

for the distribution of hole rapidities $u = 2\theta/\pi$ supported on the interval $[-a, a]$, with $B = a\pi/2$.[2] We see that the hole distribution acts as a source (inhomogeneous term) in the system of equations (B.4.3). In the following, we will determine the solution $\delta\gamma_{\pm}(t)$ to (B.4.3) assuming that the coupling is large $g \gg 1$ and that the Fermi rapidity B fulfills $B < \pi g$ (i.e. $a < 2g$).

To begin with, we observe that a particular solution $\delta\gamma_{\pm}^{\rm p}(t)$ to the equations (B.4.3) is given by

$$\delta\gamma_{+}^{\rm p}(t) = -\frac{2}{g}\left(\frac{\cosh(t/4g)}{\cosh(t/2g)}\gamma_{\rm h}(t/2g) - \frac{j}{8}J_0(t)\right), \qquad \delta\gamma_{-}^{\rm p}(t) = -\frac{2}{g}\frac{\sinh(t/4g)}{\cosh(t/2g)}\gamma_{\rm h}(t/2g). \tag{B.4.5}$$

Indeed, one easily verifies after a bit of algebra and the use of the identity ($n \geqslant 1$)

$$\int_0^{\infty}\frac{dt}{t}\, J_0(t)J_{2n}(t) = 0, \tag{B.4.6}$$

that $\delta\gamma_{\pm}^{\rm p}(t)$ is an exact solution to (B.4.3). However, the functions $\delta\gamma_{\pm}^{\rm p}(t)$ does not have the right analytic properties because it has poles along the imaginary axis. It follows that the function $\delta\gamma_{-}^{\rm p}(t)$ does not reproduce the correct scaling function $\epsilon(g, j)$. We immediately verify it since for any value of j we find

$$\epsilon(g, j)^{\rm p} = 16g^2 \lim_{t\to 0}\delta\gamma_{-}^{\rm p}(t)/t = -8\gamma_{\rm h}(0) = -j. \tag{B.4.7}$$

Here we used the fact that the hole distribution is normalized as $\gamma_{\rm h}(0) = j/8$ ($\forall g$). The result $\epsilon(g, j) = \epsilon(g, j)^{\rm p} = -j$ is certainly not consistent at weak coupling since $\epsilon(g, j) = O(g^2)$, and, indeed, the particular solution (B.4.5) is too singular to describe the perturbative scaling function of the gauge theory. The situation is better at strong coupling $g \gg 1$. Namely, we note that the result $\epsilon(g, j) = \epsilon(g, j)^{\rm p} = -j$ is exact up to non-perturbative corrections, $(\epsilon(g, j) + j) \sim mj$, at small j ($j \ll m \sim {\rm e}^{-\pi g}$). It suggests that the particular solution (B.4.5) is a good starting point to investigate the scaling function at strong coupling. As we shall see, the correction to the particular solution (B.4.5) are exponentially suppressed at strong coupling $g \gg 1$ as long as $B < \pi g$, or equivalently $a < 2g$.

Let us rewrite the particular solution as

$$\delta\gamma^{\rm p}(it) \equiv \delta\gamma_{+}^{\rm p}(it) + i\delta\gamma_{-}^{\rm p}(it) = -\frac{\sqrt{2}}{g}\frac{\gamma_{\rm h}(it/2g)}{\sin(t/4g + \pi/4)} + \frac{j}{4g}J_0(it). \tag{B.4.8}$$

The term involving the Bessel function, in the right-hand side of the last equality above, has correct analytic properties, being holomorphic with a Fourier transform supported on the interval

[2] Here the use of the variable θ and of the Fermi rapidity B is motivated by the correspondence with the O(6) sigma model where $\chi(\theta)$ is the differential rapidity distribution for a gas of massive particles carrying individual momentum $p = m\sinh\theta$ and energy $E = m\cosh\theta$.

$[-1,1]$. The term depending on $\gamma_{\rm h}(it/2g)$, however, has singularities that should be compensated. To this end, we decompose the exact solution $\delta\gamma(it) \equiv \delta\gamma_+(it) + i\delta\gamma_-(it)$ as

$$\delta\gamma(it) = \delta\gamma^{\rm p}(it) + \delta\gamma^{\rm np}(it)\,, \tag{B.4.9}$$

where $\delta\gamma^{\rm np}(it) \equiv \delta\gamma_+^{\rm np}(it) + i\delta\gamma_-^{\rm np}(it)$ is, by construction, a homogeneous solution to the magnon equations (B.4.3), or equivalently a homogeneous solution to the BES equation. We note that $\delta\Gamma^{\rm np}(it)$ should have poles located at $t = t_m = 4\pi g\big(m - \frac{1}{4}\big)$, with $m \in \mathbb{Z}$, for the function $\delta\gamma(it)$ to be holomorphic. Namely, we require that

$$\delta\Gamma^{\rm np}(it) = \frac{4\sqrt{2}}{t-t_m}(-1)^m\gamma_{\rm h}(it_m/2g) + O\left((t-t_m)^0\right)\,, \tag{B.4.10}$$

for $t \sim t_m$. Introducing the function $\delta\Gamma^{\rm np}(it)$ as

$$\delta\Gamma^{\rm np}(it) = \frac{\sin(t/4g + \pi/4)}{\sin(t/4g)\sin(\pi/4)}\delta\gamma^{\rm np}(it)\,, \tag{B.4.11}$$

the equation (B.4.10) translates into

$$\delta\Gamma^{\rm np}(it_m) = -\frac{2\sqrt{2}}{g}(-1)^m\gamma_{\rm h}(it_m/2g) = -\frac{\sqrt{2}}{4\pi g}(-1)^m\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\left((4m-1)\theta\right), \tag{B.4.12}$$

where in the last equality we introduced the Fourier transform of $\gamma_{\rm h}(t)$ as in Eq. (B.4.4). In the terminology of the Chapter 3, the set of equations (B.4.12) are the quantization conditions for the function $\delta\Gamma^{\rm np}(it)$. Before to make use of them, we need first to find a convenient parameterization of the solution $\delta\Gamma^{\rm np}(it)$. To get it, we will follow the steps of the construction of the general solution to the BES equation (see Chapter 3 and Appendix A.1).

We note that the function $\delta\Gamma^{\rm np}(it)$ has an infinite number of poles located at $t = \pm 4\pi ng$ with $n \in \mathbb{N}^*$. From the residues at these poles, we extract an infinite set of coefficients ($n \in \mathbb{N}^*$) introduced as

$$c_\pm^{\rm np}(n,g) = \mp 4g\,\delta\gamma^{\rm np}(\pm 4i\pi ng)\,{\rm e}^{-4\pi ng}\,. \tag{B.4.13}$$

As for the construction of the BES solution, we would like to make use of these coefficients to parameterize the solution and then determine them with help of the quantization conditions (B.4.12). To this end, let us introduce the Fourier transform of $\delta\Gamma^{\rm np}(t)$ as

$$\delta\widetilde{\Gamma}^{\rm np}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,{\rm e}^{ikt}\,\delta\Gamma^{\rm np}(t)\,. \tag{B.4.14}$$

The function $\delta\Gamma^{\rm np}(t)$ is a homogeneous solution to the BES equation, meaning that

$$\int_0^\infty dt\left[{\rm e}^{ikt}\,\delta\Gamma_-^{\rm np}(t) - {\rm e}^{-ikt}\,\delta\Gamma_+^{\rm np}(t)\right] = 0\,, \qquad (-1 \le k \le 1)\,, \tag{B.4.15}$$

where $\delta\Gamma^{\rm np}(t) = \delta\Gamma_+^{\rm np}(t) + i\delta\Gamma_-^{\rm np}(t)$ and $\delta\Gamma_\pm^{\rm np}(-t) = \pm\delta\Gamma_\pm^{\rm np}(t)$. In principle, if $\delta\widetilde{\Gamma}^{\rm np}(k)$ is known on the interval $k^2 > 1$, then $\delta\widetilde{\Gamma}^{\rm np}(k)$ can be found for $k^2 < 1$ by solving the equation (B.4.15). It should be sufficient, therefore, to look for a parameterization of the function $\delta\widetilde{\Gamma}^{\rm np}(k)$ in terms of the coefficients $c_\pm^{\rm np}(n,g)$ for $k^2 > 1$.

Taking into account the expression (B.4.11) for $\delta\Gamma^{\rm np}(t)$ and the relation (B.4.9), we find that the Fourier integral (B.4.14) can be written as

$$\begin{aligned}\delta\widetilde{\Gamma}^{\rm np}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, {\rm e}^{ikt}\, \frac{\sin(it/4g-\pi/4)}{\sin(it/4g)\sin(\pi/4)}\delta\gamma^{\rm np}(t) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, {\rm e}^{ikt}\, \frac{\sin(it/4g-\pi/4)}{\sin(it/4g)\sin(\pi/4)}\Big(\delta\gamma(t)-\delta\gamma^{\rm p}(t)\Big)\,,\end{aligned} \tag{B.4.16}$$

The form of the integrands above suggests to perform integration by the method of residues. To this end, we recall that the function $\delta\gamma(t)$ has Fourier transform supported on $[-1,1]$, while the function $\delta\gamma^{\rm p}(t)$ depends on $\gamma_{\rm h}(t/2g)$ which has Fourier transform supported on $[-a/2g, a/2g]$. We expect, therefore, that the functions $\delta\gamma(t)$ and $\delta\gamma^{\rm p}(t)$ will scale as ${\rm e}^{|t|}$ and ${\rm e}^{a|t|/2g}$, respectively, for $t\sim i\infty$. Thus, in order to damp the asymptotic behavior of the integrand in (B.4.16) and then compute the integral by the method of residues, we impose that the Fourier variable k satisfies both $|k|>1$ and $|k|>a/2g$. Assuming $a<2g$, we find in this way and for $k^2>1$ that

$$\delta\widetilde{\Gamma}^{\rm np}(k) = \theta(-k-1)\sum_{n\geq 1} c_-^{\rm np}(n,g)\,{\rm e}^{-4\pi n g(-k-1)} + \theta(k-1)\sum_{n\geq 1} c_+^{\rm np}(n,g)\,{\rm e}^{-4\pi n g(k-1)}\,, \tag{B.4.17}$$

where we used the definition (B.4.13) to introduce the coefficients $c_\pm^{\rm np}(n,g)$. We have thus found the desired parametrization for the solution $\delta\widetilde{\Gamma}^{\rm np}(k)$ on the interval $k^2>1$. We note finally that the result (B.4.17) would not be valid for $1<k^2<a^2/(2g)^2$ if $a>2g$. It implies that a separate study of the solution on the interval $1<k^2<a^2/(2g)^2$ is needed to completely characterize it outside the interval $k^2<1$. We will not attempt to do it here, since we are primarily interested in the regime $a<2g$ which is connected to the small j regime of the scaling function. In the following, we will thus always assume that $a<2g$ and rely on the relation (B.4.17) to continue the analysis.

The parameterization of $\delta\Gamma^{\rm np}(k)$ on the interval $k^2>1$, Eq. (B.4.17), being identical to the one of the BES solution, we can apply our previous findings (see Chapter 3 and Appendix A.1) and immediatley conclude that the function $\delta\Gamma^{\rm np}(it)$ is given by

$$\delta\Gamma^{\rm np}(it) = \delta f_0^{\rm np}(t)V_0(t) + \delta f_1^{\rm np}(t)V_1(t)\,, \tag{B.4.18}$$

where the functions $\delta f_0^{\rm np}(t)$ and $\delta f_1^{\rm np}(t)$ read

$$\begin{aligned}\delta f_0^{\rm np}(t) &= \sum_{n\geq 1} t\left[\delta c_+^{\rm np}(n,g)\frac{U_1^+(4\pi n g)}{4\pi n g - t} + \delta c_-^{\rm np}(n,g)\frac{U_1^-(4\pi n g)}{4\pi n g + t}\right]\,, \\ \delta f_1^{\rm np}(t) &= \sum_{n\geq 1} 4\pi n g\left[\delta c_+^{\rm np}(n,g)\frac{U_0^+(4\pi n g)}{4\pi n g - t} + \delta c_-^{\rm np}(n,g)\frac{U_0^-(4\pi n g)}{4\pi n g + t}\right]\,.\end{aligned} \tag{B.4.19}$$

The special functions $V_{0,1}(t), U_{0,1}^\pm(t)$ are given in Appendix A.1. We recall that for any $g>0$ and whatever are the coefficients $\delta c_\pm^{\rm np}(n,g)$, the function (B.4.18) supplemented with (B.4.19) is a homogeneous solution to the equation (B.4.15) satisfying (B.4.17) for $k^2>1$. Now, to find explicit expressions for $\delta f_{0,1}^{\rm np}(t)$, one has to solve the quantization conditions (B.4.12) for the coefficients $\delta c_\pm^{\rm np}(n,g)$, i.e. solve the system of equations

$$\delta f_0^{\rm np}(t_m)V_0(t_m) + \delta f_1^{\rm np}(t_m)V_1(t_m) = -\frac{\sqrt{2}}{4\pi g}(-1)^m\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\left((4m-1)\theta\right)\,, \tag{B.4.20}$$

with $t_m = 4\pi g\big(m - \frac{1}{4}\big)$ and $m \in \mathbb{Z}$. Once done, one can read the scaling function (B.4.2) as

$$\epsilon(g,j) + j = -8g\delta f_1^{\rm np}(0)\,. \tag{B.4.21}$$

Let us examine the quantization conditions (B.4.20). It is not possible in general to find an explicit solution to them but we expect some simplification to occur at strong coupling. In that case, we know that the functions $V_{0,1}\,(t_m)$ scale as

$$V_{0,1}\,(t_m) = V_{0,1}\,\big(4\pi g\big(m - \tfrac{1}{4}\big)\big) \sim {\rm e}^{4\pi g\left|m-\frac{1}{4}\right|}\,, \tag{B.4.22}$$

when $g \gg 1$. Dividing both sides of (B.4.20) by this asymptotic behavior, we obtain that the source terms, i.e. the right-hand side of the system of equations (B.4.20), get suppressed exponentially at large g. The leading source term is the one evaluated at $t_{m=0} = -\pi g$ and consequently the functions $\delta f_{0,1}^{\rm np}(t)$ are of order $O({\rm e}^{-\pi g})$ at strong coupling. Let us verify it explicitely by computing the contribution to the scaling function generated by the homogeneous solution (B.4.18). Restricting our analysis to the first non-perturbative $O({\rm e}^{-\pi g})$ correction permits to consider the simpler set of quantization conditions given by

$$\delta f_0^{\rm np}(t_m)V_0(t_m) + \delta f_1^{\rm np}(t_m)V_1(t_m) = -\xi\delta_{m,0}\,, \tag{B.4.23}$$

where

$$\xi = \frac{\sqrt{2}}{4\pi g}\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\theta\,. \tag{B.4.24}$$

Now, we observe that we have already encountered the system of equations (B.4.23) in Chapter 3. They appeared in the computation of the first non-pertubative correction to the BES solution, but with a different value of the constant ξ (see Eqs. (3.4.17), (3.4.18) and (3.4.19) in Section 3.4.3). Explicit expressions for $\delta f_{0,1}^{\rm np}(t)$ at strong coupling can therefore be found in Chapter 3 by adapting the value of ξ to the current situation. In particular, one can immediately obtain the contribution to the scaling function associated with the solution to (B.4.23). Indeed, we found that the relation (see Eq. (3.4.20) and Appendix B.4)

$$\delta f_1^{\rm np}(0) = -\frac{\xi m}{2\sqrt{2}}\,, \tag{B.4.25}$$

holds true ($\forall g > 0$) for a homogeneous solution, parameterized as in (B.4.18) and (B.4.19), and satisfying the quantization conditions (B.4.23). Then, plugging this result into the equation (B.4.21) and using (B.4.24), we get the scaling function as

$$\epsilon(g,j) + j = \frac{m}{\pi}\int_{-B}^{B} d\theta\,\chi(\theta)\cosh\theta\,. \tag{B.4.26}$$

This result agrees with the one obtained from the hole energy formula. Since $m \sim {\rm e}^{-\pi g}$, we check that the function $\delta\gamma^{\rm np}(it)$ generates a non-perturbative contribution to the magnon solution (B.4.9) when the coupling is large.

We stress finally that the term on the right-hand side of (B.4.26) is just the first non-perturbative contribution to the scaling function, which is obtained by replacing the quantization conditions (B.4.20) by (B.4.23). A more general non-perturbative expansion can be developed by considering the family of solutions satisfying the quantization conditions

$$\delta f_0^{\rm np}(t_m)V_0(t_m) + \delta f_1^{\rm np}(t_m)V_1(t_m) = -\xi_n \delta_{m,n}\,, \tag{B.4.27}$$

with

$$\xi_n = \frac{\sqrt{2}}{4\pi g}(-1)^n \int_{-B}^{B} d\theta\, \chi(\theta) \cosh\left((4n-1)\theta\right). \tag{B.4.28}$$

Here $t_m = 4\pi g\big(m - \frac{1}{4}\big)$ with $m \in \mathbb{Z}$, while $n \in \mathbb{N}$. Proceeding in this way and using the asymptotic behavior (B.4.22) for the $V-$functions, one finds that the $(n+1)^{\rm th}$ non-perturbative correction induces a contribution to the scaling function that behaves at strong coupling as

$$\delta\epsilon^{(n)}(g,j) \sim {\rm e}^{-|4n-1|\pi g} \int_{-B}^{B} d\theta\, \chi(\theta) \cosh\left((4n-1)\theta\right). \tag{B.4.29}$$

It indicates that the parameter controlling this non-perturbative expansion is given by $z = \exp(B - \pi g)$, which can be kept small as long as $B < \pi g$. For $B > \pi g$ $(a > 2g)$, our analysis breakdowns and, as remarked before, another parametrization/or a resummation of the solution has to be worked out.

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