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Gauge/Gravity Duality and Field Theories at Strong Coupling

Gregory C. Giecold

Institut de Physique Théorique, CNRS–URA 2306
CEA/Saclay, F–91191 Gif–sur–Yvette, France

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Iosif Bena	Directeur de Thèse
Emilian Dudas	Membre du Jury
Edmond Iancu	Directeur de Thèse
Henning Samtleben	Rapporteur
Kostas Skenderis	Rapporteur

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Résumé

L'objet de cette thèse est l'étude de certaines propriétés de théories des champs à fort couplage via la dualité avec la théorie des cordes, dans la limite de supergravité. L'analyse expérimentale du plasma de quarks et de gluons produit au RHIC et au LHC tend en effet à indiquer que cet état de la matière se comporte comme un fluide quasiment parfait. Les méthodes perturbatives de la QCD sont impuissantes à décrire ses propriétés et la chromodynamique quantique sur réseau fait face à des problèmes tant techniques que conceptuels pour calculer les observables dynamiques d'un tel système. La correspondance AdS/CFT offre par conséquent un outil unique permettant d'étudier en première approximation cette phase de la QCD. L'un des aspects de cette thèse consiste en la description par une équation stochastique de Langevin d'un parton massif se propageant dans le plasma d'une théorie de Yang–Mills maximalement supersymétrique. Bien que cette théorie semble décrire de manière satisfaisante la phase déconfinée de la QCD, il est toutefois désirable de chercher un dual en théorie des cordes rendant compte des aspects de la QCD à basse énergie. L'autre axe directeur de cette thèse propose ainsi de rendre compte de solutions de moindre supersymétrie, sans invariance conforme, et avec confinement. On obtient le dual gravitationnel d'états metastables de telles théories. En particulier, on dérive une contribution au potentiel inflationnaire dans le cadre d'un modèle cosmologique générique de la théorie des cordes.

Abstract

In this thesis, we apply the gauge/string duality in its supergravity limit to infer some properties of field theories at strong coupling. Experiments at RHIC and at the LHC indeed suggest that the quark–gluon plasma behaves as one of the most perfect fluid ever achieved in any controlled experimental setup. Perturbative approaches fail at accounting for its properties, whereas lattice QCD methods face technical as well as conceptual difficulties in computing dynamical aspects of this new state of matter. As a result, the AdS/CFT correspondence currently is the best tool at our disposal for analytically modelling this phase of QCD. One of the contributions of this thesis amounts to deriving a stochastic Langevin equation for a heavy quark moving across a maximally supersymmetric Yang–Mills plasma at strong coupling. Even though this theory seems to describe in a surprisingly satisfactory way the high–energy, deconfined phase of QCD, it is also of much interest to try and search for a string theory dual making closer contact with QCD at lower energies. As such, the other main focus of this thesis deals with supergravity solutions of lesser supersymmetry, without conformal invariance and exhibiting confinement. We build for the first time the gravity dual to metastable states of such theories. In particular, we find the contribution from anti–branes to the inflation potential in some general scenario of string cosmology.

Contents

Chapter 1

Introduction

This introduction aims at presenting the different topics included in the bulk of this thesis in a more qualitative and informal way than found in the later chapters, which mostly consist in a collection of my published work. This is primarily an opportunity to motivate work done over a two years and a half period on loosely–connected subjects, with as few equations and technicalities as possible.

Gauge/gravity duality

Let us start with introducing heuristically gauge/gravity duality. Many outstanding reviews are presently available [4, 172, 74, 194, 205] that mostly focus on traditional material. Excellent textbooks [219, 220, 164, 26, 217, 154] provide background material on string theory and D–brane physics. Here I will instead entirely focus on "guessing" a string dual to QCD–like theories. A similar trail was blazed by Polchinski in his recent TASI lectures [221]. In what follows, I have relied on a similar minimalist list of ingredients for holography and have added my own modest insight along the way.

The suggestion that there exists a string theory dual to QCD has a long history. Actually, string theory was developed in order to understand the properties of the strong interaction. Soon after Veneziano [256] proposed a model — which is now understood as corresponding to the tree–level scattering of four tachyon vertex operators — that could account for the Dolen–Horn–Schmid duality relating the sum of s–channel exchanges with that of t–channel exchanges [77, 228][1], it was realized that the physical states underlying the dual resonance model can be accounted for by what we know call String Theory (see the historical review [75] and references therein). String Theory was however later superseded by Quantum Chromodynamics for the purpose of explaining the strong interaction force.

Still, it kept on resurfacing given that QCD is not suited for understanding strongly–coupled phenomena and confinement. For example, successful jet hadronization algorithms still in use nowadays rely on the so–called phenomenological, string–like Lund model of QCD flux tubes [11]. In addition, long QCD strings appear to be accurately described by the Nambu–Goto action when one compares predictions for the quark/anti–quark potential derived from string theory to the one computed via lattice QCD. Most suggestive of all indications for an underlying string dual is 't Hooft calculation [250] that suggests Feynman diagrams in large N non–abelian gauge theories form a kind of net reminiscent of the genus expansion of a string theory with string coupling constant of order $1/N$.

What sort of string dual should we expect ? First of all, we can take for granted that four–dimensional QCD cannot admit a four–dimensional string theory dual. The reason is that bosonic or fermionic string theory is consistent only in $d = 26$, respectively $d = 10$, space–time

[1] Amusingly, their phenomenological observation is dependent on the energy levels that had been probed at the time and does not hold for currently available baryon scattering data.

dimensions. Indeed, under a rescaling of the metric

$$g_{ab} \to e^{\phi}\, g_{ab} \tag{1.1}$$

the partition function changes as

$$\delta S_{\text{eff}}(g) = \frac{26-d}{48\,\pi} \int \left[\frac{1}{2}\,(\nabla\phi)^2 + \mu^2\,\phi^2 + R^{(2)}\,\phi\right]\,, \tag{1.2}$$

due to a Weyl anomaly [223]. The above formula involves the Liouville action. In $d \leq 1$ one could start with a matrix integral and integrating over ϕ will add a new dimension and yield a string theory (the review [167] is a helpful entry to the literature on this subject). In higher dimension, one doesn't know how to quantize the Liouville action. Still, this suggests that we should include one extra dimension.

That seems quite unusual and far–fetched for a hard–nosed physicist so is there any hint or guiding principle we can find on the four–dimensional gauge theory side ? We are intendedly sketchy in this section but one could bring back to memory the fact that in the BFKL analysis of Regge scattering the pair wave–function obeys a five–dimensional equation [189, 182, 22, 48]. Four of the variables correspond to the center of mass coordinates of the pair, whereas the fifth one corresponds to the separation of the gluons constituting the pair. That is a very rough suggestion but at least it comes from the confines of where string theory was first devised, Regge scattering.

We are not going to make much progress if we are to narrow our attention to something as messy and complicated as real–world QCD. So let us add a few more attributes and invoke supersymmetry. The motivation is that there are operators which are BPS protected in the SUSY algebra. They will allow for some tests of our candidate string dual once we manage to find one. Comparing known perturbative results to conjectured ones at strong coupling predicted by our string theory dual is a sharp test that, with hindsight, is passed with success by known gauge/gravity duals. Another closely–related reason for supersymmetry is that it relates the Hamiltonian of our theory to the sum of the supercharges squared. Consequently, a supersymmetric vacuum has to be a zero of the energy. This will prevent a phase transition of the gauge theory when we go from weak to strong coupling.

Furthermore, for the same reason, the spectrum is bounded from below, which removes one generic source of instabilities that field theories experience when they reach the strong coupling regime. For instance, particle–antiparticle pairs are most likely to pop out spontaneously. Their negative potential energy will exceed their kinetic and rest mass energy. This instability is well–known for QED where it is encapsulated as the Landau pole. Similarly, the Thirring model does not exist, even as an effective theory, beyond a critical value of the coupling [64].

Another desirable ingredient is conformal symmetry. This way, we will be able to say in one single shot a lot about a wide range of scales. From the principle that the more symmetries a theory possesses the simplest it is likely to be, we will make full use of those two ingredients and consider maximally supersymmetry superconformal $\mathcal{N}=4$ Yang–Mills even before attempting to find a full–fledged string theory dual to real–world QCD, which to date is still lacking anyways.

The most general metric describing a five–dimensional background where our candidate dual string theory lives and that preserves scale invariance along with four–dimensional Lorentz invariance takes the form

$$ds^2 = \frac{R^2}{z^2}\,\left(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + dz^2\right)\,. \tag{1.3}$$

This describes a Poincaré patch of five–dimensional anti de Sitter spacetime with radius of curvature R. This is the most symmetric space with constant negative curvature. Here we need to ensure that R is much bigger than the Planck scale L_P. Otherwise Einstein gravity and supergravity would break down. Indeed, we would like to use tools we are familiar with and press them into service to test the string dual, once we will have some more precise idea of what it turns out to be. Note that if we were to relax scaling symmetry, the space where the string evolves would be a warped background of the form

$$ds^2 = w(z)^2 \left(\eta_{\mu\nu}\, dx^\mu\, dx^\nu + dz^2\right) . \tag{1.4}$$

Let us pause for a moment and see if this idea of somehow expressing a four–dimensional gauge theory in terms of a gravitational theory makes sense after all. Does not this suggestion clashes against the Weinberg and Witten no–go theorem [257] ? Under some general assumptions, the "theorem" of Weinberg and Witten precludes any theory where gravity is emergent and the graviton appears as a bound state of two spin–1 gauge boson. More precisely, it states that if the theory has massless spin one or spin two particules, then these are gauge particles and so the currents they couple to are not observable. On the face of it, we better have to find a loop–hole in their argument or throw to the dustbin the previous suggestion of a supergravity dual to a close cousin of QCD (various theories of supergravity arising as the low–energy limits of corresponding string theories).

Fortunately, there is some hope for that proposal. After all, we are advocating that a five–dimensional graviton is some bound state of four–dimensional gauge bosons, not five–dimensional ones. Furthermore, this fits nicely with the holographic principle [251, 246, 46], which is believed to be one of the few guiding principles and cornerstones on the road to quantum gravity. The holographic principle simply amounts to considering a shell of matter. If its size exceeds the Schwarzschild radius, then it will collapse to a black hole. So, in a sense black holes put a cut–off on the amount of matter and entropy that can be confined to any volume of space. But the entropy of a black hole is proportional to its area. What this then suggests is that the largest possible entropy of a system is proportional to its area and that the fundamental degrees of freedom of this system somehow live on its boundary. Gravity provides a bound on information. See also [242] for an alternative explanation. Besides, a black hole is the archetype of a graviton bound state. The holographic principle thus suggests a way out of the no–go theorem of Witten and Weinberg [257] that at the same time goes well with the sort of string dual we have broadly sketched[2].

Taking stock of the aforementioned point, we can formulate one more obvious condition that our string theory dual should obey. That is, we need strong coupling in the relevant gauge theory. Otherwise a pair of gluons would just be that and would not stand a chance of being bestowed a more suitable description as a graviton. We have already imposed that R/L_P should be much bigger than unity. Now, recall 't Hooft's suggestion that there is a natural string theory dual to large–N gauge theories. This means a large number of gauge theory degrees of freedom. Since we believe that they should live on the boundary of a black hole we have another reason for making sure that $R/L_P \gg 1$. That will ensure that there is enough room for a large black hole to fit into five–dimensional AdS space. Actually, as we will see in a more quantitative sense below, $R/L_P \gg 1$ implies $N \gg 1$.

Returning to equation (1.3), we should spell out that the radial variable z runs from $z = 0$ to $z = \infty$. The latter value of z denotes the location of an horizon that has to do with the the fact the Poincaré wedge of AdS is geodesically incomplete. It will wash out if we consider the full AdS space. $z = 0$ is the boundary of AdS and can be reached in finite time by geodesics $z = \pm\Delta t$. This boundary is where the dual field theory should somehow live and this is in agreement with our discussion on the holographic principle.

This extra dimensions parameterized by z arising in the string theory dual living on AdS_5 has the same meaning as in our starting point, the perfunctory interpretation involving gluon

[2]In other words, one does not see the massless spin two pole of a dynamical graviton emerging in SYM. The reason is that the graviton is a massless particle in five–dimensional AdS, not in four dimensions. In AdS/CFT, there is no local stress tensor in five dimensions. There is a local stress tensor but it has to do with the four–dimensional gauge dual. See the work of Kiritsis and Nitti [163] who reach this conclusion but find possible exceptions when space–time ends on a brane or a singularity far from the UV boundary.

separation. Indeed, in the BFKL analysis we have $p_z \sim 1/z$. On the other hand, from the form of the metric of a Poincaré patch of AdS_5 (1.3) we see that the following relation between the energy E of an excitation seen by an invariant observer and its Killing energy p_0 holds:

$$p_0 = \frac{R}{z} E\,. \tag{1.5}$$

For lowest–level Kaluza–Klein excitations we expect $E \sim 1/R$. Hence, we see that a bulk excitation has higher energy when it moves closer to the boundary. This is the origin of z on the gauge theory side: $p_z \sim 1/z \sim p_0$. There is some confusion arising from the dual interpretation of the radial AdS coordinate. It stems from the fact that z is frequently interpreted as the size of a state of the CFT. This is actually suited for bulk excitations corresponding to string scale energies [247, 213]. For energies given by the KK excitations this is misleading.

This is all well but so far the proposed string dual looks somewhat like pie in the sky. Isn't there some possibility of making more quantitative contact with what we know in supergravity or string theory ? If we are to succeed, we should look for solutions of string theory or its low–energy limit that admit five–dimensional anti de Sitter vacua. There is one such solution of ten–dimensional IIB string theory, the so–called black 3–brane [136] with N units of Ramond–Ramond flux for the five–form self–dual field strength. In the extremal, zero–temperature limit, its metric is given by

$$ds^2 = H^{-1/2}(r)\, ds^2_{1+3} + H^{1/2}(r)\, \delta_{ij}\, dx^i\, dx^j\,, \qquad i,j = 4, ..., 9 \tag{1.6}$$

$$H(r) = 1 + \frac{R^4}{r^4}\,, \qquad R^4 = 4\,\pi g_s\, N\, \alpha'^2\,, \qquad r^2 = \delta_{ij}\, x^i\, x^j\,. \tag{1.7}$$

Our initial guess for a string theory dual to $3+1$–dimensional $\mathcal{N}=4$ SYM did suggest a fifth extra dimension. It seems that we will have to bear with five more. If we focus on the regime where $r \ll R$ we notice that the metric (1.6) describes the warped product of a five–dimensional AdS space with a five–sphere. Is there any particular reason one should expect such highly symmetric spaces from the point of view of the gauge dual ? We have scaling symmetry there, which combines with the Lorentz group to an enhanced $SO(2,4)$ conformal symmetry. This is exactly the isometry group of AdS_5. Furthermore, $\mathcal{N}=4$ SYM has an $SU(4)_R \cong SO(6)_R$ global symmetry. It is a symmetry that does not commute with the supercharges, an instance of a so–called R–symmetry. Under this group the six adjoint scalars and four adjoint Majorana–Weyl fermions of the $\mathcal{N}=4$ SYM field content transform in the antisymmetric product of two fundamentals and in the fundamental representation, respectively. Are we to expect the same R–symmetry on the supergravity side ? In a sense yes, $SO(6)$ obviously is the symmetry group of the five–sphere we tried to make sense of on the string theory side. But we should remember the "folks theorem" that no continuous symmetry group is allowed in supergravity theories and all known trusted and consistent glimpses of quantum gravity [23, 25]. So, it is expected that we will encounter supergravity vector fields associated to the $SO(6)$. This group was a flavor symmetry of the gauge theory. It has to be gauged in the supergravity.

While we are discussing symmetries, we could mention the well–known $SL(2,\mathcal{R})$ of $\mathcal{N}=4$ SYM. Type IIB string theory has exactly the same weak–strong duality group. There is also a $SU(N)$ gauge symmetry on the boundary and a ten–dimensional bulk diffeomorphism and local supersymmetry in the bulk. What are we to make of those ? Well, we should not expect to have to match gauge symmetries from side to side. Those symmetries are redundancies and a duality is only concerned with relating gauge–invariant observables.

Let us now see if the candidate supergravity dual (1.6) is consistent with the enunciated requirements concerning strong coupling and a large number of fields. Ten–dimensional Newton's constant is given by

$$16\,\pi G_N = (2\,\pi)^7\; g_s^2\, \alpha'^4\,, \tag{1.8}$$

where g_s denotes the string coupling and α' is related to the inverse string tension or to the string length $\alpha' = l_s^2$. We thus find that

$$R/l_s \gg 1\,, \qquad R/L_P \gg 1 \tag{1.9}$$

imposes

$$\lambda^{1/4} \gg 1\,, \qquad N^{1/4} \gg 1\,, \tag{1.10}$$

with $\lambda \equiv g_s N$. We know how to relate N, which is the number of units of RR five–form flux, to a quantity on the gauge theory side, namely the rank of the gauge group or the number of fields. On the gauge theory side, we have another parameter, namely the gauge theory coupling constant g_{YM}, to match to those available from the supergravity solution.

We can make progress if we notice that the black 3–brane solution sources the same RR flux as D–branes [216], N of them more precisely. What is then the relation between the supergravity black 3–brane solution and D–branes ? D–branes are defined by their boundary conditions and they are solitonic objects on which strings can end. Consider a stack of N coincident D3–branes. They are connected by strings whose endpoints live on their worldvolume. There are also closed strings around.

If we focus on the light degrees of freedom in this picture, the open string modes reduce to a massless four–dimensional $N = 4$ SYM multiplet, which decouples from a tower of massive string excitations. On the closed string side, a quick look at Newton's constant in ten dimensions (1.8) tells us that the gravitational coupling is dimensionful or, put another way, that ten–dimensional gravity is IR–free. So, the dimensionless quantity at energy E is $G_N\, E^8$. This implies that closed string modes are non–interacting in the low–energy limit we are considering. Henceforth they also decouple from the open string sector and in particular the SYM multiplet.

Incidentally, we can now relate g_{YM} to string theory parameters. Indeed, the DBI action for D3–branes can be expanded to a term proportional to

$$-\frac{1}{2\, g_s} \int d^4x\, F_{\mu\nu}^2\,, \tag{1.11}$$

which simply comes from the tension for a single D3–brane being $T_{D3} = \frac{1}{(2\,\pi)^3\, g_s\, \alpha'^2}$. With N D3–branes we are thus led to

$$g_{YM}^2 = g_s\, N\,. \tag{1.12}$$

Let us resume trying to relate D3–branes to the black 3–brane solution we started with. We have just seen that in the low–energy limit, the D3–brane system yields a four–dimensional multiplet of super Yang–Mills theory decoupled from gravity and massive stringy modes. What about the low–energy limit of the black 3–brane solution ?

The geometry of this solution consists in an asymptotically Minkowski space–time far away from a throat. The modes that survive the low–energy limit are, on one hand, a graviton multiplet in ten–dimensional Minkowski space and, on the other hand, a whole tower of string excitations in the $AdS_5 \times S^5$ throat geometry. The reason we should consider the whole tower of massive string modes pertains to the large gravitational well that separates the throat region from the asymptotic Minkowski one. As an aside, this also implies that both regions are decoupled from each other at low energies.

Comparing the two descriptions and their distinct low–energy limits, results in the celebrated Maldacena conjecture [191] that $N = 4$ SYM in $3 + 1$ dimensions is equivalent to IIB string theory on $AdS_5 \times S^5$. We have seen in the process of deriving this duality that the gravity, black 3–brane description is valid at large N and large 't Hooft coupling λ. Besides, since each extra world–sheet boundary on the brane gives a factor of g_s and one of N from the genus and Chan–Paton trace respectively, the D3–brane picture holds as long as $g_s\, N \ll 1$, i.e. at weak 't Hooft coupling of the gauge theory. The reason this gauge/gravity duality and its cousins in other dimensions (or with some amount of supersymmetry broken, extra matter multiplets, etc.) have proven of such importance is that they relate two different limits that describe the same underlying physical theory but in two opposite ranges of the coupling constant. This brings the hope of learning about field theories at strong coupling, where perturbative Feynman diagrams or resummation methods fails, just out of classical supergravity computations in a negatively–curved higher dimensional anti de Sitter space !

This conjecture has now been developed into a widely–tested duality. We have mentioned that the symmetries on both sides match. We have also alluded to supersymmetry–protected states. Many have been compared and agreement has always been found. The high symmetry of $\mathcal{N} = 4$ supersymmetry allows for the computation of quantities such as the anomalous cusp dimension (first considered on the string theory side by Gubser, Klebanov and Polyakov [106]) for any value of the coupling g_{YM} and amazingly there is no discrepancy at this stage [3, 36, 27, 83]. There are even some checks coming from lattice computations about the predictions of the duality for gauge theories at strong coupling (see for instance [61] and further work by those authors.).

There is by now a precise dictionary linking correlations functions on the CFT side to classical calculations in the bulk[3] or relating the mass of supergravity fields to operator dimensions of dual CFT states. An illustration for bulk fermions and the subtleties that arise as compared to the usual treatment for bosonic supergravity fields appears in Chapter 2. Many such verifications and entries of dictionary are well–known and implicitly but profusely used to in the later chapters of this thesis. As such, we will not review them here.

Linear response in AdS/QCD

Having now under our belt some intuition on gauge/gravity duality, as well as, more importantly, a definite, quantitative instance of such a duality that gives us in a sense a novel perturbation theory for $\mathcal{N} = 4$ SYM that applies at strong ‘t Hooft coupling in a weakly–coupled emergent spacetime, we are ready to return to our starting point, QCD and the problems standard approach for tackling it face beyond the perturbative regime of Feynman diagram expansions. It is all the more natural to confront the AdS/CFT correspondence to QCD data. It has achieved some success in this regard. The main focus of this section however amounts to extracting new insights into processes at strong coupling, taking for granted the validity of the correspondence that has still to fail any of the test cases it has been submitted thus far.

It is well–known that experiments at RHIC and at the LHC probe the properties of the quark gluon plasma. This is a thermal state of matter where colored states are deconfined. Yet, the coupling is still strong. The ’t Hooft coupling lambda is roughly of order 10. There are various indications about that. First of all, the QGP is well described by an almost ideal fluid. Its viscosity is probably the smallest ever reached. Understanding this new state of matter calls for new tools and ideas, such as the AdS/CFT correspondence.

On the other hand, as we have seen in the previous section, near–extremal D3–branes provide a gravitational representation of $\mathcal{N} = 4$ super–Yang–Mills theory at finite temperature, large N and large ’t Hooft coupling. To recap, at weak coupling, strings stretched between the branes behave as nearly free gluons and their superpatners from a 3+1–dimensional supersymmetric gauge theory, $\mathcal{N} = 4$ super Yang–Mills. At strong coupling the curved $\mathrm{AdS}_5 \times S^5$ geometry encodes the gauge dynamics of the theory. The metric describing the AdS–Schwarzschild black hole is obtained as a small modification of the geometry described by (1.6) upon the introduction of the usual blackening factors.

There are good reasons for being cautious about replacing QCD with $\mathcal{N} = 4$ SYM. The matter content and the dynamics are clearly different. Besides, one could mention, not the least, the fact that $\mathcal{N} = 4$ SYM is exactly conformal (an easy proof stems from going to the

[3]The classic review on all the subtleties concerning the counter–terms required for a proper variational principle, and the final, ready–to–use recipe on extracting correlators from bulk content is [234]. This regularization and renormalization method can be traced to [123, 124] and [71, 233].

Coulomb branch in the brane picture. See [235] for more details on the Coulomb phase of the $d = 4$, $\mathcal{N} = 4$ theory and its dual.). Still, by now various pieces of evidence have accumulated that seem to indicate that $\mathcal{N} = 4$ SYM at large 't Hooft coupling λ and at finite temperature may capture the dynamics of the quark gluon plasma. At least in a range of temperatures high enough that confinement and the chiral condensate have disappeared, yet low enough that the 't Hooft coupling is still large.

A computation of the shear viscosity to entropy density ratio of black D3–branes reproduces the low shear viscosity that events with elliptic flow exhibit. This ground–breaking calculation [178, 179] led to a multitude of other computations that support the view that the gauge/string duality might indeed help us better understand heavy ion physics. After all, that was the whole point of the convoluted chain of arguments from the previous sections that we made in order to guess a string dual. They were grounded in QCD, Regge scattering and the BFKL analysis did stand out. See [59] for an extensive and recent overview of the literature.

In particular, one of the distinctive features of RHIC and LHC data is jet–quenching, namely, strong energy loss when a high–energy parton goes through the QGP. Within the AdS/CFT approach a heavy, external quark is represented as a string hanging from the boundary of AdS. Well, actually from a D7–brane whose asymptotic bulk radial position is related to the mass of the heavy quark. If one considers a trajectory with constant velocity, one expects the string to trail down behind the quark. This string corresponds on the boundary to the cloud of color flux, gluons and their super–partners, evolving around the quark.

From the properties of a classical string solution in AdS one can then expect some understanding of the dynamics of charm and bottom quarks propagating through the QGP. The third and fourth chapters of this thesis deal with how to derive from a bulk calculation a Langevin description for the dynamics of a relativistic heavy quark moving through a strongly–coupled plasma. The suitability of a Langevin description in this case was even challenged by Gubser who made the observation that the corresponding expressions for the drag force and momentum broadening do not appear to obey the Einstein relations. This relation is a hallmark of thermal equilibrium and must be satisfied by a Langevin equation describing thermalization. The motivation that underpinned my work on this topic is that since a Langevin description is more universal it might be that the noise terms are of a non–thermal nature. Another motivation was to gain a better understanding of the role of black holes which live on the world–sheet of strings describing heavy quarks.

Overall, most of the studies in AdS/CFT have so far focused on the mean field dynamics responsible for dissipation. On the other hand, the statistical properties of the plasma have been less investigated. Yet, when thermal noise is neglected in AdS/CFT it leads to seemingly incorrect results : for instance, it predicts zero drag on mesons. Usually, it is advocated that this is fine though, since anyways the effect of thermal noise is suppressed by either large N or large λ.

Our interest in this subject was initiated by the work on holographic brownian motion by de Boer, Hubeny, Rangamani and Shigemori [70]. So, let us first sketch their results on the brownian motion experienced by a quark that is on average at rest in a strongly–coupled plasma.

They were really interested in understanding why a heavy quark undergoes Brownian motion from the bulk perspective and why its motion is described by a Langevin equation. Indeed, while it is true that in some sense all the phenomenological data was already available — the rates of energy loss and the momentum broadening coefficients have been obtained in previous work — it is not clear what their origin is. By this it is meant how the random force acting on a quark arises from the bulk. As one might suspect, they have shown that the answer is as expected: the random force arises from Hawking radiation in the bulk.

They find interesting results for the following scales associated with Brownian motion. First of all, the relaxation time, which is the crossover time between the ballistic and the diffusive regimes goes like

$$t_{relax} \sim \frac{m}{\sqrt{\lambda}T^2}\,. \tag{1.13}$$

The collision time, which is the width of the random force correlator, goes like

$$t_{coll} \sim \frac{1}{T}\,. \tag{1.14}$$

This scale can be thought of as being associated with the duration of a single scattering process. The third scale is the mean free path time ; it is usually thought of as the characteristic time elapsed between two collisions. In kinetic theory, there is typically a hierarchy where $t_{coll} \ll t_{mfp} \ll t_{relax}$. But not anymore at strong coupling ! In point of fact, de Boer et al. find that

$$t_{mfp} \sim \frac{1}{\sqrt{\lambda}T}\,. \tag{1.15}$$

This implies that a Brownian particle interacts with many plasma particle at the same time. In a recent paper they also relate the mean free path time to four–point correlators [16].

In addition, from holography de Boer et al. derive a Langevin equation for a heavy quark globally at rest in the thermal medium. The generic features of such a Langevin description consist in a drag term η_D along with stochastic force contribution, in addition to a possible forcing term. The Einstein relation emerges from requiring that a heavy quark propagating according to the Langevin equations should eventually equilibrate to a thermal distribution with temperature T. This is nothing more than an instance of the fluctuation–dissipation theorem.

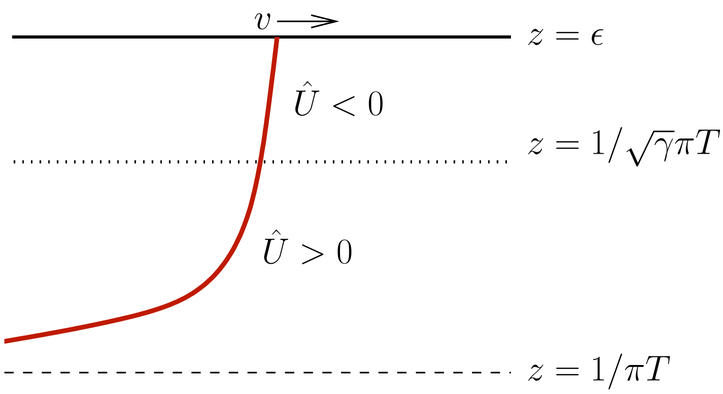


Figure 1.1: The trailing string solution and the location of its worldsheet horizon at $z_s = 1/\sqrt{\gamma}\,\pi\,T$. $\hat{U}$ denotes a Kruskal coordinate for the worldsheet.

The string configuration, for momentum pointing in the x^1 direction is given by the so–called "trailing string", depicted on the Figure 1.1. The string bends and lags behing the quark endpoint, which ends on a D7–brane whose bulk radial coordinate fixes the bare mass of the heavy quark $m_q \sim 1/z_{D7}$. The quark moves with constant average velocity v. For this to be possible it must be subjected to some external force that will compensate for the energy loss towards the plasma. The work over tension indeed leads to energy loss. When one integrates the differential equation governing this solution, it turns out that in the first integral, both numerators and denominators are positive for small radial coordinate z and negative for z near the black hole horizon. The turning point signals a new scale

$$z_s = 1/\sqrt{\gamma}\,\pi\,T\,. \tag{1.16}$$

In my work [92, 93], I have investigated the momentum broadening of a relativistic quark. This is related to fluctuations around the classical, steady trailing string solution. We therefore have to expand the Nambu–Goto action in static gauge to quadratic order in the fluctuations. The equations of motion for the longitudinal and transverse fluctuations are such that they have regular singular points. In particular, the special role played by z_s (1.16) as the location of a world–sheet horizon becomes manifest. For $z = z_s$ the value of the first derivative of a fluctuation is determined from the equations of motion. This indicates that the fluctuations of the string at $z \leq z_s$ are causally disconnected from those below the location of the world–sheet horizon.

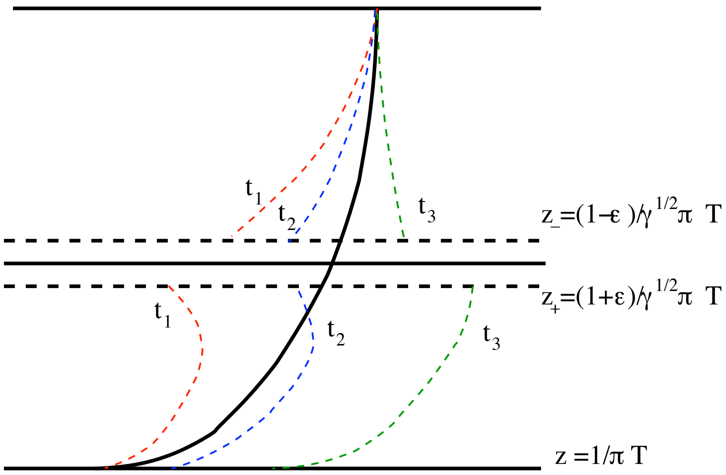


Figure 1.2: The stochastic ensemble of trailing strings.

On Figure 1.2. we see the picture of what we have substantiated with computations. The random force that appears in the Langevin description for a heavy quark is associated to an ensemble of strings which fluctuate around the average trailing string solution. Each of the strings from this ensemble has the same z–dependence as the trailing string profile but with a higher characteristic temperature $\sqrt{\gamma}T$. Notice that this is given by the inverse of the position of the world–sheet horizon (1.16). This stochastic ensemble of strings generates the random force on the boundary.

Actually, this picture contains more information than we were concerned with at the outset of our paper with Edmond Iancu and Al Mueller [92]. In fact, the parts of the string below and above the world–sheet horizon are separated by an extra random variable [58]. In [92] we just integrate out everything below the stretched membrane. However, Casalderrey–Solana and collaborators later found that since the fluctuations below the world–sheet horizon are causally connected to the boundary they are reflected by an extra random variable. The end result for the 2–point function of stochastic force is unchanged from our original result [92].

To gain more quantitative ground one should note that the retarded and advanced solutions, Ψ_{adv} and Ψ_{ret}, can be found for this problem. The full–status analysis found in Chapter 3 of this thesis reveals that the fluctuations are log–divergent close to the world–sheet horizon. Upon Fourier–transforming one can also see that $\Psi_{adv} = e^{-i\omega t}\psi_{adv}$ is an outgoing wave : with increasing time the phase remains constant while departing from the horizon. To come back to the above–mentioned picture, integrating out the modes which grow close to the world–sheet horizon will lead to stochasticity. To have an idea of how thick the stretched membrane can be, one must consider the full Nambu–Goto action for fluctuations. There is a term that grows faster than linealy close to the horizon. One must then impose that the strip thickness ϵ conforms to

$$\frac{\epsilon}{\log^2\left(\sqrt{\gamma}\,\epsilon\right)} \gg \frac{1}{\sqrt{\gamma}\,\lambda}\,, \tag{1.17}$$

where λ stands for 't Hooft coupling in general accordance of our notations.

Now, one might think that to find the Langevin equation and its coefficients one could just use the usual well–known formula relating the stochasticity to the Bose–Einstein distribution and the imaginary part of the retarded Green function. With the latter yielding the drag force η_D. Actually one must be much more careful. The relation I have just hinted between stochasticity and diffusion is named the KMS relation. It can be derived using the formalism of real–time Green's functions. In AdS/CFT this means that one must consider the trailing string solution on the Kruskal diagram of the black hole (cf. Figure 2.1.), not just in the R–region of the original coordinates.

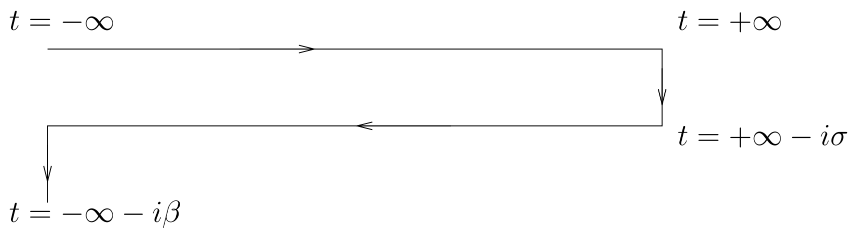
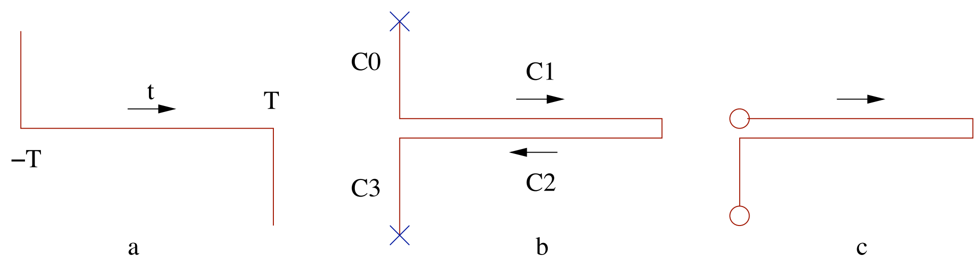


Figure 1.3: A Schwinger–Keldysh contour for real–time Green's functions.

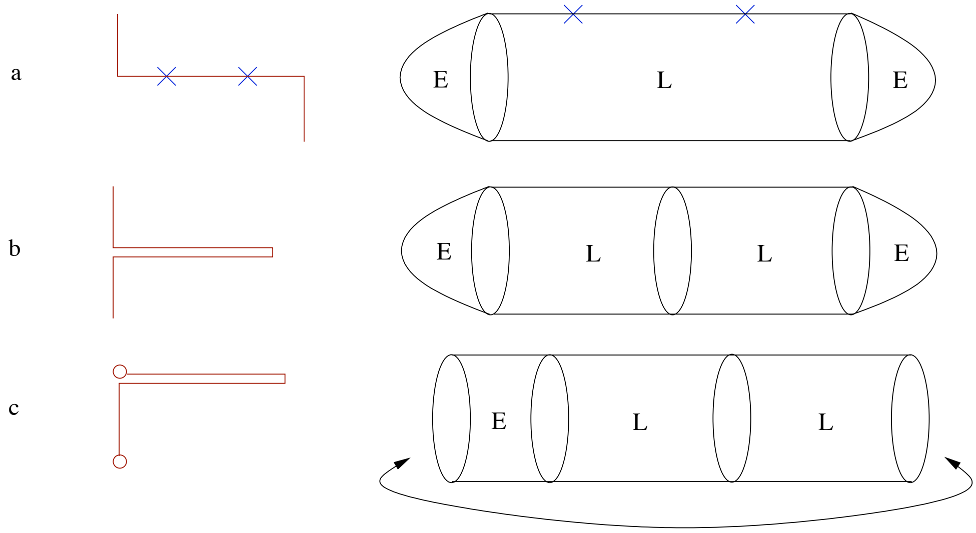
I will explain that in a moment but let us first review the link between Kruskal diagrams and Schwinger–Keldysh contours for real–time Green's functions. As a short aside, it is worth pointing to recent progress that circumvents previous involved calculations with analytic continuations or computations that *assumed* in some form thermal equilibrium. The recent work of Caron–Huot et al. [53] indeed formulates the problem of finding the coefficients of a Langevin description as a initial values problem. This is of most interest for seeing the approach to thermal equilibrium in truly out–of–equilibrium settings.

Real–time thermal field theory requires a doubling of fields. I will explain why this is so from the bulk perspective. But first one should recall the origin of the usual relations among the four Green's functions labelled by two indices indicating the position on the real–time contour in this doubled field formalism. One of those relations can be derived by inserting complete sets of states. The second and third rely on having a fully equilibrated system. Yet the heavy quark is not equilibrated, at least not at the naive temperature of the plasma T. This relation will be modified by changing the temperature here to be related to the equilibrium temperature that would be measured by a thermometer on board of a black body moving with velocity v through the plasma. For now, as advertised, we probably have to explain more the real–time formalism of correlation functions.



Consider a field configuration with initial condition $\phi_{\pm}(x^i)$ at $t = \pm T$. If we are interested in vacuum amplitudes we should multiply the path integral with fields constrained to satisfy the previous conditions by the vacuum wavefunction and next integrate over ϕ_+ and ϕ_-. This is equivalent to extending the fields in the path integral to live along a contour in complex time plane. Indeed the infinite vertical segment at $-T$ corresponds to a transition amplitude $lim_{\beta\to\infty}\langle\phi_-, -T \mid e^{-\beta H} \mid \Psi\rangle$ for some state Ψ.

Put another way, a Euclidean path integral creates an initial or final state into the Lorentzian path integral. Here, we focused on the vacuum as the initial state but more generally one can similarly use a Euclidean path integral to generate other initial/final states for the Lorentzian path integral. The beautiful prescription of Skenderis and van Rees [236, 237, 255] (from which I have borrowed the pictures immediately above and below) is to fill in the QFT contours with bulk solutions. As emphasized, this means either Euclidean or Riemannian geometries. There are matching conditions that control the behavior of the supergravity fields at the corners, the junction hypersurfaces.



These conditions can be shown to reduce to the prescription of Herzog and Son [130] under more restrictive assumptions. This latter prescription is related to the old Unruh instruction of extending solutions to the full Kruskal plane. Basically, the black hole horizon specifies the connection between solutions in the right and in the left quadrant by requiring that positive infalling modes should be positive energy ; negative energy modes should be outgoing. As already stressed, this recipe leads to the correct thermal correlators. In summary, the thermal distributions are generated via analytic continuation across the Kruskal plane. Chapter 2 of this thesis contains work I have done on extending the prescription of Herzog and Son [130] to correlators of fermionic operators. There are some delicate points involved compared to the more tradition situation for bosonic fields. My motivation for this work revolved around the current interest in applying AdS/CFT to long–standing issues in condensed matter physics.

To come back to the main issue, we are trying to find the correct correlators from which we will derive a Langevin equation with drag and stochastic coefficients. It will turn out that the correlators are thermal but not with the plasma temperature. For this purpose, the string fluctuations are then decomposed in an outgoing/infalling basis. We have one condition in each quadrant, the boundary value of the fluctuation. The other conditions are provided by the analytic continuations, both at the world–sheet and the non world–sheet horizons. Interestingly, the multiplicative factors associated with crossing the black hole horizon compensate each other. Those associated with crossing the world–sheet horizons rather enhance each other. This yields the end result ; the details can be found in the relevant chapter of this thesis.

From the Langevin equation we have derived for a fast–moving quark, the Einstein relation for longitudinal fluctuations

$$\kappa_\ell = 2\, E\, T_{plasma}\, \eta_D \tag{1.18}$$

seems to be violated by the Lorentz contraction factor squared. This led us to advocate the a "democratic" parton branching picture for energy loss and momentum broadening at strong coupling. We claim that the effective temperature is a signature that the world–sheet horizon generates quantum mechanical fluctuations associated to medium–induced radiation ; that is, instead of being associated with thermal stochasticity.

This supports previous results relating to medium–induced radiative energy loss and momentum broadening [117][4]. Indeed, Mueller et al. [78] notice in particular that the expressions for energy loss and broadening take on the same form the same at weak and at strong coupling when one expresses these equations in terms of the saturation momentum. Yet the underlying processes are quite different. Even though it is true that at both weak and strong coupling energy loss is due to medium–induced gluon emission, there is a significant difference. At weak coupling, the radiated gluon, which typically comes from a highly virtual gluon in the quark wave–function, is set free via thermal rescattering (this is the dominant mechanism). On the other hand, according to the democratic parton–branching, at strong coupling the mechanism for medium–induced radiation is different: radiation is somehow caused by a plasma force scaling like T^2 and only those quanta can be lost to the plasma, whose virtuality is lower than some number Q_s (the saturation momentum) of order of the coherence time t_{coh} for the emission of a virtual parton times T^2 ; $Q_s \sim t_{coh}\, T^2$.

Duals to confining theories

One might worry that $\mathcal{N} = 4$ SYM is not a very suitable approximation to QCD and that one should instead try to find more appropriate instance of a gauge/gravity duality that would befit QCD more. Still, one should emphasize that given the success of the primeval AdS/CFT duality those concerns should be somewhat alleviated. On the other hand, a lot of progress has been witnessed since Maldacena first formulated his conjecture and part of it has been driven by the successful search for duals to field theories without the full set of bells and whistles of maximally supersymmetric, conformal theories.

Actually, the finite–temperature extension of the AdS/CFT duality sketched in the previous section is already pushing the correspondence beyond its realm. Putting it at finite temperature breaks scale–invariance of the theory. Furthermore, it is easily seen that we lose supersymmetry at non–vanishing temperature. Consider the extremal brane metric (1.6) and take the Euclidean time to be periodic. We must then impose periodic boundary conditions on bosonic fields and anti–periodic ones on fermions. Hence, there is no fermion zero–mode at all and supersymmetry is broken. This is not quite the sort of metric we have been considering in the previous section when we were frequently referring to a string trailing in a black hole background. Actually, as shown by Hawking and Page [120], the high temperature phase of the theory that we are presently dealing with is dominated by an AdS_5 black hole.

One fact we know about non–supersymmetric, non–abelian theories is that they exhibit confinement. The geometric origin of this is seen in the bulk as the space being bounded by the black hole horizon. It was mentioned in the previous section that the chromo–electric flux lines of the gauge theory are dual to strings dangling in AdS. One can compute the quark anti–quark potential at strong coupling along this suggestion. Their separation is proportional to the maximal value of the z radial coordinate the string reaches in the bulk. The first calculations that did illustrate this idea were done for the vacuum of $\mathcal{N} = 4$ SYM. They indicate an attractive Coulomb potential for the pair [192, 226]. As an aside, this is also interesting to notice, in that it clearly shows that this dual string description concerns non–confining theories, whereas the initial motivation for a string dual mostly had to do with the string–like properties of confining

[4] The picture of Hatta, Iancu and Mueller [117] concerning parton–branching at strong coupling inferred from the UV/IR duality of AdS/CFT (see the nice reviews [141, 142, 143]) might have to be modified at vanishing temperature, although in view of accumulating other similar evidence for a democratic splitting of energy and momentum at strong coupling [21], it is quite likely that it still holds at finite temperature. As first observed in [14, 15] and by Hubeny and Hatta et al. in [139, 118, 140, 119], no off–shell virtual gluon is emitted at all in the vacuum of the theory at zero temperature. Instead the radiation profile seems to be fixed by the fact that radiation propagates at the speed of light and by the trajectory of the probe ; as such it must be as in classical electrodynamics

flux tubes in QCD. On the other hand, at finite temperature, we lose SUSY and, geometrically due to the space closing at some finite bulk radius, confinement can be seen. This is as required for agreement with the gauge theory side.

This is fine but first of all, we would like to see confinement without having to heat up the theory. An idea would be to break some amount of supersymmetry out of turning some bosonic or fermionic mass terms in the $\mathcal{N} = 4$ theory. This however is plagued by the generic fact that at strong 't Hooft coupling λ, the intrinsic scale of the pure non–abelian theory, Λ_{SYM}, is of the same order as the supersymmetry scale M:

$$\Lambda_{SYM} \simeq M\,. \tag{1.19}$$

This relation is simply a reflection of asymptotic freedom at weak coupling. But this is annoying in view of the fact that we would like the low–energy regime below the scale that corresponds to the particular way one breaks some amount of supersymmetry of the initial $\mathcal{N} = 4$ SYM to decouple from the extra fields of this theory. Yet, (1.19) guarantees that the contribution of those undesirable fields won't be suppressed. Besides, such a course of action frequently results in high curvature singularities on the supergravity side and one must switch to a description involving explicit NS5 and D5 branes (or some combination thereof) with dissolved D3's [218].

An alternative way of reducing the number of supersymmetries in AdS/CFT is to take a stack of N D3–branes of the type we have formerly considered in our sketchy derivation of the correspondence and put them instead on the tip of a six–dimensional Ricci–flat cone whose base is a five–dimensional Einstein space X. As expected, the near–horizon limit will this time yield an $AdS_5 \times X$ IIB background with N units of RR five–form flux.

The simplest examples of such a construction arise out of cones which are orbifolds of the flat six–dimensional space transverse to the D3–branes [155], i.e. of the type

$$\mathbb{R}^6/\Gamma\,, \tag{1.20}$$

where Γ is a subgroup of the isometry group of the six–sphere. Depending on the values of the generators of the orbifold group, supersymmetry can either be completely broken, or some amount of it can be preserved.

The dual gauge theory is obtained by keeping only the fields that are invariant under the orbifold action combined with a conjugation by a $U(N\,k)$ matrix acting on the gauge indices. Here k is the rank of the orbifold action. For an orbifold group that breaks all SUSY, it seems that naively the gauge theory is still conformal. Inspection of the planar beta functions for single–trace operators indeed reveals that they vanish [185, 41, 42]. Nevertheless, double–trace operators built out of single–trace ones are induced. And they cannot be ignored in the large N limit. The one–loop planar β–functions have been computed by Dymarsky, Klebanov and Roiban [79, 80] from the Coleman–Weinberg formula for the one–loop potential of the 1PI action. Their analysis leads to the conclusion that such non–supersymmetric abelian orbifold constructions contain unstable couplings.

For all that, it is not excluded that the theory could flow to a real fixed point. In fact, the analysis of Dymarsky et al. suggests a one–to–one mapping between tachyons in the twisted sector of the IIB background and a lack of real zeros for the beta functions on the gauge theory side, whenever the abelian orbifold has fixed points. For a freely acting orbifold, the background contains no tachyon at sufficiently large λ, or, alternatively, large radius. Still, we are not out of the wood. As shown by Horowitz et al. [138], a hole develops and consumes space via tunneling. The same process occurs for orbifolds with fixed points [1], but via a tachyon rather than a Kaluza–Klein "bubble of nothing".

In view of this, we should better preserve some amount of supersymmetry. There are other theories derived from branes on conical singularities on which we have good control, provided as we said they preserve some residual supersymmetry. Of particular interest for the purpose of this thesis, is the situation where the $SU(N)^k$ gauge group of the quiver theory associated to a

$\Gamma = \mathbb{Z}_k$ orbifold is replaced by $SU(N_1) \times ... \times SU(N_k)$. This can be achieved by wrapping some D5–branes on cycles within the singularity. It has been shown that the $\mathcal{N} = 1$ orbifold theory flows to the regular Klebanov–Strassler theory corresponding to branes at the tip of a non–compact Calabi–Yau whose base space is of topology $S^2 \times S^3$. The M D5–branes wrapping the S^2 at the tip of the cone break conformal invariance. As a result, the dual gauge theory displays a quasi–periodic renormalization group flow associated to a cascade of Seiberg dualities [229], corresponding to a series of interchanges of the two gauge groups from $SU(N + M) \times SU(N)$ to $SU(\hat{N}) \times SU(\hat{N} + M)$, with $\hat{N} \equiv N - M$.

Chapter 5 work starts with an extensive review of the literature relevant to the above–mentioned solution that is by now widely known as the Klebanov–Strassler (KS) theory [171]. In Chapter 5, I investigate a different mechanism of singularity resolution via a black hole horizon for a close cousin of the KS theory. Excellent reviews entirely devoted to what has now become a whole industry are [127, 129].

The Klebanov–Strassler theory is a fully regular supergravity solution[5] ; the M units of RR three–form flux created by the fractional D3–branes blow up the three–cycle of the transverse geometry and resolve the singular conifold via a geometric transition to the so–called deformed conifold

$$\sum_{i=1}^{4} z_i^2 = \epsilon^2 \,, \tag{1.21}$$

defined in a $\mathbb{C}^4$ embedding.

In addition, the warp factor can be seen to be non–vanishing at the tip of the throat. On that account, we have achieved a confining theory at zero temperature, with confinement scale proportional to $\epsilon^{2/3}$. Besides, recall that in the basic approach described at the beginning of this section that gives confinement by heating $\mathcal{N} = 4$ SYM, there was no asymptotic freedom in the UV and therefore no dimensional transmutation. For the Klebanov–Strassler theory, the geometric transition that gives a non–vanishing ϵ corresponds to a quantum deformation of the moduli space of vacua of the dual gauge theory [230]. The theory flows in the IR to an $SU(2\,M) \times SU(M)$ gauge theory and there are two kinds of baryonic operators built out of the two pairs of chiral superfields, A_1 and A_2, B_1 and B_2 respectively, transforming generally in the $(N, \overline{N + M})$, respectively, $(\overline{N}, N + M)$ representation of the KS field theory with quartic superpotential. Those baryonic operators are of the form

$$\overline{\mathcal{B}} = A_1^M\, A_2^M \,, \qquad \mathcal{B} = B_1^M\, B_2^M \,. \tag{1.22}$$

Classically, one would have $\mathcal{B}\,\overline{\mathcal{B}} = 0$. The analysis of Seiberg [230] instructs us that quantum mechanically a non–vanishing constant arises on the r.h.s. of the above identity and that it corresponds to the scale whose origin is dimensional transmutation. We can see this on the supergravity side [107]. Actually a family of supergravity solutions dual to the full baryonic branch of the KS theory (which itself is at the point $\mid \mathcal{A} \mid = \mid \mathcal{B} \mid$) has been derived in [51].

Finding this moduli space of resolved warped deformed conifolds made use of the Papadopoulos–Tseytlin ansatz [211]. My work [31] is concerned with proving that this is a consistent truncation for IIB solutions on the conifold $T^{(1,1)}$. The analysis of [31] gives the most general such reduction preserving the global $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry of $T^{(1,1)}$ and should prove helpful in determining the spectrum of linear deformations around the baryonic branch of BPS supergravity solutions. I have also carried work with Nick Halmagyi on a similar analysis for the four–fold cousin to $T^{(1,1)}$ and the corresponding analogue to the Klebanov–Strassler solution, a background with warped transverse Stenzel space. We find it surprising that there is the only regular BPS solution, while we were expecting a whole family similar to the class of solutions dual to the baryonic branch of KS.

[5] without the need of introducing extra explicit branes to cure naked singularities, as occurs in Polchinski–Strassler or the enhancon mechanism

To close this overview, it is worth pointing that all the versions of the AdS/CFT correspondence all seem to fall more or less in the same universality class. For sure, one can take a more phenomenological standpoint and derive interesting physics but it is not a fully satisfactory situation, in view of the instabilities that arise even in a situation of which we yet have good understanding, say orbifolds theories.

Indeed, in all the examples of holography that we really understand, first of all, the field content is always limited to adjoint or bi–fundamental matter representations. This is encoded in quiver diagrams and their geometric origin is that the dualities over which we have good command stem from considering D3–branes at certain singularities in the ten–dimensional critical string theory. As we have seen, supersymmetry can be broken but in all the examples that we really understand it is actually there in some disguise. Extra six–dimensional scalars are more or less there. Of course, it is possible to really go to theories that have fewer degrees of freedom, by considering deformations such that in the IR we get genuinely fewer degrees of freedom, for instance by giving a mass. But as previously mentioned quite often one either runs into some kind of a high curvature singularity on the supergravity side or the extra fields rear their heads.

A more quantitative way of formulating this issue of being in the same universality class as $\mathcal{N} = 4$ SYM hinges on considering the two anomaly structures given by some combination of curvature invariants in four dimensions. The analysis of [123] and the product space structure of spacetime apprise us of the fact that $a = c$ at leading order for $\mathcal{N} = 4$ super Yang–Mills. And all theories under which there is some analytic control share this property. We thus face some no–go theorem proclaiming that any theory that will ever be realized holographically from product geometries will have this property. If that is to be the case this is a serious limitation in getting hold of a dual to QCD. Of course, an assumption is that the supergravity description is based on the Einstein–Hilbert action. Then if higher–curvature correction are negligible $a = c$ at large N. And we certainly do not want an answer that depends on some modulus tuning higher–curvature contributions.

It might therefore be of interest to try and look for theories with genuinely fewer degrees of freedom and fewer amount of supersymmetry. There is some reason to think they should be associated with non–critical strings. $\mathcal{N} = 4$ corresponds to ten–dimensional critical string theory, as we have seen and there is a suggestion due to Polyakov that there should exist an AdS_5 solution to the non–critical type 0 string theory dual to a non–supersymmetric CFT [224]. It is a natural speculation to try and fill the slots in between, as has been attempted for example in [181, 173]. However, the curvature of the AdS space solutions of those non–critical string theories is of the order of the string scale. Those authors are well aware that higher–curvature corrections cannot be neglected and might drastically change their solutions.

So, it seems that most of the ways of ridding supergravity duals from of supersymmetry are prone to problems of their own. It would be of value to try and understand better higher–order stringy corrections if that is achievable with the current status of the field, given that there have been successful attempts at getting closer to real–world QCD modeling. The work of Kiritsis and collaborators is worth pointing in this respect [113]. They have investigated a similar problem to the one that constitutes Chapters 3 and 4 of this thesis, the Langevin description of a heavy quark moving in some strongly–coupled plasma [112]. Their model is of the phenomenological type, an Einstein–dilaton background. Yet, it apparently arises from a non–critical string theory [165].

Having motivated the need for exploring supergravity solutions beyond the original $AdS_5 \times S^5$ and its maximally supersymmetric dual in order to learn more about QCD, having explained why one must be careful in keeping some supersymmetry, let us now see how on can try and use confining theories with $\mathcal{N} = 1$ supersymmetry to investigate models of metastable supersymmetry breaking and string cosmology.

Assessing candidate supergravity duals to metastable vacua

Metastable supersymmetry breaking [85] owes much of the revival in interest it has enjoyed since the work of Intriligator Seiberg and Shih [145] (ISS) to an ability to naturally circumvent some of the problems afflicting other mechanisms for dynamic supersymmetry breaking (DSB) [6].

While the latter often require very complicated ingredients, the fact that many simpler–looking theories exhibit metastable breaking of supersymmetry suggests it is generic. Furthermore, as a rule (bent to have some exceptions), models of metastable DSB only have an approximate R–symmetry [147], whereas spontaneous breaking of supersymmetry is linked to an exact R–symmetry [206]. Thus, models of metastable DSB are relieved of the problems associated with a lack of Majorana gaugino masses (unbroken R–symmetry) or the occurrence of a light R–axion (if R–symmetry is spontaneously broken).

Given such promises and more, some of which can be found in reviews of the field [146, 166], it wasn't long before their realizations as string theory configurations were enquired. Much insight has been gained into non–perturbative effects in field theory from string theory with some amount of supersymmetry and it seems natural to extend this approach to supersymmetry–breaking situations.

Relevant constructions involve NS5–branes in IIA and D–branes suspended or inserted between them [259, 98]. One typically starts with a configuration of two NS5–branes with a stack of N_c D4–branes on which a $\mathrm{SU}(N_c)$ gauge group lives. There are also N_f hypermultiplets represented by D6–branes, with values of the masses $m_1, ..., m_{N_f}$ corresponding to the respective projection of D6–branes' coordinates onto one of the NS5–branes. The meta–stable vacuum is obtained by going to the magnetic description of this electric configuration. This is achieved by switching one of NS5 past the other [84]. With masses turned on, N_c out of the N_f D4–branes created through the Hanany–Witten effect [114] have to be tilted to connect to one NS5. This breaks supersymmetry.

Such brane realizations are of interest as instances of non–supersymmetric models over which some control is available [210, 99]. They provide an intuitive geometric understanding of various characteristics of the ISS model, such as its global symmetries and pseudo–moduli. They can also be regarded as toy models for the string landscape or reachable regions thereof [209, 183]. A loose motivation for the work on metastability in three–dimensional field theories contained in the latter part of this thesis relates with this latter point. As recently discussed in [54], there might be transitions between vacua of any dimension among the string landscape. Three–dimensional QFT's at strong coupling are also of relevance to gauge/gravity duality applied to problems in condensed matter physics, which justify further analysis of gravity duals to metastable vacua in $1+2$–dimensions. A recent model of holographic cosmology [197, 198, 199] is worth pointing, although the connection to our approach is not obvious, given that McFadden and Skenderis use three–dimensional field theories to model the strongly–coupled phase of the early universe, whereas our analysis of metastability via holography is in the opposite regime of coupling.

Although the low–energy physics matches that of the SQCD metastable vacuum, the magnetic phase of such string theory realizations actually describes a vacuum that is not part of the four–dimensional theory encompassing the original vacuum from the electric configuration [29]. This is ascribed in MQCD [258, 135, 47] to some brane bending effect for $g_s \neq 0$ which induces the asymptotics of the brane configurations in the two phases to differ by an infinite amount. The tiny differences in the fields sourced in the infrared by the D4–branes in the supersymmetric and non–supersymmetric situations then grow into a logarithmic divergence in the UV. It is perhaps not so surprising that MQCD fails to reproduce non–supersymmetric aspects of SQCD. There is indeed no guarantee that the success of MQCD in tallying with many features of supersymmetric field theory, which can be traced back to holomorphy, will continue to hold

[6] A review of dynamical supersymmetry breaking and some of the realistic models evading its indirect criteria can be found in [231]

once supersymmetry is broken.

Whether this issue arises in the IIB supergravity duals to $\mathcal{N} = 1$ four–dimensional gauge theories, such as the Klebanov–Strassler background [171], has been investigated in [30]. In [73, 72], DeWolfe et al. work out the UV asymptotics of a perturbative supergravity solution [7] claimed to describe a prominent string theory model for metastable supersymmetry–breaking vacua put forth by Kachru, Pearson and Verlinde [156]. The latter consists in adding p anti–D3 branes to the Klebanov–Strassler background [171]. They fall toward the tip of the throat, where they are screened by flux and puff up via the Myers effect [204] to a NS5 brane wrapping a two–cycle inside a three-sphere. The NS5 slowly unwraps the cycle. This holographically describes the slow decay of a SUSY–breaking metastable vacuum. More examples and extensions of such constructions with which we have some familiarity can be found in [12, 13].

Chapters 6 contains work aiming at extending the analysis of [30] for a supergravity description of metastable SUSY–breaking and considers the backreaction of supersymmetry–breaking deformation of the supersymmetric warped M–theory background with transverse Stenzel space [244] first introduced by Cvetič, Gibbons, Lü and Pope in [66]. A probe analysis of a supersymmetry–breaking configuration obtained by introducing p anti–M2 branes at the tip of the throat has recently been considered by Klebanov and Pufu [175]. We should note at this point that the CGLP solution with transverse Stenzel geometry are to the solution warped M–theory solution with transverse Stiefel space [62] what the IIB Klebanov–Strassler solution [171] and the deformed conifold [52] are to the Klebanov–Tseytlin solution [170] of fractional D3–branes on the singular conifold. The Stenzel space is a higher–dimensional generalization of the deformed conifold.

In particular, one important motivation was to investigate the occurrence of the finite–action singularity encountered in [30] and to further discuss if it is to be deemed physical or not. If physical, we have found the first backreacted supergravity solution dual to metastable susy–breaking since the work of Maldacena and Nastase [193]. If unphysical, it does not necessarily mean this solution should be discarded, though it would have significant consequences for the KKLT [157] construction of deSitter vacua in string theory ; it might be resolved from the point of view of string theory or of a localized supergravity solution.

The general course of action goes as follows. Borokhov and Gubser [45] have introduced a general framework for first–order perturbations around a supersymmetric family of solutions governed by a superpotential. This simplifies the problem of solving second–order equations for the fields of supersymmetry–breaking supergravity solutions to solving a set of coupled first–order ordinary differential equations. This amounts to dealing with a closed set of o.d.e's for the supersymmetry–breaking modes $\tilde{\xi}_i$. The first–order deformations $\tilde{\phi}_i$ of the BPS solution ϕ_i^0 obey $\tilde{\xi}_i$–dependent, first–order coupled differential equations.

One thus first has to lay down an Ansatz for a family of solutions to which the CGLP [66] background belongs, reduce the Lagrangian to a one–dimensional sigma model and find the corresponding superpotential.

Before going through involved computations, it is interesting and of much importance later when imposing boundary conditions to compute the force experienced by a probe M2–brane in the supersymmetry–breaking background. In accordance with [158], it falls off as $1/r^7$. It takes on a form similar to the force on a probe D3–brane found in [30], for it only depends on a single susy–breaking mode, $\tilde{\xi}_4$ in the case under discussion.

Unlike the work concerning anti–D3 branes, here we realize that the full set of $\tilde{\xi}_i$ equations can be solved analytically. Key to this is the observation that some combination of zeroth–order fields appearing in the homogeneous equation for $\tilde{\xi}_4$, which at first doesn't look very promising, actually bows and tidies to the derivative of the warp factor in the CGLP solution. The same simplification occurs in the case of interest for D3–branes and turns out to make the numerics plainer [35].

[7] The IR asymptotics appear in [201]

Once we have the asymptotics of the $\tilde{\phi}_i$ modes, we must impose physical boundary conditions to select the particular solution we are looking for and see if it makes sense. This was the main objective of this paper, namely building the metastable supersymmetry–breaking supergravity solution corresponding to the backreaction of anti–M2 branes in the CGLP background. We find that we unique candidate solution is marred by an infrared singularity.

This singularity stems from a quartic divergence at the origin in the energy density of the four–form flux of eleven–dimensional supergravity. A similar issue arises for backreacted anti–D3 branes in the Klebanov–Strassler background, where it could not be decided if the finite–action singularity should be kept as physically acceptable or not. My analysis of backreacted anti–M2 branes provides more evidence to settle its status.

Those results could have far–reaching implications for many models of inflation in string theory where a small number of anti–branes are typically required to uplift an AdS minimum to a metastable de Sitter ground state. As a first step, my collaborators and I have derived the complete form of the contribution of supersymmetry–breaking anti–branes to the inflaton potential in certain string theory models of inflation [33]. This appears in Chapter 7 of this thesis.

Chapter 2

Fermionic Schwinger–Keldysh Propagators from AdS/CFT

The Herzog and Son prescription [130] for computing real–time Green functions for finite temperature gauge theories from their gravity dual is generalized to fermions. These notes explain how such an extension involves properties of spinors in a curved, complexified space–time.

2.1 Introduction

The gauge–string duality relates some classes of field theories to dual string theories in specified background space–times [74]. While string theory in a curved background does not generally lends itself to tractable calculations and even to our understanding, its low–energy supergravity limit is far more compliant. In the case of asymptotically AdS spaces, the dual field theories are in the large 't Hooft coupling limit. The AdS/CFT correspondence thus provides a framework for understanding strongly–coupled gauge theories. Recent work has been devoted to computing transport coefficients [55, 109, 56, 241, 92, 70] and gaining insight into dynamic and nonequilibrium settings [63, 93] from the correspondence. Most of them required a real–time formulation of finite temperature field theory. The way real–time correlators can be derived in AdS/CFT is hinted by the following analogy. There is a doubling of the degrees of freedom in the Schwinger–Keldysh real–time prescription (reviewed in Section 2.2 of this chapter). On the other hand, the Penrose diagrams of asymptotically AdS spacetimes with a black hole[1] exhibit two boundaries (cf. Figure 2.1. below), on which the dual gauge theory fields live. This conjecture was proved by Herzog and Son [130]. They showed how the 2×2 matrix of two–point correlation functions for a scalar field and its doubler partner field is reproduced from the AdS dual supergravity action. Their work also made it clear that the thermal nature of black hole physics gives rise to the thermal nature of its dual field theory. In these notes, we would like to extend their work and find out how black hole physics gives rise to real–time correlators of fermionic operators in a dual finite–temperature field theory. While deriving real–time propagators of vector field operators from AdS/CFT is an obvious extension of the scalar field case expounded in [130], the case of fermions proves less straightforward. The analysis presented below relies on a treatment of spinor fields in curved space–times and on their transformation laws under global symmetry transformations. We begin by reviewing in the next section the Schwinger–Keldysh formalism for real–time finite temperature field theory. This section collects results usually dispersed among the literature, as this formalism is usually exposed for a scalar field. We present the 2×2 matrix of two–point correlation functions for a fermionic operator and its link with the retarded and advanced propagators. Section 2.3 reviews how the retarded Green function for a fermionic operator at strong coupling can be computed from the dual supergravity spinor

[1] whose temperature is that of the dual gauge theory, according to the AdS/CFT correspondence

classical action in AdS/CFT. Section 2.4 is devoted to a short review of spinors in curved and complexified space–times. These are important for the analysis of Section 2.5 where we review the relationship between positive– and negative–energy modes of a wave equation and analyticity conditions in the complex Kruskal planes of the underlying background. These conditions lead to the real–time propagators for fermionic operators from the dual boundary action in the gauge–gravity duality.

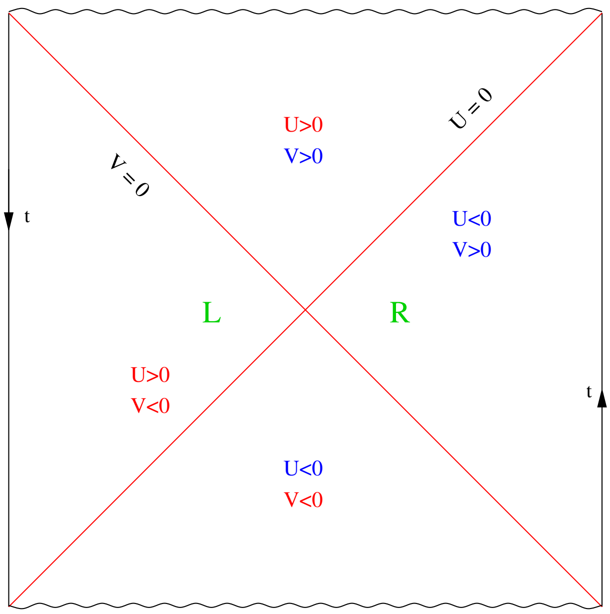


Figure 2.1: Kruskal diagram for the AdS–Schwarzschild black hole

Recently, there has been a sustained interest in fermions from theories with gravity duals. In [10] the two–point function for a fermionic operator in a non–relativistic conformal field theory is computed. The gravity dual corresponds to fermions propagating in a background with the Schrödinger isometry. The authors of [65] argue that the gauge–gravity correspondence proves a useful tool for exploring fermionic quantum phase transitions. The retarded fermion Green function is found from an analysis of the solutions to the Dirac equation and its quasi–normal modes in an AdS Reissner–Nordström black hole. Real–time correlators for non–relativistic holography have been considered recently in [187], where the construction of [236, 237] is involved. For an explanation of how their construction generalizes the one due to [130] used here to the case of distinct sources on the R and L boundaries of a Penrose diagram, see [255]. It would be interesting to apply the approach contained in the present notes to more general geometries, possibly duals to non–relativistic conformal field theories [240, 132, 195, 2]. This paper offers to explain how Schwinger–Keldysh n–point functions can be computed from string theory, thus emphasizing the latter as a relevant approach to tackle some models or phases of condensed matter physics.

2.2 Review of Schwinger–Keldysh formalism for fermions

The Schwinger–Keldysh prescription allows for a study of real–time Green functions by introducing a contour $\mathcal{C}$ in the complex time plane [186, 184], as illustrated on Figure 2.2. Fields live on this time contour. The forward and return contour of the path are labelled by indices i_1 and i_2 respectively. The idea is that the quantum dynamics does the doubling of the degrees of freedom required for describing non–equilibrium states. The starting point I at time t_i and the ending point B at $t_i - i\beta$ are identified and fermionic fields are such that $\Upsilon_I = -\Upsilon_B$. In the remainder of this paper, the conventions for the propagators are those of [159].

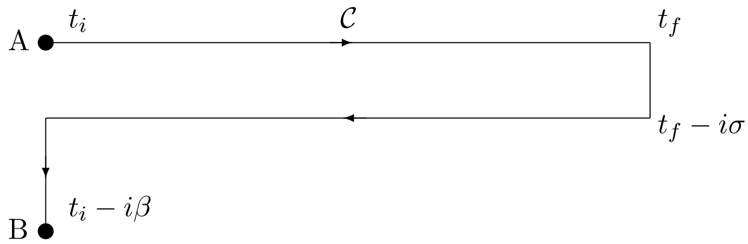


Figure 2.2: The Schwinger–Keldysh contour

The action splits into contributions from the four parts of the contour:

$$
\begin{aligned}
S &= \int_{\mathcal{C}} dt_{\mathcal{C}} L(t_{\mathcal{C}}), \\
&= \int_{t_i}^{t_f} dt L(t) - i \int_0^{\sigma} d\tau L(t_f - i\tau) - \int_{t_i}^{t_f} dt L(t - i\sigma) - i \int_{\sigma}^{\beta} d\tau L(t_i - i\tau),
\end{aligned} \tag{2.1}
$$

where

$$
L(t) = \int d^{d-1}\vec{x} \mathcal{L}\left[\Upsilon(t, \mathbf{x}), \bar{\Upsilon}(t, \mathbf{x})\right]. \tag{2.2}
$$

The generating functional is defined as

$$
Z = \int \mathcal{D}\Upsilon \mathcal{D}\bar{\Upsilon} \exp\left(iS + i\int_{t_i}^{t_f} d^d x \bar{\eta}_1 \Upsilon_1 + i\int_{t_i}^{t_f} d^d x \bar{\Upsilon}_1 \eta_1 - i\int_{t_i}^{t_f} d^d x \bar{\eta}_2 \Upsilon_2 - i\int_{t_i}^{t_f} d^d x \bar{\Upsilon}_2 \eta_2\right), \tag{2.3}
$$

where the sources $\eta_{1,2}$ and the fields are such that

$$
\begin{cases}
\eta_1(t, \mathbf{x}) = \eta(t, \mathbf{x}), & \Upsilon_1(t, \mathbf{x}) = \Upsilon(t, \mathbf{x}), \\
\eta_2(t, \mathbf{x}) = \eta(t - i\sigma, \mathbf{x}), & \Upsilon_2(t, \mathbf{x}) = \Upsilon(t - i\sigma, \mathbf{x}).
\end{cases} \tag{2.4}
$$

The same relations hold for their conjugates. The contour–ordered Green functions are mapped into a matrix whose components are indexed by the position on the contour:

$$
iG(j, k) = \frac{1}{i^2} \frac{\delta^2 \ln Z\left[\eta_{1,2}, \bar{\eta}_{1,2}\right]}{\delta\eta_j \delta\eta_k^\dagger} = i \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \tag{2.5}
$$

The time in the components of this matrix of Green function is standard time and in the operator formalism

$$
\begin{cases}
G_{11}(t, \mathbf{x}) = -i\langle T\Upsilon(t, \mathbf{x})\Upsilon^\dagger(0)\rangle, & G_{12}(t, \mathbf{x}) = +i\langle \Upsilon^\dagger(0)\Upsilon(t, \mathbf{x})\rangle, \\
G_{21}(t, \mathbf{x}) = -i\langle \Upsilon(t, \mathbf{x})\Upsilon^\dagger(0)\rangle, & G_{22}(t, \mathbf{x}) = -i\langle \hat{T}\Upsilon(t, \mathbf{x})\Upsilon^\dagger(0)\rangle.
\end{cases} \tag{2.6}
$$

Note the sign reversal in G_{12} as compared to the case where the fields are bosonic. T and $\hat{T}$ denote the time–ordering and anti–time–ordering operators. The fields are taken in the Heisenberg picture. Those Schwinger–Keldysh correlators are related to the retarded and advanced Green functions through

$$
\begin{cases}
G_R(x - y) = -i\theta(x^0 - y^0)\langle\left\{\Upsilon(x), \Upsilon^\dagger(y)\right\}\rangle, \\
G_A(x - y) = +i\theta(y^0 - x^0)\langle\left\{\Upsilon(x), \Upsilon^\dagger(y)\right\}\rangle.
\end{cases} \tag{2.7}
$$

The relation

$$G_R(x-y) = G_A^*(y-x) \tag{2.8}$$

is valid, irrespective of the fields obeying a Bose–Einstein or Fermi–Dirac statistics. Using (2.6), (2.7) and the completeness relation for a complete set of state results in

$$\begin{cases} G_{11}(k) = -\mathrm{Re}G_R(k) + i\tanh(\frac{k^0}{2T})\mathrm{Im}G_R(k), \\ G_{12}(k) = -\frac{2ie^{\sigma k^0}}{1+e^{\beta k^0}}\mathrm{Im}G_R(k), \\ G_{21}(k) = \frac{2ie^{(\beta-\sigma)k^0}}{1+e^{\beta k^0}}\mathrm{Im}G_R(k), \\ G_{22}(k) = \mathrm{Re}G_R(k) + i\tanh(\frac{k^0}{2T})\mathrm{Im}G_R(k). \end{cases} \tag{2.9}$$

When $\sigma = \frac{\beta}{2}$ — a value which will naturally appear in the following — $G_{12}(k) = -G_{21}(k)$. $\sigma = 0$ yields $G_{21}(k) = G^>(k)$ and $G_{12}(k) = G^<(k)$. Since $G_{21}(k) - G_{12}(k)\mid_{\sigma=0} = 2i\mathrm{Im}G_R(k)$, the relationship

$$G_R(k) - G_A(k) = G^>(k) - G^<(k) \tag{2.10}$$

holds as required whatever the quantum statistics of the fields under consideration. The main purpose of these notes is to show how the above relations (2.9) are derived in AdS/CFT. Herzog and Son [130] obtain analogous relations for a scalar field. As it turns out, extending their result to correlators of fermionic operators is not entirely straightforward.

2.3 Review of fermionic retarded correlators in AdS/CFT

This part reviews fermions in AdS/CFT [122, 121]. The prescription for computing retarded fermionic Green functions draws on the approach of Iqbal and Liu [149, 148], where conjugate momenta for supergravity fields are defined with respect to a r–foliation. This is suggestive of some sort of stochastic quantization interpretation of the AdS/CFT correspondence.
Consider a boundary fermionic operator $\mathcal{O}$ whose gravity dual is a spinor field Ψ. The bulk background space–time metric

$$ds^2 = g_{rr}dr^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{2.11}$$

is subjected to the asymptotically–AdS conditions

$$g_{tt}, g_{ii} \sim r^2, \quad g_{rr} \sim 1/r^2, \quad r \to \infty. \tag{2.12}$$

The d–dimensional boundary is a $r \to \infty$. The AdS/CFT prescription for computing n–point functions of a quantum field theory from a classical supergravity action goes as

$$\left\langle \exp\left[\int d^dx\left(\bar{\chi}_0\mathcal{O} + \bar{\mathcal{O}}\chi_0\right)\right]\right\rangle_{QFT} = e^{iS_{SUGRA}[\chi_0,\bar{\chi}_0]}, \tag{2.13}$$

where $\chi_0 = \lim_{r\to\infty} r^{d-\Delta}\Psi$ and Δ is the scaling dimension of $\mathcal{O}$, related to the mass m of the bulk spinor. Imposing such a Dirichlet boundary condition on spinors requires care, especially given that this condition must relate a spinor in the bulk in $d+1$ dimensions to one in d space-time dimensions on the boundary. Given that our focus is on Green functions, the quadratic part of the action for ψ meets our purpose. It is given by

$$S = i\int_M d^{d+1}x\sqrt{-g}\left(\bar{\Psi}\Gamma^M D_M\Psi - m\bar{\Psi}\Psi\right) + S_{\partial M}, \tag{2.14}$$

with $\bar{\Psi} = \Psi^\dagger\Gamma^t$. The covariant derivative is specified by the spin connection ω_{abM}:

$$D_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab}, \tag{2.15}$$

where $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$. Upper–case letters stand for abstract space–time indices, while lower–case ones denote tangent frame indices. The two are linked through a choice of vielbein e^a_M, defined by $G_{MN} = e^a_M e^b_N \eta_{ab}$, with η_{ab} a $d+1$–dimensional Minkowski metric of signature $(-,+,+,...,+)$ such that

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab}. \tag{2.16}$$

The inverse vielbein satisfy $\eta_{ab} = G_{MN} e^M_a e^N_b$. The spin connection components are given by $\omega_{abc} = e^M_a \omega_{bcM}$ and $\omega_{abc} = \frac{1}{2}(\mathfrak{C}_{bca} + \mathfrak{C}_{acb} - \mathfrak{C}_{abc})$, where $[e_a, e_b] = \nabla_{e_a} e_b - \nabla_{e_b} e_a = \mathfrak{C}^c_{ab} e_c$. Alternatively ω_{abM} can be viewed as the components of 1–forms ω^M in Cartan's structure equations [202]. Whenever a specific index appears as a label on a Gamma matrix, it refers to that particular index in the tangent frame. For d even, a convenient choice for the bulk Gamma matrices is

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^r = \gamma^{d+1}, \tag{2.17}$$

γ^μ being the boundary gamma matrices and γ^{d+1} being proportional to their product. When d is odd, it is appropriate to choose

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.18}$$

They satisfy the Clifford algebra. So, for general d

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \Gamma_\pm \Psi, \quad \Gamma_\pm = \frac{1}{2}\left(1 \pm \Gamma^r\right), \tag{2.19}$$

with $\Psi_\pm$ being opposite chirality Weyl spinors when d is even, and d–dimensional Dirac spinors for d odd. Whatever the value of d the number of components of the fermionic operator $\mathcal{O}$ is always half that of its dual Ψ. Quite generally [220], Appendix B, D–dimensional gamma matrices are constructed by noticing that for D even, increasing D by 2 doubles the size of the Dirac matrices. They can therefore be constructed iteratively, starting in $D = 2$ with

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.20}$$

to the following in $D = 2k + 2$

$$\Gamma^\mu = \gamma^\mu \bigotimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^{D-2} = I \bigotimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^{D-1} = I \bigotimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.21}$$

Here, γ^μ, $\mu = 0, ..., D-3$ are $2^k \times 2^k$ gamma matrices and I is the $2^k \times 2^k$ identity matrix. When D is odd, simply add $\Gamma^D = \Gamma$ or $-\Gamma$ to the set of $D-1$ gamma matrices. Note that from our conventions for the metric and anti–commutation relations the 0–component of gamma matrices is anti–hermitian while other matrices are hermitian. In order to solve the equations of motion near the boundary and find the scaling dimension Δ of the fermionic operator $\mathcal{O}$, one refers to the usual Frobenius procedure of trying for solutions of the type $r^{-\rho}\sum_{n=0}^{\infty}\Psi_n(t, x^i)/r^n$. Ψ_n are boundary spinors. Consider for instance the case of pure AdS, $ds^2 = r^2(-dt^2 + d\mathbf{x}^2) + \frac{dr^2}{r^2}$. Setting $e^\mu = r dx^\mu$, $e^r = \frac{dr}{r}$, the non–vanishing spin coefficients

$$\omega_{tr} = -r dt, \quad \omega_{ir} = r dx^i. \tag{2.22}$$

The Dirac equation

$$[DD - m]\,\Psi = 0, \tag{2.23}$$

with $DD = \Gamma^M D_M$, becomes

$$r\Gamma^r \partial_r \Psi + \frac{i}{r}\Gamma\,.\,k\Psi + \frac{d}{2}\Gamma^r \Psi - m\Psi = 0, \tag{2.24}$$

where $\Gamma \,.\, k = \gamma^\mu k_\mu$. Then ρ must be set to $\rho = \frac{d}{2} \pm m$ and Ψ_0 is annihilated by $\frac{1}{2}(1 \pm \Gamma^r)$, respectively. Incidentally, the scaling dimension is therefore found to be $\Delta = \frac{d}{2} + m$. The leading asymptotic behaviour of $\Psi_\pm$ is then

$$\Psi_+(k,r) = \chi_0(k) r^{m-\frac{d}{2}} + \lambda_0(k) r^{-\frac{d}{2}-m-1}, \Psi_-(k,r) = \psi_0(k) r^{-\frac{d}{2}-m} + \mu_0(k) r^{m-\frac{d}{2}-1} \tag{2.25}$$

Note that the dominant term (when $m \geq 0$) has been denoted χ_0 on purpose, as a reference to the source for the dual operator $\mathcal{O}$ in (2.13):

$$\lim_{r\to\infty} r^{d-\Delta}\Psi_+ = \chi_0. \tag{2.26}$$

Inserting this back into their equation of motion yields

$$\begin{cases} \psi_0(k) = -\frac{i\gamma\,.\,k}{k^2}(1+2m)\lambda_0(k); \\ \mu_0(k) = -\frac{i\gamma\,.\,k}{1-2m}\chi_0(k). \end{cases} \tag{2.27}$$

If one then demands that the solution be regular in the whole of AdS space, it turns out that χ_0 and ψ_0 are not independent. Note that this is not apparent from the analysis presented above where χ_0 and ψ_0 are the boundary values of the fields Ψ_+ and Ψ_-. A general solution to the Dirac equation near the boundary is a superposition of those fields. However the Dirac equation can be solved exactly in a few cases, including pure AdS. In this latter case, suitable solutions are given by

$$\Psi_+ = \begin{cases} r^{-\frac{d+1}{2}} K_{m+\frac{1}{2}}\left(\frac{\sqrt{\mathbf{k}^2-\omega^2}}{r}\right)\kappa_+, & k^2 > 0, \\ r^{-\frac{d+1}{2}} H^{(1)}_{m+\frac{1}{2}}\left(\frac{\sqrt{\omega^2-\mathbf{k}^2}}{r}\right)\kappa_+, & \omega > \sqrt{\mathbf{k}^2}, \\ r^{-\frac{d+1}{2}} H^{2}_{m+\frac{1}{2}}\left(\frac{\sqrt{\omega^2-\mathbf{k}^2}}{r}\right)\kappa_+, & \omega < -\sqrt{\mathbf{k}^2}, \end{cases} \tag{2.28}$$

κ_+ denoting a constant spinor. Regularity as $r \to 0$ then imposes that

$$\psi_0(k) = -\frac{i\gamma\,.\,k}{k^2}\frac{(\frac{k}{2})^{2m}\Gamma(\frac{1}{2}-m)}{\Gamma(\frac{1}{2}+m)}\chi_0(k). \tag{2.29}$$

More generally ψ_0 and χ_0 are related by a matrix $\mathcal{S}$:

$$\psi_0(k) = \mathcal{S}(k)\chi_0(k). \tag{2.30}$$

Regularity in the bulk and a given boundary condition χ_0 for Ψ_+ when $m \geq 0$ then uniquely determine a solution Ψ to the classical equations of motion. Similar relations apply for $\bar{\Psi}_+$ and $\bar{\Psi}_-$. Note that relations such as (2.29) or (2.30) are on–shell relations. Other off–shell histories contributing to the variational principle can be constructed as superpositions of independent Ψ_+ and Ψ_-. We now turn to a discussion of the variational principle. The boundary term in (2.14) will be determined from stationarity of the action. That one cannot fix all the components of Ψ and $\bar{\Psi}$ but must rather set conditions on, say, χ_0, $\bar{\chi}_0$, and leave ψ_0, $\bar{\psi}_0$ free to vary[2], stems from the Dirac equation being first order in derivatives. Varying with respect to Ψ, $\bar{\Psi}$ the Euclidean action

$$S = -\int_M d^{d+1}x\sqrt{g}\bar{\Psi}\left(\frac{1}{2}\left(\overrightarrow{\not{D}} - \overleftarrow{\not{D}}\right) - m\right)\Psi, \tag{2.31}$$

one finds a surface term, i.e. $\delta S = C_{\partial M} +$ bulk term, where the bulk term involves radial derivatives and is proportional to the equations of motions, while

$$\begin{aligned} C_{\partial M} =& \frac{1}{2}\int_{\partial M} d^d x\,(\delta\bar{\psi}_0\chi_0 + \bar{\chi}_0\delta\psi_0)(x), \\ =& \delta\left\{\frac{1}{2}\int_{\partial M} d^d x\,(\bar{\psi}_0\chi_0 + \bar{\chi}_0\psi_0)(x)\right\}, \end{aligned} \tag{2.32}$$

[2] On–shell they become functions of the boundary conditions χ_0, $\bar{\chi}_0$.

given that $\delta\chi_0 = 0$ and $\delta\bar{\chi}_0 = 0$, as explained above. Since $C_{\partial M}$ does not vanish, that the action must be stationary requires that one adds to it a boundary term $S_{\partial M}$ such that $C_{\partial M} = -\delta S_{\partial M}$. Actually, since they are fixed, one may add any function of χ_0 and $\bar{\chi}_0$ to the action without breaking the stationarity condition. However conditions such as locality, absence of derivatives and invariance under the asymptotically AdS symmetries seem to uniquely select a boundary term [121]. Hence, after another Wick–rotation to go back to a Lorentzian signature,

$$S_{\partial M} = -i \int_{\partial M} d^d x \sqrt{g g^{rr}} \bar{\Psi}_+ \Psi_-. \tag{2.33}$$

The factor of g^{rr} entering the square root comes from the vielbein. We now review the prescription derived by Iqbal and Son [149, 148] for computing retarded propagators. It amounts to taking Euclidean canonical momenta conjugate to $\Psi_\pm$ with respect to a r–foliation:

$$\Pi_+ = -\sqrt{g g^{rr}} \bar{\Psi}_-, \quad \Pi_- = -\sqrt{g g^{rr}} \bar{\Psi}_+. \tag{2.34}$$

Then (2.25) and (2.26) result in

$$\begin{aligned} \langle \mathcal{O} \rangle_{\chi_0} &= -\lim_{r\to\infty} r^{\Delta-d} \Pi_+, \\ &= \bar{\psi}_0. \end{aligned} \tag{2.35}$$

From (2.30), i.e. $\psi_0 = \mathcal{S}\chi_0$, and analytic continuation, one obtains retarded correlators in Lorentzian signature:

$$G_R(k) = i\mathcal{S}(k)\gamma^t. \tag{2.36}$$

The γ^t gamma matrix arises since $G_R \sim \langle \mathcal{O}\mathcal{O}^\dagger \rangle$, rather than $\langle \mathcal{O}\bar{\mathcal{O}} \rangle$.

2.4 Spinors in complexified space–time

In order to generalize the work of Herzog and Son [130] to fermions, it is necessary to consider spinors in curved space–time. Besides, a potential difficulty arises given that [130] relies crucially on analycity of complexified Kruskal coordinates. [215], section 6.9, has a few pages devoted to spinors in complex space–times. [214, 207] provide a complementary treatment of spinors and twistors. A spin space $\aleph$ of complex dimension two comes with each point of the underlying space–time manifold. The members of such a space are negative–chirality, dotted, say, Weyl spinors. Undotted spinors are members of the complex conjugate space $\bar{\aleph}$. The manifold being complexified, $\aleph$ and $\bar{\aleph}$ must be viewed as independent spaces, so that pairs of spinors ξ^α and $\bar{\xi}^{\dot{\alpha}}$ which previously determined one another under complex conjugation, are replaced by a pair of independent such spinors. A complexified space–time originates from a real underlying space–time by allowing its coordinates to take on complex values and by extending the metric coefficients analytically to the complex domain. Note that this is distinct from a complex space–time where generally no subspace can be singled out as real. Defining a spinor basis or dyad $\{\varsigma^\alpha, \imath^\alpha\}$ for $\aleph$ and $\{\bar{\varsigma}^{\dot{\alpha}}, \bar{\imath}^{\dot{\alpha}}\}$ for $\bar{\aleph}$, each comes bestowed with its own indices–lowering $\epsilon_{\alpha\beta}$ spinor or $\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}$ spinor, such that $\epsilon(\varsigma, \imath) = \gamma$ and $\bar{\epsilon}(\bar{\varsigma}, \bar{\imath}) = \bar{\gamma}$. When $\gamma = 1$, the dyad is called a spin–frame. Associated with any spin frame is a null tetrad

$$l = \varsigma\bar{\varsigma}, \quad m = \varsigma\bar{\imath}, \quad n = \imath\bar{\imath}, \quad \bar{m} = \imath\bar{\varsigma}, \tag{2.37}$$

which spans the tensor product space $\aleph \bigotimes \bar{\aleph}$. This illustrates the standard connection between world–tensor indices a as a pair of spinor indices, one dotted and one undotted. From ϵ and $\bar{\epsilon}$, a symmetric metric on T is built such that

$$g(l, n) = 1, \quad g(m, \bar{m}) = -1, \tag{2.38}$$

with all other scalar products vanishing, i.e.

$$g_{ab} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{2.39}$$

$\aleph \bigotimes \bar{\aleph}$ endowed with this scalar product has the structure of a tangent space to a complexified manifold. In the Newman–Penrose formalism a basis for the tangent space is a null tetrad consisting of two real vectors and one complex–conjugate pair of vectors. Consider the complexified manifold obtained from a metric of type (2.11) describing a geometry with a horizon. Let U and V label its Kruskal coordinates. In the context of an asymptotically AdS black hole geometry they are introduced below around (2.45). Staying general for now, a basis for the tangent space is given by four null vectors

$$\frac{\partial}{\partial U}, \quad \frac{\partial}{\partial V}, \quad \frac{\partial}{\partial \zeta} = e^{i\phi} \cot\frac{\theta}{2}, \quad \frac{\partial}{\partial \bar{\zeta}} = e^{-i\phi} \cot\frac{\theta}{2}, \tag{2.40}$$

with $\frac{\partial}{\partial \bar{\zeta}}$ parameterizing the anti–celestial sphere. It is related to $\frac{\partial}{\partial \zeta}$ by an antipodal map. Let us map l, n, m and $\bar{m}$ to $\frac{\partial}{\partial U}$, $\frac{\partial}{\partial V}$, $\frac{\partial}{\partial \zeta}$ and $\frac{\partial}{\partial \bar{\zeta}}$, respectively. The spinors ς and $\imath$ are then associated with the null vectors $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$, respectively.
The combinations $\sqrt{V}\imath$ and $\sqrt{-U}\varsigma$ or $\sqrt{-V}\imath$ and $\sqrt{U}\varsigma$ are parallely transported across the full U and V complex planes. Indeed, parallel transport of $U\frac{\partial}{\partial U}$ along $\frac{\partial}{\partial V}$ stems from

$$\begin{aligned} \nabla_{\frac{\partial}{\partial V}} \left[U\frac{\partial}{\partial U}\right] &= \frac{\partial}{\partial V}\left[U\right]\frac{\partial}{\partial U} + U\nabla_{\frac{\partial}{\partial V}}\frac{\partial}{\partial U} \\ &= U\left(\Gamma^{U}_{UV}\frac{\partial}{\partial U} + \Gamma^{V}_{UV}\frac{\partial}{\partial V}\right), \end{aligned} \tag{2.41}$$

where $\Gamma^{U}_{UV} = \frac{1}{2}g^{UV}\left(\partial_U g_{VV} + \partial_V g_{UV} - \partial_V g_{UV}\right) = 0$ and $\Gamma^{V}_{UV} = 0$. As for the covariant derivative with respect to $U\frac{\partial}{\partial U}$

$$\nabla_{U\frac{\partial}{\partial U}} \left[U\frac{\partial}{\partial U}\right] = \left(\frac{\partial}{\partial U}\left[U\right] + U\,\Gamma^{U}_{UU}\right)\, U\frac{\partial}{\partial U}, \tag{2.42}$$

it is directed along $U\frac{\partial}{\partial U}$. This corresponds to the weaker definition of a curve $\mathcal{C}(s)$ with tangent vector T being a geodesic if $\nabla_T T = \alpha T$, α an arbitrary function on the curve. This definition agrees with the notion of $\mathcal{C}(s)$ being among the straightest curves in a Riemannian manifold if the change of T is parallel to T. The modification with respect to the more familiar criterion $\nabla_T T = 0$ for geodesics and parallel transport lies in the length of T being generally not conserved under $\nabla_T T = \alpha T$. However, in the case at hand, the two conditions are directly equivalent since $U\frac{\partial}{\partial U}$ is null. In any case it is always possible to parametrize the curve so that the geodesic condition takes on its customary form, provided $\frac{d^2 s'}{ds^2} = \alpha\frac{ds'}{ds}$ under $s \to s'$. With such a parameter change, $U\frac{\partial}{\partial U}$ is parallel transported along two families of curves spanning the whole Kruskal plane and associated respectively with the vector field $\frac{\partial}{\partial V}$ and the one obtained from $U\frac{\partial}{\partial U}$. The same holds for $V\frac{\partial}{\partial V}$. The attendant statement on the related spinor basis follows.
This choice of spinor basis will appear naturally in Section 2.5. $\sqrt{-U}$ and the likes are pivotal in generating Fermi–Dirac distribution functions. We should also note that even though the AdS setting presented below involves spinors in a bulk geometry of dimension five, the present discussion on spinors in a complexified four–dimensional space–time is of relevance due to the decomposition (2.19) of a general bulk spinor evolving in $d+1$ dimensions into d–dimensional spinors. The latter are used in the following as in, e.g., [122, 121, 149], and while in five dimensions an extra fifth basis vector should appear in (2.40), it is irrelevant for the present purpose. $\Psi_\pm$ spinors will be expanded in the basis constructed out of Kruskal coordinates and $\imath, \varsigma$.

2.5 Real–time correlators from gravity

While the results about to be derived below should be applicable to a broader class of finite–temperature field theories with a gravity dual, we focus on theories arising from non–extremal D3 branes. Boundary values of supergravity fields in the resulting AdS_5 background provide the $N \to \infty$, $g_{YM}^2 N \to \infty$ limit of $\mathcal{N} = 4$ SU(N) supersymmetric Yang–Mills correlators. In the near–horizon limit the metric on a stack of non–extremal D3 branes reads[3]

$$ds^2 = (\pi T R r)^2 \left(-f(r)dt^2 + d\mathbf{x}^2\right) + R^2 \frac{dr^2}{r^2 f(r)}, \tag{2.43}$$

where $f(r) = 1 - \frac{1}{r^4}$. The boundary is a $r \to \infty$ and the horizon at $r = 1$. Here, T is the Hawking temperature of this AdS–Schwarzschild black hole and R is the radius of AdS. The analysis of [130] relies on the behaviour of a scalar field in this background. Near the horizon, $\frac{r}{\pi T} = 1 + \epsilon$, solutions to the wave equation behave as $e^{ik^0 r^*}$ and its conjugate, with $k^0 = \omega$ and r^* being the tortoise coordinate:

$$\frac{dr^*}{dr} = \frac{1}{\pi T} \frac{1}{r^2 f(r)}, \quad r^* = \frac{1}{2\pi T}\left(\arctan(r) + \log\sqrt{\frac{r-1}{r+1}}\right). \tag{2.44}$$

The Kruskal coordinates are defined as

$$\begin{cases} U = -\frac{e^{-2\pi T(t+r^*)}}{2\pi T}, \\ V = \frac{e^{2\pi T(t-r^*)}}{2\pi T}. \end{cases} \tag{2.45}$$

The Penrose diagram of Figure 2.1. is constructed from these coordinates. The retarded and advanced solutions comport themselves as

$$\begin{cases} e^{-i\omega t} f(k,r) \sim e^{-\frac{i\omega}{2\pi T}\ln(V)}, & \text{in–falling}, \\ e^{-i\omega t} f^*(k,r) \sim e^{\frac{i\omega}{2\pi T}\ln(-U)}, & \text{out–going}. \end{cases} \tag{2.46}$$

When we considered solutions to the wave equation, we were working in the R–quadrant, $U < 0$, $V > 0$. However, as explained in [130] if one extends the mode functions to the complex U and V planes, one finds that positive–frequency solutions to the wave equation are analytic in the lower U and V complex planes. A solution is composed of only negative–frequency modes provided it is analytic in the upper U and V planes. With regard to the modes of (2.46) one then requires that the solution be analytic in the lower V plane and the upper U plane. Since in the $r - t$ coordinates one can solve the wave equation independently in the R and L regions of the Penrose diagram one obtains the following set of mode functions in each quadrants:

$$\begin{array}{ll} u_{R,i}(k) = \begin{cases} e^{-i\omega t} f(k,r), & \text{in R} \\ 0, & \text{in L} \end{cases} & u_{L,i}(k) = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t} f(k,r), & \text{in L} \end{cases} \\ u_{R,o}(k) = \begin{cases} e^{-i\omega t} f^*(k,r), & \text{in R} \\ 0, & \text{in L} \end{cases} & u_{L,o}(k) = \begin{cases} 0, & \text{in R} \\ e^{-i\omega t} f^*(k,r), & \text{in L} \end{cases} \end{array} \tag{2.47}$$

Only two linear combinations can be built which meet the above criterium on holomorphicity. These are

$$\begin{cases} u_o = u_{R,o} + \alpha_o u_{L,o}, \\ u_i = u_{R,i} + \alpha_i u_{L,i}. \end{cases} \tag{2.48}$$

From the behaviour close to the horizon of the solutions and the analyticity requirement, the in–going and out–going cross–connecting functions α_i and α_o are constrained to be

$$\begin{cases} \alpha_o = e^{\frac{\pi\omega}{2}}, \\ \alpha_i = e^{-\frac{\pi\omega}{2}}. \end{cases} \tag{2.49}$$

[3]Setting $R = 1$ for convenience.

In order to carry a similar analysis to the case of fermions, one must first check that solutions to the Dirac equation in an AdS–Schwarzschild background (2.43) behave as $e^{-i\omega r*}$ and its conjugate. In such a background[4] the spin coefficients are

$$\omega_{tr} = -r\left(1+\frac{1}{r^4}\right)dt, \quad \omega_{ir} = r\sqrt{f}dx^i. \tag{2.50}$$

The Dirac equation then reads

$$\begin{cases} \left[-\frac{i\omega}{\sqrt{f}}\gamma^t + i\vec{\gamma}\,.\,\vec{k}\right]A(m)\Psi_+(k,r) = \left[-\frac{\omega^2}{f}+\mathbf{k}^2\right]\Psi_-(k,r), \\ \left[-\frac{i\omega}{\sqrt{f}}\gamma^t + i\vec{\gamma}\,.\,\vec{k}\right]A(-m)\Psi_-(k,r) = -\left[-\frac{\omega^2}{f}+\mathbf{k}^2\right]\Psi_+(k,r), \end{cases} \tag{2.51}$$

where $A(m) = r\left[r\sqrt{f(r)}\partial_r + \frac{d-1}{2}\sqrt{f(r)} + \frac{(1+\frac{(\pi T)^4}{r^4})}{2\sqrt{f(r)}} - m\right]$. Focusing on the terms relevant for the near–horizon behaviour, solutions of the type $\frac{r}{\sqrt[4]{f(r)}}e^{\pm i\omega r^*}$ satisfy these equations. Note that it is crucial that the Γ^0 matrix be anti–hermitian. This leading near–horizon behaviour of solutions to the Dirac equation in a curved background is reminiscent of the forms of the solutions found in [252] in the course of this study of second–quantization for neutrino fields in a Kerr background. From the previous discussion on parallely–transported spinors in the complex U and V planes, one is led to consider the following set of mode functions in each quadrant:

$$\begin{aligned} \psi_{R,i} &= \begin{cases} e^{-i\omega t}\sqrt{V}f(k,r)\imath, \text{ in R} \\ 0, \text{ in L} \end{cases} & \psi_{L,i} &= \begin{cases} 0, \text{ in R} \\ e^{-i\omega t}\sqrt{-V}f(k,r)\imath, \text{ in L} \end{cases} \\ \psi_{R,o} &= \begin{cases} e^{-i\omega t}\sqrt{-U}f^*(k,r)\varsigma, \text{ in R} \\ 0, \text{ in L} \end{cases} & \psi_{L,o} &= \begin{cases} 0, \text{ in R} \\ e^{-i\omega t}\sqrt{U}f^*(k,r)\varsigma, \text{ in L} \end{cases} \end{aligned} \tag{2.52}$$

The mergers $f(k,r)\begin{Bmatrix}\sqrt{-U} \\ \sqrt{V}\end{Bmatrix}$ behave as $\frac{r}{\sqrt[4]{f(r)}}e^{\pi Tt}e^{i\omega r^*}$, as required for solutions to the Dirac equation, except for the extra $e^{\pm\pi Tt}$ term, which could be inserted in the definitions of the modes in (2.55) below. As for the scalar case reviewed above, the conditions that positive–frequency solutions are analytic in the lower U and V complex planes and negative–energy modes are analytic in their upper counterparts leads to the following linear combinations

$$\begin{cases} \psi_o = \psi_{R,o} + \beta_o\psi_{L,o}, \\ \psi_i = \psi_{R,i} + \beta_i\psi_{L,i}. \end{cases} \tag{2.53}$$

The behaviour of the solutions close to the horizon fixes

$$\begin{cases} \beta_o = ie^{\frac{\pi\omega}{2}}, \\ \beta_i = -ie^{-\frac{\pi\omega}{2}}. \end{cases} \tag{2.54}$$

The out–going and in–going solutions in (2.53) form a basis for a spinor field defined over the full Kruskal plane of the AdS–Schwarzschild geometry

$$\Psi_-(r) = \sum_k \left[a(\omega,\mathbf{k})\psi_o(k,r) + b(\omega,\mathbf{k})\psi_i(k,r)\right]. \tag{2.55}$$

In accordance with our discussion on the variational principle for spinor fields in AdS/CFT we do not expand the Ψ_+ field. Its leading–order part in an expansion near the boundary is fixed. One must fix the "position" and leave the "momentum" free to vary in a set of canonically

[4]Setting $R = 1$, $\pi T = 1$ for convenience.

conjugate pairs given by $\bar{\chi}_0$ and ψ_0. The coefficients $a(\omega, \mathbf{k})$, $b(\omega, \mathbf{k})$ are determined by requiring that (2.55) approaches $\Psi_-^R(k)$ and $\Psi_-^L(k)$ on their respective boundaries:

$$\begin{cases} \left[a(\omega,\mathbf{k})\sqrt{-U}\varsigma + b(\omega,\mathbf{k})\sqrt{V}\imath\right]_{r_{\partial M}} = \Psi_-^R(k), \\ -\left[a(\omega,\mathbf{k})e^{\pi\omega}\sqrt{U}(-)\varsigma - b(\omega,\mathbf{k})\sqrt{-V}(-)\imath\right]_{r_{\partial M}} = -ie^{-\frac{\pi\omega}{2}}\Psi_-^L(k), \end{cases} \tag{2.56}$$

The function $f(k,r)$ is normalized such that $f(k, r_{\partial M}) = 1$ at the boundary. The overall minus sign on the r.h.s. of the second equation results from the effect on spinor fields of time reversal from going to the R–quadrant to the L one[5]:

$$T^{-1}\Psi_{+,\alpha}(x)T = -\Psi_{+,\alpha}(\mathcal{T}x), \quad T^{-1}\Psi_-^{\dot{\alpha}}T = +\Psi_{-,\dot{\alpha}}(\mathcal{T}x). \tag{2.57}$$

Also, recall that raising or lowering a spinor index comes with a minus sign. The meaning of (2.56) is as a set of two equations for two unknown spinors, $a(\omega,\mathbf{k})\sqrt{-U}\varsigma$ and $b(\omega,\mathbf{k})\sqrt{V}\imath$. The second equation in (2.56) involves the same unknown spinors but in the L quadrant, which introduces extra $\sqrt{-1}$. Taking care of those additional factors of i which occur in going from $\begin{Bmatrix}\sqrt{U}\\ \sqrt{-V}\end{Bmatrix}$ to $\begin{Bmatrix}\sqrt{-U}\\ \sqrt{V}\end{Bmatrix}$, (2.56) leads to

$$\begin{cases} a(\omega,\mathbf{k})\sqrt{-U}\mid_{r_{\partial M}}\varsigma = \frac{1}{e^{\pi\omega}+1}\left[\Psi_-^R(k) + e^{\frac{\pi\omega}{2}}\Psi_-^L(k)\right], \\ b(\omega,\mathbf{k})\sqrt{V}\mid_{r_{\partial M}}\imath = \frac{1}{e^{\pi\omega}+1}\left[e^{\pi\omega}\Psi_-^R(k) - e^{\frac{\pi\omega}{2}}\Psi_-^L(k)\right]. \end{cases} \tag{2.58}$$

Here, $n(\omega) = \frac{1}{e^{\beta\omega}+1}$ is the Fermi–Dirac distribution. One may think that using (2.58) to replace , say, ς in the identification of l with $\frac{\partial}{\partial U}$ would lead to a constraint on the boundary data, which cannot be. Rather, the constraints are on the modes $a(\omega,\mathbf{k})$ and $b(\omega,\mathbf{k})$. The same happens for the scalar case first dealt with in [130], as is most apparent in equations (3.48) and (3.49) of [56] where the modes in an expansion of the scalar field in an in–going and out–going basis are related to the boundary values of the field in the right and left quadrants of a Kruskal diagram. A computation of real–time Green functions from the standard AdS/CFT prescription is now in order. The classical boundary action in $r-t$ coordinates is

$$\begin{aligned} S_{\partial M} &= -i\int_{\partial M}\frac{d^dk}{(2\pi)^d}\sqrt{-g}g^{rr}\bar{\Psi}_+(-k,r)\Psi_-(k,r) \\ &= -i\int\frac{d^dk}{(2\pi)^d}\sqrt{-g}g^{rr}\bar{\Psi}_+(-k,r)\Psi_-(k,r)\mid_{r_R} -i\int\frac{d^dk}{(2\pi)^4}\sqrt{-g}g^{rr}\bar{\Psi}_+(-k,r)\Psi_-(k,r)\mid_{r_L} \end{aligned} \tag{2.59}$$

For a scalar field the second integral would have come with the opposite sign. Here, however one must recall that spinor fields behave non–trivially when they cross the L quadrant where time ordering is reversed. While $\bar{\Psi}\Psi$ is invariant under time–reversal, what we are really considering instead is an expression where $\bar{\Psi}_+$ is fixed whereas Ψ_- is free to vary. The unusual sign is associated with the latter's transformation under time reversal, (2.57). Using (2.55) and (2.58)

[5]Our conventions for the Clifford algebra differ from those of [243]. This affects in particular the Γ^5 matrix. Hence the overall sign flip in (2.57) as compared to the more familiar equation (40.32) from [243].

the boundary action becomes

$$\begin{aligned} S_{\partial M} =& - i \int \frac{d^4k}{(2\pi)^4} \sqrt{-g g^{rr}} n(\omega) u_{R,o}(k) \left[\bar{\Psi}^R_+(-k) \Psi^R_-(k) + e^{\frac{\beta\pi}{2}} \bar{\Psi}^R_+(-k) \Psi^L_-(k) \right] \\ & - i \int \frac{d^4k}{(2\pi)^4} \sqrt{-g g^{rr}} n(\omega) u_{R,i}(k) \left[e^{\frac{\beta\omega}{2}} \bar{\Psi}^R_+(-k) \Psi^R_-(k) - e^{\beta\omega} \bar{\Psi}^R_+(-k) \Psi^L_-(k) \right] \\ & + i \int \frac{d^4k}{(2\pi)^4} \sqrt{-g g^{rr}} n(\omega) u_{L,o}(k) e^{\frac{\beta\omega}{2}} \left[\bar{\Psi}^L_+(-k) \Psi^R_-(k) + e^{\frac{\beta\omega}{2}} \bar{\Psi}^L_+(-k) \Psi^L_-(k) \right] \\ & - i \int \frac{d^4k}{(2\pi)^4} \sqrt{-g g^{rr}} n(\omega) u_{L,i}(k) e^{-\frac{\beta\omega}{2}} \left[e^{\beta\omega} \bar{\Psi}^L_+(-k) \Psi^R_-(k) - e^{\frac{\beta\omega}{2}} \bar{\Psi}^L_+(-k) \Psi^L_-(k) \right] . \end{aligned} \quad (2.60)$$

Equations (2.36) and (2.8), which for fermionic operators reads $G_{A,ab}(k) = G^*_{R,ba} = -iS^*(k)\gamma^t_{ba} = iS^*(k)\gamma^t_{ab}$ – from Γ^0 being anti–hermitian, cf., e.g., (2.21) – and the near–boundary expansions of $\Psi^{R,L}_\pm$ yield

$$\begin{cases} G_{RR}(k) = -i \left[n(\omega) e^{\beta\omega} (-i) G_R(k) + n(\omega)(-i) G_A(k) \right] = -\mathrm{Re} G_R(k) + i \tanh(\frac{\omega}{2T}) \mathrm{Im} G_R(k), \\ G_{LL}(k) = -i \left[-n(\omega)(-i) G_R(k) - n(\omega) e^{\beta\omega} (-i) G_A(k) \right] = \mathrm{Re} G_R(k) + i \tanh(\frac{\omega}{2T}) \mathrm{Im} G_R(k), \\ G_{RL}(k) = -in(\omega) e^{\frac{\beta\omega}{2}} \left[(-i) G_R(k) - (-i) G_A(k) \right] = -\frac{2ie^{\frac{\beta\omega}{2}}}{1+e^{\beta\omega}} \mathrm{Im} G_R(k), \\ G_{LR}(k) = -in(\omega) e^{\frac{\beta\omega}{2}} \left[-(-i) G_R(k) + (-i) G_A(k) \right] = \frac{2ie^{\frac{\beta\sigma}{2}}}{1+e^{\beta\omega}} \mathrm{Im} G_R(k). \end{cases} \quad (2.61)$$

These are the Schwinger–Keldysh propagators (2.9) with $\sigma = \frac{\beta}{2}$ for a fermionic operator dual to the supergravity field Ψ. One can redefine $\Psi^L(k)$ to obtain real–time propagators with arbitrary $0 < \sigma < \beta$. Let us illustrate how (2.61) is obtained and focus on $G_{RR}(k)$.
The relevant terms from the boundary action are
$-in(\omega)\sqrt{-gg^{rr}} \left[u_{R,i}(k) e^{\beta\omega} \bar{\Psi}^R_+(-k) \Psi^R_-(k) + u_{R,o}(k) \bar{\Psi}^R_+(-k) \Psi^R_-(k) \right]$.
The first one comes with an in–going mode. One then uses the near–boundary expansion for $\Psi^R_\pm$ and the relation (2.30), i.e. $\psi_0(k) = \mathcal{S}(k)\chi_0(k)$, to translate this term into an expression proportional to the retarded propagator. On the other hand, the second term in brackets involves an out–going mode. It must be associated with an advanced Green function [238]. One then writes $\sqrt{-gg^{rr}} \bar{\Psi}^R_+(-k) \Psi^R_-(k) = \sqrt{-gg^{rr}} \bar{\Psi}^R_-(k) \Psi^R_+(-k)$ which is equal to $\mathcal{S}^*(k) \bar{\chi}^R_0(k) \chi^R_0(-k) = \mathcal{S}^*(k) \bar{\chi}^R_0(-k) \chi^R_0(k)$. This finally leads to the stated result. We have thus checked that Schwinger–Keldysh correlators for fermionic operators and the standard relations among them and the retarded, advanced and symmetric propagators hold in AdS/CFT by taking functional derivatives of boundary part of the dual supergravity action. This prescription can be generalized to higher–point functions of a fermionic operator, provided the non–quadratic parts of the action for its dual supergravity spinor field are known.

Chapter 3

Stochastic Trailing String and Langevin Dynamics from AdS/CFT

In this chapter, using the gauge/string duality, we derive a set of Langevin equations describing the dynamics of a relativistic heavy quark moving with constant average speed through the strongly–coupled $\mathcal{N} = 4$ SYM plasma at finite temperature. We show that the stochasticity arises at the string world–sheet horizon, and thus is causally disconnected from the black hole horizon in the space–time metric. This hints at the non–thermal nature of the fluctuations, as further supported by the fact that the noise term and the drag force in the Langevin equations do not obey the Einstein relation. We propose a physical picture for the dynamics of the heavy quark in which dissipation and fluctuations are interpreted as medium–induced radiation and the associated quantum–mechanical fluctuations. This picture provides the right parametric estimates for the drag force and the (longitudinal and transverse) momentum broadening coefficients.

3.1 Introduction

Motivated by possible strong–coupling aspects in the dynamics of ultrarelativistic heavy ion collisions, there have been many recent applications of the AdS/CFT correspondence to the study of the response of a strongly coupled plasma — typically, that of the $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) theory at finite temperature — to an external perturbation, so like a "hard probe" — say, a heavy quark, or an electromagnetic current (see the review papers [239, 141, 110] for details and more references). Most of these studies focused on the mean field dynamics responsible for dissipation (viscosity, energy loss, structure functions), as encoded in retarded response functions — typically, the 2–point Green's function of the $\mathcal{N}=4$ SYM operator perturbing the plasma. By comparison, the statistical properties of the plasma (in or near thermal equilibrium) have been less investigated. Within the AdS/CFT framework, such investigations would require field quantization in a curved space–time — the $AdS_5 \times S^5$ Schwarzschild geometry dual to the strongly–coupled $\mathcal{N}=4$ SYM plasma —, which in general is a very difficult problem. Still, there has been some interesting progress in that sense, which refers to a comparatively simpler problem: that of the quantization of the small fluctuations of the Nambu–Goto string dual to a heavy quark immersed into the plasma.

Several noticeable steps may be associated with this progress: In Ref. [130], a prescription was formulated for computing the Schwinger–Keldysh Green's functions at finite temperature within the AdS/CFT correspondence. With this prescription, the quantum thermal distributions are generated via analytic continuation across the horizon singularities in the Kruskal diagram for the AdS_5 Schwarzschild space–time. Using this prescription, one has computed the diffusion coefficient of a non–relativistic heavy quark [55], and the momentum broadening for a relativistic heavy quark which propagates through the plasma at constant (average) speed [56, 109]. Very recently, in Refs. [70, 241], a set of Langevin equations has been constructed which describes the Brownian motion of a non–relativistic heavy quark and of the attached Nambu–Goto string. Within these constructions, the origin of the 'noise' (the random force in the Langevin equations) in the supergravity calculations lies at the black hole horizon, as expected for thermal fluctuations.

The Langevin equations in Refs. [70, 241] encompass previous results for the drag force [131, 108] and the diffusion coefficient [55] of a *non–relativistic* heavy quark. But to our knowledge, no attempt has been made so far at deriving corresponding equations for a *relativistic* heavy quark, whose dual description is a trailing string [131, 108]. In fact, the suitability of the Langevin description for the stochastic trailing string was even challenged by the observation that the respective expressions for the drag force and the momentum broadening do not to obey the Einstein relation [109]. The latter is a hallmark of thermal equilibrium and must be satisfied by any Langevin equation describing thermalization. However, Langevin dynamics is more general than thermalization, and as a matter of facts it does apply to the stochastic trailing string, as we will demonstrate in this chapter.

Specifically, our objective in what follows is twofold: (i) to show how the Langevin description of the stochastic trailing string unambiguously emerges from the underlying AdS/CFT formalism, and (ii) to clarify the physical interpretation of the associated noise term, in particular, its non–thermal nature.

Our main conclusion is that the stochastic dynamics of the relativistic quark is fundamentally different from the Brownian motion of a non–relativistic quark subjected to a thermal noise. Within the supergravity calculation, this difference manifests itself via the emergence of an event horizon on the string world–sheet [56, 109], which lies in between the Minkowski boundary and the black hole horizon, and which governs the stochastic dynamics of the fast moving quark. With our choice for the radial coordinate z in AdS_5, the Minkowski boundary lies at $z = 0$, the black hole horizon at $z_H = 1/T$, and the world–sheet horizon at $z_s = z_H/\sqrt{\gamma}$, where $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz factor of the heavy quark. (We assume that the quark is pulled by an external force in such a way that its average velocity remains constant.) The presence of the world–sheet horizon means that the dynamics of the upper part of the string at $z < z_s$ (including the heavy quark at $z \simeq 0$) is causally disconnected from that of its lower part at $z_s < z < z_H$, and thus cannot be influenced by thermal fluctuations originating at the black hole horizon.

This conclusion is supported by the previous calculations of the momentum broadening for the heavy quark [56, 109], which show that the relevant correlations are generated (via analytic continuation in the Kruskal plane) at the *world–sheet* horizon, and not at the black hole one. Formally, these correlations look as being thermal (they involve the Bose–Einstein distribution), but with an effective temperature $T_{\rm eff} = T/\sqrt{\gamma}$, which is the Hawking temperature associated to the world–sheet horizon. Thus, no surprisingly, our explicit construction of the Langevin equations will reveal that the corresponding noise terms arise from this world–sheet horizon.

The Langevin equations for the relativistic heavy quark will be constructed in two different ways: (1) by integrating out the quantum fluctuations of the upper part of the string, from the world–sheet horizon up to the boundary, and (2) by integrating out the string fluctuations only within an infinitesimal strip in z, from the world–sheet horizon at $z = z_s$ up to the 'stretched' horizon at $z = z_s(1-\epsilon)$ with $\epsilon \ll 1$; this generates a 'bulk' noise term at the stretched horizon, whose effects then propagate upwards the string, via the corresponding classical solutions. Both procedures provide exactly the same set of Langevin equations, which encompass the previous results for the drag force [131, 108] and for the (longitudinal and transverse) momentum broadening [56, 109]. In these manipulations, the lower part of the string at $z > z_s$ and, in particular, the black hole horizon, do not play any role, as expected from the previous argument on causality.

If the relevant fluctuations are not of thermal nature, then why do they *look* as being thermal

? What is their actual physical origin ? And what is the role played by the thermal bath ? To try and answer such questions, we will rely on a physical picture for the interactions between an energetic parton and the strongly–coupled plasma which was proposed in Refs. [116, 117, 78, 141], and that we shall here more specifically develop for the problem at hand. In this picture, both the energy loss ('drag force') and the momentum broadening ('noise term') are due to medium–induced radiation. This is reminiscent of the mechanism of energy loss of a heavy, or light, quark at weak coupling [17, 19, 177, 57], with the main difference being in the cause of the medium–induced radiation. At weak coupling, multiple scattering off the plasma constituents frees gluonic fluctuations in the quark wavefunction, while at strong coupling the plasma exerts a force, proportional to T^2, acting to free quanta from the heavy quark as radiation. In the gravity description, this appears as a force pulling energy in the trailing string towards the horizon. At either weak or strong coupling, quanta are freed when their virtuality is smaller than a critical value, the *saturation momentum* Q_s; at strong coupling and for a fast moving quark, this scales like $Q_s \sim \sqrt{\gamma}T$. Within this picture, the world–sheet horizon at $z_s \sim 1/Q_s$ corresponds to the causal separation between the highly virtual quanta ($Q \gg Q_s$), which cannot decay into the plasma and thus are a part of the heavy quark wavefunction, and the low virtuality ones, with $Q \lesssim Q_s$, which have already been freed, thus causing energy loss. The recoil of the heavy quark due to the random emission of quanta with $Q \lesssim Q_s$ is then responsible for its momentum broadening.

From his perspective, the noise terms in the Langevin equations for the fast moving quark reflect quantum fluctuations in the emission process. Of course, the presence of the surrounding plasma is essential for this emission to be possible in the first place (a heavy quark moving at constant speed through the vacuum could not radiate), but the plasma acts merely as a background field, which acts towards reducing the virtuality of the emitted quanta and thus allows them to decay. The genuine thermal fluctuations on the plasma are unimportant when $\gamma \gg 1$, although when $\gamma \simeq 1$ they are certainly the main source of stochasticity, as shown in [70, 241]. Besides, we see no role for Hawking radiation of supergravity quanta at any value of γ.

This picture is further corroborated by the study of a different physical problem, where the thermal effects are obviously absent, yet the mathematical treatment within AdS/CFT is very similar to that for the problem at hand: this is the problem of a heavy quark propagating with constant acceleration a through the vacuum of the strongly–coupled $\mathcal{N} = 4$ SYM theory [78, 261, 212]. The accelerated particle can radiate, and this radiation manifests itself through the emergence of a world–sheet horizon, leading to dissipation and momentum broadening. The fluctuations generated at this horizon are once again thermally distributed, with an effective temperature $T_{\text{eff}} = a/2\pi$. In that context, it is natural to interpret the induced horizon as the AdS dual of the Unruh effect [253] : the accelerated observer perceives the Minkowski vacuum as a thermal state with temperature $a/2\pi$. For an inertial observer, this is interpreted as follows [254]: the accelerated particle can radiate and the correlations induced by the backreaction to this radiation are such that the excited states of the emitted particle are populated according to a thermal distribution. Most likely, a similar interpretation holds also for the thermal–like correlations generated at the world–sheet horizon in the problem at hand — that of a relativistic quark propagating at constant speed through a thermal bath. It would be interesting to identify similar features in other problems which exhibit accelerated motion, or medium–induced radiation, or both, so like the rotating string problem considered in Ref. [86].

This chapter of the thesis is organized as follows: In Sect. 2 we construct the Langevin equations describing the stochastic dynamics of the string endpoint on the boundary of AdS_5, i.e., of the relativistic heavy quark. Our key observation is that, in the Kruskal–Keldysh quantization of the small fluctuations of the trailing string, the stochasticity is generated exclusively at the world–sheet horizon. This conclusion is further substantiated by the analysis in Sect. 3 where we follow the progression of the fluctuations along the string, from the world–sheet horizon up to the string endpoint on the boundary. We thus demonstrate that the noise correlations are faithfully transmitted from the stretched horizon to the heavy quark, via the fluctuations of the string. Finally, Sect. 4 contains our physical discussion. First, in Sect. 4.1, we argue that the Langevin equations do not describe thermalization, although they do generate thermal–like momentum distributions, but at a fictitious temperature which is not the same as the temperature of the plasma, and is moreover different for longitudinal and transverse fluctuations. Then, in Sect. 4.2, we develop our physical picture for medium–induced radiation and parton branching, which emphasizes the quantum–mechanical nature of the stochasticity.

3.2 Boundary picture of the stochastic motion

In this section we will construct a set of Langevin equations for the stochastic dynamics of a relativistic heavy quark which propagates with uniform average velocity through a strongly–coupled $\mathcal{N}=4$ SYM plasma at temperature T. To that aim, we will follow the general strategy in Ref. [241], that we will extend to a fast moving quark and the associated trailing string. In this procedure, we will also rely on previous results in the literature [56, 109] concerning the classical solutions for the fluctuations of the trailing string and their quantization via analytic continuation in the Kruskal plane.

3.2.1 The trailing string and its small fluctuations

The AdS dual of the heavy quark is a string hanging down in the radial direction of AdS_5, with an endpoint (representing the heavy quark) attached to a D7–brane whose radial coordinate fixes the bare mass of the quark. The string dynamics is encoded in the Nambu–Goto action,

$$S \,=\, -\frac{1}{2\pi\ell_s^2}\int \mathrm{d}^2\sigma\sqrt{-\det h_{\alpha\beta}}\,, \qquad h_{\alpha\beta} \,=\, g_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu\,, \tag{3.1}$$

where σ^α, $\alpha = 1, 2$, are coordinates on the string world–sheet, $h_{\alpha\beta}$ is the induced world–sheet metric, and $g_{\mu\nu}$ is the metric of the AdS_5–Schwarzschild space–time, chosen as

$$\mathrm{d}s^2 = \frac{R^2}{z_H^2 z^2}\left(-f(z)\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}z^2}{f(z)}\right), \tag{3.2}$$

where $f(z) = 1 - z^4$ and $T = 1/\pi z_H$ is the Hawking temperature. (As compared to the Introduction, we have switched to a dimensionless radial coordinate.)

The quark is moving along the longitudinal axis x^3 with constant (average) velocity v in the plasma rest frame. For this to be possible, the quark must be subjected to some external force, which compensates for the energy loss towards the plasma. The profile of the string corresponding to this steady (average) motion is known as the 'trailing string'. This is obtained by solving the equations of motion derived from (3.1) with appropriate boundary conditions, and reads [131, 108]

$$x_0^3 \,=\, vt + \frac{vz_H}{2}\left(\arctan z - \mathrm{arctanh} z\right). \tag{3.3}$$

In what follows we shall be interested in small fluctuations around this steady solution, which can be either longitudinal or transverse: $x^3 = x_0^3 + \delta x_\ell(t,z)$ and $x_\perp = \delta x_\perp(t,z)$. To quadratic order in the fluctuations, the Nambu–Goto action is then expanded as (in the static gauge $\sigma^\alpha = (t,z)$)

$$S = -\frac{\sqrt{\lambda}Tz_s^2}{2}\int \mathrm{d}t\mathrm{d}z\frac{1}{z^2} + \int \mathrm{d}t\mathrm{d}zP^\alpha\partial_\alpha\delta x_\ell - \frac{1}{2}\int \mathrm{d}t\mathrm{d}z\left[T_\ell^{\alpha\beta}\partial_\alpha\delta x_\ell\,\partial_\beta\delta x_\ell + T_\perp^{\alpha\beta}\partial_\alpha\delta x_\perp^i\partial_\beta\delta x_\perp^i\right], \tag{3.4}$$

where $z_s \equiv \sqrt[4]{1-v^2} = 1/\sqrt{\gamma}$ and[1] [109]

$$P^\alpha = \frac{\pi v \sqrt{\lambda} T^2}{2 z_s^2} \begin{pmatrix} \frac{z_H}{z^2(1-z^4)} \\ 1 \end{pmatrix}, \tag{3.5}$$

$$T_\perp^{\alpha\beta} = z_s^4 T_\ell^{\alpha\beta} = -\frac{\pi\sqrt{\lambda}T^2}{2z_s^2} \begin{pmatrix} \frac{z_H}{z^2}\frac{1-(zz_s)^4}{(1-z^4)^2} & \frac{v^2}{1-z^4} \\ \frac{v^2}{1-z^4} & \frac{z^4-z_s^4}{z_H z^2} \end{pmatrix}. \tag{3.6}$$

The quantities $T_{\ell,\perp}^{\alpha\beta}$ have the meaning of local stress tensors on the string. At high energy, the components $T_\ell^{\alpha\beta}$ of the longitudinal stress tensor are parametrically larger, by a factor $\gamma^2 \gg 1$, than the corresponding components $T_\perp^{\alpha\beta}$ of the transverse stress tensor. This difference reflects the strong energy–dependence of the gravitational interactions.

Using $\partial_\alpha P^\alpha = 0$, one sees that the term linear in the fluctuations in (4.4) does not affect the equations of motion, which therefore read

$$\partial_\alpha (T^{\alpha\beta}\partial_\beta \psi) = 0, \qquad \psi = \delta x_\ell,\ \delta x_\perp, \tag{3.7}$$

in compact notations which treat on the same footing the longitudinal and transverse fluctuations. Upon expanding in Fourier modes,

$$\psi(t,z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \psi(\omega,z)\, \mathrm{e}^{-i\omega t}, \tag{3.8}$$

this yields

$$\left\{a(z)\partial_z^2 - 2b(\omega,z)\partial_z + c(\omega,z)\right\}\psi(\omega,z) = 0, \tag{3.9}$$

where

$$\begin{aligned} a(z) &= z(1-z^4)^2(z_s^4 - z^4), \\ b(\omega,z) &= (1-z^4)\left[1 - z^8 - v^2(1 - z^2 + i\omega z_H z^3)\right], \\ c(\omega,z) &= \omega z_H z \left[\omega z_H(1-z^4) + v^2 z^4(\omega z_H + 4iz)\right]. \end{aligned} \tag{3.10}$$

The zeroes of $a(z)$ determine the regular singular points of this equation. In particular, the special role played by the point z_s as a world–sheet horizon becomes manifest at this level: for $z = z_s$ the value of $\partial_z\psi(\omega, z_s)$ is determined from the equation of motion. This means that fluctuations of the string at $z < z_s$ are causally disconnected from those below the location of the world–sheet horizon.

3.2.2 Keldysh Green function in AdS/CFT

In what follows we construct solutions to Eqs. (3.9)–(3.10) for the string fluctuations which are well defined everywhere in the Kruskal diagram for the AdS_5 Schwarzschild space–time (see Fig. 3.1). These solutions are uniquely determined by their boundary conditions at the two Minkowski boundaries — in the right (R) and, respectively, left (L) quadrants of the Kruskal diagram —, together with the appropriate conditions of analyticity in the Kruskal variables U and V (as explained in [130]). The latter amount to quantization prescriptions which impose infalling conditions on the positive–frequency modes and outgoing conditions on the negative–frequency ones. These prescriptions ultimately generate the quantum Green's functions at finite–temperature and in real time, which are time–ordered along the Keldysh contour [130]. Specifically, the time variables on the R and, respectively, L boundary in Fig. 3.1 correspond to the chronological and, respectively, antichronological branches of the Keldysh time contour.

[1] In (3.6), we have corrected an overall sign error in Eq. (22) of Ref. [109].

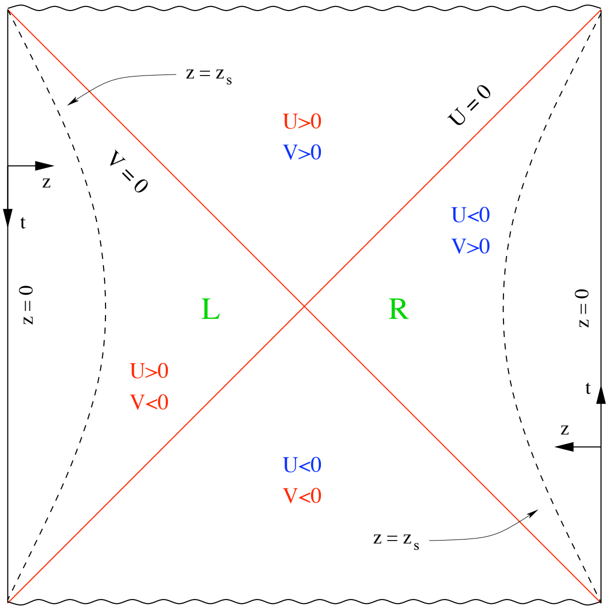


Figure 3.1: *Kruskal diagram for AdS_5 Schwarzschild metric. The Kruskal coordinates U and V are defined by $UV = \frac{z-1}{z+1}\, e^{-2\arctan(z)}$, $V/U = -e^{4\pi Tt}$. Hence, the black horizon at $z = 1$ corresponds to $UV = 0$, whereas $V = 0$ ($U = 0$) corresponds to $\mathrm{Re}\,t \to -\infty$ ($\mathrm{Re}\,t \to \infty$). The position of the induced world–sheet horizon is shown with dashed lines in both R and L quadrants.*

As usual in the framework of AdS/CFT, we are interested in the classical action expressed as a functional of the fields on the boundary. We denote by $\psi_R(t_R, z)$ and $\psi_L(t_R, z)$ the classical solutions in the R and L quadrant, respectively. Making use of the equations of motion and integrating by parts, the classical action reduces to its value on the boundary of the Kruskal diagram, i.e., the R and L Minkowski boundaries:

$$S_{\rm bndry} = \int {\rm d}t_R \Big[-P^z\psi_R + \frac{1}{2}\,\psi_R T^{z\beta}\partial_\beta\psi_R\Big]_{z_R=z_m} - \int {\rm d}t_L \Big[-P^z\psi_L + \frac{1}{2}\,\psi_L T^{z\beta}\partial_\beta\psi_L\Big]_{z_L=z_m}, \tag{3.11}$$

where it is understood that the terms involving P_z exist only in the longitudinal sector. $z_m \ll 1$ is the radial location of the D7–brane on which the string ends.

In (3.11), the world–sheet index β can take *a priori* both values t and z, but the contribution corresponding to $\beta = t$ is in fact zero, since the respective integrand is an odd function of t. This is worth noticing since in Ref. [109] it was found that the dominant contribution to the imaginary part of the retarded propagator at high energy ($\gamma \gg 1$) comes from the piece proportional to T^{zt}. We will later see how that contribution arises in the present calculation, where only the piece proportional to T^{zz} survives in (3.11).

Switching to the frequency representation, we introduce a basis of retarded and advanced solutions, $\psi_{ret}(\omega, z)$ and $\psi_{adv}(\omega, z)$, which are normalized such that $\psi_{ret}(\omega, 0) = \psi_{adv}(\omega, 0) = 1$. They obey $\psi_{ret}(\omega, z) = \psi^*_{ret}(-\omega, z)$, and similarly for ψ_{adv}. These solutions are truly boundary–to–bulk propagators in Fourier space. They have been constructed in Ref. [109] (see also [56]) from which we quote the relevant results.

Note first that, unlike what happens for a static (or non–relativistic) quark [241], the retarded and advanced solutions are not simply related to each other by complex conjugation: one rather has

$$\psi_{adv}(\omega, z) \;=\; [g(z)]^{i\omega/2}\,[g(z/z_s)]^{-i\omega z_H/2z_s}\;\psi^*_{ret}(\omega, z)\,, \tag{3.12}$$

with $g(z) = \frac{1+z}{1-z}\,\mathrm{e}^{-2\arctan z}$. Near the boundary ($z \ll 1$), these solutions behave as follows

$$\psi_{ret}(\omega, z) = \left(1 + \frac{z_H^2\omega^2}{2z_s^4} z^2 + \mathcal{O}(z^4)\right) + C_{ret}(\omega)\big(z^3 + \mathcal{O}(z^5)\big) \tag{3.13}$$

$$\psi_{adv}(\omega, z) = \left(1 + \frac{z_H^2\omega^2}{2z_s^4} z^2 + \mathcal{O}(z^4)\right) + C_{adv}(\omega)\big(z^3 + \mathcal{O}(z^5)\big), \tag{3.14}$$

where the expansion involving even (odd) powers of z is that of the non–normalizable (normalizable) mode. The coefficients of the normalizable mode are related by

$$C_{adv}(\omega) = C^*_{ret}(\omega) - iX(\omega), \tag{3.15}$$

with

$$X(\omega) \;=\; \frac{2\omega z_H}{3}\, v^2\gamma^2\,, \qquad \mathrm{Im}\, C_{ret}(\omega) \;=\; i\,\frac{\omega z_H}{3}\,. \tag{3.16}$$

The real part of coefficient $C_{ret}(\omega)$ has been numerically evaluated in Ref. [109]. Here, we only need to know that, at small frequency $\omega \ll z_s T$, its real part is comparatively smaller: $\mathrm{Re}\, C_{ret}(\omega) \sim \mathcal{O}(\omega^2 z_H^2/z_s^2)$. Note that, at high energy ($\gamma \gg 1$), X dominates over $\mathrm{Im}\, C_{ret}$ in (3.15).

Consider also the approach towards the world–sheet horizon ($z = z_s$) from the above ($z < z_s$): there, ψ_{ret} remains regular, whereas ψ_{adv} has a branching point:

$$\psi_{adv}(\omega, z) \;\propto\; (z_s - z)^{\frac{i\omega z_H}{2z_s}}\,\left[1 + \mathcal{O}(z_s - z)\right]\,. \tag{3.17}$$

This shows that $\Psi_{adv}(\omega, t, z) \;\equiv\; \mathrm{e}^{-i\omega t}\psi_{adv}(\omega, z)$ is an outgoing wave: with increasing time, the phase remains constant while departing from the horizon. One can similarly argue that $\Psi_{ret}(\omega, t, z) = \mathrm{e}^{-i\omega t}\psi_{ret}(\omega, z)$ is an infalling solution [109].

We now expand the general solution in the right and left quadrants of the Kruskal diagram in this retarded/advanced basis :

$$\begin{aligned}
\psi_R(\omega, z) &= A(\omega)\psi_{ret}(\omega, z) + B(\omega)\psi_{adv}(\omega, z),\\
\psi_L(\omega, z) &= C(\omega)\psi_{ret}(\omega, z) + D(\omega)\psi_{adv}(\omega, z).
\end{aligned} \tag{3.18}$$

We need four conditions to determine the four unknown coefficients A, B, C, and D. Two of them are provided by the boundary values at the R and L Minkowski boundaries, that we denote as $\psi_R^0(\omega)$ and $\psi_L^0(\omega)$, respectively. The other two are determined by analyticity conditions in the Kruskal plane, which allows one to connect the solution in the L quadrant to that in the R quadrant. With reference to Fig. 3.1, one sees that this requires crossing two types of horizons: the world–sheet horizons in both R and L quadrants, and the black hole horizons at $U = 0$ and $V = 0$. The detailed matching at these horizons, following the prescription of Ref. [130], has been performed in Appendix B of Ref. [109], with the following results: the multiplicative factors associated with crossing the black hole horizons precisely compensate each other, unlike those associated with crossing the world–sheet horizons, which rather enhance each other. Hence, the net result comes from the world–sheet horizons alone[2], and reads

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\frac{\omega}{z_s T}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{3.19}$$

The two independent coefficients can now be determined from the boundary values $\psi_R^0(\omega)$ and $\psi_L^0(\omega)$. This eventually yields

$$A(\omega) = (1+n)(\omega)\psi_R^0(\omega) - n(\omega)\psi_L^0(\omega), \tag{3.20}$$

[2]This point is even more explicit in the analysis in Ref. [56], where a different set of coordinates was used, in which the world–sheet metric is diagonal. With those coordinates, the only horizons to be crossed when going from the R to the L boundary in the respective Kruskal diagram are the *world–sheet* horizons.

$$B(\omega) = n(\omega)\left[\psi_L^0(\omega) - \psi_R^0(\omega)\right]. \tag{3.21}$$

Here $n(\omega) = 1/(e^{\omega/z_s T} - 1)$ is the Bose–Einstein thermal distribution with the effective temperature $T_{\rm eff} = z_s T = T/\sqrt{\gamma}$. As it should be clear from the previous manipulations, this effective thermal distribution has been generated via the matching conditions at the R and L world–sheet horizons, cf. (3.19).

The equations simplify if one introduces 'average' (or 'classical') and 'fluctuating' variables, according to $\psi_r \equiv (\psi_R + \psi_L)/2$ and $\psi_a \equiv \psi_R - \psi_L$, and similarly for the boundary values. One then finds

$$\begin{aligned} \psi_r(\omega, z) &= \psi_r^0(\omega)\psi_{ret}(\omega, z) + \frac{1+2n(\omega)}{2}\,\psi_a^0(\omega)(\psi_{ret}(\omega, z) - \psi_{adv}(\omega, z)), \\ \psi_a(\omega, z) &= \psi_a^0(\omega)\psi_{adv}(\omega, z), \end{aligned} \tag{3.22}$$

and the boundary action takes a particularly simple form:

$$S_{\rm bndry} = \frac{1}{2}\int \frac{d\omega}{2\pi}\, T^{zz}(z)\big[\psi_r(-\omega, z)\partial_z\psi_a(\omega, z) + (r \leftrightarrow a)\big]_{z=z_m}. \tag{3.23}$$

(We have here omitted the term linear in the fluctuations, since this does not matter for the calculation of the 2–point Green's functions. This term will be reinserted in the next subsection.) By combining the above equations, we finally deduce

$$iS_{\rm bndry} = -i\int \frac{d\omega}{2\pi}\,\psi_a^0(-\omega)G_R^0(\omega)\psi_r^0(\omega) - \frac{1}{2}\int \frac{d\omega}{2\pi}\,\psi_a^0(-\omega)G_{\rm sym}(\omega)\psi_a^0(\omega), \tag{3.24}$$

with the retarded and symmetric Green's functions defined as

$$G^0_{\perp,R}(\omega) = z_s^4 G^0_{\ell,R}(\omega) \;=\; -\frac{1}{2}\,T_\perp^{zz}(z)\,\partial_z\Big[\psi_{ret}(\omega, z)\psi_{adv}(-\omega, z)\Big]_{z=z_m}, \tag{3.25}$$

and, respectively,

$$G_{\rm sym}(\omega) = -(1 + 2n(\omega))\,{\rm Im}\,G_R^0(\omega). \tag{3.26}$$

Note that (3.26) is formally the same as the fluctuation–dissipation theorem (or 'KMS condition') characteristic of thermal equilibrium, but with an effective temperature $T_{\rm eff} = z_s T$. By also using Eqs. (3.13)–(3.14) together with the expression of $T_\perp^{zz}$ given in (3.6), one finally deduces

$$G^0_{\perp,R}(\omega) \;=\; z_s^4 G^0_{\ell,R} \;=\; G_R(\omega) - \gamma M_Q\omega^2\,, \tag{3.27}$$

where

$$\begin{aligned} G_R(\omega) &\equiv -Y\big(C_{ret}(\omega) + C^*_{adv}(\omega)\big), \\ Y &\equiv \frac{3\sqrt{\lambda}}{4\pi\gamma z_H^3}\,, \\ M_Q &\equiv \frac{\sqrt{\lambda}T}{2z_m} \;=\; \frac{\sqrt{\lambda}\,r_m}{2\pi R^2}\,. \end{aligned} \tag{3.28}$$

M_Q is the (bare) rest mass of the heavy quark and is independent of temperature, as manifest in his last rewriting. (r_m denotes the position of the D7–brane in the usual radial coordinate r, which is related to z as $z/\pi T = R^2/r$.) At finite temperature, this mass receives thermal corrections, as encoded in the contribution of $\mathcal{O}(\omega^2)$ to ${\rm Re}\,C_{ret}(\omega)$; such corrections are however negligible at high energy, since their contribution to $G_R(\omega)$ is not enhanced by a factor of γ (unlike M_Q).

Note that the previous formulae fully specify the imaginary part of the retarded propagator, and hence also $G_{\rm sym}(\omega)$. Namely, by using (cf. Eqs. (3.15)–(3.16))

$${\rm Im}\,C_{ret}(\omega) + {\rm Im}\,C^*_{adv}(\omega) \;=\; \frac{2\omega z_H}{3}(1 + v^2\gamma^2) \;=\; \frac{2\omega z_H}{3}\,\gamma^2, \tag{3.29}$$

one immediately finds

$$\mathrm{Im}\, G_R(\omega) \;=\; -\omega\gamma\eta\,, \qquad \text{with} \qquad \eta \equiv \frac{\pi\sqrt{\lambda}}{2}\, T^2\,. \tag{3.30}$$

Remarkably, this exact result involves just a term linear in ω. On the other hand, we expect $\mathrm{Re}\, G_R(\omega)$ to receive contributions to all orders in ω starting at $\mathcal{O}(\omega^2)$.

The above expression for G_R^0, (3.27), coincides with that originally derived in Ref. [109], although the definition used there for the retarded propagator was different, namely

$$G_R^0(\omega) \;\equiv\; -\Psi^*_{ret} T^{z\beta} \partial_\beta \Psi_{ret}|_{z=z_m}\,. \tag{3.31}$$

(Recall that $\Psi_{ret}(\omega, t, z) = \mathrm{e}^{-i\omega t}\psi_{ret}(\omega, z)$.) With this definition, the dominant contribution to the imaginary part at high energy — the term proportional to $X(\omega)$ — arises from the time derivative of the retarded solution. Although it does not naturally emerge when constructing the boundary action in the Kruskal plane, this formula (3.31) has another virtue, which will be useful later on: with this definition, the imaginary part of the retarded propagator,

$$\mathrm{Im}\, G_R^0(\omega) \;=\; \frac{1}{2i}\, T^{z\beta}\Big(\Psi^*_{ret}\partial_\beta\Psi_{ret} - \Psi_{ret}\partial_\beta\Psi^*_{ret}\Big)\,, \tag{3.32}$$

can be evaluated at any z, since the r.h.s. of (3.32) is independent of z. Indeed, as noticed in Ref. [109], the world–sheet current

$$J^\alpha \;=\; \frac{1}{2i}\, T^{\alpha\beta}\Big(\Psi^*_{sol}\partial_\beta\Psi_{sol} - \Psi_{sol}\partial_\beta\Psi^*_{sol}\Big)\,, \tag{3.33}$$

($\Psi_{sol}(t, z)$ is an arbitrary solution to the classical EOM (4.6)) is conserved by the equations of motion, $\partial_\alpha J^\alpha = 0$. When $\Psi_{sol}(t, z) = \Psi_{ret}(\omega, t, z)$, this conservation law reduces to $\partial_z J^z = 0$. As we shall shortly see, $\mathrm{Im}\, G_R^0(\omega)$ is a measure of the energy loss of the heavy quark towards the plasma. Thus the fact this quantity is independent of z is a statement about the conservation of the energy flux down the string in the present, steady, situation.

3.2.3 A Langevin equation for the heavy quark

Following the general strategy of AdS/CFT, the boundary action (3.24) can be used to generate the correlation functions of the $\mathcal{N} = 4$ SYM operator which couples to the boundary value of the field — in this case, the Schwinger–Keldysh 2–point functions of the force operator acting on the heavy quark [130, 56, 109, 241]. Alternatively, in what follows, this action will be used to derive stochastic equations for the string endpoint, in the spirit of the Feynman–Vernon ‘influence functional’ [87] (see also Ref. [241]).

To that aim we start with the following path integral which encodes the (quantum and thermal) dynamics of the string fluctuations in the Gaussian approximation of interest:

$$Z = \int \big[D\psi^0_R D\psi_R\big]\,\big[D\psi^0_L D\psi_L\big]\, e^{iS_R - iS_L}\,. \tag{3.34}$$

This involves two types of functional integrations: `(i)` those with measure $[D\psi_R D\psi_L]$, which run over all the string configurations $\psi_{R,L}(t, z)$ (in the corresponding quadrants of the Kruskal diagram) with given boundary values $\psi^0_{R,L}(t)$, and `(ii)` those with measure $[D\psi^0_R D\psi^0_L]$, which run over all the possible paths $\psi^0_{R,L}(t)$ for these endpoint values.

Performing the Gaussian path integral over the bulk configurations amounts to evaluating the action in the exponent of (3.34) with the classical solutions computed in the previous section. This leaves us with the boundary action in (3.24), which determines the dynamics of the string endpoints — i.e., of the heavy quark —, and which is itself Gaussian. To perform the corresponding path integral it is convenient to 'break' the quadratic term for the fluctuating fields ψ_a^0, by introducing an auxiliary stochastic field $\xi(t)$. Then the partition function becomes

$$Z = \int \left[D\psi_r^0\right] \left[D\psi_a^0\right] \left[D\xi\right] \ \mathrm{e}^{-\int \mathrm{d}t\mathrm{d}t' \, \frac{1}{2}\left[\xi(t) G_{\mathrm{sym}}^{-1}(t,t')\xi(t')\right]} \times$$
$$\times \exp\left\{-i\int \mathrm{d}t\mathrm{d}t' \, \psi_a^0(t)\left[G_R^0(t,t')\psi_r^0(t') + \delta(t-t')\big(P^z - \xi(t')\big)\right]\right\}, \tag{3.35}$$

where we recall that the term involving P^z appears only in the longitudinal sector. The integral over ψ_a acts as a constraint which enforces a Langevin equation for the 'average' field ψ_r. This equation reads

$$\int \mathrm{d}t' G_R^0(t,t')\psi_r^0(t') + P^z - \xi(t) = 0, \qquad \langle \xi(t)\xi(t')\rangle = G_{\mathrm{sym}}(t,t')\,, \tag{3.36}$$

and is generally non–local in time. At this point it is convenient to focus on the large time behaviour, as controlled by the small–frequency expansion of the Green's functions G_R and G_{sym}, and also distinguish between longitudinal and transverse fluctuations. As discussed in Sect. 2.2, for $\omega \ll z_s T$, the retarded propagator reduces to its imaginary part, (3.30) (in addition to the bare mass term). In the same limit, one can use $1 + 2n(\omega) \simeq 2z_s T/\omega$ to simplify the expression of $G_{\mathrm{sym}}(\omega)$, which then becomes independent of ω :

$$\begin{aligned} G_{\perp,\,\mathrm{sym}}(\omega) \ &\simeq\ \pi\sqrt{\lambda}\,\gamma^{1/2}T^3 \ \equiv\ \kappa_\perp\,, \\ G_{\ell,\,\mathrm{sym}}(\omega) \ &\simeq\ \pi\sqrt{\lambda}\,\gamma^{5/2}T^3 \ \equiv\ \kappa_\ell\,. \end{aligned} \tag{3.37}$$

This in turn implies that, when probed over large time separations $t - t' \gg 1/z_s T$, the retarded propagator can be replaced by a local time derivative ('friction force'), while the noise–noise correlator looks local in time ('white noise'). The we can write

$$\gamma M_Q\,\frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2} = -\gamma\eta\,\frac{\mathrm{d}\delta x_\perp}{\mathrm{d}t} + \xi_\perp(t), \qquad \langle \xi_\perp(t)\xi_\perp(t')\rangle = \kappa_\perp\delta(t-t')\,, \tag{3.38}$$

for the transverse modes and, respectively (note that $P^z = \gamma\eta v$),

$$\gamma^3 M_Q\,\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2} = -\gamma^3\eta\,\frac{\mathrm{d}\delta x_\ell}{\mathrm{d}t} - \gamma\eta v + \xi_\ell(t), \qquad \langle \xi_\ell(t)\xi_\ell(t')\rangle = \kappa_\ell\,\delta(t-t')\,, \tag{3.39}$$

for the longitudinal one. The physical interpretation of these equations becomes more transparent if they are first rewritten in terms of the respective momenta $p_\perp = \gamma M_Q v_\perp$ and $p_\ell = \gamma M_Q v_\ell$, with $v_\perp = \mathrm{d}\delta x_\perp/\mathrm{d}t$ and $v_\ell = v + \mathrm{d}\delta x_\ell/\mathrm{d}t$.

At this point, we come across a rather subtle point: in all the equations written so far, the Lorentz factor γ is evaluated with the *average* velocity v of the heavy quark — the one which enters the trailing string solution (3.3). However, the event–by–event fluctuations of the velocity turn out to be significantly large (especially in the longitudinal sector; see below), and then it becomes appropriate to define the *event–by–event* (or 'fluctuating') momenta $p_\perp$ and p_ℓ by using the respective, event–by–event, Lorentz factor, as evaluated with the instantaneous velocity. For more clarity, let us temporarily denote by v_0 and γ_0 the average velocity and the associated Lorentz factor, $\gamma_0 \equiv 1/\sqrt{1-v_0^2}$, and reserve the notations v and γ for the respective fluctuating quantities:

$$v^2 = v_\ell^2 + v_\perp^2 = \left(v_0 + \frac{\mathrm{d}\delta x_\ell}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\delta x_\perp}{\mathrm{d}t}\right)^2, \qquad \gamma\,=\,\frac{1}{\sqrt{1-v^2}}\,. \tag{3.40}$$

When taking the time derivatives of $p_\perp$ and p_ℓ, as associated with variations in $v_\perp$ and, respectively, v_ℓ, one must also take into account the corresponding change in the γ–factor. Consider the longitudinal sector first:

$$\frac{1}{M_Q}\,\frac{\mathrm{d}p_\ell}{\mathrm{d}t} = \left(\gamma + v_\ell\frac{\partial\gamma}{\partial v_\ell}\right)\frac{\mathrm{d}v_\ell}{\mathrm{d}t} = \big(\gamma + v_\ell^2\gamma^3\big)\,\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2} \simeq \gamma_0^3\,\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2}\,, \tag{3.41}$$

where the last, approximate, equality follows since the fluctuations are assumed to be small, hence $v_\ell \simeq v_0$ and $\gamma \simeq \gamma_0$. The final result above is recognized as the expression in the l.h.s. of (3.39). To the same accuracy, we can write (with $\delta v_\ell = \mathrm{d}\delta x_\ell/\mathrm{d}t$)

$$\gamma v_\ell \simeq \left(\gamma_0 + \frac{\partial \gamma}{\partial v_\ell}\,\delta v_\ell\right)(v_0 + \delta v_\ell) \simeq \gamma_0 v_0 + (\gamma_0 + \gamma_0^3 v_0^2)\delta v_\ell = \gamma_0 v_0 + \gamma_0^3 \delta v_\ell\,, \tag{3.42}$$

in which we recognize the terms multiplying η in the r.h.s. of (3.39).

Consider similarly the transverse sector. The analog of (3.41) reads

$$\frac{1}{M_Q}\frac{\mathrm{d}p_\perp}{\mathrm{d}t} = \left(\gamma + v_\perp \frac{\partial \gamma}{\partial v_\perp}\right)\frac{\mathrm{d}v_\perp}{\mathrm{d}t} = \gamma\left(1 + v_\perp^2\gamma^2\right)\frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2} \simeq \gamma_0 \frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2}\,, \tag{3.43}$$

where we made the additional assumption that $v_\perp^2 \ll 1 - v_0^2$. (This can be always ensured by taking the quark mass M_Q to be sufficiently large.) Similarly, in the r.h.s. of (3.38), one can replace $\gamma_0 v_\perp \simeq p_\perp/M_Q$.

To summarize, to the accuracy of interest, we have derived the following Langevin equations for the dynamics of the heavy quark

$$\frac{\mathrm{d}p_\perp^i}{\mathrm{d}t} = -\eta_D p_\perp^i + \xi_\perp^i(t), \qquad \langle \xi_\perp^i(t)\xi_\perp^j(t')\rangle = \kappa_\perp \delta^{ij}\delta(t-t')\,, \tag{3.44}$$

$$\frac{\mathrm{d}p_\ell}{\mathrm{d}t} = -\eta_D p_\ell + \xi_\ell(t), \qquad \langle \xi_\ell(t)\xi_\ell(t')\rangle = \kappa_\ell\,\delta(t-t')\,, \tag{3.45}$$

where the upper index $i = 1, 2$ in (3.44) distinguishes between the two possible transverse directions, $\kappa_\perp$ and κ_ℓ are given in (3.37), and

$$\eta_D \equiv \frac{\eta}{M_Q} = \frac{\pi\sqrt{\lambda}}{2M_Q}T^2\,. \tag{3.46}$$

The general structure of these equations — with a friction term (or 'drag force') describing dissipation and a noise term describing momentum broadening — is as expected, and so are the above expressions for η_D, $\kappa_\perp$ and κ_ℓ, which agree with previous calculations in the literature [55, 56, 109]. It is however important to keep in mind that Eqs. (3.44)–(3.46) have been derived here only for the situation where the fluctuations in the velocity of the heavy quark remain small as compared to its average velocity v_0. To ensure that this is indeed the case, (3.45) for the longitudinal motion must be supplemented with an external force which is tuned to reproduce the average motion. (Without such a term, (3.45) would describe the rapid deceleration of the heavy quark due to its interactions in the plasma. Such a deceleration may entail additional phenomena, like bremsstrahlung, which are not encoded in the above equations; see the discussion in Refs. [78, 261, 86].) Namely, we shall add to the r.h.s. of (3.45) a term $F_{ext} = \eta\gamma_0 v_0$ which for large times equilibrates the average friction force and thus enforces a constant average velocity v_0. Further consequences of these equations will be discussed in Sect. 4.

3.3 Bulk picture of the stochastic motion

In the previous section we have obtained a set of Langevin equations for the heavy quark by integrating out the fluctuations of the upper part of the string, from the world–sheet horizon up to the boundary. The noise terms in these equations have been generated via boundary conditions at the world–sheet horizon, cf. (3.19). This suggests that, within the context of the supergravity calculation, quantum fluctuations are somehow encoded in the world–sheet horizon. To make this more explicit, we shall follow Refs. [241, 70] and construct a set of equations describing the stochastic dynamics of the upper part of the string, in which the noise term is acting on a point on the string which is infinitesimally close to the world–sheet horizon.

More precisely, we introduce a 'stretched' horizon at $z_h \equiv z_s - \epsilon$ and integrate out the fluctuations of the part of string lying between z_s and z_h. The procedure is quite similar to the one described in the previous section except that one has to fix the boundary values for the fluctuations also on the stretched horizon, rather than just on the Minkowski boundary. Denoting the respective values by ψ^h, where as before ψ stands generically for either δx_ℓ or $\delta x_\perp$, this procedure yields an effective action $S^h_{\rm eff}$ for ψ^h with the same formal structure as exhibited in Eq. (3.24), that is,

$$iS^h_{\rm eff} = -i\int \frac{{\rm d}\omega}{2\pi}\psi^h_a(-\omega)G^h_R(\omega)\psi^h_r(\omega) - \frac{1}{2}\int \frac{{\rm d}\omega}{2\pi}\psi^h_a(-\omega)G^h_{\rm sym}(\omega)\psi^h_a(\omega). \tag{3.47}$$

(We temporarily omit the term linear in the fluctuations; this will be restored in the final equations.) The horizon Green's functions G^h_R and $G^h_{\rm sym}$ will be shortly constructed. The calculations being quite involved, it is convenient to start with a brief summary of our main results:

The r–fields $\psi_r(\omega, z)$ describing the string fluctuations within the bulk ($z_m \le z \le z_h$) obey the equations of motion (4.6) with Neumann boundary conditions at $z = z_m$ — meaning that the string endpoint on the boundary is freely moving (except for the imposed longitudinal motion with velocity v_0) — and with Dirichlet boundary conditions at $z = z_h$: $\psi_r(\omega, z_h) = \psi^h_r(\omega)$. This boundary field $\psi^h_r(\omega)$ is however a stochastic variable, whose dynamics is described by the effective action (3.47). Via the classical solutions, this stochasticity is transmitted to the upper endpoint of the string, i.e., to the heavy quark. As a result, the latter obeys the same Langevin equations as previously derived in Sect. 2.

We start with the partition function encoding the quantum dynamics of the upper part of the string ($z_m \le z \le z_h$) in the Gaussian approximation:

$$Z = \int \left[D\psi^0_R D\psi_R D\psi^h_R\right]\left[D\psi^0_L D\psi_L D\psi^h_L\right]\, {\rm e}^{iS_R - iS_L + iS^h_{\rm eff}}. \tag{3.48}$$

The different measures $D\psi^0$, $D\psi$ and $D\psi^h$ correspond, respectively, to the path integral over the string endpoint on the Minkowski boundary, over the bulk part of the string, and over the point of the string on the stretched horizon (separately for the left and right quadrants of the Kruskal plane). Also, S_R and S_L are defined as in (4.4), but with the integral over z restricted to $z_m < z < z_h$. In what follows we shall construct the various pieces of the action which enter the exponent in (3.48).

(I) The effective action at the stretched horizon, $S^h_{\rm eff}$. As anticipated, this is obtained by integrating out the string fluctuations within the infinitesimal strip $z_h < z < z_s$. To that aim, we need the classical solutions $\psi_R(\omega, z)$ and $\psi_L(\omega, z)$ in the Kruskal plane which take the boundary values $\psi^h_R(\omega)$ and $\psi^h_L(\omega)$ at $z = z_h$ and are related by the condition (3.19). Clearly, the respective solutions read (in the (r, a) basis, for convenience)

$$\begin{aligned}
\psi_r(\omega, z) &= \psi^h_r(\omega)\psi^h_{ret}(\omega, z) + \frac{1 + 2n(\omega)}{2}\,\psi^h_a(\omega)(\psi^h_{ret}(\omega, z) - \psi^h_{adv}(\omega, z)),\\
\psi_a(\omega, z) &= \psi^h_a(\omega)\psi^h_{adv}(\omega, z),
\end{aligned} \tag{3.49}$$

where ψ^h_{ret} and ψ^h_{adv} are rescaled versions of the retarded and advanced solutions introduced in Sect. 2.2 which are normalized to 1 at $z = z_h$; e.g., $\psi^h_{ret}(\omega, z) = \psi_{ret}(\omega, z)/\psi_{ret}(\omega, z_h)$. For z close to z_s (and hence to z_h as well), these functions can be expanded as

$$\begin{aligned}
\psi^h_{\rm ret}(\omega, z) &= 1 + O(z_s - z)\,,\\
\psi^h_{\rm adv}(\omega, z) &= \left(\frac{z_s - z}{z_s - z_h}\right)^{\frac{i\omega z_H}{2z_s}} [1 + \mathcal{O}(z_s - z)]\,.
\end{aligned} \tag{3.50}$$

Substituting these classical solutions into the action produces the boundary action shown in (3.47), with G^h_R defined by the horizon version of (3.25). Given the near–horizon behaviour of the solutions (3.50) and of the local tension $T^{zz}(z)$ (which vanishes at $z = z_s$, cf. (3.6)), it is clear that only $\partial_z \psi^h_{adv}$ contributes to G^h_R in the limit $\epsilon \to 0$. This yields the following, purely imaginary, result:

$$G^h_{\perp,R}(\omega) \,=\, z_s^4 G^h_{\ell,\,R}(\omega) \,=\, -\frac{1}{2} T^{zz}_{\perp}(z_h)\partial_z \psi^h_{adv}(-\omega, z)\big|_{z=z_h} \,=\, -i\omega\gamma\eta\,. \tag{3.51}$$

This coincides, as it should, with the imaginary part of the respective boundary propagator[3], (3.30) (cf. the discussion at the end of Sect. 2.2). Then the symmetric Green's function $G^h_{\rm sym}$, which is related to $\,{\rm Im}\, G^h_R$ via the KMS relation (3.26), is exactly the same as the corresponding function on the boundary.

If the string point on the stretched horizon was a free endpoint endowed with the action (3.47), it would obey Langevin equations similar to those derived in Sect. 2.3. However, this is an internal point on the string, and as such it is also subjected to a tension force from the upper side of the string at $z < z_h$. This force is encoded in the bulk action $S_R - S_L$, to which we now turn.

(II) The bulk piece of the action $S_R - S_L$. This is defined as

$$\begin{aligned} S_R - S_L &= -\frac{1}{2}\int_{z_m}^{z_h} {\rm d}z \int {\rm d}t\, T^{\alpha\beta}(z)\Big[\partial_\alpha\psi_R\partial_\beta\psi_R - \partial_\alpha\psi_L\partial_\beta\psi_L\Big] \\ &= \frac{1}{2}\int_{z_m}^{z_h} {\rm d}z \int {\rm d}t\, \Big[\psi_R\partial_\alpha\Big(T^{\alpha\beta}\partial_\beta\psi_R\Big) - \psi_L\partial_\alpha\Big(T^{\alpha\beta}\partial_\beta\psi_L\Big)\Big] \\ &\quad -\frac{1}{2}\int {\rm d}t\, T^{zz}(z)\Big(\psi_R\partial_z\psi_R - \psi_L\partial_z\psi_L\Big)\Big|_{z=z_m}^{z=z_h} \end{aligned} \tag{3.52}$$

or, after going to Fourier space and to the (r, a)–basis,

$$\begin{aligned} S_R - S_L &= \int \frac{{\rm d}\omega}{2\pi}\int {\rm d}z\,\psi_a(-\omega, z)\partial_\alpha\Big[T^{\alpha\beta}(z)\partial_\beta\psi_r(\omega, z)\Big] \\ &\quad -\frac{1}{2}\int \frac{{\rm d}\omega}{2\pi}\, T^{zz}(z)\Big(\psi_a(-\omega, z)\partial_z\psi_r(\omega, z) + (r \leftrightarrow a)\Big)\Big|_{z=z_m}^{z=z_h}. \end{aligned} \tag{3.53}$$

We were so explicit here about the integration by parts, because this operation turns out to be quite subtle. First, notice that the contributions proportional to T^{zt} have cancelled in the boundary terms, for the same reason as discussed below (3.11), i.e., because they are odd functions of t (or ω). To ensure this property, it has been important to perform the previous operations in the order indicated above, that is, to first integrate by parts, as in (3.52), and only then change to the (r, a)–basis, as in (3.53). (Reversing this order would have affected the symmetry properties of the integrand, and then the terms $\propto T^{zt}$ would not cancel anymore.)

Second, there are some subtleties about the boundary value of $S_R - S_L$ at the stretched horizon, that we rewrite here for more clarity:

$$(S_R - S_L)^h_{\rm bndry} = -\frac{1}{2}\int \frac{{\rm d}\omega}{2\pi}\, T^{zz}(z_h)\Big(\psi_a(-\omega, z)\partial_z\psi_r(\omega, z) + (r \leftrightarrow a)\Big)\Big|_{z=z_h} \tag{3.54}$$

If we were to evaluate this action with the classical solutions (3.49), the result would precisely cancel the effective action (3.47) in the exponent of (3.48). Indeed, up to a sign, (3.54) has exactly the structure that has been used to build the effective action by inserting the classical solutions (compare to (3.23)). However, in the present context, (3.54) must be rather seen as the boundary value of the bulk action when approaching the stretched horizon from the *above* (i.e., from $z < z_h$), and as such it provides boundary conditions for the dynamics of the upper

[3] Incidentally, this calculation of $\,{\rm Im}\, G^h_R$, which is exact, together with the conservation law $\partial_z J^z = 0$, cf. (3.33), can be used to check, or even derive, the expressions for ${\rm Im}\, C_{ret}(\omega)$ and ${\rm Im}\, C_{adv}(\omega)$ given in Eqs. (3.15)–(3.16).

side of the string (see below). This being said, it is nevertheless possible, and also convenient, to use the appropriate piece of (3.54) in order to cancel the dissipative piece $\propto G_R^h$ in the effective action (3.47). This is simply the statement that the totality of the energy which crosses the stretched horizon coming from the above flows further down along the string.

Specifically, the relevant piece of (3.54) is that proportional to $\partial_z \psi_a$, which after using (3.49) can be evaluated as

$$-\frac{1}{2}T^{zz}(z_h)\partial_z \psi^h_{adv}(-\omega, z)\big|_{z=z_h} = G^h_R(\omega) \tag{3.55}$$

where we have recognized the expression (3.51) for G_R^h. One thus obtains:

$$(S_R - S_L)^h_{\text{bndry}} = -\int \frac{\mathrm{d}\omega}{2\pi}\, \psi^h_a(-\omega) \left[\frac{1}{2}T^{zz}(z)\partial_z\psi_r(\omega, z) - G^h_R(\omega)\psi^h_r(\omega)\right]_{z=z_h}. \tag{3.56}$$

As anticipated, the last term in (3.56) compensates the piece involving G_R^h in (3.47), and then the total action reads

$$\begin{aligned} iS_R - iS_L + iS^h_{\text{eff}} =& i\int \frac{\mathrm{d}\omega}{2\pi}\, \psi^0_a(-\omega) \left[\frac{1}{2}T^{zz}(z)\Big(\partial_z\psi_r(\omega, z) + \psi^0_r(\omega)\partial_z\psi_{adv}(-\omega, z)\Big) - P^z\right]_{z=z_m} \\ &+ i\int_{z_m}^{z_h} \mathrm{d}z \int \frac{\mathrm{d}\omega}{2\pi}\, \psi_a(-\omega, z)\partial_\alpha \left[T^{\alpha\beta}(z)\partial_\beta\psi_r(\omega, z)\right] \\ &- i\int \frac{\mathrm{d}\omega}{2\pi}\, \psi^h_a(-\omega) \left[\frac{1}{2}T^{zz}(z)\partial_z\psi_r(\omega, z) - \xi^h(\omega)\right]_{z=z_h} \end{aligned} \tag{3.57}$$

where the differences between longitudinal and transverse fluctuations are kept implicit (in particular, it is understood that the term proportional to P^z appears only in the longitudinal sector). Two additional manipulations have been necessary to write the action in its above form: (i) In the first line of (3.57) we have used $\psi_a(\omega, z) = \psi^0_a(\omega)\psi_{adv}(\omega, z)$, cf. (3.22). (ii) The piece involving $G^h_{\text{sym}}(\omega)$ in (3.47) have been reexpressed as a Gaussian path integral over the noise variables ξ^h, which therefore obey

$$\left\langle \xi^h(\omega)\xi^h(\omega')\right\rangle = 2\pi\delta(\omega+\omega')\,(1+2n)\omega\gamma\eta\,, \quad \text{with} \quad n(\omega) = \frac{1}{\mathrm{e}^{\omega/z_sT} - 1}\,. \tag{3.58}$$

Once again, (3.58) involves the effective temperature $T_{\text{eff}} = z_sT$.

By integrating over the fluctuating fields $\psi^0_a(-\omega)$, $\psi_a(-\omega, z)$, and $\psi^h_a(-\omega)$, we are finally left with the following set of equations of motion and boundary conditions:

(1) A modified Neumann boundary condition for the string endpoint string at the boundary (we temporarily reintroduce the polarization label p with $p = \ell$ or $\perp$) :

$$\frac{1}{2}\,T^{zz}_p(z)\left[\partial_z\psi^p_r(\omega, z) + \psi^{0,p}_r(\omega)\partial_z\psi_{adv}(-\omega, z)\right]_{z=z_m} = \eta\gamma v\,\delta_{p\ell}\,. \tag{3.59}$$

(2) The standard equations of motion for the fluctuations $\psi_r(\omega, z)$ of the string in the bulk at $z_m < z < z_h$ (cf. (4.6)).

(3) A stochastic equation for the point of the string on the stretched horizon :

$$\frac{1}{2}T^{zz}(z)\partial_z\psi_r(\omega, z)\big|_{z=z_h} = \xi^h(\omega). \tag{3.60}$$

We now analyze the consequences of these equations and, in particular, emphasize the differences w.r.t. the corresponding analysis in Ref. [241].

(3.60) is a Langevin equation of a special type: the noise term is precisely compensating the pulling force $T^{zz}(z_h)\partial_z\psi_r$ due to the string tension. By taking the expectation value of this equation and recalling that $T^{zz}_p(z_h) \sim \epsilon$ vanishes in the limit $\epsilon \to 0$, we conclude that

$\partial_z\langle\psi_r^p(\omega,z)\rangle$ is regular near the world–sheet horizon; this implies that the average value of the classical solution is proportional to ψ_{ret} :

$$\langle\psi_r(\omega,z)\rangle \;=\; \langle\psi_r^0(\omega)\rangle\,\psi_{ret}(\omega,z)\,. \tag{3.61}$$

The normalization is fixed by the expectation value of $\psi_r^0(\omega)$ — the boundary value of $\psi_r(\omega,z)$ at $z=z_m\ll 1$.

We will now construct the solution $\psi_r(\omega,z)$ to the EOM (3.7) by specifying its boundary values, $\psi_r^0(\omega)$ and $\psi_r^h(\omega)$, at the points $z=z_m$ and $z=z_h$, respectively. After simple algebra, the respective solution can be written as

$$\psi_r(\omega,z)=\psi_r^0(\omega)\psi_{ret}(\omega,z)+\left[\psi_r^h(\omega)-\psi_r^0(\omega)\psi_{ret}(\omega,z_h)\right]\frac{\psi_{ret}(\omega,z)-\psi_{adv}(\omega,z)}{\psi_{ret}(\omega,z_h)-\psi_{adv}(\omega,z_h)}\,.$$

The reason why this particular writing is natural is as follows: when taking the expectation value according to (3.61), we find $\langle\psi_r^h(\omega)\rangle=\langle\psi_r^0(\omega)\rangle\,\psi_{ret}(\omega,z_h)$, which shows that the coefficient $\psi_r^h(\omega)-\psi_r^0(\omega)\psi_{ret}(\omega,z_h)$ in front of the second term in (3.62) is a random variable with zero expectation value. Clearly, this term plays the role of a noise. The statistics of this noise is determined by the horizon Langevin equation (3.60), and in turn it implies a boundary Langevin equation for $\psi_r^0(\omega)$, via the condition (3.59). Let's see how all that works in detail. We will first rewrite (3.62) as

$$\psi_r(\omega,z)=\psi_r^0(\omega)\psi_{ret}(\omega,z)+i\,\xi^0(\omega)\frac{\psi_{ret}(\omega,z)-\psi_{adv}(\omega,z)}{\operatorname{Im}G_R(\omega)}\,, \tag{3.62}$$

thus fixing the normalization of the noise term $\xi^0(\omega)$. After inserting (3.62) in the Neumann boundary condition (3.59) (say, in the transverse sector), one finds[4]

$$G_R^0(\omega)\,\psi_r^0(\omega)\;=\;\xi^0(\omega)\,, \tag{3.63}$$

which is the standard form of a Langevin equation (compare to (3.36)).

It remains to check that the statistics of ξ^0, as inferred from (3.60), is indeed the same as previously derived in Sect. 3.2.3. To that aim, we insert the form (3.62) of the solution into (3.60); as already explained, the regular piece of the solution $\propto\psi_{ret}(\omega,z)$ does not contribute in the limit $\epsilon\to 0$, so we are left with

$$-\,\frac{i}{2}\,\frac{\xi^0(\omega)}{\operatorname{Im}G_R(\omega)}\,T^{zz}(z)\partial_z\psi_{adv}(\omega,z)\big|_{z=z_h}\;=\;\xi^h(\omega). \tag{3.64}$$

Using $T^{zz}(z)\partial_z\psi_{adv}(\omega,z)|_{z=z_h}\;=\;-2i\omega\gamma\eta\psi_{adv}(\omega,z_h)$, cf. (3.55), together with $\operatorname{Im}G_R(\omega)=-\omega\gamma\eta$, this finally becomes

$$\psi_{adv}(\omega,z_h)\,\xi^0(\omega)\;=\;\xi^h(\omega)\,. \tag{3.65}$$

This relation is in fact natural, as we argue now: from the transformation connecting ξ to ψ_a, or directly by comparing (3.62) with the standard expression (3.22) for ψ_r, one can see that the strength of the noise term scales like $\xi(\omega)\sim\psi_a(\omega)\,G_{\rm sym}(\omega)$. On the other hand, (3.22) implies $\psi_a^h(\omega)=\psi_{adv}(\omega,z_h)\psi_a^0(\omega)$. Hence one can write

$$\xi^h(\omega)\;\sim\;\psi_a^h(\omega)\,G_{\rm sym}^h(\omega)\;\simeq\;\psi_{adv}(\omega,z_h)\psi_a^0(\omega)\,G_{\rm sym}(\omega)\;\sim\;\psi_{adv}(\omega,z_h)\,\xi^0(\omega)\,. \tag{3.66}$$

[4]The following identities, which can be checked from (3.25), are useful in this respect:

$$G^0_{\perp,\,R}(\omega)=-\frac{1}{2}\,T_\perp^{zz}(z_m)\left[\partial_z\psi_{ret}(\omega,z)+\partial_z\psi_{adv}(-\omega,z)\right]_{z=z_m}.$$

$$\operatorname{Im}G_R(\omega)=\frac{i}{2}\,T_\perp^{zz}(z_m)\,\partial_z\left[\psi_{ret}(\omega,z)-\psi_{adv}(\omega,z)\right]_{z=z_m}.$$

Remarkably, the relative factor between ξ^0 and ξ^h in (3.65) does not spoil the normalization of the noise–noise correlator, because $|\psi_{adv}(\omega, z_h)| = 1$, as we now demonstrate. To that aim we rely on the observation at the end of Sect. 3.2.2 that the r.h.s. of (3.32) is independent of z. Clearly, this remains true after replacing $\psi_{ret} \to \psi_{adv}$ in (3.32). (Indeed, the current (3.33) is conserved for an arbitrary solution $\Psi_{sol}(t, z)$.) Writing $\psi_{adv}(\omega, z) = C(\omega)\psi^h_{adv}(\omega, z)$, so that $\psi_{adv}(\omega, z_h) = C(\omega)$, and evaluating the r.h.s. of (3.32) separately at $z = z_m$ and $z = z_h$, one deduces that $|C(\omega)| = 1$, as anticipated. Thus, the 2–point function $\langle\xi^0(\omega)\xi^0(\omega')\rangle$ is indeed the same as in Sect. 3.2.3. Note also that our previous argument is independent of the precise value of ϵ (the distance between the world–sheet and the stretched horizons), so long as ϵ is small enough for the near–horizon expansions to make sense. This strongly suggests that the strength of the noise remains constant along the string, from the stretched horizon up to the boundary.

3.4 Discussion and physical picture

In this section, we will first discuss some consequences of the previously derived Langevin equations, which support the idea that the noise terms in these equations are of non–thermal nature, and then propose a physical picture in which these fluctuations are interpreted as quantum mechanical fluctuations associated with medium–induced radiation.

3.4.1 Momentum distributions from the Langevin equations

An important property of the Langevin equations (3.44)–(3.45) that we would like to emphasize is that, except in the non–relativistic limit $\gamma \simeq 1$, these equations do not describe the thermalization of the heavy quark. There are several arguments to support this conclusion. For instance, in thermal equilibrium the momentum distributions should be isotropic, but this is clearly not the case for the large–time distributions generated by Eqs. (3.44)–(3.45), because of the mismatch between κ_ℓ and $\kappa_\perp$ when $\gamma > 1$. Besides, in order to generate the canonical distribution for a relativistic particle, $P \propto \exp\{-\sqrt{\mathbf{p}^2 + M_Q^2}/T\}$, the noise correlations must not only be isotropic, but also obey the relativistic version of the Einstein relation, which reads $\kappa = 2ET\eta_D$ [69]. Using Eqs. (3.37) and (3.46), it is easily seen that this condition is not satisfied for either transverse, or longitudinal, fluctuations (except if $\gamma = 1$, once again).

The Einstein relation is a particular form of the fluctuation–dissipation theorem, so its failure might look surprising given that the Green's functions at the basis of our Langevin equations obey the KMS condition (3.26). Recall, however, that this peculiar KMS condition involves an effective temperature $T_{\rm eff} = T/\gamma^{1/2}$; and indeed, in the transverse sector at least, the Einstein relation appears to be formally satisfied, but with $T \to T_{\rm eff}$. But this does not hold in the longitudinal sector, where κ_ℓ involves an additional factor γ^2. Hence, the present equations cannot lead to thermal distributions.

It is then interesting to compute the actual momentum distributions generated by these Langevin equations at large times. Consider first the transverse sector, and introduce the probability distribution $P(\mathbf{p}_\perp, t)$ for the transverse momentum $\mathbf{p}_\perp = (p_1, p_2)$ at time t:

$$P(\mathbf{p}_\perp, t) \equiv \int [D\xi_i]\, \delta\Big(\mathbf{p}_\perp - \mathbf{p}_\perp[\xi_i](t)\Big)\, \mathrm{e}^{-\frac{1}{2\kappa_\perp}\int \mathrm{d}t\, \xi_i(t)\xi_i(t)}\,. \tag{3.67}$$

Here, $\mathbf{p}_\perp[\xi_i](t)$ is the solution to (3.44) corresponding to a given realization of the noise[5], and reads (with $i = 1, 2$; we assume $p_i(0) = 0$ so that $\langle\mathbf{p}_\perp(t)\rangle = 0$ at any time)

$$p_i(t) = \int_0^t \mathrm{d}t'\, \mathrm{e}^{-\eta_D(t-t')}\, \xi_i(t')\,. \tag{3.68}$$

[5]In general, this solution will depend on our prescription for discretizing the time axis; this is so since the noise–noise correlator depends itself on the momentum ('multiplicative noise'). But to the accuracy of interest, we can treat γ in (3.37) as the fixed quantity γ_0, and then one can safely use continuous notations.

This implies $\langle p_1^2 \rangle = \langle p_2^2 \rangle \equiv \langle p_\perp^2 \rangle$ with

$$\langle p_\perp^2(t) \rangle \,=\, \frac{\kappa_\perp}{2\eta_D}\left(1-\mathrm{e}^{-2\eta_D t}\right) \,\simeq\, \frac{\kappa_\perp}{2\eta_D}\,, \tag{3.69}$$

where the last, approximate equality holds for large times $\eta_D t \gg 1$. Returning to (3.67), this gives

$$P(\mathbf{p}_\perp, t) \,=\, \frac{1}{2\pi\langle p_\perp^2(t)\rangle}\,\exp\left\{-\frac{p_1^2+p_2^2}{2\langle p_\perp^2(t)\rangle}\right\}. \tag{3.70}$$

A similar expression holds in the longitudinal sector, but only after subtracting away the global motion with velocity v_0, which one can do by writing $\delta p_\ell \equiv p_\ell - p_0$ with $p_0 = M_Q\gamma_0 v_0$.

For large times $t \gg 1/\eta_D$, the transverse and longitudinal momentum distributions for the heavy quark approach the following, stationary, forms

$$\begin{aligned} P(\mathbf{p}_\perp, t) \;&\simeq\; \frac{1}{2\pi\gamma^{1/2}TM_Q}\,\exp\left\{-\frac{p_1^2+p_2^2}{2\gamma^{1/2}TM_Q}\right\}, \\ P(\delta p_\ell\,, t) \;&\simeq\; \frac{1}{\sqrt{2\pi\gamma^{5/2}TM_Q}}\,\exp\left\{-\frac{\delta p_\ell^2}{2\gamma^{5/2}TM_Q}\right\}, \end{aligned} \tag{3.71}$$

which formally look like thermal, Maxwell–Boltzmann, distributions for *non–relativistic* particles, but with different temperatures in the transverse and longitudinal sector — $T_\perp = \gamma^{1/2}T$ and $T_\ell = \gamma^{5/2}T$ —, none of them equal to the plasma temperature T.

3.4.2 Physical picture: Medium–induced radiation

In this section, we propose a physical picture for the dynamics of the heavy quark, as encoded in the Langevin equations (3.44)–(3.46). The general picture is that of medium–induced parton branching, as previously developed in Refs. [116, 117, 78, 141], that we shall here adapt to the problem at hand. As we will see, this qualitative and admittedly crude picture provides the right parametric estimates for both the drag force and the (transverse and longitudinal) momentum broadening. Besides, it supports the non–thermal nature of the noise terms in the Langevin equations.

Due to its interactions with the strongly–coupled plasma, a heavy quark can radiate massless $\mathcal{N} = 4$ SYM quanta (gluons, adjoint scalars and fermions) which then escape in the medium, thus entailing energy loss towards the plasma and momentum broadening (due to the recoil of the heavy quark associated with successive parton emissions). This dynamics is illustrated in Fig. 3.2. The main ingredients underlying our physical picture are as follows:

`(i)` The emission of a virtual parton with energy ω and (space–like) virtuality $Q^2 = \mathbf{k}^2 - \omega^2 > 0$ requires a formation time $t_{\rm coh} \sim \omega/Q^2$. ($\mathbf{k}$ is the parton 3–momentum, and we assume high–energy kinematics: $|\mathbf{k}| \simeq \omega \gg Q$.) This follows from the uncertainty principle: in a comoving frame where the parton has zero momentum, its formation time is of order $1/Q$; this becomes ω/Q^2 after boosting by the parton Lorentz factor $\gamma_p = \omega/Q$. Note also that, when the parent heavy quark is highly energetic ($\gamma \gg 1$), the momentum $\mathbf{k}$ of the emitted parton is predominantly longitudinal: $|\mathbf{k}| \simeq k_\ell \simeq \omega$, whereas $k_\perp \sim Q \ll k_\ell$.

`(ii)` During the formation time $t_{\rm coh} \sim \omega/Q^2$, the heavy quark does not radiate just a single parton, but rather a large number of quanta, of $\mathcal{O}(\sqrt{\lambda})$, whose emissions are uncorrelated with each other. This is merely an assumption, which as we shall see provides the right λ–dependence for the final results.

`(iii)` Only those quanta can be lost towards the plasma, whose virtualities are small enough — smaller than the *saturation momentum* $Q_s \sim t_{\rm coh} T^2$ corresponding to the parton formation time. This follows from the analysis in Ref. [116] which shows that an energetic parton propagating through the strongly coupled plasma feels the latter as a constant force $\sim T^2$ which acts towards reducing its transverse momentum (or virtuality). Then, a space–like parton, which would be stable in the vacuum, can decay inside the plasma provided the lifetime $t_{\rm coh}$ of its virtual fluctuations is large enough for the mechanical work $\sim t_{\rm coh}T^2$ done by the plasma to compensate the energy deficit $\sim Q$ of the parton. This condition amounts to $Q \lesssim Q_s$, with the upper limit given by

$$Q_s \,\sim\, t_{\rm coh}T^2 \,\sim\, (\omega T^2)^{1/3} \,\sim\, \sqrt{\gamma_p}\, T\,, \tag{3.72}$$

where we have also used $t_{\rm coh} \sim \omega/Q^2$ and $\gamma_p = \omega/Q$.

(iv) The rapidities of the radiated quanta are bounded by the rapidity of the heavy quark: $\gamma_p \lesssim \gamma$. This is again motivated by the uncertainty principle and at least at weak coupling it is confirmed by the explicit construction of the heavy quark wavefunction [76].

We shall now use this picture to compute the rate for energy loss and momentum broadening of the heavy quark. The latter radiates energy $\Delta E \sim \sqrt{\lambda}\omega$ over a time interval $\Delta t \sim \omega/Q^2$, where ω and Q are constrained by $Q \lesssim Q_s(\omega, T)$. The dominant contribution to the rate $|\Delta E/\Delta t|$ comes from those quanta carrying the maximal possible energy $\omega \simeq \gamma Q$ and also the maximal corresponding virtuality $Q \simeq Q_s(\gamma, T) \sim \sqrt{\gamma}\, T$ (to minimize the emission time). Therefore,

$$-\,\frac{{\rm d}E}{{\rm d}t} \,\simeq\, \frac{\sqrt{\lambda}\,\omega}{(\omega/Q_s^2)} \,\simeq\, \sqrt{\lambda}\,Q_s^2 \,\sim\, \sqrt{\lambda}\,\gamma\, T^2\,, \tag{3.73}$$

in qualitative agreement with the estimate for the drag force $F_{\rm drag} = \eta\gamma v \sim \gamma\sqrt{\lambda}T^2$ in (3.45). (Recall that we consider the relativistic case $v \simeq 1$.)

Consider similarly momentum broadening: being uncorrelated with each other, the $\sqrt{\lambda}$ quanta emitted during a time interval $t_{\rm coh}$ have transverse momenta which are randomly oriented, so their emission cannot change the *average* transverse momentum of the heavy quark. However, the changes in the *squared* momentum add incoherently with each other, thus yielding (once again, the dominant contribution comes from quanta with $Q \sim Q_s(\gamma, T)$ and $\omega \simeq \gamma Q$)

$$\frac{{\rm d}\langle p_\perp^2\rangle}{{\rm d}t} \,\sim\, \frac{\sqrt{\lambda}\,Q_s^2}{(\omega/Q_s^2)} \,\sim\, \sqrt{\lambda}\,\frac{Q_s^4}{\gamma Q_s} \,\sim\, \sqrt{\lambda}\,\sqrt{\gamma}\,T^3\,, \tag{3.74}$$

which is parametrically the same as the estimate for $\kappa_\perp$ in the first equation (3.37). The random emissions also introduce fluctuations in the energy (or longitudinal momentum) of the heavy quark, in addition to the average energy loss. The dispersion associated with such fluctuations is estimated similarly to (3.74) (below, $\delta p_\ell \equiv p_\ell - \langle p_\ell\rangle$)

$$\frac{{\rm d}\langle \delta p_\ell^2\rangle}{{\rm d}t} \,\sim\, \frac{\sqrt{\lambda}\,\omega^2}{(\omega/Q_s^2)} \,\sim\, \sqrt{\lambda}\,\sqrt{\gamma}\,\gamma^2\,T^3\,, \tag{3.75}$$

in qualitative agreement with the previous result, (3.37), for κ_ℓ. Note that, with this interpretation, the relative factor γ^2 in between κ_ℓ and $\kappa_\perp$ is simply the consequence of the relation $\omega \simeq \gamma Q$ between the energy and the virtuality (or transverse momentum) of an emitted parton.

This physical picture also clarifies the role of the world–sheet horizon in the dual gravity calculation: via the UV/IR correspondence, the radial position $z_s = 1/\sqrt{\gamma}$ of this horizon (in units of $z_H = 1/\pi T$) is mapped onto the saturation momentum $Q_s \sim \sqrt{\gamma}\,T$ in the boundary, gauge, theory. Hence the emergence of the noise terms from the near–horizon dynamics of the string reflects quantum–mechanical fluctuations in the emission of quanta with virtualities $Q \sim Q_s$, which as we have just seen control momentum broadening.

It is furthermore interesting to compare the above physical picture to the corresponding one at weak coupling [17, 19, 177, 57]. Note first that the mechanism for momentum broadening is different in the two cases: at weak coupling, this is dominated by thermal rescattering, i.e., by successive collisions with the plasma constituents which are thermally distributed (see Fig. 3.3). In that case, the rate ${\rm d}\langle p_\perp^2\rangle/{\rm d}t \equiv \hat{q}$ defines a genuine transport coefficient — the "jet–quenching parameter" —, i.e. a local quantity which depends only upon the local density of thermal constituents (quarks and gluons) together with the gluon distribution produced via their high–energy evolution. By contrast, at strong coupling, the dominant mechanism at work is medium–induced radiation, which is intrinsically non–local (it requires the formation time $t_{\rm coh}$) and hence cannot be expressed in terms of a local transport coefficient. Medium–induced radiation is of course possible at weak coupling too (see Fig. 3.4), but the respective contribution is suppressed by a factor g^2N_c as compared to the thermal rescattering. We see that, formally, it is the replacement $g^2N_c \to \sqrt{\lambda}$ (i.e., the coherent emission of a large number of quanta) which makes the medium–induced radiation become the dominant mechanism for momentum broadening at strong coupling.

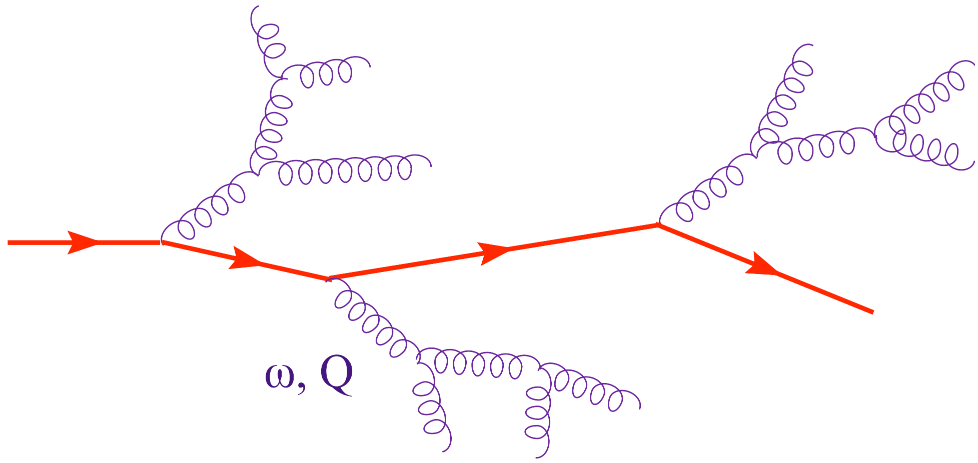


Figure 3.2: *Energy loss and momentum broadening via medium–induced parton emission at strong coupling. It is understood that the radiated partons feel a plasma force which allows them to be liberated from the parent heavy quark (see text for details).*

On the other hand, energy loss is predominantly due to medium–induced radiation at both weak and strong coupling, but important differences occur between the detailed mechanisms in the two cases (compare Figs. 3.2 and 3.4): At weak coupling, the radiated gluon, which typically comes from a highly virtual gluon in the quark wavefunction, is freed (radiated) via thermal rescattering. At strong coupling, radiation is caused by the plasma force $\sim T^2$. After being emitted, the parton undergoes successive medium–induced branchings, thus producing a system of partons with lower and lower energies and transverse momenta, down to values of $\mathcal{O}(T)$, when the partons cannot be distinguished anymore from the thermal bath.

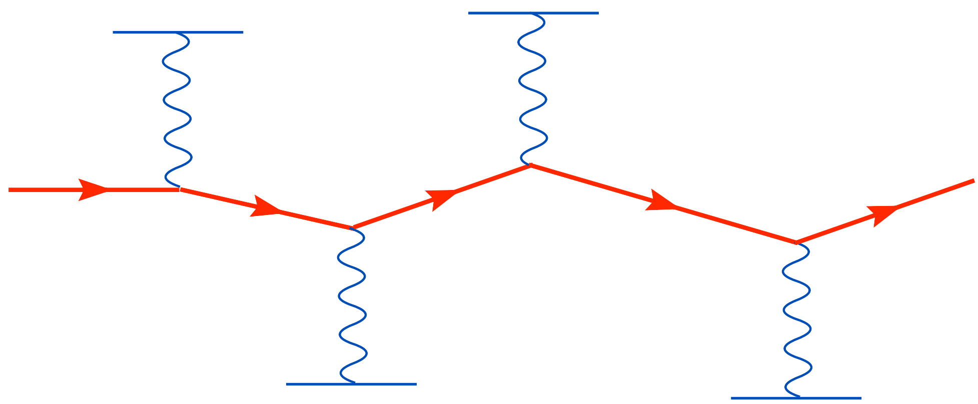


Figure 3.3: *Momentum broadening via thermal rescattering at weak coupling.*

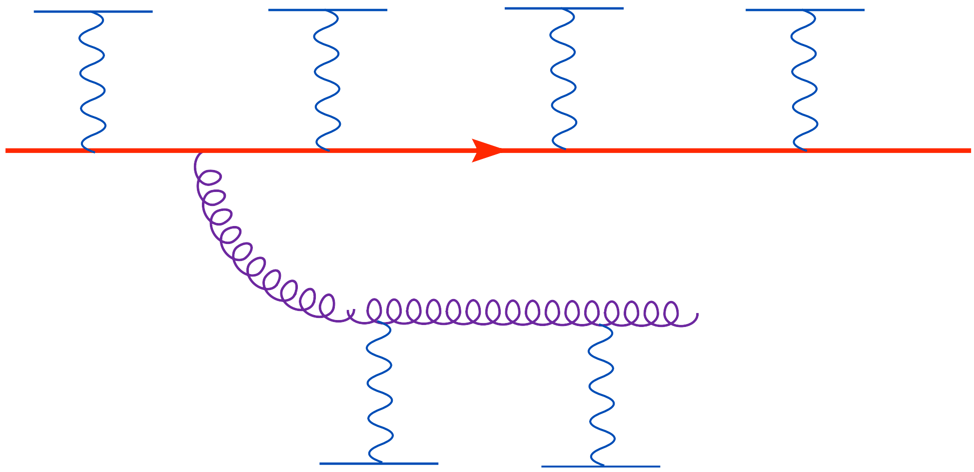


Figure 3.4: *Energy loss via medium–induced gluon emission at weak coupling*

It is finally interesting to notice that, in spite of such physical dissymmetry, the formula for energy loss at weak coupling can be written in a form which ressembles (3.73), namely

$$-\frac{\mathrm{d}E}{\mathrm{d}t} \simeq g^2 N_c\, Q_s^2 \qquad \text{(weak coupling)}\,, \tag{3.76}$$

where however Q_s is now the saturation momentum to lowest order in perturbative QCD and is related to the respective jet–quenching parameter via $Q_s^2 \simeq \hat{q} t_{\rm coh}$. Energy loss involves a coherent phenomenon at both weak and strong coupling.

Chapter 4

Heavy Quark in an Expanding Plasma in AdS/CFT

Using the Janik–Peschanski dual to a Bjorken flow, a Langevin equation is derived for a heavy quark in an expanding $\mathcal{N} = 4$ supersymmetric Yang–Mills plasma. Such a plasma is characterized by a proper–time dependence of the temperature and corresponds to a system out of equilibrium. The analysis first focuses on a quark at rest in the plasma comoving frame. The case of a quark moving across a longitudinally expanding plasma is then considered. The two–point functions for the random noise acting on such heavy quark probes are computed.

4.1 Introduction

Many problems in physics require going beyond standard Feynman diagrams and S–matrices calculations. In non–equilibrium settings, interactions generally take place in a short time interval and cannot be switched adiabatically as is done e.g. in the LSZ reduction formula for scattering experiments. An asymptotic state might also be out of grasp due to an inherent instability of the system. The initial state is known though, so that $\langle in \mid in \rangle$ matrix elements still provide valuable data. This is at the core of the Schwinger–Keldysh method where the amplitudes are calculated along a path extended in the complex time plane [161, 184]. While in non–equilibrium statistical physics the response of a system to a disturbance can often be reduced to real–time Green functions for thermal equilibrium systems, the method of Keldysh Green functions was historically first developed to directly tackle systems out of equilibrium. Equilibrium and non–equilibrium statistical physics are actually formally equivalent when one introduces a contour–ordering to replace the usual time–ordering. See section 2.1.3 of [184] and references therein for a more detailed discussion. Non–equilibrium statistical physics is concerned with correlators of the type $\langle \mathcal{O}(t) \rangle = \mathrm{Tr}\,[\rho \mathcal{O}(t)]$ for $t > t_i$, where ρ denotes the distribution for an equilibrium hamiltonian but $\mathcal{O}(t)$ is an operator in a Heisenberg representation with respect to an hamiltonian with an interactions part. Here, t_i refers to an initial time. The standard procedure for obtaining a non–equilibrium state is to consider a state which until t_i was in equilibrium with a reservoir and was thus prepared in some initial conditions. At $t > t_i$ the state is disconnected from the reservoir and interactions are switched on. In fact, unless for fleeting properties of a system out of equilibrium, the dependence on the initial state is rapidly lost due to interactions and the distribution ρ is arbitrary in this case.
In [130] a prescription for computing Keldysh Green functions in the AdS/CFT correspondence was found and later implemented in [109] for computing transverse and longitudinal momentum broadening for a heavy quark, from variations of the underlying Wilson line. Recent works [70, 92, 241] explore the Langevin description for a heavy quark from the gauge–string duality. The present chapter of this thesis aims at generalizing this to an expanding plasma. This is a non–equilibrium situation. In particular the medium is characterized by a local temperature

whose proper–time dependence obeys a scaling law first devised by Bjorken [43]. Actually, the recipe of [130] for computing real–time correlators in AdS/CFT was later justified in a series of papers by Skenderis and van Rees [236, 237, 255]. See [255] for a review and further explanations on how their results reduce to the ingoing boundary condition for bulk fields of [130] when the sources are set equal on both boundaries of the Penrose diagrams used in such calculations. Besides, their work is amenable to all sorts of initial states and ensembles by switching additional sources in the Euclidean segments of the devised construction.
The authors of [63] studied horizon formation and thermalization in a non–Abelian plasma resulting from turning on background fields, described by gravitational waves. It would be very interesting to derive transport coefficients such as momentum broadening coefficients for a hard probe from the numerical analysis presented in [63] but how this might be achieved is obscured by a lack of hindsight for an evolution out of equilibrium at strong coupling in AdS/CFT. The approach presented in the present work relies on the leading–order expansion at large times to the Janik–Peschanski dual [152, 153] to a Bjorken flow and it allows for explicit results.
The next section first reviews the work of Kim, Sin and Zahed [162] and explains how to derive, for a quark at rest in the expanding plasma comoving frame, a Langevin equation. The correlators of the random forces thereof are computed. Section 4.3. is concerned with a fast quark moving transversally in a strongly coupled $\mathcal{N}=4$ supersymmetric Yang–Mills plasma experiencing Bjorken flow. The gravity dual corresponds to a string trailing in the Janik–Peschanski background. The dispersions relations, energy loss parameter and momentum broadening coefficients are derived. They exhibit the expected scaling behaviour for the temperature, with no other dependence on the initial thermalization temperature. The method used to compute those quantities in a non–equilibrium, expanding plasma relies on a coordinate change and a particular Fourier–like mode–expansion to map the problem to a situation where the background has a fixed, global temperature.

4.2 Transverse and rapidity fluctuations in an expanding plasma and the Langevin description

In [152, 153] the gravity dual to a Bjorken flow [43] was derived in a $\tau^{-2/3}$ expansion to the bulk metric in Fefferman-Graham coordinates. The proper–time of an expanding plasma τ is related, along with the rapidity y, to the physical laboratory time t_{lab} and direction of expansion x^3 as $t_{lab} = \cosh(y)\tau$, $x^3 = \sinh(y)\tau$. Those parameters are convenient for describing the hydrodynamic regime which takes over after a scenario where typically at proper time $\tau = 0$ two gold nuclei collide at high enough energy that their subsequent evolution leads to a quark–gluon plasma. At proper time τ_0 the resulting plasma is thermalized and its properties are described by Bjorken's hydrodynamic model [43]. The plasma expands along the collision axis. Most useful to the remainder of this work is the scaling law

$$T^3\tau^\alpha = const, \tag{4.1}$$

where $\alpha = 3v_S^2$. From conformal invariance the sound velocity v_S is set to $1/\sqrt{3}$ and then $\alpha = 1$. In this chapter, especially in Section 4.3. thereof, it is assumed that the plasma expands for a sufficiently long period of time that the quark probes a large enough distance L of the quark–gluon plasma. No other possible phase will be considered.
The leading order result in the JP expansion reads

$$ds^2 = \frac{R^2}{z_H^2 z^2}\left[-\frac{(1-w^4)^2}{(1+w^4)}d\tau^2 + (1+w^4)\left[\tau^2 dy^2 + dx_\perp^2\right] + z_H^2 dz^2\right], \tag{4.2}$$

with $w = \frac{z}{(\tau/\tau_0)^{1/3}}\epsilon^{1/4}$, $\epsilon = (\pi T_0)^4/4$ and where $T_0 = \frac{1}{\pi z_H}$ is the Hawking temperature.
The picture that emerges is that of a black hole whose horizon is moving away from the boundary.

The coordinate change, $\frac{t}{t_0} = \frac{3}{2}(\frac{\tau}{\tau_0})^{2/3}$, $u(t,z) = \frac{2w^2}{1+w^4}$ – after discarding the non–diagonal components which ensue, as they are subleading in the τ expansion – yields [162]

$$ds^2 = \frac{R^2}{z_H^2 u}\left(-f dt^2 + \frac{4t^2}{9}dy^2 + \frac{3t_0}{2t}dx_\perp^2 + z_H^2\frac{du^2}{4uf}\right), \tag{4.3}$$

where $f(u) = 1 - u^2$.

The above form of the metric proves convenient as it converts a time–dependent problem into a setting where the usual recipe for extracting dual gauge theory correlators from fields in a AdS–Schwarzschild black hole background applies.

The time–dependent transverse and rapidity components of the metric are accounted for by Fourier–Hankel transformations. The study of transverse string fluctuations was carried out in [162] where the corresponding momentum broadening coefficient and diffusion coefficients were found for a heavy quark probe at rest in the plasma co–moving frame.

The remainder of this section generalizes this to the rapidity fluctuations as well. Moreover, the Kubo–Martin–Schwinger formula relating the retarded and symmetric correlators is derived. The construction of Schwinger–Keldysh propagators in AdS/CFT first devised in [130] and later justified in [236, 237, 255] thus holds. This then ensures for the existence of a Langevin description – which was merely postulated in [162].

The Nambu–Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-g}, \qquad g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{4.4}$$

can be expanded to quadratic order in the transverse and rapidity fluctuations $\delta X^{1,2} = \xi^{1,2}(t,u)$ and $\delta y(t,u)$ in the background specified by the target–space metric components $G_{\mu\nu}$ of (4.3). This provides

$$S_{NG} = -\frac{\sqrt{\lambda}T_0}{4}\int dt\, du \frac{1}{u^{3/2}} + \frac{\sqrt{\lambda}T_0}{8}\int dt\, du\left(\frac{3t_0}{2t}\right)\sum_{i=1,2}\left[\frac{(\partial_t\xi^i)^2}{u^{3/2}f(u)} - (2\pi T_0)^2\frac{f(u)}{u^{1/2}}(\partial_u\xi^i)^2\right]$$
$$+\frac{\sqrt{\lambda}T_0}{8}\int dt\, du\left(\frac{4t^2}{9}\right)\left[\frac{(\partial_t\delta y)^2}{u^{3/2}f(u)} - (2\pi T_0)^2\frac{f(u)}{u^{1/2}}(\partial_u\delta y)^2\right]. \tag{4.5}$$

Here $\lambda = R^2/\alpha' >> 1$, so that string loop corrections are negligible at this order and computations at the two–derivatives supergravity level are reliable. The action (4.5) is the same as in a static black hole background apart from overall time–dependent factors. The equations of motion are

$$\left[\partial_t^2 - \frac{1}{t}\partial_t + 2\pi^2T_0^2 f(u)(1+3u^2)\partial_u - (2\pi T_0)^2 u f(u)^2\partial_u^2\right]\xi^i = 0, \quad i = 1,2 \tag{4.6}$$

along with

$$\left[\partial_t^2 + \frac{2}{t}\partial_t + 2\pi^2T_0^2 f(u)(1+3u^2)\partial_u - (2\pi T_0)^2 u f(u)^2\partial_u^2\right]\delta y = 0. \tag{4.7}$$

Expanding in a basis defined by Hankel functions

$$\begin{cases} \xi^i(t,u) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{i\pi\omega}{2}}tH_1^{(2)}(\omega t)\Psi_\omega(u)\tilde{\xi_0}^i(\omega); \\ \delta y(t,u) = \int_{\infty}^{\infty}\sqrt{\frac{i\pi\omega}{2}}(\frac{-i}{\sqrt{t}})H_{1/2}^{(2)}(\omega t)\Phi_\omega(u)\tilde{\delta y_0}(\omega), \end{cases} \tag{4.8}$$

yields

$$\left[\partial_u^2 - \frac{3u^2+1}{2uf(u)}\partial_u + \frac{\mathfrak{w}^2}{4uf(u)^2}\right]\begin{pmatrix}\Psi_\omega\\ \Phi_\omega\end{pmatrix}(u) = 0, \quad \mathfrak{w} = \frac{\omega}{\pi T_0}, \tag{4.9}$$

where Ψ_ω and Φ_ω are normalized to unity at $u = 0$. Inserting (4.8) into (4.5), using the equations of motion and integrating by parts gives

$$S_{bndry} = \frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{4}\sum_{i=1,2}\int dt\ \frac{f(u)}{\sqrt{ut}}\xi^i\partial_u\xi^i(t,u)\ |_{u=0}^{u=1} + \frac{2\pi^2\sqrt{\lambda}T_0^3}{9}\int dt\ t^2\frac{f(u)}{\sqrt{u}}\delta y\partial_u\delta y(t,u)\ |_{u=0}^{u=1}, \tag{4.10}$$

One then appeals to the approximate completeness relation

$$-\frac{1}{4}\int_{-\infty}^{\infty} dt\ tH_\nu^{(2)}(\omega t)H_\nu^{(2)}(-\omega' t) \simeq \frac{1}{\omega}\delta(\omega-\omega'), \tag{4.11}$$

which stems from the exact relation $\int_0^\infty dt\ tJ_\nu(\omega t)J_\nu(\omega' t) = \frac{1}{\omega}\delta(\omega-\omega')$ for Bessel functions. One can argue that (4.11) is a fair approximation given that the dominant contributions in the integrals come from the late time region and that the Janik–Peschanski metric is defined as a large τ inverse expansion.
As a result, the following expressions for the retarded Green functions hold

$$\begin{cases} G_{R,\perp}(\omega) = \left[-\frac{3\pi^2\sqrt{\lambda}T_0^3t_0}{2}\right]\left[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0}; \\ G_{R,\delta y}(\omega) = \left[\frac{4\pi^2\sqrt{\lambda}T_0^3}{9}\right]\left[\frac{f(u)}{\sqrt{u}}\Phi_{-\omega}(u)\partial_u\Phi_\omega(u)\right]_{u=0}. \end{cases} \tag{4.12}$$

In the following, it is checked explicitly that the symmetrized Wightman functions $G_{sym}(\omega)$ are related to the corresponding retarded correlators by a Kubo–Martin–Schwinger (KMS) relation [186, 184]

$$G_{sym}(\omega) = -\coth(\frac{\omega}{2T_0})Im\ G_R(\omega). \tag{4.13}$$

It involves the temperature T_0, which is the initial, thermalization temperature in the original Bjorken frame. The following illustrates how the proper–time dependent temperature appears in the 2–point functions for this frame, from the Green functions computed in the $\{t-u\}$ system. For this purpose let us follow the usual prescription as it appears in [55, 56, 92, 109, 130, 241] and expand a general solution in the right and left quadrants of the black hole background (4.3), whose Kruskal diagram is the same as for a AdS–Schwarzschild black hole :

$$\begin{cases} \Upsilon_{R,\ \omega}(u) = A(\omega)\Psi^H_{\omega,\ in}(u) + B(\omega)\Psi^H_{\omega,\ out}(u); \\ \Upsilon_{L,\ \omega}(u) = C(\omega)\Psi^H_{\omega,\ in}(u) + D(\omega)\Psi^H_{\omega,\ out}(u), \end{cases} \tag{4.14}$$

$\Upsilon_\omega(u)$ denoting collectively $\Psi_\omega(u)$ or $\Phi_\omega(u)$ from (4.9).
$\Psi^H_{\omega,\ in}(u)$ and $\Psi^H_{\omega,\ out}(u)$ form a basis of two independent wave–functions whose expansion near the horizon at $u = 1$ is, up to $\mathcal{O}(\omega^2)$ terms

$$\begin{cases} \Psi^H_{\omega,\ in} = (1-u^2)^{-i\frac{\mathfrak{w}}{4}}\left[1+\frac{i\mathfrak{w}}{8}(\pi - 4\tan^{-1}(\sqrt{u}) - 6\log(2) + 2\log(1+u)(1+\sqrt{u})^2)\right]\ ; \\ \Psi^H_{\omega,\ out} = \Psi^{H*}_{\omega,\ in}. \end{cases} \tag{4.15}$$

The Kruskal coordinates are cast in the form $U = -\frac{1}{2\pi T_0}e^{-2\pi T_0(t-r_*)}$, $V = \frac{1}{2\pi T_0}e^{2\pi T_0(t+r_*)}$. $r_* = \frac{1}{4\pi T_0}\left[1+\log(\frac{1}{u}-1)\right]$ denotes the tortoise coordinate.
From

$$\begin{cases} (-U)^{\frac{i\mathfrak{w}}{2\pi T_0}} \simeq (1-u)^{i\mathfrak{w}/4}e^{-i\omega t}(\frac{1}{2\pi T_0})^{i\mathfrak{w}/2}e^{-i\mathfrak{w}/4}\left[1+(1-u)\frac{i\mathfrak{w}}{2}\right]\ ; \\ (V)^{-\frac{i\mathfrak{w}}{2\pi T_0}} \simeq (1-u)^{-i\mathfrak{w}/4}e^{-i\omega t}(\frac{1}{2\pi T_0})^{-i\mathfrak{w}/2}e^{-i\mathfrak{w}/4}\left[1-(1-u)\frac{i\mathfrak{w}}{2}\right]\ ; \\ H_\nu^{(2)}(\omega t) \simeq \sqrt{\frac{2}{\pi\omega t}}e^{-i(\omega t - \pi\nu/2 - \pi/4)}, \quad |\ \omega t\ |\to\infty, \end{cases} \tag{4.16}$$

near the horizon, one obtains the following behaviour for the modes satisfying the equations of motion :

$$\left\{ \begin{array}{l} \sqrt{\frac{i\pi\omega}{2}} t H_1^{(2)}(\omega t)\Psi^H_{\omega,\ in}(u \simeq 1) \simeq \sqrt{\log(\frac{-V}{U})} e^{-\frac{i\mathfrak{w}}{2}\log(V)} \ ; \\ \sqrt{\frac{i\pi\omega}{2}} t H_1^{(2)}(\omega t)\Psi^H_{\omega,\ out}(u \simeq 1) \simeq \sqrt{\log(\frac{-V}{U})} e^{-\frac{i\mathfrak{w}}{2}\log(-U)}, \end{array} \right. \tag{4.17}$$

and

$$\left\{ \begin{array}{l} \sqrt{\frac{i\pi\omega}{2}} (\frac{-i}{\sqrt{t}}) H_1^{(2)}(\omega t)\Psi^H_{\omega,\ in}(u \simeq 1) \simeq \frac{1}{\log(\frac{-V}{U})} e^{-\frac{i\mathfrak{w}}{2}\log(V)} \ ; \\ \sqrt{\frac{i\pi\omega}{2}} (\frac{-i}{\sqrt{t}}) H_1^{(2)}(\omega t)\Psi^H_{\omega,\ out}(u \simeq 1) \simeq \frac{1}{\log(\frac{-V}{U})} e^{-\frac{i\mathfrak{w}}{2}\log(-U)}, \end{array} \right. \tag{4.18}$$

The conditions at the horizon used in [130] amount to the analyticity of the infalling modes in the lower V complex plane (which guarantees that such modes carry positive energy). Similarly, they guarantee that the outgoing solutions are of negative energy, hence analytic in the upper U plane. A full justification of this recipe and a generalization to a broader framework for computing real–time correlators in the gauge/gravity correspondence appears in [236, 237, 255]. One can generalize and extend the transformation from the right quadrant ($U < 0$, $V > 0$) to the left quadrant ($U > 0$, $V < 0$) to $V \to\mid V \mid e^{-i\theta}$, $-U \to\mid U \mid e^{-i(2\pi-\theta)}$ [241], where θ was naturally set to π in [130]. In the case at hands, $\theta = 0 \ mod \ [\pi]$ is most convenient. $\theta = 0$ leads to a treatment in terms of retarded and advanced wave–functions $\Upsilon_a = \Upsilon_R - \Upsilon_L$, $\Upsilon_r = \frac{\Upsilon_R + \Upsilon_L}{2}$. The current problem is thus amenable to the same discussion as in [92, 241]. Following the analysis expounded in those references,

$$\left(\begin{array}{c} C \\ D \end{array} \right)(\omega) = \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{\omega/T_0} \end{array} \right) \left(\begin{array}{c} A \\ B \end{array} \right)(\omega), \tag{4.19}$$

and from here on the recipe for obtaining a Langevin equation applies :

$$\begin{aligned} iS_{bndry,\perp} = & - i\int \frac{d\omega}{2\pi} x^h_{a,\perp}(-\omega)\left[G^h_{R,\perp}(\omega)\right] x^h_{r,\perp}(\omega,\perp) \\ & - \frac{1}{2}\int \frac{d\omega}{2\pi} x^h_{a,\perp}(-\omega)\left[G_{sym,\perp}(\omega)\right] x^h_{a,\perp}(\omega), \end{aligned} \tag{4.20}$$

and similarly for the rapidity sector with – e.g. for transverse fluctuations –

$$\begin{aligned} G^h_{R,\perp}(\omega) &= -\frac{3\pi\sqrt{\lambda}T_0^2 t_0}{4} i\omega, \\ &= -i\omega\eta_\perp, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} G^h_{sym,\perp}(\omega) &= \frac{3\pi^2\sqrt{\lambda}T_0^3 i t_0}{2}\frac{(1+2n(\omega))}{2}\left[\frac{f(u)}{\sqrt{u}}\partial_u(\Psi^H_{\omega,\ in} - \Psi^H_{\omega,\ out})\right]_{u=1}, \\ &= -(1+2n(\omega)) Im\ G_{R,\perp}(\omega). \end{aligned} \tag{4.22}$$

$n(\omega)$ denotes the thermal distribution at temperature T_0. In the original Bjorken variable, this is the temperature at the thermalization time τ_0. In the Bjorken frame, the temperature subsequently decreases according to the scaling law (4.1). Yet, the above analysis was performed at a single temperature T_0. How should one possibly expect to gain knowledge of 2–point functions in an expanding plasma ? The change of coordinates that we made allows for an analysis where the plasma local temperature is kept at T_0. The time dependence is indeed transferred only to the transverse and velocity coordinates components of the metric, while the time and radial components take on the same form as for an AdS–Schwarzschild black hole with temperature T_0. In the following, we show how the physical temperature $T(\tau)$ at proper–time τ makes its way in the coefficients of the Langevin description.

The bulk picture of Brownian motion [92, 241] leads to a stochastic equation with random noise ξ for the horizon endpoint of the string :

$$\left\{ \begin{array}{l} T_\perp(u_h)\partial_u x_{r,\perp}(\omega,u) + \xi^h_\perp(\omega) = -i\omega\eta_\perp x^h_{r,\perp}(\omega), \\ \langle \xi^h_\perp(-\omega)\xi^h_\perp(\omega)\rangle = \eta_\perp\omega[1+2n(\omega)]. \end{array} \right. \tag{4.23}$$

$T_\perp(u) = \frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{2}\frac{1-u^2}{\sqrt{u}}$ is the local tension in the string. In the long time limit where the relevant scales are large with respect to the heavy quark relaxation time, this term is negligible as the string appears straight and the bulk has no effect on the stretched horizon. Therefore the equation of motion for the horizon endpoint is

$$\frac{dx^h_\perp}{dt} \simeq \frac{\xi^h}{\eta_\perp} \tag{4.24}$$

and similarly for the boundary endpoint. Going from (4.23) to (4.24) requires the completeness relation. Also, terms of order $\mathcal{O}(1/\sqrt{t})$ were discarded in $d(\eta_\perp\sqrt{t}x^h_\perp)/dt$.
Besides, using the inverse of (4.11),

$$\begin{aligned} \langle \xi^h_\perp(t_1)\xi^h_\perp(t_2)\rangle &= -\frac{1}{4}\int_{-\infty}^{\infty} d\omega\omega H_1^{(2)}(\omega t_1)H_1^{(2)}(-\omega t_2)\langle \xi^h_\perp(\omega)\xi^h_\perp(-\omega)\rangle \\ &\simeq \frac{3\pi\sqrt{\lambda}T_0^3 t_0}{2}\frac{1}{(t_1+t_2)/2}\delta(t_1-t_2), \\ &= K_\perp(t_1,t_2). \end{aligned} \tag{4.25}$$

Use has been made of $t_{1,2} >> 1$, as appropriate from the JP asymptotic condition. Besides, $\sqrt{t_1t_2} = \sqrt{\mathcal{T} - \frac{s^2}{4}}$, where $\mathcal{T} = \frac{t_1+t_2}{2}$, $s = t_1 - t_2$, and the conditions $\mathcal{T} >> 1$, $s << 1$ were then invoked.
In a Langevin description which gives the dynamics of the heavy quark propagating in the expanding plasma

$$\begin{array}{l} \frac{dp_i}{dt} = F_i^L + F_i^T, \\ \langle F_i^L(t_1)F_j^L(t_2)\rangle = \hat{p}_i\hat{p}_j K_L(t_1,t_2), \\ \langle F_i^T(t_1)F_j^T(t_2)\rangle = (\delta_{ij} - \hat{p}_i\hat{p}_j)K_T(t_1,t_2). \end{array} \tag{4.26}$$

Switching to the proper–time coordinate, each force component comes with an additional factor of $\sqrt{3t_0/2t}$: $F_i^{L,T}(\tau_{1,2}) = \sqrt{\frac{3t_0}{2t_{1,2}}}F_i^{L,T}(t(\tau_{1,2}))$.
Taking care of the Dirac distribution transformation law under coordinate change, this yields, e.g. for the transverse force,

$$\begin{aligned} \langle F_i^T(\tau_1)F_j^T(\tau_2)\rangle &= (\frac{3t_0}{2t_1})^2\pi\sqrt{\lambda}T_0^3(\frac{\tau_1}{\tau_0})^{1/3}\delta(\tau_1-\tau_2)(\delta_{ij}-\hat{p}_i\hat{p}_j) \\ &= \pi\sqrt{\lambda}T^3(\tau_1)\delta(\tau_1-\tau_2)(\delta_{ij}-\hat{p}_i\hat{p}_j). \end{aligned} \tag{4.27}$$

In the Bjorken frame the force correlator thus exhibits a simple scaling law on the temperature with no explicit dependence on the initial temperature T_0. The initial condition on the temperature is then partially washed out. As a landmark of adiabatic evolution, though, it is still hidden in the scaling law for the local temperature.

4.3 Trailing string in the BF background

We now turn to the case of a heavy quark probe moving transversally at some average velocity through an expanding strongly–coupled $\mathcal{N} = 4$ SYM plasma. Suppose that after the hydrodynamic regime has settled, a heavy quark is created among the debris of the collision and starts propagating through the thermalized state of matter with vanishing longitudinal momentum, which means it lies at rapidity $y = 0$. Hence, the proper time parameter τ in the comoving frame measures the physical time elapsed since the probe departed. The quark will hit subsequent layers of matter at different cooling temperatures and densities. In particular, the temperature is described by the scaling law (4.1).

In the context of weakly–coupled quantum chromodynamics, the authors of [18] studied the energy loss and momentum broadening for such a probe created either inside or coming from outside of such an expanding plasma. Their analysis relied on perturbation theory. From $\hat{q}(\tau) = \rho(\tau) \int d^2\vec{q}_\perp \vec{q}_\perp^2 \frac{d\sigma}{d^2\vec{q}_\perp}$, with $\rho(\tau)$ the position–dependent density of the medium, which entails $\hat{q}(\tau) = \hat{q}(\tau_0)(\frac{\tau_0}{\tau})^\alpha$, they found an increase in the rate of energy loss compared to their results in a static medium [18] :

$$-\frac{dE}{dx_\perp} = \frac{2}{2-\alpha}\left(-\frac{dE}{dx_\perp}\right)_{|static}, \tag{4.28}$$

in case the quark is produced inside the medium. It should also be noted that theirs is a finite–extent plasma, unlike the one described by the JP dual that is investigated below.

As pointed out in [19], an expanding medium amounts to an effective transport coefficient $\hat{q}_{eff}(L)$ which would be equivalent to a jet–quenching coefficient in a static plasma :

$$\begin{aligned} \hat{q}_{eff}(L) &= \frac{2}{L^2}\int_{\tau_0}^{L} d\tau\ (\tau-\tau_0)\ \hat{q}(\tau) \\ &\simeq \frac{2}{2-\alpha}\hat{q}(L), \end{aligned} \tag{4.29}$$

as the limit $\tau_0 \to 0$ is taken in much of these studies. The coefficient $\hat{q}(L)$ is evaluated at the temperature $T(L)$ probed by the quark after it has travelled a distance L through the cooling medium.

We would like to learn what happens at strong coupling, despite the difference in the mechanism for energy loss from the one that prevails at weak coupling, in a static plasma, as emphasized, e.g., in [78]. The discussion focuses on the rate of energy loss and, in the final part, on the momentum broadening coefficients.

The starting point is the gravitational dual to the Bjorken flow, the JP metric [152, 153]. One would like to check if a similar enhancement exists and, besides if independence of the transport and momentum broadening coefficients on the thermalization temperature T_0, which was indeed qualified as 'remarkable' by [18], is observed.

It would seem appropriate to start with the Ansatz $X^1(\tau, z) = v\tau + \zeta(z)$ $(*)$ for the trajectory of the string and its quark boundary endpoint, so as to gather information on the drag force and momentum broadening coefficients experienced by a heavy quark moving a velocity v in the plasma proper frame at strong coupling. For convenience, we defined z as $\sqrt{u}$. It should not be mistaken with the z–variable from the starting JP metric. In the following, the background is provided by the tamed form of the metric (4.3). The initial JP metric is far less pliable to tractable computations.

However consider instead a different Ansatz

$$X^1(t, z) = vt + \zeta(z), \tag{4.30}$$

and momentarily defer a discussion on the difficulties one would have run into, had one chosen to work with the proper–time parameter directly. The Ansatz (4.30) yields

$$\sqrt{-g} = \frac{R^2}{z_H^2 z^2}\sqrt{-\tilde{g}}, \quad \sqrt{-\tilde{g}} = \sqrt{z_H^2(1 - \frac{3t_0}{2t}\frac{v^2}{f(z)}) + \frac{3t_0}{2t}f(z)(\partial_z\zeta)^2}, \tag{4.31}$$

and the equation of motion

$$\frac{3t_0}{2t}\frac{f}{z^2}\partial_z\zeta = C\sqrt{-\tilde{g}}. \tag{4.32}$$

Inserting this implicit expression for the derivative of ζ and solving for $\sqrt{-g}$ gives

$$(\sqrt{-\tilde{g}})^2 = z_H^2 \frac{1-\frac{3t_0}{2t}v^2 - z^4}{1-\left[1+C^2/\frac{3t_0}{2t}\right]z^4}. \tag{4.33}$$

In order to ensure that $(-g)$ stays positive everywhere on a string that extends from the horizon to the boundary, both numerator and denominator must change sign at the same point (note that t starts at t_0). Hence

$$C = \pm\frac{\frac{3t_0}{2t}v}{\sqrt{1-\frac{3t_0}{2t}v^2}}, \tag{4.34}$$

and $\partial_z\zeta = \pm v z_H \frac{z^2}{f}$, which is integrated to

$$X^1(t,z) = X^1_0(t,z) = x^1_0 + vt \mp \frac{v z_H}{2}\left[\tan^{-1}(z) + \log\sqrt{\frac{1-z}{1+z}}\right]. \tag{4.35}$$

In the subsequent discussion the $+$ sign in (4.35) is always assumed. Starting with $(*)$ would result in $X^1(\tau,z) = x^1_0 + v\tau \mp \frac{v z_H}{2}(\frac{\tau}{\tau_0})^{1/3}\left[\tan^{-1}(z) + \log\sqrt{\frac{1-z}{1+z}}\right]$. An extra proper time dependence is forced on $\zeta(z)$ from the value taken by C in the process. This is in contradiction with $(*)$. Note that having to work with (4.35) instead of a linear motion in the plasma proper frame raises no problem if one accepts to momentarily set aside a picture of the quark in Bjorken variables ; actually the following discussion establishes how a trajectory for a quark moving with constant velocity with respect to the proper–time variable appears.

4.3.1 Dispersion relations and drag force

This section investigates the way the dispersion relations and the drag acting on the quark are modified by the changing properties of the plasma. A similar analysis was performed for a string trawling an AdS–Schwarzschild black hole in [131].
It is shown that the dispersions relations take on their usual expressions only after a change of reference frame to the starting Bjorken variables is performed. This then leads to the identification of a term responsible for energy loss.
The general expressions for the canonical momentum densities to an open string in a background specified by $G_{\mu\nu}$ are

$$\pi^0_\mu = -\frac{1}{2\pi\alpha'}G_{\mu\nu}\frac{(\dot{X}.X')(X^\nu)' - (X')^2(\dot{X}^\nu)}{\sqrt{-g}}, \tag{4.36}$$

$$\pi^1_\mu = -\frac{1}{2\pi\alpha'}G_{\mu\nu}\frac{(\dot{X}.X')(\dot{X}^\nu) - (\dot{X})^2(X^\nu)'}{\sqrt{-g}}. \tag{4.37}$$

For a string trailing in a JP background massaged to the metric (4.3) this reduces to

$$\begin{cases} \pi^0_t = -\frac{\sqrt{\lambda}T_0}{2}\frac{1}{\sqrt{1-\frac{3t_0}{2t}v^2}}\frac{\left[1-(1-\frac{3t_0}{2t}v^2)z^4\right]}{z^2 f(z)}, \\ \pi^0_{x^1} = \frac{\sqrt{\lambda}T_0}{2}\frac{\frac{3t_0}{2t}v}{\sqrt{1-\frac{3t_0}{2t}v^2}}\frac{1}{z^2 f(z)}, \end{cases} \tag{4.38}$$

$$\begin{cases} \pi^1_t = \frac{\pi\sqrt{\lambda}T_0^2}{2}\frac{\frac{3t_0}{2t}v^2}{\sqrt{1-\frac{3t_0}{2t}v^2}} = \frac{\pi\sqrt{\lambda}T(\tau)^2}{2}\frac{v^2}{\sqrt{1-\frac{3t_0}{2t}v^2}}, \\ \pi^1_{x^1} = \frac{\pi\sqrt{\lambda}T(\tau)^2}{2}\frac{v}{\sqrt{1-\frac{3t_0}{2t}v^2}}. \end{cases} \tag{4.39}$$

Integrating along the string, the resulting total energy and momentum, $E = -\int d\sigma\pi^0_t$, $p = \int d\sigma\pi^0_{x^1}$, read

$$E = \frac{\sqrt{\lambda}T_0}{2}\frac{1}{\sqrt{1-\frac{3t_0}{2t}v^2}}\left[z_m^{-1} - z_h^{-1} + \frac{3t_0}{2t}v^2\Lambda(z_h)\right], \tag{4.40}$$

$$p = \frac{\sqrt{\lambda}T_0}{2}\frac{\frac{3t_0}{2t}v}{\sqrt{1-\frac{3t_0}{2t}v^2}}\left[z_m^{-1} - z_h^{-1} + \Lambda(z_h)\right], \tag{4.41}$$

where

$$\Lambda(z_h) = \frac{1}{4}\left[2\ \tan^{-1}(z_m) - 2\ \tan^{-1}(z_h) + \log\frac{(1-z_m)(1+z_h)}{(1+z_m)(1-z_h)}\right]. \tag{4.42}$$

This compares with eq.(3.21) in [131].
The total energy and momentum diverge due to their contributions close to the horizon, i.e. as the cut–off $z_h \to 1$.
The total energy exhibits a contribution $\gamma(t)E_{straight} = 1/\sqrt{1-\frac{3t_0}{2t}v^2}E_{straight}$ identified with the boosted static energy to a frame moving at velocity v, where $E_{straight} = \frac{R^2}{2\pi\alpha' z_H}(z_m^{-1} - z_h^{-1})$.
Hence the dispersion relation

$$\begin{cases} E = \gamma(t)\frac{\sqrt{\lambda}T_0}{2}\left[z_m^{-1} - z_h^{-1}\right] + \frac{1}{v}\frac{dE}{dt}\Delta x^1(z_h), \\ p = \gamma(t)\frac{\sqrt{\lambda}T(\tau)}{2}\sqrt{\frac{3t_0}{2t}}v\left[z_m^{-1} - z_h^{-1}\right] + \frac{1}{v}\frac{dp}{dt}\Delta x^1(z_h). \end{cases} \tag{4.43}$$

$\Delta x^1(z_h)$ is defined as

$$\Lambda(z_h) = \frac{1}{z_H} \mid \frac{\Delta x^1(z_h)}{v} \mid, \tag{4.44}$$

with $dE/dt = \pi^1_t$, $dp/dt = -\pi^1_x$.
The square root appearing in the expression for p could potentially spoil the interpretation of these formulas as providing the energy and momentum for a quark moving at velocity v through the plasma. Note however that going to the co–moving frame, (4.43) reads

$$\begin{cases} \tilde{E} \simeq \gamma(\tilde{v})\frac{\sqrt{\lambda}T(\tau)}{2}\left[z_m^{-1} - z_h^{-1}\right] + \frac{1}{\tilde{v}}\frac{d\tilde{E}}{d\tau}\Delta x^1(z_h), \\ \tilde{p} = \gamma(\tilde{v})\frac{\sqrt{\lambda}T(\tau)}{2}\tilde{v}\left[z_m^{-1} - z_h^{-1}\right] + \frac{1}{\tilde{v}}\frac{d\tilde{p}}{d\tau}\Delta x^1(z_h), \end{cases} \tag{4.45}$$

where $\tilde{v} = \frac{\partial X^1}{\partial\tau} = \sqrt{\frac{3t_0}{2t}}v$ is the speed of a particle moving with constant velocity in the plasma co–moving frame, with the same trajectory as the heavy quark probe described according to (4.35). Terms of order $\mathcal{O}(\tau^{-4/3})$ have been discarded. This is legitimate given the JP asymptotic condition and the background metric coefficients being actually leading order contributions to an expansion in $\tau^{-2/3}$.
It is now straightforward to derive the drag coefficient :

$$\frac{d\tilde{p}}{d\tau} = -\eta\tilde{p}, \quad \eta = \frac{\pi\sqrt{\lambda}T^2(\tau)}{2M}, \tag{4.46}$$

which displays the same form as in [131], with the proper–time dependence of the temperature in an expanding plasma now taken into account.
This marks a difference in the energy loss mechanism in QCD from the one in a $\mathcal{N} = 4$ SYM plasma at strong coupling. In perturbative QCD the energy loss is dominated by induced radiation of gluons. The transverse momentum of those gluons is high enough that the coupling α_s at this scale is weak, allowing for a perturbative calculation for a parton energy loss :

$$\Delta E = \frac{1}{4}\alpha_s C_R\hat{q}\frac{L^{-2}}{2}. \tag{4.47}$$

L^- stands for the path length of the parton in the plasma. $\hat{q}$ keeps track of the nonperturbative soft interactions between emitted gluons and the medium and between the emitting parton and the plasma.
While in QCD the average loss of energy (4.47) from a fast parton has at most a logarithmic dependence on the latter's momentum and is proportional to the square of its path–length, (4.46) is linear in p.
As illustrated in previous works [78, 109, 92, 117], the mechanisms for momentum broadening appear to differ at weak and strong coupling, if $\mathcal{N} = 4$ provides any hint on QCD in the latter regime. We now explore how the momentum broadening coefficients are modified in an expanding plasma at strong coupling.

4.3.2 Fluctuating trailing string and momentum diffusion

This section is concerned with deriving the momentum broadening coefficients from fluctuations of the trailing string (4.35). This was done in [55, 56, 92, 109] for the case of a static medium. For a review of jet quenching and momentum broadening in perturbative QCD and in AdS/CFT, the review [57] is particularly recommended.
Writing

$$X^1(t,z) = X^1_0(t,z) + \delta\xi^1(t,z) \quad X^2(t,z) = \delta\xi^2(t,z) \quad Y(t,z) = \delta y(t,z), \tag{4.48}$$

with fluctuating terms in the transverse and velocity directions, and inserting in the Nambu–Goto action after some algebra ultimately leads to the following expansion at quadratic order of the action:

$$\begin{aligned} S_{NG} = &-\frac{R^2/z_H}{2\pi\alpha'}\int dt\, dz \frac{\sqrt{1-\frac{3t_0}{2t}v^2}}{z^2} + \int dt\, dz P^\alpha \partial_\alpha \xi^1 \\ &-\frac{1}{2}\int dt\, dz T^{\alpha\beta}_{\delta y}\partial_\alpha \delta y \partial_\beta \delta y - \frac{1}{2}\int dt\, dz \sum_{i=1,2} T^{\alpha\beta}_{\xi^i}\partial_\alpha \delta\xi^i \partial_\beta \delta \xi^i, \end{aligned} \tag{4.49}$$

where

$$P^\alpha = -\frac{R^2/z_H^2}{2\pi\alpha'}\frac{\frac{3t_0}{2t}v}{\sqrt{1-\frac{3t_0}{2t}v^2}}\begin{pmatrix} z_H/(z^2(1-z^4)) \\ 1 \end{pmatrix}, \tag{4.50}$$

$$T^{\alpha\beta}_{\delta y} = -\frac{R^2/z_H^2}{2\pi\alpha'}\frac{4t^2/9}{\sqrt{1-\frac{3t_0}{2t}v^2}}\begin{pmatrix} \frac{z_H}{z^2}\frac{\left[1-(1-\frac{3t_0}{2t}v^2)z^4\right]}{(1-z^4)^2} & \frac{3t_0}{2t}\frac{v^2}{1-z^4} \\ \frac{3t_0}{2t}\frac{v^2}{1-z^4} & \frac{\left[z^4-(1-\frac{3t_0}{2t}v^2)\right]}{z_H z^2} \end{pmatrix}, \tag{4.51}$$

and

$$\begin{aligned} T^{\alpha\beta}_{\xi^2} &= \left[1-\frac{3t_0}{2t}v^2\right] T^{\alpha\beta}_{\xi^1} \\ &= -\frac{R^2/z_H^2}{2\pi\alpha'}\frac{3t_0/2t}{\sqrt{1-\frac{3t_0}{2t}v^2}}\begin{pmatrix} \frac{z_H}{z^2}\frac{\left[1-(1-\frac{3t_0}{2t}v^2)z^4\right]}{(1-z^4)^2} & \frac{3t_0}{2t}\frac{v^2}{1-z^4} \\ \frac{3t_0}{2t}\frac{v^2}{1-z^4} & \frac{\left[z^4-(1-\frac{3t_0}{2t}v^2)\right]}{z_H z^2} \end{pmatrix}. \end{aligned} \tag{4.52}$$

Making use of reparametrization–invariance on the world-sheet, these results translate into the following expressions :

$$\begin{aligned} S_{NG} = &-\frac{\sqrt{\lambda}}{2}\int d\tau\, dz \frac{\sqrt{1-\tilde{v}^2}T(\tau)}{z^2} + \int d\tau\, dz \tilde{P}^\alpha \partial_\alpha \xi^1 \\ &-\frac{1}{2}\int d\tau\, dz \tilde{T}^{\alpha\beta}_{\delta y}\partial_\alpha \delta y \partial_\beta \delta y - \frac{1}{2}\int d\tau\, dz \sum_{i=1,2} \tilde{T}^{\alpha\beta}_{\xi^i}\partial_\alpha \delta\xi^i \partial_\beta \delta \xi^i, \end{aligned} \tag{4.53}$$

with α, β running over z, τ and

$$\tilde{P}^{\alpha} = -\frac{\pi\sqrt{\lambda}T^2(\tau)}{2}\frac{\tilde{v}}{\sqrt{1-\tilde{v}^2}}\begin{pmatrix} \frac{1}{\pi T(\tau)(z^2(1-z^4))} \\ 1 \end{pmatrix}, \tag{4.54}$$

$$\tilde{T}^{\alpha\beta}_{\delta y} = -\frac{R^2/z_H^2}{2\pi\alpha'}\frac{(\tau_0\tau^2)^{\frac{2}{3}}}{\sqrt{1-\tilde{v}^2}}\begin{pmatrix} \frac{1}{\pi T(\tau)z^2}\frac{\left[1-(1-\tilde{v}^2)z^4\right]}{(1-z^4)^2} & \frac{\tilde{v}^2}{1-z^4} \\ \frac{\tilde{v}^2}{1-z^4} & \frac{\pi T(\tau)\left[z^4-(1-\frac{3t_0}{2t}v^2)\right]}{z^2} \end{pmatrix}, \tag{4.55}$$

and

$$\begin{aligned}\tilde{T}^{\alpha\beta}_{\xi^2} &= \left[1-\tilde{v}^2\right]\tilde{T}^{\alpha\beta}_{\xi^1} \\ &= -\frac{\pi\sqrt{\lambda}T^2(\tau)}{2}\frac{1}{\sqrt{1-\tilde{v}^2}}\begin{pmatrix} \frac{1}{\pi T(\tau)z^2}\frac{\left[1-(1-\tilde{v}^2)z^4\right]}{(1-z^4)^2} & \frac{\tilde{v}^2}{1-z^4} \\ \frac{\tilde{v}^2}{1-z^4} & \frac{\pi T(\tau)\left[z^4-(1-\tilde{v}^2)\right]}{z^2} \end{pmatrix}.\end{aligned} \tag{4.56}$$

Recall that τ is related to t through $\frac{t}{t_0} = \frac{3}{2}(\frac{\tau}{\tau_0})^{\frac{2}{3}}$. In the above, the temperature appears only through its local, proper–time dependent expression. Let us now show that the momentum broadening coefficients are then formally the same as in [56, 109].
Indeed, if one uses the second set $\tilde{T}^{\alpha\beta}$ of tensor densities, proper time derivatives of the temperature are discarded in the equations of motion, $\partial_\alpha T^{\alpha\beta}\partial_\beta\phi = 0$, given that they imply sub–leading $\mathcal{O}(\tau^{-4/3})$ contributions. Therefore, independent solutions to the equations of motion look the same as in [109][1], with $T_0 \to T(\tau)$.
In z, τ coordinates the location z_S of world–sheet horizon[2] is $z_S = \sqrt[4]{1-\tilde{v}^2}$.
We are interested in the form of the Kruskal diagram in the z, τ coordinates with the JP asymptotic condition on the latter variable.
Keeping only the z, τ components, this reads $ds^2 = (R\pi T(\tau))^2/z^2\left[-fd\tau^2 + dz^2/\pi^2T(\tau)^2f\right]$, i.e. $ds^2 \simeq (R\pi T(\tau))^2/z^2\left[-fd\tau^2 + \left[d(z/\pi T(\tau))\right]^2/f\right]$.
At this order of the JP expansion the Kruskal coordinates are found as follows.
The null condition leads to $(\pi T(\tau))^2(d\tau)^2 = \frac{(dz)^2}{f(z)^2}$, hence

$$\tau^{\frac{2}{3}} = \pm z_* + C. \tag{4.57}$$

where C labels a constant c–number and

$$z_* = \frac{1}{3\pi T_0\tau_0^{1/3}}[\arctan(z) + \frac{1}{2}\log(\frac{1+z}{1-z})]. \tag{4.58}$$

Introducing $\nu_+ = \tau^{2/3} - z_*$ and $\nu_- = \tau^{2/3} + z_*$, the metric is written as

$$ds^2 = -(\frac{3}{2}\pi RT_0\tau_0^{1/3})^2\frac{f(z)}{z^2}d\nu_-d\nu_+. \tag{4.59}$$

z and τ are given through the implicit equations

$$\begin{cases} \tau = (\frac{\nu_-+\nu_+}{2})^{\frac{3}{2}}, \\ \arctan(z) + \frac{1}{2}\log(\frac{1+z}{1-z}) = \frac{3}{2}\pi T_0\tau_0^{1/3}(\nu_- - \nu+). \end{cases} \tag{4.60}$$

[1] See also [92] where they are labelled $\psi_{ret}(\omega, z)$ and $\psi_{adv}(\omega, z)$.
[2] A world–sheet horizon is generally determined from the zeroes of the polynomial factor appearing in front of the second AdS–radial derivative in the equations of motion. They determine the regular singular points of this equation. When $z = z_S$ the value of $\partial_z\phi$ at $z = z_S$ is determined from the equation of motion. This means that fluctuations of the string at $z < z_S$ are causally disconnected from those away from the location of the world-sheet horizon.

It is then natural to introduce the variables U and V, the Kruskal coordinate for this setting :

$$U = -e^{-3\pi T_0 \tau_0^{1/3} \nu_-}, \quad V = e^{3\pi T_0 \tau_0^{1/3} \nu_+}. \tag{4.61}$$

z and τ are then defined implicitly in those coordinates as

$$\begin{cases} -UV = \frac{1-z}{1+z} e^{-2\arctan(z)}, \\ -\frac{V}{U} = e^{6\pi T(\tau)\tau} = e^{4\pi T_0 t}. \end{cases} \tag{4.62}$$

The Kruskal diagram is split into four quadrants by the curves $U = 0$ and $V = 0$. $UV = 0$ still yields $z = 1$ and τ is given by $\log(-\frac{V}{U}) = 6\pi T(\tau)\tau$, so that $V = 0$, resp. $U = 0$, still corresponds to $\tau = -\infty$ or $t = -\infty$, resp. $+\infty$. This conclusion is supported by [88], Fig. 1, where they show that the BF geometry is a regular black hole spacetime. The apparent and event horizons were found at various orders in the JP metric and they tend to a common slowly varying line in the z–τ plane when $\tau >> 1$.
The trailing string solution (4.35) is cast in the form

$$X_0^1(t, z) = x_0^1 + \frac{v}{2\pi T_0} \log(V) + \frac{v}{\pi T_0} \arctan(z), \tag{4.63}$$

which explicitly shows that the trailing string is regular at the horizon between the upper and the right quadrants. A state of the system is prepared at $Re\ t = -\infty$, which corresponds to the singularity at $V = 0$, is propagated along $Im\ t = 0$, and back along $Im\ t = -\sigma$, for some constant σ in the Schwinger–Keldysh path.
This suggests that all the analysis exposed in [92, 109] is directly applicable to the current problem, with the proviso that the world–sheet horizon in the $t - z$ coordinates is now time–dependent, $z_S = \sqrt[4]{1 - \frac{3t_0}{2t} v^2}$. Indeed the equations of motion obtained from (4.50), (4.51) and (4.52) are the same as those found in the above references, given that the additional time factors should be neglected at the order of the JP expansion one is dealing with. The additional time–dependence of the location to the world–sheet horizon is accounted for by noticing that it turns out to be subleading compared to the decorrelation time.
All in all, after going back to the Bjorken frame as was done at the end of Section 4.2., this yields

$$\hat{q}_{\xi^2}(\tau) = \pi\sqrt{\lambda} T^3(\tau) \frac{\sqrt{\gamma(\tilde{v})}}{\tilde{v}}, \qquad \hat{q}_{\xi^1}(\tau) = \pi\sqrt{\lambda} T^3(\tau) \frac{\gamma(\tilde{v})^{5/2}}{\tilde{v}}, \qquad \gamma(\tilde{v}) = \frac{1}{\sqrt{1-\tilde{v}^2}}. \tag{4.64}$$

This corresponds to a stochastic force in a Langevin equation satisfying

$$\frac{d\tilde{p}_i}{d\tau} = F_i \quad \langle F_i(\tau_1) F_j(\tau_2) \rangle = \delta_{ij} K_i(\tau_1, \tau_2), \quad i, j = 1, 2 \tag{4.65}$$

From $\hat{q} = \langle p^2 \rangle / l$ — l, the path length travelled by the quark in the plasma proper frame, being large enough that the memory short range correlations is not taken into account but large enough that the quark has not departed significantly from its initial trajectory — this gives

$$\hat{q}_i = \frac{1}{\tilde{v}} \int d\tau K_i(\tau, 0), \tag{4.66}$$

as defined in, e.g. [109].
Note however that (4.64) bears no relation to jet–quenching. The longitudinal and transverse parts do not match at finite momentum. The momentum broadening coefficients have been written in this varnished form to suggest a similarity to the local jet–quenching parameter appearing in perturbative QCD calculations with a high energy quark moving in an expanding plasma.

Chapter 5

Finite–Temperature Fractional D2–Branes and the Deconfinement Transition in 2+1 Dimensions

The supergravity dual to N regular and M fractional D2–branes on a cone over $\mathbb{CP}^3$ has a naked singularity in the infrared. One can resolve this singularity and obtain a regular fractional D2–brane solution dual to a confining 2+1 dimensional $\mathcal{N} = 1$ supersymmetric field theory. The confining vacuum of this theory is described by the solution of Cvetic, Gibbons, Lu and Pope [67]. In this chapter, we explore the alternative possibility for resolving the singularity – the creation of a regular horizon. The black–hole solution we derive corresponds to the deconfined phase of this dual gauge theory in three dimensions. This solution is derived in perturbation theory in the number of fractional branes. We argue that there is a first–order deconfinement transition. Connections to Chern–Simons matter theories, the ABJM proposal and fractional M2–branes are presented.

5.1 Introduction

Since its inception, the AdS/CFT correspondence [191, 103, 260] and its various extensions have provided valuable information on gauge theories at strong coupling. In the present work we investigate the deconfinement transition of a $2 + 1$ dimensional gauge theory by constructing a black hole solution in supergravity.

In order to reach closer connection with QCD or condensed–matter gauge theories there exist different techniques to break some amount of the supersymmetry or conformal invariance involved in the gauge/gravity dualities [150, 160]. Putting for instance a stack of branes at a singularity in the transverse space results in a dual field theory with lower supersymmetry. More generally, singular points in the compactifying space lead to interesting behaviour in the scaling limit. The geometrical identification of symmetries of the corresponding gauge theory and its amount of supersymmetry appears in [203]. The Klebanov–Witten construction is a particularly interesting example arising from placing N D3–branes at a conical singularity [168]. The base of the cone is the Einstein manifold $T^{1,1} = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{U(1)}$ with topology $S^2 \times S^3$. The cone over $T^{1,1}$, known as the conifold [52], is defined as the locus $\sum_{i=1}^4 z_i^2 = 0$ in $\mathbb{C}^4$. It is a Calabi–Yau manifold with Kähler potential $K = \left(\sum_{i=1}^4 |z_i|^2\right)^{2/3}$ and as such indeed preserves 1/4 supersymmetry. The dual four–dimensional field theory is then $\mathcal{N} = 1$ supersymmetric. The matter field content consists of chiral superfields A_1, A_2 and B_1, B_2 in the $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ representations, respectively. Each pair forms a doublet under SU(2). The theory has a superpotential $\mathcal{W} = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}A_iB_kA_jB_l$ which preserves the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times U(1)$ isometry of the Einstein metric on $T^{1,1}$.

As a rule, for certain cones it is possible to consider fractional branes which are stuck at the apex and wrap some cycle of the base manifold. For example, going back to the conifold and adding M fractional D3–branes to it changes the gauge group to $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ [104, 169, 170]. Matching the two gauge couplings to the moduli of IIB string theory on this background leads to a non–vanishing NSVZ beta function [208, 232] for $\frac{4\pi}{g_1^2}-\frac{4\pi}{g_2^2}$. The M fractional D3–branes indeed are sources of the magnetic RR 3–form flux through the S^3 of $T^{1,1}$ and the RR 3-form field strength's Poincare dual is proportional to the NSNS 3–form field strength. The effective number of D3–branes varies logarithmically with the AdS radius r. The gauge theory intepretation is in terms of a cascade of Seiberg dualities [144, 5, 81], i.e. $\mathrm{SU}(N+M)\times\mathrm{SU}(N)\rightarrow\mathrm{SU}(N)\times\mathrm{SU}(N-M)$.

The Klebanov–Tseytlin solution [170] is well–behaved at large r but exhibits a naked singularity in the infrared. It was shown in [171] that in order to remove this singularity the conifold could be replaced by its deformation $\sum_{i=1}^{4} z_i^2=\epsilon^2$. This corresponds to blowing–up the S^3 of $T^{1,1}$. The resulting theory is confining and the deformation is the geometrical realization of chiral symmetry breaking. The $U(1)_R$ symmetry is broken to $\mathbb{Z}_{2M}$ by instanton effects. For large M however, $\mathbb{Z}_{2M}\sim U(1)$ and this corresponds to acting as $z_i\rightarrow z_i e^{i\theta}$ on the $\mathbb{C}^4$ embedding coordinates. The deformation breaks this action down to $\mathbb{Z}_2$ while preserving $\mathcal{N}=1$ supersymmetry.

Another mechanism for removing the singularity of the Klebanov–Tseytlin solution was developed in [49, 50, 105]. The idea is that a non–extremal generalization of the KT solution is expected to develop a regular Schwarzschild horizon which will remove the naked singularity. Unlike the Klebanov–Strassler solution, the KT solution preserves the $U(1)$ symmetry of the Einstein metric of $T^{1,1}$. A non–extremal solution breaks supersymmetry but chiral symmetry is restored in this instance. Reference [105] finds a regular black hole solution via a perturbative expansion in the number of fractional D3–branes. This work is suggestive of a critical temperature T_c where the number of ordinary and fractional branes vanishes at the horizon. This corresponds to an expected reduction in the effective number of degrees of freedom of the dual gauge theory at the phase transition.

It is our purpose to understand this mechanism for a three–dimensional gauge theory. In [67], Cvetic, Gibbons, Lu and Pope (CGLP) derive a regular fractional D2–brane solution. The metric appearing in the CGLP solution involves an asymptotically conical G_2 manifold. It is an $\mathbb{R}^3$ bundle over S^4. This solution has two supercharges and is then dual to an $\mathcal{N}=1$ supersymmetric gauge theory in three dimensions. At large distance (small u in our subsequent notation), the geometry becomes a cone over the squashed Einstein metric of the three–dimensional complex projective space $\mathbb{CP}^3$. The resolved solution of [67] is on par with the Klebanov–Strassler deformed conifold solution in that both are regular solutions. The CGLP solution cures the naked IR singularity caused by flux wrapping a shrinking cycle in [125]. The CGLP solution describes the confining phase of a three–dimensional gauge theory. Since the space ends, the warp factor is finite and so is the tension of a string hanging in this background [128]. This is a hallmark of confinement. The spectrum of minimally–coupled scalars is discrete which is another hint of confinement.

In this chapter, following analogous work [49] for fractional D3–branes and fractional D1–branes [126], we show how the singularity appearing in the Herzog–Klebanov solution [125] is shielded by a regular horizon. We start with non–extremal ordinary and fractional D2–branes probing a cone over $\mathbb{CP}^3$ and from there on build a solution with a horizon. On the gauge theory side, this solution describes the deconfined phase of the underlying three–dimensional field theory. We argue that below some critical temperature, the regular black–hole solution that we derive should be replaced, for thermodynamic reasons, by the CGLP solution. The latter corresponds to the confining vacuum of the dual field theory.

It is not totally clear what are the gauge group and the field content of the gauge theory whose dual supergravity solution we consider in the present work. According to [67], the dual asymptotic field theory should be the same as for regular D2–branes, with gauge group of the special unitary type but with a charge determined by the additional fluxes coming from fractional branes. However, the space transverse to the branes is not $\mathbb{R}^7$ but a cone over $\mathbb{CP}^3$. It is then more appropriate to look for an $\mathcal{N} = 1$ field theory in three dimensions with this space as its classical moduli space of vacua. This analysis is carried in [190] for no fractional branes and suggests that the gauge group is $SU(N) \times SU(N)$. The field content consits in an $\mathcal{N} = 1$ vector multiplet and four $\mathcal{N} = 2$ chiral (eight $\mathcal{N} = 1$ scalar) superfields, one pair in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation, another in the conjugate. We have more to say on this in the concluding section. In any case, the gauge group is a product of special unitary groups. If field theory results are of any hint, let us see what one might expect for the gauge dual of the supergravity transition.

All gauge theories with a low–temperature confining phase are thought to possess a non–confining high–temperature phase. This is supported by various pieces of evidence such as lattice studies [222, 245] or perturbative methods [101]. In [248] the nature of the phase transition for lattice gauge theories with various gauge groups and in various dimensions are presented. Those results rely on dynamical arguments and renormalization group methods to connect the critical behaviour of gauge theories with lower–dimensional spin systems[1]. Of particular interest to us is the case of three–dimensional gauge theories with gauge group SU(N) where the critical behaviour should be equivalent to that of a two–dimensional $\mathbb{Z}_N$ spin system. For $N = 2$, 3 or 4 the general arguments of [248] could not rule between a first or second–order phase transition. Of more relevance to our purpose, for $N \geq 4$ [2] the transition was predicted to be either first–order or, if continuous for sufficiently strong coupling, of the Kosterlitz–Thouless type. For the latter kind of transition, thermodynamic quantities display essential singularities at the critical temperature. Relatively recent work on the deconfining phase transition for SU(N) theories in 2+1 dimensions with $N = 4,\ 5,\ 6$ however suggests that the transition is first order for $N \geq 5$ [188].

Whereas there is some debate concerning the nature of the phase transition from lattice QCD calculations, our supergravity solution shows that the transition between the supersymmetric CGLP solution and the black hole solution we obtain consists in a first–order deconfinement transition for a three–dimensional gauge theory at strong coupling. There is no evidence of a Kosterlitz–Thouless transition. As explained in the Conclusion, it would be interesting within this supergravity solution to study the free energy and numerically determine the critical temperature at which it vanishes.

The remainder of this chapter is organized as follows. In the next section we propose an Ansatz for the metric and p–form field strengths for ordinary and fractional D2–branes at finite temperature. We then derive in Section 5.3., from the IIA equations of motion and the Bianchi identities, a system of mixed, non–linear, second–order differential equations for the functions in our Ansatz. We remark on a significant difference from an analysis for D3 or D1-branes, which pertains to the squashing function on the fiber to the transverse space. In Section 5.4., we present three simple solutions to these equations. Two of them will be used in Sections 5.5. and 5.6. as UV and IR boundary conditions for the interpolating non–extremal fractional D2–brane solution we construct from perturbation theory in the number of fractional D2–branes. This perturbative solution is the main result of this part of the thesis. We conclude with several suggestions for extension. We mention especially a potential connection to fractional M2–branes and the ABJM proposal.

[1] See [249] for a review and further references.

[2] This constitutes the case of interest for comparison with theories with a supergravity dual, where $N >> 1$ in order to ignore α' stringy corrections.

5.2 Non–extremal generalization of the fractional D2–brane Ansatz

To construct the non–extremal fractional D2–branes we start by explaining the general Ansatz we begin with. It is similar to those described in [49, 50, 105] and [126] for obtaining non–extremal generalizations of fractional D3–branes and D–strings. They involve adding extra warping functions to the metric which preserve the underlying symmetry of the space transverse to the worldvolume of the branes.

The metric of space transverse to the worldvolume of the D2–branes is the squashed $\mathbb{CP}^3$ Einstein metric. There is a reason why we start with $\mathbb{CP}^3$ instead of one of the simpler manifolds with requisite Betti numbers, such as $S^2 \times S^4$. It has to do with the fact that the cone over a manifold of $\mathbb{CP}^3$ topology admits a smooth resolution. This is the resolution which was used to build the regular supersymmetric fractional D2–brane solution [67]. So, if one is to find a transition from the confined phase whose supergravity dual is the CGLP solution to the deconfined phase of the underlying dual field theory, the UV limits of both supergravity solutions must have the same transverse manifold. For this reason the dependence of the p–forms appearing on the cycles of the geometry is almost the same in our Ansatz as in the singular Herzog–Klebanov solution [125]. There is a significant difference, though : the 3–form field strength considered in [125] is proportional to $d\rho\wedge\omega_2$. Yet, turning on squashing functions from the transverse space metric of our Ansatz below prevents the possibility of an harmonic three–cycle of this type. Instead, H_3 picks an additional contribution, which however does not resolve the singularity. We discuss this issue further in Section 5.4.2.

The general Ansatz for a 10–d Einstein–frame metric consistent with the underlying symmetries of the squashed three–dimensional complex projective space involves four functions x, y, z and w of a radial coordinate u:

$$ds^2_{10E} = e^{\frac{5}{2}z}\left(e^{-4x}dX_0^2 + e^{2x}dX_idX_i\right) + e^{-\frac{3}{2}z}ds_7^2 \tag{5.1}$$

where

$$ds_7^2 = e^{12y}du^2 + e^{2y}\left(dM_6\right)^2, \tag{5.2}$$

$$\left(dM_6\right)^2 = \lambda^2 e^{-4w}\left(D\mu^i\right)^2 + e^{2w}d\Omega_4^2 \tag{5.3}$$

with $d\Omega_4^2$ the metric on the unit 4–sphere. The usual and the squashed Einstein metrics on $\mathbb{CP}^3$ correspond to $\lambda^2 = 1$ and $\lambda^2 = 1/2$, respectively. From here on we work with the second case. The coordinates μ^i on $\mathbb{R}^3$ are subject to $\mu^i\mu^i = 1$, $i = 1,\ 2,\ 3$. Their covariant derivatives are $D\mu^i \equiv d\mu^i + \epsilon_{ijk}A^j\mu^k$, where A^j refer to $\mathfrak{su}(2)$ Yang–Mills instanton potentials : $A^j = A^j_aT^a = iU\partial^jU$, where T^a stand for $\mathfrak{su}(2)$ generators. The special unitary 2×2 matrix U can be expressed in terms of Pauli matrices as $U = a_4 + ia^j\sigma^j$. The field strength components $J^i = dA^i + \frac{1}{2}\epsilon_{ijk}A^jA^k$ satisfy the algebra of the unit quaternions: $J^i_{\alpha\gamma}J^j_{\gamma\beta} = -\delta_{ij}\delta_{\alpha\beta} + \epsilon_{ijk}J^k_{\alpha\beta}$. X_0 is the Euclidean time and X_i are the longitudinal D2–brane directions. It should be emphasized that although we switched to Euclidean time in (5.2) and for the external components of Einstein's equations below, intermediate calculations are carried in real–time.

The metric can be brought into a more familiar form:

$$ds^2_{10E} = H(\rho)^{-5/8}\left[A(\rho)dX_0^2 + dX_idX_i\right] + H(\rho)^{3/8}\left[\frac{d\rho^2}{B(\rho)} + \rho^2\left(dM_6\right)^2\right], \tag{5.4}$$

with the redefinitions

$$H(\rho) \equiv e^{-4z-\frac{16}{5}x}, \quad \rho \equiv e^{y+\frac{3}{5}x}, \quad A(\rho) \equiv e^{-6x}, \quad B(\rho)^{-1}d\rho^2 \equiv e^{12y+\frac{6}{5}x}du^2. \tag{5.5}$$

When $w = 0$ and $e^{5y} = \rho^5 = \frac{1}{5u}$, the transverse seven–dimensional space is the cone over the squashed $\mathbb{CP}^3$. Small u corresponds to large distances (we indeed assume that A, B and H all approach unity as $\rho \to \infty$). The function $w(\rho)$ squeezes the S^2 fiber relative to the base

4–sphere. It does not affect the symmetries of the $\mathbb{CP}^3$ transverse to fractional branes[3]. The extremal D2–brane solution and the more general fractional D2–brane solution on the cone over $\mathbb{CP}^3$ have $x = 0$, $w = 0$. Adding a non–trivial $x(u)$ corresponds to going away from extremality. Our aim is to understand how this changes the extremal D2–brane solution.

It should be noted at this point that the A and B turn out to be equal at leading order in the solution we derive in Section 5.6. This matches the accustomed expectation for a black–hole solution. However, B receives corrections in perturbation theory whereas A is not affected. The solution we find is then different from the usual black D–branes in that $A \neq B$. It still has a regular horizon with a corresponding Hawking temperature and qualifies as a black hole.

As previously explained, the Ansatz for the p–form fields is such that in the UV it is of the same form as for extremal fractional D2–branes [125]:

$$\tilde{F}_4 = K(u) e^{\frac{15}{2}z - \frac{\Phi}{2}} du \wedge d^3x + P\ \Omega_4, \tag{5.6}$$

$$H_3 = g_s P\ \Omega_3. \tag{5.7}$$

Like the other fields, the dilaton is a function of the radial variable u alone. Here, $\Omega_3 = \star_7 \Omega_4$ a harmonic three-form, the Hodge dual being defined with respect to the metric on the cone (5.2), and

$$\Omega_3 = f(u)\omega_1 + g(u)\omega_2 + h(u)\omega_3 \tag{5.8}$$

is a combination of three–forms which are invariant under the SO(5) isometry group of the base manifold and the SO(3) isometry group of the S^2 fibers [91, 67]. Explicitly,

$$\omega_1 = D\mu_i \wedge J^i, \quad \omega_2 = du \wedge J, \quad \omega_3 = du \wedge X_2, \tag{5.9}$$

where $J \equiv \mu_i J^i$ and $X_2 \equiv \frac{1}{2}\epsilon_{ijk}\mu^i D\mu^j \wedge D\mu^k$. They satisfy

$$d\omega_1 = 0, \quad d\omega_2 = -du \wedge \omega_1, \quad d\omega_3 = -du \wedge \omega_1, \tag{5.10}$$

along with

$$\star_7\omega_1 = e^{6y}\epsilon_{ijk}\mu^i du \wedge D\mu^j \wedge J^k,\ \star_7\omega_2 = \frac{1}{2}e^{-4y-4w} X_2 \wedge J,\ \star_7\omega_3 = e^{-4y+8w} J \wedge J. \tag{5.11}$$

To guarantee that the NSNS three–form field strength is harmonic one must have

$$f' = g + h, \quad e^{6y} f = \frac{1}{4}\left(e^{-4w-4y} g\right)', \quad e^{6y} f = \frac{1}{2}\left(e^{-4y+8w} h\right)'. \tag{5.12}$$

These are equations for a single independent function f, once expressions for y and w are obtained:

$$\begin{aligned} f &= \frac{1}{4} e^{-6y}\left(e^{-4w-4y}\left(1 + 1/2\ e^{-12w}\right)^{-1} f'\right)', \\ g &= \left(1 + 1/2\ e^{-12w}\right)^{-1} f', \\ h &= \frac{1}{2} e^{-12w} g. \end{aligned} \tag{5.13}$$

Note that $H_3 = dB_2$, with

$$B_2 = g_s P\left[\left(\int_0^u h(\rho)d\rho\right) X_2 + \left(\int_0^u g(\rho)d\rho\right) J\right], \tag{5.14}$$

[3]While the metric on $\mathbb{CP}^3$ is usually presented as deriving from a S^7 Hopf fibration, it can alternatively be written as a twistor space.

up to an exact form.

The M fractional D2–branes (D4–branes wrapping a 2–cycle of the space transverse to the ordinary D2–branes) thus correspond to M units of magnetic flux through the four–cycle of the six–manifold with $P \sim g_s^{3/4} M$. This scaling is derived after (5.42) in the next section. Ordinary D2–branes are charged electrically under $\tilde{F}_4$ and the function $K(u)$ in (5.6) corresponds to the number of ordinary and fractional D2–branes at the scale associated to u. The equation of motion for $\tilde{F}_4$

$$d \star e^{\frac{\Phi}{2}} \tilde{F}_4 = -g_s^{1/2} F_4 \wedge H_3 \tag{5.15}$$

implies

$$K(u) = Q - 8 g_s^{3/2} P^2 f(u) \int_0^u e^{6y} f(\rho) d\rho. \tag{5.16}$$

For the purpose of the calculations leading to (5.16), the constraint $\mu^i \mu^i = 1$ allows to take $\mu^i = (0, 0, 1)$ and we consider a consistent choice for the quaternionic Kähler forms [67] :

$$J^1_{12} = J^1_{34} = J^2_{13} = J^2_{42} = J^3_{14} = J^3_{23} = -1, \quad i = 1, 2, 3. \tag{5.17}$$

At this point, we should note that it is generally not consistent to ask for identically vanishing f and non–trivial g and h. However, as long as $w \equiv 0$, which will happen at zeroth–order in the perturbative approach of Section 5.6., (5.12) gives $g = -h = g_0 e^{4y}$, with non–necessarily vanishing g_0. As a result, the equation of motion for $\tilde{F}_4$ yields

$$K(u) = Q - 6 g_s^{3/2} P^2 g_0 \int_0^u g(\rho) d\rho, \tag{5.18}$$

instead of (5.16).

In what follows we use the Ansatz (5.1)–(5.3), (5.6) and (5.7) to reduce the remaining equations of motion of IIA supergravity to a system of non–linear, coupled ordinary differential equations describing the radial evolution of x, y, z, w, K and Φ.

5.3 Derivation of the equations of motion

Six independent scalars appear in the Ansatz (5.1), (5.6), (5.7) and we will then need a system of as many ordinary differential equations. Einstein's equation provide five independent equations. The one involving R_{uu} stands apart as it provides a zero energy constraint on integration constants. The equation of motion for the dilaton and the one derived from $H_3 = dB_2$ being (co–)closed provide two nontrivial equations. Like (5.15), they are derived from the bosonic part of the IIA superstring theory action [227] in the Einstein frame:

$$\begin{aligned}
\mathcal{S}_{IIA} &= \mathcal{S}_{NS} + \mathcal{S}_R + \mathcal{S}_{CS}, \\
\mathcal{S}_{NS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \Big(R - \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} e^{-\Phi} |H_3|^2 \Big), \\
\mathcal{S}_R &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \Big(e^{\frac{3\Phi}{2}} \left|\tilde{F}_2\right|^2 + e^{\frac{\Phi}{2}} \left|\tilde{F}_4\right|^2 \Big), \\
\mathcal{S}_{CS} &= -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4
\end{aligned} \tag{5.19}$$

where $F_2 = dA_1$, $H_3 = dB_2$, $\tilde{F}_4 = dA_3 - A_1 \wedge H_3$ and $2\kappa_{10}^2 \equiv (2\pi)^7 \alpha'^4 g_s^2$. Let us specify that

$$\int d^d x \sqrt{-G} \, |F_p|^2 \equiv \int d^d x \sqrt{-G} \frac{1}{p!} G^{M_1 N_1} ... G^{M_p N_p} F_{M_1 ... M_p} F_{N_1 ... N_P}. \tag{5.20}$$

Whenever $P \neq 0$, the H_3 equation of motion reduces to the first–order differential equation

$$\left(e^{3z - \Phi}\right)' = -e^{\frac{15}{2} z - \frac{\Phi}{2}} K, \tag{5.21}$$

while the dilaton equation of motion

$$\nabla^2\Phi = -\frac{g_s}{12}e^{-\Phi}(H_3)_{MNP}(H_3)^{MNP} + \frac{g_s^{3/2}}{96}e^{\frac{\Phi}{2}}(\tilde{F}_4)_{MNPQ}(\tilde{F}_4)^{MNPQ} \tag{5.22}$$

yields

$$\Phi'' = P^2\Big(-g_s^3e^{3z-\Phi} + \frac{g_s^{3/2}}{2}\,e^{\frac{9}{2}z+\frac{\Phi}{2}}\Big)\Big(e^{-4y-4w}g^2 + 2e^{-4y+8w}h^2 + 4e^{6y}f^2\Big) - \frac{g_s^{3/2}}{4}e^{\frac{15}{2}z-\frac{\Phi}{2}}K^2. \tag{5.23}$$

Einstein's equations are $R_{MN} = T_{MN}$ where R_{MN} is the Ricci curvature and the energy–momentum tensor for the relevant field content of IIA supergravity is

$$\begin{aligned} T_{MN} =& \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{g_s}{4}e^{-\Phi}\left(H_M{}^{PQ}H_{NPQ} - \frac{1}{12}G_{MN}H^{PQR}H_{PQR}\right) \\ &+ \frac{g_s^{3/2}}{12}e^{\frac{\Phi}{2}}\left(\tilde{F}_M{}^{PQR}\tilde{F}_{NPQR} - \frac{3}{32}G_{MN}\tilde{F}^{PQRS}\tilde{F}_{PQRS}\right). \end{aligned} \tag{5.24}$$

In order to write down these equations in a convenient form, we will work in an orthonormal frame basis:

$$\begin{aligned} &\hat{e}^0 = e^{\frac{5}{4}z-2x}dX_0, \quad \hat{e}^{x_i} = e^{\frac{5}{4}z+x}dX_i, \quad \hat{e}^u = e^{-\frac{3}{4}z+6y}du, \\ &\hat{e}^{\mu^i} = \frac{1}{\sqrt{2}}e^{-\frac{3}{4}z+y-2w}D\mu^i, \quad \hat{e}^\alpha = e^{-\frac{3}{4}z+y+w}g^\alpha, \quad i=1,2,3, \quad \alpha = 1,...,4. \end{aligned} \tag{5.25}$$

At this stage, it might not yet be obvious that Einstein's equations are diagonal in this basis. Actually, they will turn out to be so, once we use the Gauss–Codazzi equation (5.30) that we discuss below, to impose the hypersurface condition $\mu^i\mu^i = 1$. The equations corresponding to $R_{\mu^i\mu^i}$ are identical and similarly for the equations corresponding to $R_{\alpha\alpha}$. The strategy will consist in dealing with the other four Einstein's equations separately from R_{uu} at first. Together with the two field strength equations we will derive thence a second order, nonlinear system of ordinary differential equations in the six warping functions. Finally, we will find out that the R_{uu} relation provides a zero energy constraint, as in [49, 50, 105, 126].

Let us first compute Ricci's tensors in the above orthonormal basis. We list the non–vanishing components of the curvature two–form $\hat{R}_{MN} = d\hat{\omega}_{MN} + \hat{\omega}_{MP}\wedge\hat{\omega}_{PN}$ found by applying Cartan's structure equations $d\hat{e}^M = -\hat{\omega}_{MN}\wedge\hat{e}^N$, $\hat{\omega}_{MN} = -\hat{\omega}_{NM}$. The Riemann tensor is obtained from $\hat{R}_{MN} = \frac{1}{2}\hat{R}_{MNPQ}\hat{e}^P\wedge\hat{e}^Q$.

$$\begin{aligned} \hat{R}_{0u} &= \left(2x'' - \frac{5}{4}z'' + 2\left(\frac{5}{4}z' - 2x'\right)(x' + 3y' - z')\right)e^{\frac{3}{2}z-12y}\hat{e}^0\wedge\hat{e}^u, \\ \hat{R}_{0x_i} &= \left(2x' - \frac{5}{4}z'\right)\left(x' + \frac{5}{4}z'\right)e^{\frac{3}{2}z-12y}\hat{e}^0\wedge\hat{e}^{x_i}, \\ \hat{R}_{0\mu^i} &= \left(2x' - \frac{5}{4}z'\right)\left(y' - \frac{3}{4}z' - 2w'\right)e^{\frac{3}{2}z-12y}\hat{e}^0\wedge\hat{e}^{\mu^i}, \\ \hat{R}_{0\alpha} &= \left(2x' - \frac{5}{4}z'\right)\left(y' + w' - \frac{3}{4}z'\right)e^{\frac{3}{2}z-12y}\hat{e}^0\wedge\hat{e}^\alpha, \end{aligned} \tag{5.26}$$

$$
\begin{aligned}
\hat{R}_{x_i u} &= -\left(x'' + \frac{5}{4}z'' + \left(\frac{5}{4}z' + x'\right)(2z' + x' - 6y')\right) e^{\frac{3}{2}z-12y}\hat{e}^{x_i} \wedge \hat{e}^u, \\
\hat{R}_{x_i x_j} &= -\left(\frac{5}{4}z' + x'\right)^2 e^{\frac{3}{2}z-12y}\hat{e}^{x_i} \wedge \hat{e}^{x_j}, \\
\hat{R}_{x_i \mu^j} &= \left(\frac{5}{4}z' + x'\right)\left(\frac{3}{4}z' + 2w' - y'\right) e^{\frac{3}{2}z-12y}\hat{e}^{x_i} \wedge \hat{e}^{\mu^j}, \\
\hat{R}_{x_i \alpha} &= -\left(\frac{5}{4}z' + x'\right)\left(y' + w' - \frac{3}{4}z'\right) e^{\frac{3}{2}z-12y}\hat{e}^{x_i} \wedge \hat{e}^{\alpha},
\end{aligned} \tag{5.27}
$$

$$
\begin{aligned}
\hat{R}_{\mu^i u} = &\left(\frac{3}{4}z'' + 2w'' - y'' - \left(\frac{3}{4}z' + 2w' - y'\right)(5y' + 2w')\right) e^{\frac{3}{2}z-12y}\hat{e}^{\mu^i} \wedge \hat{e}^u \\
&- \frac{3}{\sqrt{2}}w' e^{-5y-2w}\epsilon_{ijk}J^j\mu^k,
\end{aligned}
$$

$$
\hat{R}_{\mu^i \mu^j} = -\frac{1}{2}\left(\frac{3}{4}z' + 2w' - y'\right)^2 e^{\frac{3}{2}z-12y}\hat{e}^{\mu^i} \wedge \hat{e}^{\mu^j} + \left(\frac{1}{4}e^{-6w} - 1\right)\epsilon_{ijk}J^k,
$$

$$
\begin{aligned}
\hat{R}_{\mu^i \alpha} = &-\frac{3}{2\sqrt{2}}w' e^{\frac{3}{2}z-7y-4w}\epsilon_{ijk}J^j_{\alpha\beta}\mu^k\hat{e}^u \wedge \hat{e}^\beta + \frac{1}{2}e^{\frac{3}{2}z-2y-2w}\left(\frac{1}{4}e^{-6w} - 1\right)\epsilon_{ijk}J^k_{\alpha\beta}\hat{e}^{\mu^j} \wedge \hat{e}^\beta \\
&+ \left(\frac{1}{8}e^{\frac{3}{2}z-2y-8w} + \frac{1}{\sqrt{2}}\left(\frac{3}{4}z' + 2w' - y'\right)\left(y' + w' - \frac{3}{4}z'\right) e^{\frac{3}{2}z-12y}\right)\hat{e}^{\mu^i} \wedge \hat{e}^\alpha,
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
\hat{R}_{\alpha u} = &\left(\frac{3}{4}z'' - y'' - w'' + \left(y' + w' - \frac{3}{4}z'\right)(5y' - w')\right) e^{\frac{3}{2}z-12y}\hat{e}^\alpha \wedge \hat{e}^u \\
&- \frac{3}{2\sqrt{2}}w' e^{\frac{3}{2}z-7y-4w}\epsilon_{ijk}J^j_{\alpha\beta}\mu^k\hat{e}^{\mu^i} \wedge \hat{e}^\beta,
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_{\alpha\beta} = &R_{\alpha\beta} - \frac{1}{8}e^{\frac{3}{2}z-2y-8w}\left(\delta_{ij} - \mu^i\mu^j\right)\left(J^i_{\alpha\beta}J^j_{\rho\sigma} + J^i_{\alpha\rho}J^j_{\beta\sigma}\right)\hat{e}^\rho \wedge \hat{e}^\sigma \\
&- \left(y' + w' - \frac{3}{4}z'\right)^2 e^{\frac{3}{2}z-12y}\hat{e}^\alpha \wedge \hat{e}^\beta + \frac{1}{2}\left(\frac{1}{4}e^{-6w} - 1\right) e^{\frac{3}{2}z-2y-2w}\epsilon_{ijk}J^k_{\alpha\beta}\hat{e}^{\mu^i} \wedge \hat{e}^{\mu^j} \\
&+ \frac{3}{\sqrt{2}}w' e^{\frac{3}{2}z-7y-4w}\epsilon_{ijk}J^j_{\alpha\beta}\mu^k\hat{e}^u \wedge \hat{e}^{\mu^i},
\end{aligned} \tag{5.29}
$$

where $R_{\alpha\beta} = 3e^{\frac{3}{2}z-2y-2w}\delta_{\alpha\beta}$ is the curvature two–form on the base manifold S^4. Some of the components listed above were already derived in [91]. To calculate the curvature with the hypersurface condition imposed we use the Gauss–Codazzi equation [202]

$$
\hat{R}^{MNPQ}_{u^2=1} = \hat{R}_{STUV}h^{MS}h^{NT}h^{PU}h^{QV} + \chi^{MP}\chi^{NQ} - \chi^{MQ}\chi^{NP}, \tag{5.30}
$$

where h_{MN} is the orthonormal frame metric on the projected space:

$$
h_{MN} = \delta_{MN} - \mu^M\mu^N, \tag{5.31}
$$

with $\mu^M = \left(\mu^0, \mu^{x_i}, \mu^u, \mu^{\mu^i}, \mu^\alpha\right) = (0,0,0,\mu^i,0)$ the orthonormal frame components of the unit vector orthogonal to the hypersurface. $\chi_{MN} = h_{MP}h_{NQ}\hat{\nabla}_P\mu_Q$ denote the components of the second fundamental form of the hypersurface. The second fundamental form corresponds to the projection of the gradient to the normal of the hypersurface onto this hypersurface. The only non–vanishing components of h_{MN} and χ_{MN} are

$$h_{00} = 1,\ h_{x_ix_j} = \delta_{ij},\ h_{uu} = 1,\ h_{\mu^i\mu^j} = \delta_{ij} - \mu^i\mu^j,\ h_{\alpha\beta} = \delta_{\alpha\beta},\ \chi_{\mu^i\mu^j} = \sqrt{2}e^{\frac{3}{4}z+2w-y}h_{\mu^i\mu^j}. \tag{5.32}$$

From the field strengths, it is straightforward to check that $T_{00} = T_{x_1x_1}$[4]. However $R_{00} = e^{\frac{3}{2}z-12y}\left(2x'' - \frac{5}{4}z''\right)$ and $R_{x_{1,2}x_{1,2}} = -e^{\frac{3}{2}z-12y}\left(x'' + \frac{5}{4}z''\right)$. The first two non–redundant Einstein's equations then allow us to solve for $x(u)$ exactly:

$$x'' = 0, \quad x = au, \quad a > 0. \tag{5.33}$$

The same behaviour was found for the function $x(u)$ in the case of non–extremal fractional D3–branes and D–strings [49, 50, 105, 126]. The factor a was identified with the degree of non–extremality. Having solved for $x(u)$, we can use either of R_{00} or $R_{x_ix_i}$ to derive an equation for $z(u)$:

$$\begin{aligned}20z'' = P^2\Big(4g_s^3e^{3z-\Phi} + 6g_s^{3/2}e^{\frac{9}{2}z+\frac{\Phi}{2}}\Big)\Big(e^{-4y-4w}g^2 + 2e^{-4y+8w}h^2 + 4e^{6y}f^2\Big)\\ +5g_s^{3/2}e^{\frac{15}{2}z-\frac{\Phi}{2}}K^2.\end{aligned} \tag{5.34}$$

Note that in the extremal case $x = 0$, $z = -\Phi$ and $h^{-5/8} = e^{5/2z} = h^{-1/2}e^{-\Phi/2}$. This means that the Einstein frame metric in the extremal case can be obtained from the string frame metric through the standard procedure of multiplying by $e^{-\Phi/2}$.

In order to find the differential equations for $y(u)$ and $w(u)$, we must consider linear combinations of the Einstein's equations involving

$$\begin{aligned}\hat{R}_{\mu^i\mu^j} &= \left(e^{\frac{3}{2}z-12y}\left(\frac{3}{4}z'' + 2w'' - y''\right) + 2e^{\frac{3}{2}z-2y+4w}\left(1 + \frac{1}{4}e^{-12w}\right)\right)h_{\mu^i\mu^j},\\ \hat{R}_{\alpha\beta} &= \left(e^{\frac{3}{2}z-12y}\left(\frac{3}{4}z'' - y'' - w''\right) + e^{\frac{3}{2}z-2y-2w}\left(3 - \frac{1}{2}e^{-6w}\right)\right)\delta_{\alpha\beta}.\end{aligned} \tag{5.35}$$

Those are easily computed from the curvature two–forms (5.26)–(5.29) and the Gauss–Codazzi condition (5.30). We set $\Phi_n \equiv \Phi + z$ and $z_n \equiv 15z - \Phi$. Computing the field strength contribution to Einstein's equations then results in a system of second order differential equations, where we also list below those found previously in (5.13), (5.23) and (5.34):

$$\begin{aligned}15y'' = P^2\Big(g_s^3e^{1/4(z_n-3\Phi_n)} - g_s^{3/2}e^{1/4(z_n+3\Phi_n)}\Big)\Big(e^{-4y-4w}g^2 + 2e^{-4y+8w}h^2 - 6e^{6y}f^2\Big)\\ + 5e^{10y}\Big(2e^{4w} + 6e^{-2w} - \frac{1}{2}e^{-8w}\Big),\end{aligned} \tag{5.36}$$

$$\begin{aligned}6w'' = P^2\Big(-g_s^3e^{1/4(z_n-3\Phi_n)} + g_s^{3/2}e^{1/4(z_n+3\Phi_n)}\Big)\Big(e^{-4y-4w}g^2 - 4e^{-4y+8w}h^2\Big)\\ - 2e^{10y}\Big(2e^{4w} - 3e^{-2w} + e^{-8w}\Big),\end{aligned} \tag{5.37}$$

$$5\Phi_n'' = P^2\Big(-4g_s^3e^{1/4(z_n-3\Phi_n)} + 4g_s^{3/2}e^{1/4(z_n+3\Phi_n)}\Big)\Big(e^{-4y-4w}g^2 + 2e^{-4y+8w}h^2 + 4e^{6y}f^2\Big), \tag{5.38}$$

[4] We recall that, once every contribution is computed, we revert to a Euclidean frame.

$$\frac{1}{4}z_n'' = P^2\Big(g_s^3 e^{1/4(z_n-3\Phi_n)} + g_s^{3/2} e^{1/4(z_n+3\Phi_n)}\Big)\Big(e^{-4y-4w}g^2 + 2e^{-4y+8w}h^2 + 4e^{6y}f^2\Big) \\ + g_s^{3/2} e^{z_n/2} K^2, \tag{5.39}$$

$$\Big(e^{-4w-4y}\left(1+1/2\ e^{-12w}\right)^{-1} f'\Big)' = 4e^{6y}f, \tag{5.40}$$

We have left out thus far the u components of Einstein's equation. For our metric Ansatz

$$\hat{R}_{uu} = e^{\frac{3}{2}z-12y}\Big(\frac{3}{4}z'' - 6y'' - 12w'^2 - 6x'^2 + 30y'^2 - \frac{15}{2}z'^2\Big) \tag{5.41}$$

which, using (5.36), (5.39), (5.38) and (5.33), provides the zero energy constraint

$$30(y')^2 - \frac{1}{32}(z_n')^2 - \frac{15}{32}(\Phi_n')^2 - 12(w')^2 + P^2\Big(- g_s^3 e^{1/4(z_n-3\Phi_n)} + g_s^{3/2} e^{1/4(z_n+3\Phi_n)}\Big) \\ + \frac{1}{2}g_s^{3/2}K^2 e^{\frac{1}{2}z_n} - 2e^{10y}\Big(2e^{4w} + 6e^{-2w} - \frac{1}{2}e^{-8w}\Big) = 6a^2. \tag{5.42}$$

Later it will be important to keep track of the dimensions of the parameters involved in this set of equations. Looking at the form of the metric (5.1) it is natural that e^y and $u^{-1/5}$ should have dimension of length, while x, z and w stay dimensionless. Since we have set the 10–d gravitational constant $\kappa_{10}^2/8\pi$ to unity (i.e. all scales are measured in units of the 10–d Planck scale $L_P \sim \left(g_s\alpha'^2\right)^{1/4}$), from (5.42) we conclude that K and Q have dimension $(\text{length})^5$. Using (5.6) and (5.38), P scales as $(\text{length})^2$ and f as $(\text{length})^{-1}$. The dependence on the Planck length can be restored by rescaling $Q \to L_P^5 Q$, $P \to L_P^2 P$ and so on. Denoting the number of ordinary and fractional D2–branes by N and M, this means that $Q \sim g_s^{5/4}\alpha'^{5/2}N$, $P \sim g_s^{3/4}\alpha'^{3/2}M$.

5.4 Three simple solutions

Aside from the extremal D2–brane solution, there exist three other simple solutions to the set of equations (5.36)–(5.40), (5.42). First of all, there is the extremal fractional D2–brane solution which was mentioned in Section 5.2. It is the analog of the Klebanov–Tseytlin solution for fractional D3–branes. The second one is the non–extremal ordinary D2–brane solution.
Later on, in Section 5.6., we will derive a non–extremal fractional D2–brane solution from perturbation theory in the number P of fractional D2–branes. The solution we find interpolates between the extremal fractional D2–brane solution in the UV and the ordinary black D2–brane Horowitz–Strominger–like solution [136] in the IR.
The third solution is the analog of the singular, non–extremal D3–brane solution found by Buchel [49]. This solution is non–extremal but has a naked singularity in the IR.

5.4.1 Singular non–extremal fractional D2–brane

A natural attempt at finding a non–extremal solution is to first switch off the stretching function $w(u)$. One will find that the solution is singular. So, this motivates the necessity of keeping a non–trivial squashing function for regular solutions. It will also happen that this solution approaches the extremal fractional D2–brane solution of the next subsection as the non–extremality parameter $a \to 0$.

Upon examination of the system of differential equations (5.36)–(5.40), (5.42), we notice as soon as we impose that $\Phi_n' = p$, a constant, (5.38) requires that $\Phi_n = 0$, up to a constant related to the string coupling g_s. The condition $\Phi_n' = p$ is a natural one to impose. The Herzog–Klebanov singular solution of Section 5.4.2. is indeed such that $e^{\Phi} = g_s H(\rho)^{1/4}$, i.e. $e^{\Phi_n} = g_s e^{-\frac{4}{5}au}$ with $a = 0$, from (5.5) and (5.33). It is worth noticing that in a similar analysis for fractional D3–branes or D1–branes [50, 105, 126], $w = 0$ automatically implies $\Phi_n'' = 0$.

Alternatively, the analogues of the IR asymptotic conditions which derive from the solution of subsection 5.4.3. below for fractional D2–branes, lead to a source term for Φ_n which in turn prevents a non–trivial squashing function w. For fractional D2–branes, the two conditions are disconnected.

We thus enforce $w = 0$ and $\Phi_n = 0$. From (5.40), z_n must then satisfy the first order equation

$$\left(e^{-\frac{z_n}{4}}\right)' = \left(Q - 8g_s^{3/2}P^2 f(u)\int_0^u e^{6y} f(\rho)d\rho\right). \tag{5.43}$$

Equation (5.36) gives $y'' = \frac{5}{2}e^{10y}$. From the zero–energy constraint (5.42)

$$y' = -\sqrt{b^2 + \frac{1}{2}e^{10y}}, \qquad 5b^2 \equiv a^2 \tag{5.44}$$

one of the integration constants of this second–order differential equation for y is already determined. Equation (5.44) integrates to $e^{5y} = \frac{\sqrt{2/5}\, a}{\sinh(\sqrt{5}au)}$, with $a > 0$ without loss of generality. As a consequence, (5.40) gives a massaged equation for $f(u)$:

$$\left(\sinh\left(\sqrt{5}au\right)^{4/5} f'\right)' = \frac{12}{5}\frac{a^2}{\sinh\left(\sqrt{5}au\right)^{6/5}} f, \tag{5.45}$$

which in turn, once inserted into (5.43), gives an expression for z_n.

Consider now the Ricci scalar for the metric Ansatz (5.1)–(5.3) specialized to the solution above:

$$R = e^{\frac{11}{32}z_n - 12y}\left[-\frac{3}{32}g_s^{3/2}e^{\frac{z_n}{4}}K^2 + \frac{3}{4}g_s^{9/4}P^2\left(e^{-4y}g^2 + 2e^{-4y}h^2 + 4e^{6y}f^2\right)\right] \tag{5.46}$$

As for the Buchel solution in the case of D3–branes or the singular non–extremal fractional D–string solution [126], this solution turns out to have a naked singularity in the far infrared, i.e. at large u. This is apparent once we write an asymptotic expansion for $f(u)$ and $z_n(u)$ in the infrared region:

$$f = f_0\left(1 + \frac{4}{5}e^{-2\sqrt{5}au}\right) + \mathcal{O}\left(e^{-2\sqrt{5}au}\right), \tag{5.47}$$

from which a development for z_n is found by integrating (5.43):

$$e^{-\frac{z_n}{4}} = C + Qu - \frac{8\ 2^{4/5}a^{1/5}f_0^2 g_s^{3/2}P^2}{3\ 5^{1/10}}u - \frac{4\ 2^{4/5}\ 5^{2/5}f_0^2 g_s^{3/2}P^2}{9a^{4/5}}e^{-\frac{6au}{\sqrt{5}}} + \mathcal{O}\left(e^{-\frac{6au}{\sqrt{5}}}\right), \tag{5.48}$$

where f_0 and C are constants.

The constant and linear terms in (5.47) and (5.48) dominate in the far infrared: $f \sim f_0$ and $e^{-\frac{z_n}{4}} \sim u$. Furthermore, $e^{-y} \sim e^{bu}$. Consequently, the e^{-y} terms will dominate the Ricci scalar which will blow up at large u, even if we consider the limit where no fractional D2–branes are present $P \to 0$. The $P \to 0$ limit of this singular non–extremal solution does not correspond to the black D2–brane solution. In Subsection 5.4.3. we demonstrate how the ordinary non–extremal D2–brane solution is encompassed within our Ansatz.

We define, as is standard, the horizon (if present at all) via $G_{00} \equiv e^{\frac{5}{32}z_n + \frac{5}{32}pu - 4au} = 0$. From the asymptotics of our solutions and the constraint imposed by the zero–energy condition, a horizon can possibly develop only as $u \to \infty$. This non–BPS solution does not develop an horizon shielding the naked singularity. We are thus led to conclude that keeping constant $\Phi_n'(u)$ cannot prevent a naked singularity. It is remarkable that in the case at hand it is still consistent to keep $w = 0$ with a non–trivial Φ_n. On the other hand, in the analysis pertaining to D3–branes, a source for Φ_n implies that a distorting function has to be switched on if non–singular solutions are to be found.

5.4.2 The extremal Herzog–Klebanov fractional D2–brane solution

We come by the extremal fractional D2–brane solution by taking the limit $a \to 0$ of the singular non–extremal D2–brane solution described in the previous subsection. Explicitly, this yields

$$\Phi_n \to 0, \quad e^{5y} \to \frac{\sqrt{2}}{5u}, \quad f \to f_0\, u^{4/5},$$
$$e^{-\frac{z_n}{4}} \to C + Qu - \frac{2^{8/5}5^{4/5}}{9} g_s^{3/2} f_0^2 P^2 u^{12/5}. \tag{5.49}$$

For this solution to be well–behaved in the UV limit $u \to 0$, we have set to zero the coefficient in $f(u)$ of a mode growing as $u^{-3/5}$. The metric (5.1) becomes asymptotically flat at large distances. However, as explained in the following subsection, we will rather take $C = 0$ which amounts to focusing on the low–energy dynamics of the gauge theory dual to this supergravity background. This solution also develops a naked infrared singularity at u_{cr} given by $u_{cr}^{7/5} = \frac{9}{2^{8/5}\,5^{4/5}} \frac{Q}{g_s^{3/2} f_0^2 P^2}$. We denote $\tilde{L}$ the value $u_{cr}^{-1/5}$ at which the singularity occurs for this extremal fractional D2–brane Herzog–Klebanov solution [125]. The singularity arises from the flux associated with fractional D2–branes, which is supported on a shrinking 4–cycle. Allowing for a non–trivial Φ_n and further taking a non–vanishing squashing function w, which corresponds to resolving the base of the cone to a non–trivial bundle, should lead to a regular supergravity solution. Actually, as discussed at the beginning of Section 5.2., this singular solution differs from Herzog–Klebanov solution [125] due to the contribution $f(u) D\mu^i \wedge J^i$ to H_3. The other contribution is of the type $du \wedge \omega_2$, where ω_2 is proportional to the fundamental form of $\mathbb{CP}^3$ when $f_0 \to 0$. In order to recover an asymptotic solution of the HK type, we set $f_0 = 0$ in (5.49) and take into account the remark expressed below (5.17). Equation (5.49) is replaced by

$$\Phi_n \to 0, \quad e^{5y} \to \frac{\sqrt{2}}{5u}, \quad f \to 0, \quad g = -h \to g_0 \left(\frac{\sqrt{2}}{5}\right)^{4/5} u^{-4/5},$$
$$e^{-\frac{z_n}{4}} \to C + Qu - 2^{2/5}5^{6/5} g_s^{3/2} g_0^2 P^2 u^{6/5}. \tag{5.50}$$

We take $C = 0$. The solution is endowed with a naked IR singularity at

$$u_{cr}^{1/5} = \frac{Q}{2^{2/5}5^{6/5} g_s^{3/2} g_0^2 P^2}. \tag{5.51}$$

From there on, we define $\tilde{L} \equiv u_{cr}^{-1/5}$.

5.4.3 The non–extremal ordinary black D2–brane

In this case we impose that there are no fractional D2–branes: $f = 0$, $g = 0$ and $P = 0$. From knowledge of this solution in [136], we also set $w = 0$. The system of differential equations (5.36)–(5.40), (5.42) simplifies to

$$y'' = \frac{5}{2} e^{10y}, \quad z_n'' = 4 g_s^{3/2} Q^2 e^{\frac{z_n}{2}}, \quad x'' = 0, \quad \Phi_n'' = 0. \tag{5.52}$$

This integrates to

$$x' = a, \quad \Phi_n' = p, \quad y'^2 = b^2 + \frac{1}{2} e^{10y}, \quad z_n'^2 = c^2 + 16 g_s^{3/2} Q^2 e^{\frac{z_n}{2}}. \tag{5.53}$$

The zero–energy constraint (5.42) further restrains the integration constants to satisfy $6a^2 - 30b^2 + \frac{c^2}{32} + \frac{15}{32}p^2 = 0$. Integrating one last time, we obtain

$$e^{5y} = \frac{\sqrt{2}b}{\sinh(5bu)}, \quad e^{\frac{z_n}{4}} = \frac{c}{4 g_s^{3/4} Q \sinh\left(\frac{c}{4}(u+k)\right)}. \tag{5.54}$$

This can be written in the variables (5.5) of the more familiar D2–brane form of the metric (5.4) as

$$\rho^5 = e^{5y+3x} = 2^{3/2} b \frac{e^{-(5b-3a)u}}{1-e^{-10bu}}, \tag{5.55}$$

$$A(u) = e^{-6x} = e^{-6au} \tag{5.56}$$

and

$$H(u) = e^{-\frac{z_n}{4}-\frac{\Phi_n}{4}-\frac{16}{5}x} = 4 g_s^{3/4} \frac{Q}{c} \sinh\left(\frac{c}{4}(u+k)\right) e^{-\frac{16}{5}au - \frac{p}{4}u}. \tag{5.57}$$

The decoupling limit in use for the AdS/CFT correspondence and its various extensions in 2+1 dimensions [150] leads to $k = 0$ so that H vanishes as $u \to 0$. This decoupling limit removes the asymptotic Minkowski region of curved geometry created by a stack of branes. This way we focus only on the throat–like, near–horizon region, i.e. on the low–energy dynamics of the gauge theory on this stack of branes in the dual picture.

We take $5b = \frac{1}{4}c = 3a$, $p = -\frac{4}{5}a$, which satisfy the zero–energy constraint. We thus recover the usual non–extremal D2–brane solution as (5.55)–(5.57) take the form

$$H(\rho) = \frac{2^{1/2} g_s^{3/4} Q}{5\rho^5}, \quad A(\rho) = B(\rho) = 1 - \frac{6\sqrt{2}a}{5\rho^5}. \tag{5.58}$$

The non–extremal ordinary D2–brane solution is then given by $w = 0$ together with

$$\Phi_n = -\frac{4}{5}au, \quad e^{6x} = e^{6au}, \quad e^{-5y} = \frac{5\sinh(3au)}{3\sqrt{2}a}, \quad e^{-\frac{z_n}{4}} = g_s^{3/4}\frac{Q}{3a}\sinh(3au). \tag{5.59}$$

5.5 Asymptotics of the regular non–extremal fractional D2–branes

Given that no analytic solution to the system (5.36)–(5.40), (5.42) of second–order differential equations with the required properties (i)–(ii) outlined below could be found, one has to content with a solution in perturbation theory or from numerical integration. As a first step, we present below numerical solutions to the differential equations found at first–order in P^2.

Regardless of the method used to tackle the system (5.36)–(5.40), (5.42), a solution must satisfy two natural conditions in the IR $u \to \infty$ and the UV $u \to 0$ regions:

(i) it must be a one–parameter (the non–extremality parameter $x' = a$ or, see below, the Hawking temperature) generalization of the extremal fractional D2–brane solution (5.50). We must thus ensure that it approaches the latter solution in the UV and impose the following boundary conditions at $u \to 0$:

$$\begin{aligned} &x, w, \Phi_n, f \to 0, \quad K(u) \to 32^{7/5} 5^{1/5} g_s^{3/2} g_0^2 P^2 \left(\frac{5}{6}\tilde{L}^{-1} - u^{1/5}\right), \\ &e^{5y} \to \frac{\sqrt{2}}{5u}, \quad e^{-\frac{z_n}{4}} \to 2^{2/5} 5^{6/5} g_s^{3/2} g_0^2 P^2 \; u \left(\tilde{L}^{-1} - u^{1/5}\right). \end{aligned} \tag{5.60}$$

It should be emphasized that $\tilde{L}$ scales as P^2 so that the leading term in the $u^{1/5}$ expansion is also of leading order in P^2.

(ii) For $P \to 0$ it must reduce to the black D2–brane solution. The latter has a regular Schwarzschild horizon, which if preserved as fractional D2–branes are added leads to the following infrared constraints as $u \to \infty$:

$$\begin{aligned} &x = au, \quad w \to w^\star, \quad \Phi_n \to -\frac{4}{5}au + \Phi_n^\star, \quad K \to K^\star, \\ &y \to -\frac{3}{5}au + y^\star, \quad z_n \to -12au + z_n^\star. \end{aligned} \tag{5.61}$$

These asymptotics ensure the existence of a regular Schwarzschild horizon for $u \gg 1$. The constants $y^\star$ and $z^\star$ encode the information on the Hawking temperature and entropy of this horizon. Indeed, the metric for $u \to \infty$ in the U–X_0 plane, where $U \equiv e^{-3au}$ is the natural near–horizon variable, takes the form

$$ds^2 = U^2 e^{\frac{5}{32}z_n^\star + \frac{5}{32}\Phi_n^\star} dX_0^2 + \frac{1}{9a^2} e^{-\frac{3}{32}z_n^\star - \frac{3}{32}\Phi_n^\star + 12y^\star} dU^2. \tag{5.62}$$

The Hawking temperature T_H is fixed from the periodicity of the Euclidean time X_0 that guarantees there is no conical singularity in the U–X_0 plane:

$$T_H = \frac{3a}{2\pi} e^{\frac{z_n^\star}{8} + \frac{\Phi_n^\star}{8} - 6y^\star}. \tag{5.63}$$

In the canonical ensemble where temperature and volume are independent quantities the Hawking temperature (5.63) of the event horizon is identified with the temperature of the dual gauge theory in its deconfined phase. There are generally different possibilities for the physics of the large u asymptotics. The first one (a) is that the $u \to \infty$ solution develops a regular horizon as will be the case from (5.61). (b) It is also possible that $H(\rho)$ in (5.4) vanishes at some finite u before $u = \infty$. This corresponds to a naked singularity. (c) Still another possibility is that $H(\rho)$ vanishes at $u = \infty$. The singularity and the horizon coincide in this case. The next section deals with the first option where our Ansatz should be appropriate at sufficiently high temperature.

The effective D2–brane charge $K(u)e^{-\frac{\Phi}{2}}$ corresponds in the gauge theory dual to an effective number of unconfined colour degrees of freedom. As we integrate towards large u $K(u)e^{-\frac{\Phi}{2}}$ decreases. On the gauge theory side this matches the expected behaviour that as we run the scale of theory towards the infrared the number of colours degree of freedom should decrease. There is an significant difference from the D3–brane case [49, 50, 105] due to the dependence of the D2–brane flux on the dilaton. Note however that $e^{-\frac{\Phi}{2}} \sim \frac{e^{\frac{3}{8}au}}{(\sinh(3a(u+k)))^{1/8}}$ is decreasing for all $u \geq 0$. Thus the flux will still decrease, with the proviso that fractional D2–branes add a small enough perturbation on the variation of the dilaton. The string coupling should be written in terms of gauge theory variables as $e^\Phi \sim g_{YM}^{5/2} N^{1/4}/\Lambda^{5/4}$ [150]. We denote Λ the energy scale on the gauge theory side. $N \sim Ke^{-\frac{\Phi}{2}}$ is the number of ordinary D2–branes at the radial variable corresponding to this scale. Then $e^{\frac{9}{8}\Phi} \sim g_{YM}^{5/2} K^{1/4}/\Lambda^{5/4}$. K should decrease in the IR as we add fractional D2–branes. Therefore with perturbative corrections included e^Φ still decreases as $u \to \infty$.

We expect that for low–enough temperatures the effective number of D2–branes will reach zero at some finite u. Above the critical temperature we expect though that the flux will stabilize at some finite value $K^\star e^{-\frac{\Phi^\star}{2}}$ for large u. This number should vanish in the limit that the temperature reaches its critical value. It would be interesting to study the free energy. It should vanish at the critical temperature, marking the transition where the CGLP confining solution is thermodynamically favoured. This would also make it clear that the transition is first–order, instead of the other possibility in 2+1 dimensions, namely Kosterlitz–Thouless.

5.6 Perturbation theory in P

To construct the regular non–extremal fractional D2–brane solution, we shall start with the non–extremal ordinary D2–brane solution (5.59). We recall that it corresponds to no fractional branes, $P = 0$. In the singularity–shielding analysis of [105, 126] for D3–branes and D1–branes respectively, Q is replaced by $K^\star$ from a set of conditions similar to (5.61) in order to match onto the asymptotics for the infrared, near–horizon boundary conditions. Presently, we keep Q fixed as the constant f_0 is introduced from the UV asymptotic condition. We will then attempt to find the deformation of this starting solution order by order in P^2. Actually the relevant expansion parameter happens to be $\lambda = g_0^2 P^2 Q^{-1} a^{-2/5}$, with $g_0 = \mathcal{O}(1)$ appearing in the large distance asymptotic condition (5.60). This ratio is dimensionless and for perturbation theory to apply, the effective D2–brane charge must thus be large enough. It will also become clear that the solution we will build in perturbation theory will correspond to the dual gauge theory at a temperature above the expected critical temperature. As in [105, 126], it turns out that a nice feature of perturbation theory around the extremal ($a = 0$) D2–brane background is that already the first–order correction in P^2 yields the exact extremal fractional D2–brane solution (5.50). This is strong evidence that an interpolating non–extremal solution that we only managed to approach in perturbation theory indeed exists.

It is convenient [105, 126] to rescale the fields by relevant powers of P^2:

$$\begin{aligned}
&f(u) = PF(u), \quad g(u) = g_0 \left(\frac{\sqrt{2}}{5}\right)^{4/5} u^{-4/5} + PG(u), \\
&\Phi_n(u) = -\frac{4}{5}au + P^2\phi(u), \quad w(u) = P^2\omega(u), \\
&y(u) \to y(u) + P^2\xi(u), \quad z_n(u) \to z_n(u) + P^2\eta(u),
\end{aligned} \tag{5.64}$$

where $y(u)$ and $z_n(u)$ on the right-hand side are the corresponding functions appearing in the non–extremal ordinary D2–brane solution (5.59), which we transcribe here as well:

$$e^{-5y} = \frac{5\sinh(3au)}{3\sqrt{2}a}, \quad e^{-\frac{z_n}{4}} = g_s^{3/4}\frac{Q}{3a}\sinh(3au). \tag{5.65}$$

The first–order correction to $g(u)$ and $h(u)$ is found from $F(u)$ and (5.13). We must impose further conditions on the correction functions F, ϕ, ω, ξ and η so as to match onto the extremal fractional D2–brane asymptotics (5.60) near $u = 0$:

$$\phi(0) = \omega(0) = \xi(0) = 0, \quad \eta \to 4\tilde{L}u^{1/5}. \tag{5.66}$$

Furthermore, $F(u)$ must tend to zero in the UV. In order to avoid excessive cluttering, from now on we set $g_s = 1$ by absorbing it into P. The system of mixed, non–linear second–order differential equations (5.36)–(5.40) along with the constraint (5.42) becomes

$$15\xi'' - 3g_0^2 e^{\frac{z_n}{4}+4y}\left(e^{\frac{3}{5}au} - e^{-\frac{3}{5}au}\right) - 375e^{10y}\xi + \mathcal{O}(P^2) = 0, \tag{5.67}$$

$$\omega'' + 2e^{10y}\omega + \mathcal{O}(P^2) = 0, \tag{5.68}$$

$$5\phi'' + 4e^{\frac{z_n}{4}}\left(e^{\frac{3}{5}au} - e^{-\frac{3}{5}au}\right)\left(6e^{6y}F^2 - \frac{2}{3}e^{-4y}F'^2\right) + \mathcal{O}(P^2) = 0, \tag{5.69}$$

$$\begin{aligned}
&\eta'' - 2e^{\frac{z_n}{2}}Q^2\eta - 4e^{\frac{z_n}{4}}\left(e^{\frac{3}{5}au} + e^{-\frac{3}{5}au}\right)\left(4e^{6y}F^2 + \frac{2}{3}e^{-4y}F'^2\right) \\
&\quad + 64e^{\frac{z_n}{2}}QF\int_0^u e^{6y}F(\rho)d\rho + \mathcal{O}(P^2) = 0,
\end{aligned} \tag{5.70}$$

$$\left(e^{-4y}F'\right)' - 6e^{6y}F + \mathcal{O}(P^2) = 0 \tag{5.71}$$

and

$$\begin{aligned}
&60y'\xi' - \frac{1}{16}z_n'\eta' + \frac{3}{4}a\phi' + \frac{1}{4}e^{\frac{z_n}{2}}Q^2\eta - 2e^{\frac{z_n}{4}}\left(e^{\frac{3}{5}au} - e^{-\frac{3}{5}au}\right) \\
&\quad - 8e^{\frac{z_n}{2}}QF\int_0^u e^{6y}F(\rho)d\rho - 150e^{10y}\xi + \mathcal{O}(P^2) = 0.
\end{aligned} \tag{5.72}$$

5.6.1 Leading–order solution for K

In terms of $v \equiv 1 - e^{-6au}$, the differential equation for the first–order solution to the condition that H_3 be harmonic (5.71) is

$$F'' + \frac{4/5 \ - \ 7/5\ v}{v(1-v)} F' - \frac{12}{25} \frac{1}{v^2(1-v)^2} F = 0. \tag{5.73}$$

The general solution is of the form

$$F(u) = \frac{{}_2F_1\left[-\frac{3}{5}, -\frac{1}{5}, -\frac{2}{5}, v\right]}{v^{3/5}} C_1 + v^{4/5}\, {}_2F_1\left[\frac{4}{5}, \frac{6}{5}, \frac{12}{5}, v\right] C_2, \tag{5.74}$$

where ${}_2F_1[a,b;c;z]$ denotes Gauss' hypergeometric function. The integration constant C_1 and C_2 must be such that $F(u)$ increases as degrees of freedom are integrated out down the infrared. This condition yields $C_1 = 0$. The behaviour of $F(v)$ appears on Figure 5.1.

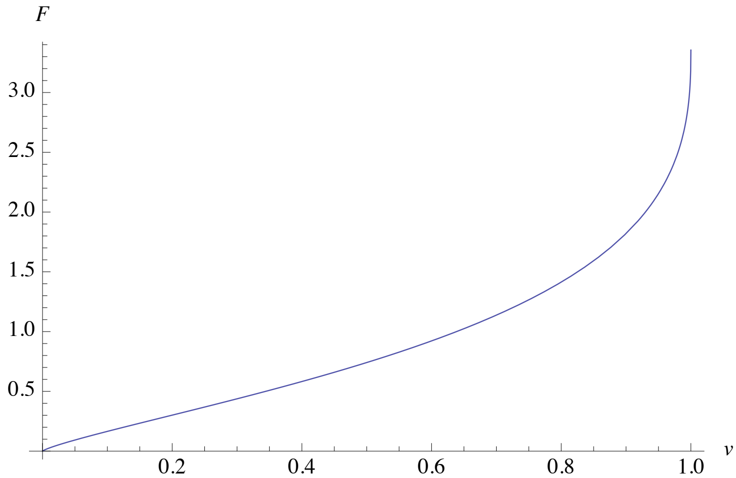


Figure 5.1: *Plot of the first–order solution for $F(u)$, with $C_2 = 1$. Even though the conditions needed to derive this solutions are imposed in the IR, it behaves in the UV as the corresponding solution for extremal fractional D2–branes found in Section 5.4.2.*

Figure 5.2. displays the generic shape of the effective number of ordinary and fractional D2–branes

$$K(v) = Q\Big(1 - \frac{4}{5}\frac{1}{3^{2/5}5^{6/5}}\frac{f_0^2 P^2}{a^{2/5}Q} v^{4/5}\, {}_2F_1\left[4/5, 6/5, 12/5, v\right] \int_0^v \frac{\rho^{4/5}}{\left(\sinh\left(-\frac{1}{2}\log(1-v)\right)\right)^{6/5}}\, {}_2F_1\left[4/5, 6/5, 12/5, \rho\right] d\rho\Big) + \mathcal{O}(P^2). \tag{5.75}$$

We see that, as advertised, the appropriate expansion parameter is $\lambda = g_0^2 P^2 Q^{-1} a^{-2/5}$. All boundary conditions on $F(u)$ are imposed at $u \to \infty$ but its $u \to 0$ limit has a surprise in store. Expanding (5.74) for small u

$$F = f_0 u^{4/5} + \mathcal{O}(u^{8/5}), \tag{5.76}$$

identifying C_2 with f_0, matches, up to 1/length corrections, the exact extremal fractional D2–brane solution (5.49). This is strong hint that an exact interpolating solution does exist. We did not manage to find such an analytic solution and have to resort to pertubation theory. Equation (5.75) suggests that this approach is reliable as long as $g_0^2 P^2 Q^{-1} a^{-2/5} << 1$. As

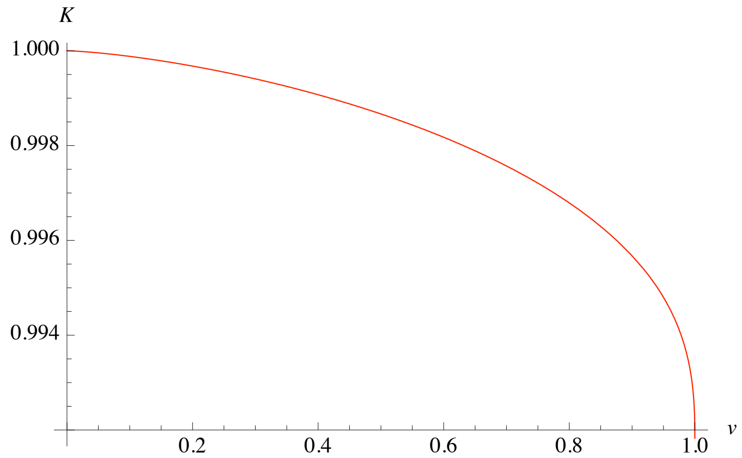


Figure 5.2: *The leading–order solution for $K(u)$ computed from perturbation theory in P^2. Here we put $K^\star = 1$ and $\lambda = 0.01$. As expected, the effective number of degrees of freedom decreases as we integrate them out flowing to the IR, $v \to 1$.*

already mentioned, this implies the Hawking temperature is high. Taking the IR limit of (5.75) results in

$$K^\star = Q - \frac{28}{15}\frac{(2/5)^{1/5}}{3^{2/5}}\frac{\Gamma[2/5]\Gamma[12/5]}{\Gamma[6/5]\Gamma[8/5]}\left(\frac{2 - {}_3F_2\left[\{3/5,4/5,1\},\{9/5,11/5\},1\right]}{1+\sqrt{5}}\right)\frac{f_0^2 P^2}{a^{2/5}}. \tag{5.77}$$

Expressed in terms of a and $K^\star$, (5.63) gives

$$T_H \sim a^{3/10} K_\star^{-1/2}. \tag{5.78}$$

The critical temperature T_c corresponds to situation where $K^\star$ vanishes. This means $a^{2/5} \sim \tilde{L}^7$ at the critical regime. From (5.77) it is apparent that our pertubative analysis is valid for $T >> T_c$.

5.6.2 Solutions for other fields

To determine the corrections to the other fields, the Lagrange method of variation of parameters seems particularly suited at first to solve the linear second–order differential equations (5.67)–(5.70). The final product of this recipe guarantees that a differential equation of the type

$$\frac{d^2\psi}{dx^2} + a(x)\frac{d\psi}{dx} + b(x)\psi = c(x) \tag{5.79}$$

admits a general solution

$$\Psi(x) = -\Psi_1(x)\int dx' \frac{c\Psi_2}{W}(x') + \Psi_2(x)\int dx' \frac{c\Psi_1}{W}(x') + c_1\Psi_1(x) + c_2\Psi_2(x) \tag{5.80}$$

in terms of two linearly independent solutions Ψ_1 and Ψ_2 for the corresponding homogeneous equation (i.e. (5.79) with $c(x) = 0$). $c_{1,2}$ are arbitrary constants. $W \equiv \Psi_1 \frac{d\Psi_2}{dx} - \Psi_2 \frac{d\Psi_1}{dx}$ is the Wronskian. In terms of $v \equiv 1 - e^{-6au}$, equations (5.67) to (5.70) can be reshuffled into the general form

$$\frac{d^2\psi}{dv^2} - \frac{1}{1-v}\frac{d\psi}{dv} - \frac{a}{v^2(1-v)}\psi = c(v), \tag{5.81}$$

where a and $c(v)$ denote an arbitrary constant and a function. A solution to the homogeneous part of this equation is $\psi(v) = v^\nu \, {}_2F_1[\nu,\nu;2\nu;v]$ with $a = \nu(\nu-1)$. The $a = 2$ case is somewhat aside in that the two linearly independent solutions reduce to $\frac{1}{v} - \frac{1}{2}$ and $-2 + \frac{v-2}{v}\ln(1-v)$.

We met considerable hindrance to deriving analytical solutions to (5.67)–(5.70) when trying to integrate hypergeometric functions. From (5.74) and (5.59), we then choose instead to numerically solve the equation of motion (5.69) for the order–P^2 corrections to the fields in our Ansatz. As far as the dilaton is concerned, we impose the large u boundary condition that $\phi' \to 0$. The small u asymptotics for its part requires that $\phi \to 0$ near $u = 0$. Integration constants for ξ are fixed from the boundary conditions that $\xi = 0$ near $u = 0$ and $\xi'(v)$ vanishes as $u \to \infty$. The solution for η is determined from the boundary conditions that $\eta'(u)$ is zero in the small distance region and η scales as $u^{7/5}$ for $u \to 0$. The behaviour of those fields for a common set of values of a is displayed on Figure 5.3., 5.4. and 5.5.

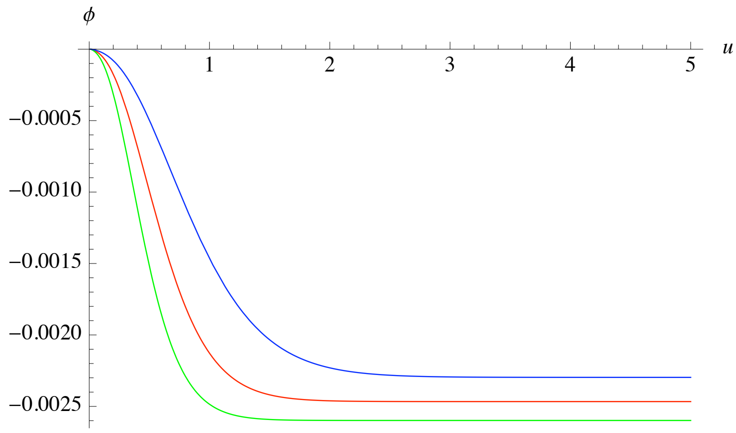


Figure 5.3: *A sample of the first–order solution for $\phi(u)$, with $a = 0.7$, $a = 1.0$ and $a = 1.3$. We code of colours we use to distinguish those is to associate the lowest value of a to the blue curve. The highest value of a corresponds to the red curve. We have set $Q = 100$, $f_0 = 1$.*

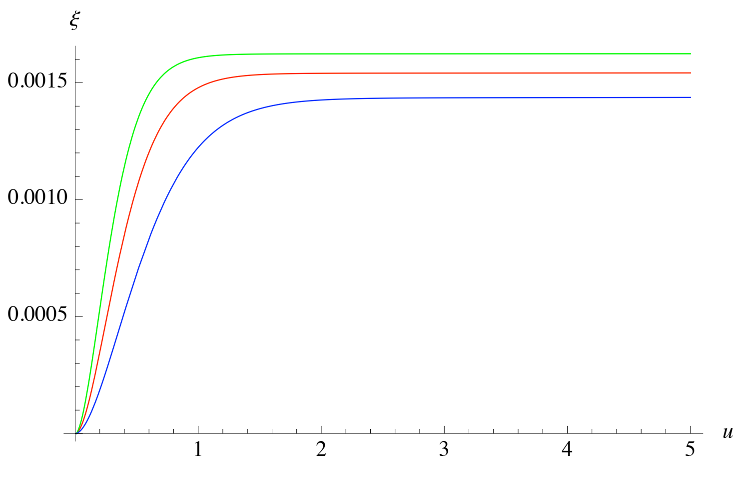


Figure 5.4: *$\xi(u)$ versus the radial coordinate u of the metric Ansatz. Three sample curves are drawn with $a = 0.7$, $a = 1.0$ and $a = 1.3$ following the conventions explained on Figure 5.3.*

Since the P^2 corrections to the entropy and the Hawking temperature do not depend on $\omega(u)$ we will just deliver its asymptotics here. For small u (5.68) yields

$$\omega = pu^{1/5} + qu^{4/5}. \tag{5.82}$$

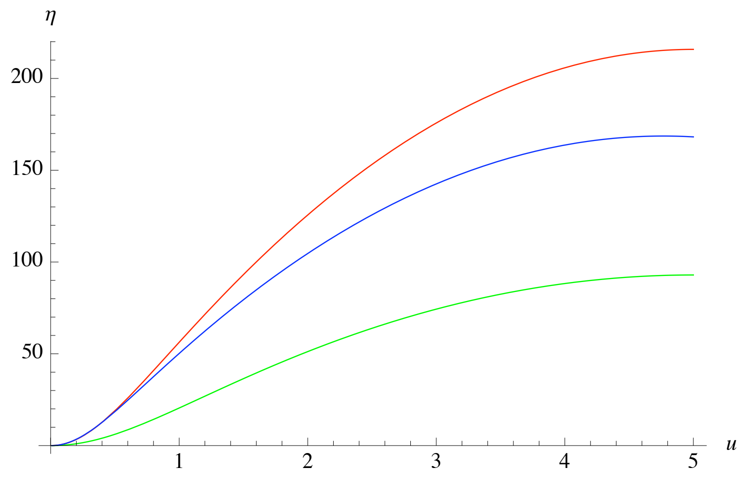


Figure 5.5: *Aspect of the first–order solutions for $\eta(u)$ with $a = 0.7$, $a = 1.0$ and $a = 1.3$. The Cauchy conditions for the second–order differential equation governing this first–order solution are such that it matches with the extremal Herzog–Klebanov solution in the UV and decreases smoothly to a constant value of $z_n \equiv 15z - \Phi$ near the horizon, in the IR.*

The constant p and q are to be found numerically from the conditions $\omega(0) = 0$ (5.66) and ω stays finite when $u \to \infty$. At large u (5.68) gives

$$\omega = \omega^\star - \frac{4}{25}\omega^\star e^{-6au} + \mathcal{O}(e^{-12au}). \tag{5.83}$$

The corrections to the Hawking temperature and the regular horizon entropy are extracted from the particular form that the metric (5.1) takes when specialized to the perturbative solutions we have just derived. From

$$\begin{aligned} ds^2_{10E} =& \left(\frac{6a}{Q}\right)^{5/8} e^{\frac{5}{32}P^2(\eta+\phi)} v^{-5/8}\Big((1-v)dX_0^2 + dX_i dX_i\Big) \\ &+ (6a)^{1/40}\left(\frac{2}{25}\right)^{1/5} Q^{3/8} v^{-1/40}\Big[\frac{1}{6^{1/40}}\frac{2}{25} e^{P^2\left(-\frac{3}{32}(\eta+\phi)+12\xi\right)} \frac{dv^2}{v^2(1-v)} \\ &+ e^{P^2\left(-\frac{3}{32}(\eta+\phi)+2\xi\right)}\left(\frac{1}{2}e^{-4P^2\omega}\left(D\mu^i\right)^2 + e^{2P^2\omega} d\Omega_4^2\right)\Big], \end{aligned} \tag{5.84}$$

the dependence on the near–horizon asymptotic developments of the entropy density over the temperature squared equals

$$\frac{S}{VT^2} = \alpha e^{3P^2\left(6\xi^\star - \frac{\eta^\star+\phi^\star}{8}\right)}, \tag{5.85}$$

where $\alpha \equiv 4\pi^2\left(\frac{2}{25}\right)^{9/5}(6a^2)^{1/40}Q^{3/2}$. From (5.77) and (5.78) and using numerical values of the fields at the horizon, the entropy density over the temperature squared (5.85) tends to a limiting value when the dual deconfined gauge theory is heated up. From the thermodynamic relation $dE = TdS$, the energy of the field theory at strong coupling is about 2/3 its the free field value, $E \sim \frac{2}{3}TS$. This is to be compared with the celebrated 25% discrepancy for $\mathcal{N} = 4$ SYM in four dimensions [102]. There is an additional contribution from fractional branes to the ratio $\frac{S}{VT^2}$ of the Horowitz–Strominger black D2–brane solution [136].

Further numerical work is required, in particular to settle the fate in the IR of the squashing function $w(u)$ or at least its first–order correction in P^2. Starting with the UV $u \to 0$ conditions (5.60), the numerical procedure would consist in integrating either the full set of equations or those derived in perturbation theory for the functions in the Ansatz. On should then vary the temperature (5.63) and find the solutions satisfying (5.61) at a regular horizon shielding the singularity without coinciding with it.

5.7 Conclusion

We have built to order P^2 in the number of fractional branes a regular non–extremal fractional D2–brane perturbative solution. This solution is the supergravity dual of the high–temperature, deconfined phase of the three–dimensional theory whose confined phase supergravity dual was constructed by Cvetic, Gibbons, Lu and Pope [67]. There are several reasons why one might be interested in this solution and the corresponding field theory.

The high–temperature limit of QCD in four dimensions may be dominated by the physics of the static theory in three space dimensions, i.e. Euclidean QCD_3. Within the context of attempting to find an appropriate supergravity dual to QCD, it might then be of interest to start in one lower dimension and see what this can tell about the high–temperature phase.

Three–dimensional field theories are also of interest for gaining better understanding of the properties of strongly–coupled systems of electrons in condensed matter physics. References [115], [133] and [200] review what supergravity theories have to teach about condensed matter systems belonging to the same universality class of the gauge duals. On this regard, we would like to mention [134] which works out the solution for a baryonic black 3–brane, allowing for a new contribution to the R–R field strengths from which the baryonic $U(1)_B$ gauge field under which their black hole solution is charged stems from upon truncation to five dimensions.

In [6] a numerical approach was performed for constructing a black hole solution from fractional D3–branes dual to cascading gauge theories. The free energy becomes positive below some critical temperature. This vindicates the suggestion that the supergravity solution is associated to one of the phases separated by a transition which is indeed first–order, as it should from field theory arguments. It would be interesting to carry a similar analysis for the case of the 2+1 dimensional cascading gauge theory dual to fractional D2–branes. This would rule out the possibility of a Kosterlitz–Thouless transition, which the arguments of [248] alone cannot prefer over a first–order transition.

It would be interesting to understand the spectrum of the field theory dual to the resolved fractional D2–brane solution of [67] of which we have described the supergravity solution corresponding to the high–temperature, deconfined phase. How this might be guessed from the geometry of the transverse space is described in [68]. The focus is on the gauge theory dual to the resolved D2–branes and wrapped NS5–branes solution found in [67]. The base of the cone is $S^3 \times S^3$. Embedding the transverse space with topology $\mathbb{R}^4 \times S^3$ into $\mathbb{R}^8$, one writes the constraint associated to this locus in terms of quaternionic coordinates. Identification of these variables with the matter field of the dual field theory[5] provides the representations under which they transform. The most recent attempt to understand the UV regime of the gauge theory we are aware of appears in [190]. This work starts with M2–branes in particular Spin(7) holonomy backgrounds, A_8 and B_8, and considers the flow to D2–branes on manifolds of G_2 holonomy. A further study of M–theory on the B_8 manifold appears in [111]. Reference [190] find that the UV field theory for N D2–branes on the background $\mathcal{M}$, where $\mathcal{M}$ is a cone over $\mathbb{CP}^3$, is $\mathcal{N} = 1$ supersymmetric with $U(N) \times U(N)$ gauge group. The field content fits into an $\mathcal{N} = 1$ vector multiplet and four $\mathcal{N} = 2$ chiral superfields, one pair in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation, another in the conjugate. Based on how the gauge groups change from adding fractional branes to the Klebanov–Witten theory, adding M fractional D2–branes might change the gauge group to $U(N) \times U(N + M)$. It would be interesting to verify this in more detail.

Recently, there has been much interest in $\mathcal{N} = 6$ superconformal Chern–Simons matter theories [7]. See [174] for an excellent review with additional references. When the level k of this

[5] The same procedure is used for the conifold [168].

$U(N)_{-k} \times U(N)_k$ theory describing N coincident M2–branes is such that $k << N << k^5$, the M–theory circle becomes small and it becomes suitable to describe the gravitational dual using a weakly–curved IIA string theory on a $\mathrm{AdS}_4 \times \mathbb{CP}^3$ geometry orientifolded under the initial M–theory $\mathbb{Z}_k$ projection. A generalization of the ABJM theory to account for the possibility of fractional M2–branes was proposed in [8]. The possibility of duality cascades was further studied in [9]. Further evidence should be provided from dual string theory constructions and this is reported in [9] as work in progress. It might be interesting to build a non–extremal solution from the IIA description of the 't Hooft limit of the ABJM theory. A difference from our work lies in the occurrence of a non–vanishing 2–form field strength in this IIA background.

More generally, there is a connection between the D2–branes theories we have considered and $\mathcal{N} = 1$ Chern–Simons theories. Indeed, starting from M2–branes on Spin(7) cones, one can obtain Chern–Simons theories by an orientifolding procedure [89][6]. From [190] there is then a flow from C–S theories to D2–branes probing a G_2 holonomy manifold. It might also be of interest to study the thermodynamics of those more general flows, with or without fractional branes, which might provide information on the deconfined phase of C-S matter theories. This might hint at how N^2 degrees of freedom on D2–branes relate to $N^{3/2}$ on N coincident M2–branes.

[6]The link between Chern–Simons terms and fluxes in M–theory on a Spin(7) manifold is discussed in [111].

Chapter 6

The Backreaction of Anti–M2 Branes on a Warped Stenzel Space

We find the superpotential governing the supersymmetric warped M–theory solution with a transverse Stenzel space found by Cvetič, Gibbons, Lü and Pope [66], and use this superpotential to extract and solve the twelve coupled equations underlying the first–order backreacted solution of a stack of anti–M2 branes in this space. These anti–M2 branes were analyzed recently in a probe approximation by Klebanov and Pufu, who conjectured that they should be dual to a metastable vacuum of a supersymmetric 2+1 dimensional theory. We find that the would–be supergravity dual to such a metastable vacuum must have an infrared singularity and discuss whether this singularity is acceptable or not. Given that a similar singularity appears when placing anti–D3 branes in the Klebanov–Strassler solution, our work strengthens the possibility that anti–branes in warped throats do not give rise to metastable vacua.

6.1 Introduction and discussion

The recent revival of interest in metastable supersymmetry breaking in quantum field theory is largely due to the work of Intriligator, Seiberg and Shih [145] (ISS). This work presents a mechanism to naturally circumvent some of the problems afflicting other models for dynamic supersymmetry breaking (DSB) [231, 206, 147, 146]. A natural question that was posed immediately after [145] is whether metastable vacua also exist in string realizations of supersymmetric field theories.

For type IIA brane–engineering models of supersymmetric field theories, the answer to this question is negative [29]. Indeed, these models are constructed using D4 branes ending on codimension–two defects inside NS5 branes [210, 90, 29], which source NS5 worldvolume fields that grow logarithmically at infinity. In supersymmetric vacua this logarithmic growth encodes the running of the gauge theory coupling constant with the energy [258, 135, 47, 98], but these logarithmic modes are different in the candidate metastable brane configuration and in the supersymmetric one. This implies that the candidate metastable brane configuration and the supersymmetric one differ by an infinite amount, and hence cannot decay into each other. Hence, the type IIA brane construction does not describe a metastable vacuum of a supersymmetric theory, but instead a nonsupersymmetric vacuum of a nonsupersymmetric theory.

Another arena where one might try to find string theory realizations of metastable vacua are IIB holographic duals of certain supersymmetric gauge theories. The best–known example in this class was proposed by Kachru, Pearson and Verlinde [156, 72], who argued that a background with anti–D3 branes at the bottom of the Klebanov–Strassler warped deformed conifold [171] is dual to a metastable vacuum of the dual supersymmetric gauge theory. Since the Klebanov–Strassler solution has positive D3 brane charge dissolved in flux, the anti–D3 branes can annihilate against this charge (this annihilation happens via the polarization of the anti–D3 branes into an NS5 brane [204, 218]), and this bulk process is argued to correspond to the decay of the metastable vacuum to the supersymmetric one in the dual field theory.

Another proposal for a metastable vacuum obtained by putting anti–branes at the bottom of a smooth warped throat with positive brane charge dissolved in flux has recently been made by Klebanov and Pufu [175], who argued that probe anti–M2 branes at the tip of a supersymmetric warped M–theory background with transverse Stenzel space [244], give rise to a long–lived metastable vacuum. The supersymmetric solution, first found by Cvetič, Gibbons, Lü and Pope (CGLP) in [66] has M2 charge dissolved in fluxes and a large S^4 in the infrared. The anti–branes can annihilate against the charge dissolved in fluxes by polarizing into M5 branes [28] wrapping three–spheres inside the S^4.

The probe brane analyses described above, while indicative that a metastable vacuum might exist, are however not enough to establish this. One possible issue which can cause the backreacted solution to differ significantly from the probe analysis is the presence of non-normalizable modes. If the anti-branes indeed source such modes then the candidate metastable configuration is not dual to a non–supersymmetric vacuum of a supersymmetric theory, but to a non–supersymmetric vacuum of a non–supersymmetric theory, and the supersymmetry breaking is not dynamical but explicit. The existence of non–normalizable modes is not visible in the probe approximation (much like the existence of type IIA log–growing modes was not visible in $g_s = 0$ brane constructions [210, 90]), but only upon calculating the backreaction of the probe branes – a not too easy task.

In [30] was found the possible first–order backreacted solution sourced by a stack of anti–D3 branes smeared on the large S^3 at the bottom of the Klebanov–Strassler (KS) solution, and found two very interesting features: first, of the 14 physical modes describing $SU(2) \times SU(2) \times Z_2$–invariant perturbations of the warped deformed conifold, only one mode enters in the expression of the force that a probe D3 brane feels in this background. Hence, since anti–D3 branes attract probe branes, if the perturbed solution is to have any chance to describe backreacted anti–D3 branes, this mode must be present[1]. The second feature of this solution is that if the force mode is present, the infrared[2] must contain a certain singularity, which has *finite* action[3]. Note that having a finite action does not automatically make a singularity acceptable – negative–mass Schwarzschild is an obvious counterexample [137]. As discussed in [30], if this singularity is unphysical, then the solution sourced by the anti–D3 branes cannot be thought of as a small perturbation of the KS solution, and therefore does not describe a metastable vacuum of the dual theory. If this singularity is physical, the first–order solution does describe anti–D3 branes at the bottom of the KS solution, and work is in progress to determine what are the features of this solution, and whether the perturbative anti–D3 brane solution describes or not metastable vacua of the dual theory.

The purpose of this part of the thesis is to calculate the first–order backreaction of the other proposed metastable configuration with anti–branes in a background with charge dissolved in fluxes: the anti–M2 branes in the Stenzel–CGLP solution [66]. In order to do this we smear the anti–M2 branes on the large S^4 at the bottom of the Stenzel–CGLP solution, and solve for all possible deformations of this background that preserve its SO(5) symmetry. We consider an Ansatz for these deformations ; the space of deformations is parameterized by 6 functions of one variable satisfying second–order differential equations. However, when perturbing around a supersymmetric solution, Borokhov and Gubser [45] have observed that these second–order equations factorize into first–order ones, that are much easier to solve. Nevertheless, in order to apply the Borokhov–Gubser method, one needs to find the superpotential underlying the

[1] The asymptotic behavior of the force matches the one argued for in [158], and the existence of this mode was first intuited in [73] which set out to study the UV asymptotics of the perturbations corresponding to anti–D3 branes in the KT background[170].

[2] An IR analysis of some of the non–supersymmetric isometry–preserving perturbations of the Klebanov–Strassler background can also be found in [201].

[3] This was first observed by I. Klebanov.

supersymmetric solution, which for the warped fluxed Stenzel–CGLP solution was not known until now. The first result of this chapter, presented in Section 6.2., is to find this superpotential[4], and derive two sets of first–order equations governing the space of deformations.

We then show in Section 6.3. that the force felt by a probe M2 brane in the most general perturbed background depends on only one of the "conjugate–momentum" functions that appear when solving the first–order system, and hence on only one of the 10 constants parameterizing the deformations around the supersymmetric solution. We then solve in Section 6.4. the two sets of first–order differential equations. Amazingly enough, the solutions for the first set of equations (for the conjugate–momentum functions) can be found explicitly in terms of incomplete elliptic integrals (a huge improvement on the situation in [30]). We also find the homogeneous solutions to the other equations and give implicitly the full solution to the system in terms of integrals. We also provide the explicit UV and IR expansions of the full space of deformations, and find which deformations correspond to normalizable modes and which deformations correspond to non–normalizable modes.

In Section 6.5. we then use the machinery we developed to recover the perturbative expansion of the known solution sourced by BPS M2 branes smeared on the S^4 at the tip of the Stenzel–CGLP solution [66], and analyze the infrared of the possible solution sourced by anti–M2 branes. After removing some obviously unphysical divergences and demanding that in the first–order backreacted solution a probe M2 brane feels a nonzero force, we find that the only backreacted solution that can correspond to anti–M2 branes must have an infrared singularity, coming from a four–form field strength with two or three legs on the three–sphere that is shrinking to zero size at the tip of the Stenzel space.

Hence, the first–order backreacted solution for the anti–M2 branes has the same two key features as the anti–D3 branes in KS: the force felt by a probe M2 brane in this background depends only on one of the 10 physical perturbation modes around this solution, and the solution where the force–carrying mode is turned on must have an infrared singularity coming from a divergent energy in the M–theory four–form field strength. Nevertheless, unlike in the "anti–D3 in KS" solution, the action of this infrared singularity also diverges. Again, if this singularity is physical, our first–order backreacted solution describes anti–M2 branes in the CGLP background, and, to our knowledge, would be the first backreacted supergravity solution dual to metastable susy–breaking in 2+1 dimensions since the work of Maldacena and Năstase [193]. This may be of interest both in the programme of using the AdS/CFT correspondence to describe strongly-interacting condensed–matter systems, and also in view of the relevance of three–dimensional QFT's at strong coupling to a recent holographic model of four–dimensional cosmology [197]. On the other hand, if the singularity is not physical then the backreaction of the anti–M2 branes cannot be taken into account perturbatively; this indicates that the only solution with proper anti–M2 brane boundary conditions in the infrared is the solution for anti–M2 branes in a CGLP background with anti–M2 brane charge dissolved in flux, and hence the anti–M2 branes flip the sign of the M2 brane charge dissolved in flux.

Given the similarity of the results of the "anti–D3 in KS" and of the "anti–M2 in CGLP" analyses and the drastically–different calculations leading to them, it is rather natural to expect that the underlying physics of the two setups is the same: either both singularities are physical, which indicates that anti–branes in backgrounds with charge dissolved in fluxes give rise to metastable vacua, or they are both unphysical, which supports the idea that anti–branes in such backgrounds cannot be treated as a perturbation of the original solution, and may flip the sign of the charge dissolved in flux. Furthermore, our analysis suggests that one cannot use the finiteness of the action as a criterion for accepting a singularity. This would allow the anti–D3 singularity and exclude the anti–M2 one, which would be rather peculiar, given the striking resemblance of the two systems.

There are a few possible explanations for the singularities we encounter in the anti–M2

[4] This is the equivalent of the Papadopoulos–Tseytlin superpotential for the KS solution [211, 60, 31].

and anti–D3 solutions. One is that these singularities are accompanied by stronger, physical singularities, coming from the smeared anti–M2 or anti–D3 sources, and one can hope that whatever mechanism renders the stronger singularities physical may cure the subleading ones as well. Another explanation is that the subleading singularities are a result of smearing the antibranes. This is a difficult argument to support with calculational evidence, as the unsmeared solution is a formidable problem even for BPS branes in Stenzel spaces [176, 225]. Furthermore, a naive comparison of the anti–M2 and anti–D3 solutions indicates that the stronger the physical singularity associated with the brane sources is, the stronger the subleading singularity will be. Hence, it is likely that unsmearing will make things worse, not better, as is illustrated for special cases in [44]. Note also that one cannot link the divergent four–form field strength with the M5 branes into which the anti–M2 branes at the tip of the Stenzel–CGLP solution polarize – they have incompatible orientations.

It is also interesting to remember that when one attempts to build string realisations of four-dimensional metastable vacua, either via brane constructions [29] or via AdS-CFT [30], the non–normalizable modes one encounters are log–growing modes, which one could in hindsight have expected from the generic running of coupling constants of four–dimensional gauge theories with the energy.

For anti–M2 branes there is no such link. There exist both AdS/CFT duals of metastable vacua of 2+1 dimensional gauge theories [193], as well as brane–engineering constructions of such metastable vacua (using D3 branes ending on codimension–three defects inside NS5 branes) [99]. The nonexistence of an anti–M2 metastable vacuum could only be seen in supergravity, and comes from the way the fields of the anti–M2 brane interact with the magnetic fields that give rise to the charge dissolved in fluxes. This may indicate there is a problem with trying to construct metastable vacua in string theory by putting antibranes in backgrounds with charge dissolved in fluxes. In an upcoming paper [97] we will also argue that anti–D2 branes in backgrounds with D2 brane charge dissolved in fluxes [67] that I have investigated in [95], which appears on Chapter 5 of this thesis, have similar problems.

6.2 Perturbations around a supersymmetric solution

We are interested in the backreaction of a set of anti–M2 branes spread on a four–sphere at the bottom of the warped Stenzel geometry [244] with nontrivial fluxes. Smearing the anti–M2's is necessary in order for the perturbed solution to have the same SO(5) global symmetry as the supersymmetric solution of Cvetič, Gibbons, Lü and Pope (CGLP) [66]. The perturbed metric and flux coefficients are then functions of only one radial variable, and generically satisfy n second–order differential equations.

However, when perturbing around a supersymmetric solution governed by a superpotential, Borokhov and Gubser [45] have observed that these n second–order equations factorize into n first–order equations for certain momenta and n first–order equations for the metric and flux coefficients, and that furthermore the n equations for the momenta do not contain the metric and flux coefficients, and hence can be solved independently. This technique has been used in several related works [30, 45, 180] and we consider this to be the technique of choice for deformation problems that depend on just one coordinate.

6.2.1 The first–order Borokhov–Gubser formalism

While the following summary can be found by now in several sources, we include it here for completeness. When the equations of motion governing the fields ϕ^a of a certain supersymmetric solution come from the reduction to a one–dimensional Lagrangian

$$\mathcal{L} = -\frac{1}{2} G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau} - V(\phi) \tag{6.1}$$

whose potential $V(\phi)$ comes from a superpotential,

$$V(\phi) = \frac{1}{8} G^{ab} \frac{\partial W}{\partial \phi^a} \frac{\partial W}{\partial \phi^b}, . \tag{6.2}$$

The Lagrangian is written as

$$\mathcal{L} = -\frac{1}{2} G_{ab} \left(\frac{d\phi^a}{d\tau} - \frac{1}{2} G^{ac} \frac{\partial W}{\partial \phi^c} \right) \left(\frac{d\phi^a}{d\tau} - \frac{1}{2} G^{ac} \frac{\partial W}{\partial \phi^c} \right) - \frac{dW}{d\tau} , \tag{6.3}$$

and the supersymmetric solutions satisfy

$$\frac{d\phi^a}{d\tau} - \frac{1}{2} G^{ab} \frac{\partial W}{\partial \phi^b} = 0 . \tag{6.4}$$

We now want to find a perturbation in the fields ϕ^a around their supersymmetric background value ϕ_0^a

$$\phi^a = \phi_0^a + \phi_1^a(X) + \mathcal{O}(X^2) , \tag{6.5}$$

where X represents the set of perturbation parameters in which ϕ_1^a is linear. The deviation from the gradient flow equations for the perturbation ϕ_1^a is measured by the conjugate momenta ξ_a

$$\xi_a \equiv G_{ab}(\phi_0) \left(\frac{d\phi_1^b}{d\tau} - M^b{}_d(\phi_0) \phi_1^d \right) , \tag{6.6}$$

$$M^b{}_d \equiv \frac{1}{2} \frac{\partial}{\partial \phi^d} \left(G^{bc} \frac{\partial W}{\partial \phi^c} \right) . \tag{6.7}$$

The ξ_a are linear in the expansion parameters X, hence they are of the same order as the ϕ_1^a. When all the ξ_a vanish the deformation is supersymmetric.

The main point of this construction is that the second–order equations of motion governing the perturbations reduce to a set of first–order linear equations for (ξ_a, ϕ^a):

$$\frac{d\xi_a}{d\tau} + \xi_b M^b{}_a(\phi_0) = 0 , \tag{6.8}$$

$$\frac{d\phi_1^a}{d\tau} - M^a{}_b(\phi_0) \phi_1^b = G^{ab} \xi_b . \tag{6.9}$$

Note that equation (6.9) is just a rephrasing of the definition of the ξ_a in (6.6), while (6.8) implies the equations of motion. Since one considers these perturbations in a metric Ansatz in which the reparametrization invariance of the radial variable is fixed, in addition to these equations one must enforce the zero–energy condition

$$\xi_a \frac{d\phi_0^a}{dr} = 0 . \tag{6.10}$$

6.2.2 The perturbation Ansatz

Using the analysis of the CGLP solution in [175], one can easily see that the Ansatz for the SO(5)–invariant eleven–dimensional supergravity solution we are looking for is

$$\begin{aligned} ds^2 &= e^{-2z(r)} dx_\mu dx^\mu + e^{z(r)} \left[e^{2\gamma(r)} dr^2 + e^{2\alpha(r)} \sigma_i^2 + e^{2\beta(r)} \tilde{\sigma}_i^2 + e^{2\gamma(r)} \nu^2 \right] \\ G_4 &= dK(\tau) \wedge dx^0 \wedge dx^1 \wedge dx^2 + m F_4 , \end{aligned} \tag{6.11}$$

where $F_4 = dA_3$ and

$$A_3 = f(r)\, \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 + h(r)\, \epsilon^{ijk} \sigma_i \wedge \sigma_j \wedge \tilde{\sigma}_k \tag{6.12}$$

$$\begin{aligned} \Rightarrow F_4 &= f'\, dr \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 + h'\, \epsilon^{ijk}\, dr \wedge \sigma_i \wedge \sigma_j \wedge \tilde{\sigma}_k \\ &\quad + \frac{1}{2} (4h - f)\, \epsilon^{ijk}\, \nu \wedge \sigma_i \wedge \tilde{\sigma}_j \wedge \tilde{\sigma}_k - 6\, h\, \nu \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 . \end{aligned} \tag{6.13}$$

Our notation for the one–forms on the Stenzel space is by now standard [175], in the sense that with the definitions

$$\sigma_i = L_{1i}\,, \qquad \tilde{\sigma}_i = L_{2i}\,, \qquad \nu = L_{12}\,, \tag{6.14}$$

they satisfy

$$\begin{aligned} d\sigma_i &= \nu \wedge \tilde{\sigma}_i + L_{ij} \wedge \sigma_j\,, && (6.15)\\ d\tilde{\sigma}_i &= -\nu \wedge \sigma_i + L_{ij} \wedge \tilde{\sigma}_j\,, && (6.16)\\ d\nu &= -\sigma_i \wedge \tilde{\sigma}_i\,, && (6.17)\\ dL_{ij} &= L_{ik} \wedge L_{kj} - \sigma_i \wedge \sigma_j - \tilde{\sigma}_i \wedge \tilde{\sigma}_j\,. && (6.18) \end{aligned}$$

Integrating one particular component of the equation of motion for the flux

$$d * G_4 = \frac{1}{2}\, G_4 \wedge G_4 \tag{6.19}$$

gives

$$K' = 6\, m^2 \left[h\, (f - 2\, h) - \frac{1}{54} \right] e^{-3(\alpha+\beta)-6z}\,, \tag{6.20}$$

where we have chosen the integration constant such that the BPS solution [66] is regular, i.e. there are no explicit source M2 branes.

Performing a standard dimensional reduction on this Ansatz down to one dimension, we obtain the following Lagrangian

$$L = (T_{gr} + T_{mat}) - (V_{gr} + V_{mat}) \tag{6.21}$$

with the gravitational and matter sectors given by

$$\begin{aligned} T_{gr} &= 3\, e^{3\,(\alpha+\beta)} \left[\alpha'^{\,2} + \beta'^{\,2} - \frac{3}{4}\, z'^{\,2} + 3\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' \right], && (6.22)\\ V_{gr} &= \frac{3}{4}\, e^{\alpha+\beta} \left[e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2\, e^{2\alpha+2\beta} - 6\, e^{2\alpha+2\gamma} \right] && (6.23) \end{aligned}$$

and

$$\begin{aligned} T_{mat} &= -\frac{m^2}{4}\, e^{3\,\alpha+\beta-3\,z} \left(f'^{\,2}\, e^{-4\beta} + 12\, h'^{\,2}\, e^{-4\alpha} \right), && (6.24)\\ V_{mat} &= 3\, m^2\, e^{\alpha+3\,\beta-3\,z} \left[3\, h^2\, e^{-4\alpha} + \frac{1}{4}\,(4h - f)^2\, e^{-4\beta} \right] \\ &\quad + 9\, m^4\, e^{-3\,(\alpha+\beta+2\,z)} \left[h\,(f - 2\,h) - \frac{1}{54} \right]^2. && (6.25) \end{aligned}$$

The superpotential is given by

$$W = -3\, e^{2\alpha+2\beta}\, (e^{2\alpha} + e^{2\beta} + e^{2\gamma}) - 6\, m^2\, e^{-3z} \left[h\,(f - 2\,h) - \frac{1}{54} \right]. \tag{6.26}$$

It is worth noting that equation (6.2) only defines the superpotential up one independent minus sign which can then be absorbed in (6.8) and (6.9) by changing the sign of the radial variable and the ξ_a. However, with the wisdom of hindsight, we choose a radial variable such that fields decay at infinity and not minus infinity, thus simultaneously fixing the sign of the superpotential.

6.2.3 The supersymmetric background

Here we summarize the expressions that the fields in our Ansatz take when specialized to the zeroth–order CGLP solution [66] around which we endeavor to study supersymmetric and non–supersymmetric perturbations.

We should note that the CGLP solution with transverse Stenzel geometry is to the warped M–theory solution with transverse Stiefel space [62] what the IIB Klebanov–Strassler solution [171] and the deformed conifold [52] are to the Klebanov–Tseytlin solution [170] and the singular conifold. The Stenzel space is a higher–dimensional generalization of the deformed conifold. A useful summary of many details of the supergravity solution can be found in [196] and proposals for the dual field theory can be found in [196, 151]

The supersymmetric solution around which we will perturb was found in [66]. It can be summarized in our Ansatz by

$$e^{2\,\alpha_0} = \frac{1}{3}\,(2+\cosh(2\,r))^{1/4}\,\cosh(r)\,, \tag{6.27}$$
$$e^{2\,\beta_0} = \frac{1}{3}\,(2+\cosh(2\,r))^{1/4}\,\sinh(r)\tanh(r)\,, \tag{6.28}$$
$$e^{2\,\gamma_0} = (2+\cosh(2\,r))^{-3/4}\,\cosh^3(r)\,, \tag{6.29}$$
$$f_0 = \frac{1}{3^{3/2}}\frac{\left(1-3\cosh^2(r)\right)}{\cosh^3(r)}\,, \tag{6.30}$$
$$h_0 = -\frac{1}{3^{3/2}}\frac{1}{2\cosh(r)}\,, \tag{6.31}$$
$$e^{3z_0(y)} = 2^{5/2}\,3\,m^2\int_y^\infty\frac{du}{(u^4-1)^{5/2}}\,, \tag{6.32}$$

where

$$y^4 \equiv 2+\cosh(2\,r)\,. \tag{6.33}$$

With this change of coordinate we can write

$$e^{3\,z_0} = \sqrt{2}\,m^2\frac{y\,\left(7-5\,y^4\right)}{(y^4-1)^{3/2}}+5\sqrt{2}\,m^2 F\left(\arcsin\left(\frac{1}{y}\right)\mid -1\right)\,, \tag{6.34}$$

where the incomplete elliptic integral of the first kind is

$$F(\phi\mid q) = \int_0^\phi \left(1-q\,\sin(\theta)^2\right)^{-1/2}d\theta \tag{6.35}$$

and we have fixed the integration constant (denoted c_0 in [66]) by requiring $e^{3z_0}\to 0$ as $r\to\infty$.

6.2.4 Explicit equations

We now write out explicitly the two sets of equations (6.8) and (6.9). In both cases a particular field redefinition simplifies things substantially.

ξ_a equations

The ξ^a equations (6.8) simplify in the basis

$$\tilde{\xi}_a = (\xi_1+\xi_2+\xi_3, \xi_1-\xi_2+3\,\xi_3, \xi_1+\xi_2-3\,\xi_3, \xi_4, \xi_5, \xi_6)\;. \tag{6.36}$$

In the order which we solve them, the equations are

$$\tilde{\xi}_4' = 6\,m^2\,e^{-3(\alpha_0+\beta_0+z_0)}\left((f_0-2\,h_0)\,h_0-\frac{1}{54}\right)\tilde{\xi}_4\,, \quad (6.37)$$

$$\tilde{\xi}_1' = 12\,m^2\,e^{-3(\alpha_0+\beta_0+z_0)}\left((f_0-2\,h_0)h_0-\frac{1}{54}\right)\tilde{\xi}_4\,, \quad (6.38)$$

$$\tilde{\xi}_5' = \frac{1}{2}\,e^{\alpha_0-\beta_0}\,\tilde{\xi}_6-2\,m^2\,h_0\,e^{-3(\alpha_0+\beta_0+z_0)}\,\tilde{\xi}_4\,, \quad (6.39)$$

$$\tilde{\xi}_6' = 6\,e^{-3(\alpha_0-\beta_0)}\,\tilde{\xi}_5-2\,e^{\alpha_0-\beta_0}\,\tilde{\xi}_6-2\,m^2\,e^{-3(\alpha_0+\beta_0+z_0)}\,(f_0-4\,h_0)\,\tilde{\xi}_4\,, \quad (6.40)$$

$$\tilde{\xi}_3' = \frac{2}{9}\,e^{-3(\alpha_0+\beta_0+z_0)}\left[18\,e^{2(\alpha_0+\beta_0+\gamma_0)+3z_0}\,\tilde{\xi}_3+m^2\,\left(54\,h_0\,(f_0-2\,h_0)-1\right)\,\tilde{\xi}_4\right]\,, \quad (6.41)$$

$$\tilde{\xi}_2' = \frac{1}{2}\,e^{-3\alpha_0-\beta_0}\left[2\,e^{2(\alpha_0+\beta_0)}\tilde{\xi}_2-6\,e^{2(\alpha_0+\gamma_0)}\tilde{\xi}_3-72\,h_0\,e^{4\beta_0}\,\tilde{\xi}_5 + e^{4\alpha_0}\left(-3\,\tilde{\xi}_1+2\,\tilde{\xi}_2+3\,\tilde{\xi}_3+2\,(f_0\,-4\,h_0)\,\tilde{\xi}_6\right)\right]\,, \quad (6.42)$$

where we remind the reader that a prime denotes a derivative with respect to r not y (6.33).

ϕ^a equations

The ϕ_a equations benefit from a field redefinition as well,

$$\phi^a = (\alpha,\beta,\gamma,z,f,h)\ , \quad (6.43)$$

$$\tilde{\phi}_a = (\phi_1-\phi_2,\phi_1+\phi_2-2\,\phi_3,\phi_3,\phi_4,\phi_5,\phi_6) \quad (6.44)$$

and we find

$$\tilde{\phi}_1' = \frac{1}{12}\,e^{-3(\alpha_0+\beta_0)}\left[-3\,\tilde{\xi}_1+4\,\tilde{\xi}_2+3\,\left(\tilde{\xi}_3-4\,e^{2(\alpha_0+\beta_0)}\,\left(e^{2\alpha_0}+e^{2\beta_0}\right)\,\tilde{\phi}_1\right)\right]\,, \quad (6.45)$$

$$\tilde{\phi}_2' = \frac{1}{12}\,e^{-3(\alpha_0+\beta_0)}\left[-3\,\tilde{\xi}_1+7\,\tilde{\xi}_3+12\,e^{2(\alpha_0+\beta_0)}\,\left(3\,\left(e^{2\beta_0}-e^{2\alpha_0}\right)\,\tilde{\phi}_1-4\,e^{2\gamma_0}\,\tilde{\phi}_2\right)\right]\,, \quad (6.46)$$

$$\tilde{\phi}_3' = \frac{1}{12}\,e^{-3(\alpha_0+\beta_0)}\left[\tilde{\xi}_1-3\,\left(\tilde{\xi}_3+6\,e^{2(\alpha_0+\beta_0)}\,\left(\left(e^{2\beta_0}-e^{2\alpha_0}\right)\,\tilde{\phi}_1-e^{2\gamma_0}\,\tilde{\phi}_2\right)\right)\right]\,, \quad (6.47)$$

$$\tilde{\phi}_5' = \frac{2}{m^2}\,e^{-3(\alpha_0-\beta_0)}\left[e^{3z_0}\,\tilde{\xi}_5+3\,m^2\,(3\,h_0\,\tilde{\phi}_1-\tilde{\phi}_6)\right]\,, \quad (6.48)$$

$$\tilde{\phi}_6' = \frac{1}{6\,m^2}\,e^{\alpha_0-\beta_0}\left[e^{3z_0}\,\tilde{\xi}_6-3\,m^2\,(f_0\,\tilde{\phi}_1-4\,h_0\,\tilde{\phi}_1+\tilde{\phi}_5-4\,\tilde{\phi}_6)\right]\,, \quad (6.49)$$

$$\tilde{\phi}_4' = \frac{1}{9}\,e^{-3(\alpha_0+\beta_0+z_0)}\left[2\,e^{3z_0}\,\tilde{\xi}_4+m^2\left(\left[1-54\,h_0\,(f_0-2\,h_0)\right]\tilde{\phi}_4+18\,f_0\,\tilde{\phi}_6 +\tilde{\phi}_2+2\,\tilde{\phi}_3+18\,h_0\,\left[\tilde{\phi}_5-4\,\tilde{\phi}_6-3\,(f_0-2\,h_0)\,(\tilde{\phi}_2+2\,\tilde{\phi}_3)\right]\right)\right]\,. \quad (6.50)$$

6.3 The force on a probe M2

Before solving the above equations, we compute the force on a probe M2–brane in the perturbed solution space. As was found in the analogous IIB scenario [30], the force turns out to benefit from remarkable cancellations and is ultimately quite simple.

The membrane action for a probe M2 brane (which by abusing notation we refer to as the DBI action) is

$$V^{DBI} = \sqrt{-\,g_{00}\,g_{11}\,g_{22}}\,, \qquad = e^{-3z} \quad (6.51)$$

and, in the first–order approximation, its derivative with respect to r is

$$F^{DBI} = -\frac{dV_0^{DBI}}{dr} + 3\,e^{-3z_0}\left(\tilde\phi_4' - 3\,z_0'\,\tilde\phi_4\right)\,. \tag{6.52}$$

We next consider the derivative of the WZ action with respect to r, which gives the force exerted on the M2–brane by the $G^{(4)}$ field :

$$\begin{aligned} F^{WZ} &= -\frac{dV^{WZ}}{dr}\,, \\ &= G^{(4)}_{012r}\,, \\ &= -6\,m^2\left[h\,(f-2\,h) - \frac{1}{54}\right]\,e^{-3(\alpha+\beta)-6z}\,. \end{aligned} \tag{6.53}$$

The zeroth–order and first–order WZ forces thus are

$$F_0^{WZ} = -6\,m^2\left[h_0\,(f_0-2\,h_0) - \frac{1}{54}\right]\,e^{-3(\alpha_0+\beta_0)-6z_0} \tag{6.54}$$

and

$$\begin{aligned} F_1^{WZ} =& -6\,m^2\Big[h_0\,(\tilde\phi_5 - 2\,\tilde\phi_6) + \tilde\phi_6\,(f_0-2\,h_0) \\ & - 3\,(\tilde\phi_2 + 2\,\tilde\phi_3 + 2\,\tilde\phi_4)\left(h_0\,(f_0-2\,h_0) - \frac{1}{54}\right)\Big]\,e^{-3(\alpha_0+\beta_0)-6z_0}\,. \end{aligned} \tag{6.55}$$

Combining these two contributions to the force we see that the zeroth–order contributions cancel as expected. Then using the explicit ϕ^a equations from Section 6.2.4 we find the beautiful result

$$\begin{aligned} F &= F_1^{DBI} + F_1^{WZ} \\ &= \frac{2}{3}\,e^{-3\,(\alpha_0+\beta_0+z_0)(r)}\,\tilde\xi_4(r)\,. \end{aligned}$$

At this point it is worthwhile to preemptively trumpet the result (6.61) from Section 6.4 where the exact solution for the mode $\tilde\xi_4$ is found:

$$\begin{aligned} F &= \frac{2}{3}\,e^{-3\,(\alpha_0+\beta_0)(r)}\,Z_0\,X_4 \\ &= \frac{18\,Z_0\,X_4}{(2+\cosh 2r)^{3/4}\,\sinh^3 r}\,, \end{aligned} \tag{6.56}$$

where Z_0 is some numerical factor which we found convenient not to absorb into the X_4 integration constant,

$$Z_0 \equiv e^{-3z_0(0)}\,. \tag{6.57}$$

So, the UV expansion of the force felt by a probe M2 brane in the first–order perturbed solution is always

$$F_r \sim X_4\,e^{-9r/2} + \mathcal{O}(e^{-17r/2})\,. \tag{6.58}$$

In terms of ρ, the "standard" radial coordinate[5], this force comes from a potential proportional to ρ^{-6}, which agrees with a straightforward extension of the brane–antibrane force analysis of [158] to this system. This will be further discussed in a forthcoming publication [34].

[5]Related to r via $\cosh(2\,r) \sim \rho^{8/3}$.

6.4 The space of solutions

In this section we find the generic solution to the system (6.37)–(6.50). This solution space has twelve integration constants of which ten are physical. We have managed to solve the $\tilde{\xi}_a$ equations exactly whereas for the ϕ_a equations we have resorted to solving them in the IR and UV limits.

6.4.1 Analytic solutions for the $\tilde{\xi}$'s

The first equation (6.37) is solved by

$$\tilde{\xi}_4 = X_4 \exp\left(6\, m^2 \int_0^r dr'\, e^{-3(\alpha_0+\beta_0+z_0)} \left[(f_0 - 2\, h_0)\, h_0 - \frac{1}{54}\right]\right) , \tag{6.59}$$

which appears to be a double integral. However, using a standard notation for the warp factor $H_0 = e^{3z_0}$, since we have

$$\frac{dH_0}{dr} = -2^3 3\, m^2 \frac{e^{2\gamma_0}}{\sinh^3 2r} \tanh^4 r\,, \tag{6.60}$$

we actually find

$$\begin{aligned} \tilde{\xi}_4 &= X_4 \exp\left(\int_0^r dr'\, \frac{1}{H_0} \frac{dH_0}{dr'}\right) , \\ &= X_4\, e^{3(z_0(r)-z_0(0))} . \end{aligned} \tag{6.61}$$

It immediately follows that

$$\tilde{\xi}_1 = X_1 + 2\, X_4\, e^{3(z_0(r)-z_0(0))} . \tag{6.62}$$

We find convenient not to include $e^{-3z_0(0)}$ into the integration constant X_4, and will use the notation

$$Z_0 \equiv e^{-3z_0(0)} . \tag{6.63}$$

We were also able to find exact analytic expressions for $\tilde{\xi}_3$ and $\tilde{\xi}_{5,6}$, in term of $y^4 \equiv 2 + \cosh(2\,r)$:

$$\begin{aligned} \tilde{\xi}_3 = &\, y^4\, \left(y^4-3\right)^2 X_3 - \frac{m^2\, Z_0\, X_4}{18\sqrt{2}}\, \frac{y\, \left(y^4-3\right)}{\left(y^4-1\right)^{3/2}} \Bigg[-96 + 599\, y^4 - 550\, y^8 + 119\, y^{12} \\ & - y^3 \sqrt{y^4-1}\, \left(3 - 4\, y^4 + y^8\right) \left(163\, F\left(\arcsin\left(\frac{1}{y}\right) \mid -1\right)\right. \\ & \left.\left. + 22 \left[\Pi\left(-\sqrt{3}; -\arcsin\left(\frac{1}{y}\right) \mid -1\right) + \Pi\left(\sqrt{3}; -\arcsin\left(\frac{1}{y}\right) \mid -1\right)\right]\right)\Bigg] , \end{aligned} \tag{6.64}$$

where $F(\phi \mid q)$ is given in (6.35) and $\Pi(n; \phi \mid m)$ is an incomplete elliptic integral of the third kind

$$\Pi(n;\phi|m) = \int_0^\phi \frac{d\theta}{\left(1 - n\, \sin(\theta)^2\right) \sqrt{1 - m\, \sin(\theta)^2}} . \tag{6.65}$$

The expressions for $\tilde{\xi}_{5,6}$ are as follows :

$$\tilde{\xi}_5 = \frac{1}{4\sqrt{2}\,(y^4-3)\,\sqrt{y^4-1}}\Bigg[\sqrt{6}\,Z_0\,X_4\,m^2\,y\,\left(13-11\,y^4\right)\,\sqrt{y^4-1}$$
$$+\,4\left[\left(y^4-1\right)^2\,X_5+\left(y^4-3\right)\,\left(1+y^4\right)\,X_6\right]$$
$$+\,\sqrt{6}\,Z_0\,m^2\,X_4\left[\left(19+7y^4\left(y^4-2\right)\right)\,F\left(\arcsin\left(\frac{1}{y}\right)\mid -1\right)\right.$$
$$\left.-\,2\,\left(y^4-3\right)\,\left(1+y^4\right)\,\left(\Pi\left(-\sqrt{3};-\arcsin\left(\frac{1}{y}\right)\mid -1\right)+\Pi\left(\sqrt{3};-\arcsin\left(\frac{1}{y}\right)\mid -1\right)\right)\right]\Bigg], \tag{6.66}$$

$$\tilde{\xi}_6 = \frac{\sqrt{2}}{(y^4-3)\,(y^4-1)^{3/2}}\Bigg[\left(y^4-7\right)\,\left(y^4-1\right)^2\,\Bigg[X_5+\sqrt{\frac{3}{2}}\,Z_0\,m^2\,X_4\left(\frac{7\,y-5\,y^5}{(y^4-1)^{3/2}}\right.$$
$$\left.\left.+\,5\,F\left(\arcsin\left(\frac{1}{y}\right)\mid -1\right)\right)\Bigg]+\frac{1}{4}\,\left(y^4-3\right)^2\,\Bigg[-\sqrt{6}\,Z_0\,m^2\,X_4\,y\,\sqrt{y^4-1}$$
$$+\,4\,\left(y^4-3\right)\,X_6-\sqrt{6}\,Z_0\,m^2\,X_4\,\left(y^4-3\right)\,\left(3\,F\left(\arcsin\left(\frac{1}{y}\right)\mid -1\right)\right.$$
$$\left.+\,2\,\left(\Pi\left(-\sqrt{3};-\arcsin\left(\frac{1}{y}\right)\mid -1\right)+\Pi\left(\sqrt{3};-\arcsin\left(\frac{1}{y}\right)\mid -1\right)\right)\right)\Bigg]\Bigg]. \tag{6.67}$$

Lastly, $\tilde{\xi}_2$ is given by the zero–energy condition (6.10) but its explicit form does not appear to be too enlightening. The IR and UV series expansions of the above solutions for $\tilde{\xi}^i$ are as follows:

IR behavior of $\tilde{\xi}$

The IR behavior of the $\tilde{\xi}_a$'s is the following :

$$\tilde{\xi}_1^{IR} = X_1+2\,X_4\left[1-\frac{3^{1/4}}{2}\,m^2\,e^{-3z_0(0)}\,r^2\right]+\mathcal{O}(r^4)\,,$$

$$\tilde{\xi}_2^{IR} = \left[\frac{3}{2}\,X_1-\frac{4}{3\sqrt{3}}\,X_5+\frac{7}{3}\,X_4\right]+\left[\frac{3}{2}\,X_1+\frac{8}{3\sqrt{3}}\,X_5\right.$$
$$\left.+\,\frac{1}{3}\,X_4\,\left(13-10\,3^{1/4}\,e^{-3z_0(0)}\,m^2\right)\right]r^2+\mathcal{O}(r^4)\,,$$

$$\tilde{\xi}_3^{IR} = 3^{1/4}\,e^{-3z_0(0)}\,m^2\,X_4\,r^2+\mathcal{O}(r^4)\,, \tag{6.68}$$

$$\tilde{\xi}_4^{IR} = X_4\left[1-\frac{3^{1/4}}{2}\,m^2\,e^{-3z_0(0)}\,r^2\right]+\mathcal{O}(r^4)\,,$$

$$
\begin{aligned}
\tilde{\xi}_5^{IR} = \frac{1}{r^2}\, &\left[X_5 + X_4\,\left(\frac{\sqrt{3}}{2} - \frac{3^{3/4}}{2}\,e^{-3z_0(0)}\,m^2\right)\right] \\
&+ \left[\frac{1}{6}\,(7\,X_5 + 12\,X_6) + X_4\left[\frac{17}{20\sqrt{3}} - \frac{97}{12}\,3^{3/4}\,e^{-3z_0(0)}\,m^2\right.\right. \\
&\left.\left. - \sqrt{6}\,e^{-3z_0(0)}\,m^2\,\Pi\left(-\sqrt{3}; -\arcsin\left(\frac{1}{3^{1/4}}\right) \mid -1\right)\right] - 3^{3/4}\,e^{-3z_0(0)}\,m^2\,X_4\log(r)\right] \\
&+ \left[\frac{53}{120}\,X_5 + \frac{1}{48}\,X_4\,\left(\frac{53}{5}\sqrt{3} + \frac{47}{5}\,3^{3/4}\,e^{-3z_0(0)}\,m^2\right)\right] r^2 + \mathcal{O}(r^4)\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\xi}_6^{IR} = -\frac{2}{r^2}\,&\left[2\,X_5 + \sqrt{3}\,X_4\right] + \left[\frac{4}{3}\,X_5 + X_4\,\left(\frac{2}{\sqrt{3}} + 3^{3/4}\,e^{-3z_0(0)}\,m^2\right)\right] \\
&+ \left[\frac{37}{30}\,X_5 + X_4\,\left(\frac{37}{20\sqrt{3}} - 2\,3^{3/4}\,e^{-3z_0(0)}\,m^2\right)\right] r^2 + \mathcal{O}(r^4)\,.
\end{aligned}
$$

UV behavior of $\tilde{\xi}$

The UV behavior of the $\tilde{\xi}_a$'s is as follows :

$$\tilde{\xi}_1^{UV} = X_1 + \frac{32}{3}\, 2^{3/4}\, m^2\, X_4\, e^{-3z_0(0)}\, e^{-\frac{9}{2}r} + \mathcal{O}(e^{-13r/2})\,,$$

$$\begin{aligned}
\tilde{\xi}_2^{UV} = &-\frac{3}{32}\, X_3\, e^{6r} + \frac{3}{16}\, X_3\, e^{4r} + \left[\frac{3}{8}\, X_1 + \frac{3}{32}\, X_3 + \frac{2}{3\sqrt{3}}\,(X_5 + X_6)\right] e^{2r}\\
&+ \left[\frac{3}{4}\, X_1 - \frac{3}{8}\, X_3 - \frac{8}{3\sqrt{3}}\,(X_5 + X_6)\right]\\
&+ \left[\frac{3}{8}\, X_1 + \frac{3}{32}\, X_3 + \frac{2}{3\sqrt{3}}\,(X_5 + X_6)\right] e^{-2r}\\
&+ \left[\frac{3}{16}\, X_3 + \frac{64}{3\sqrt{3}}\, X_6\right] e^{-4r} + \frac{32}{7}\, 2^{3/4}\, e^{-3z_0(0)}\, m^2\, X_4\, e^{-9r/2}\\
&- \left[\frac{3}{32}\, X_3 + \frac{256}{3\sqrt{3}}\, X_6\right] e^{-6r} + \mathcal{O}(e^{-13r/2})\,,
\end{aligned}$$

$$\begin{aligned}
\tilde{\xi}_3^{UV} = &\frac{1}{8}\, X_3\, e^{6r} - \frac{9}{8}\, X_3\, e^{2r} + 2\, X_3 - \frac{9}{8}\, X_3\, e^{-2r}\\
&+ \frac{32}{7}\, 2^{3/4}\, e^{-3z_0(0)}\, m^2\, X_4\, e^{-9r/2} + \frac{1}{8}\, X_3\, e^{-6r} + \mathcal{O}(e^{-13r/2})\,,
\end{aligned}$$

$$\tilde{\xi}_4^{UV} = \frac{16}{3}\, 2^{3/4}\, m^2\, X_4\, e^{-3z_0(0)}\, e^{-\frac{9}{2}r} + \mathcal{O}(e^{-13r/2})\,, \tag{6.69}$$

$$\begin{aligned}
\tilde{\xi}_5^{UV} = &\frac{1}{2}\,(X_5 + X_6)\, e^{r} + \frac{5}{2}\,(X_5 + X_6)\, e^{-r} + 2\,(3\, X_5 - X_6)\, e^{-3r}\\
&+ 2\,(5\, X_5 + X_6)\, e^{-5r} - \frac{96}{13}\, 2^{3/4}\, \sqrt{3}\, e^{-3z_0(0)}\, m^2\, X_4\, e^{-11r/2} + \mathcal{O}(e^{-13r/2})\,,
\end{aligned}$$

$$\begin{aligned}
\tilde{\xi}_6^{UV} = &(X_5 + X_6)\, e^{r} - 7\,(X_5 + X_6)\, e^{-r} - 24\,(X_5 - X_6)\, e^{-3r}\\
&- 8\,(5\, X_5 + 7\, X_6)\, e^{-5r} - \frac{192}{13}\, 2^{3/4}\, \sqrt{3}\, m^2\, X_4\, e^{-3z_0(0)}\, e^{-11r/2} + \mathcal{O}(e^{-13r/2})\,.
\end{aligned}$$

6.4.2 Solving the ϕ^i equations

The space of solutions

We now solve the system of equations for ϕ^i (6.45)–(6.49) using the Lagrange method of variation of parameters.

Equation (6.45) is solved by

$$\tilde{\phi}_1 = \frac{\tilde{\lambda}^1(r)}{\sinh(2\,r)}\,, \tag{6.70}$$

with

$$\tilde{\lambda}^1 = \frac{9}{2}\int \frac{\cosh(r)}{\sinh(r)^2\,(2 + \cosh(2\,r))^{3/4}}\left[-3\,\tilde{\xi}_1 + 4\,\tilde{\xi}_2 + 3\,\tilde{\xi}_3\right] + Y_1^{IR}\,. \tag{6.71}$$

$\tilde{\xi}_2$ and $\tilde{\xi}_3$ are given in Section 6.4.1. above and $\sinh(2\,r)^{-1}$ is the homogeneous solution to the $\tilde{\phi}_1$ equation.

The same Lagrange method is used for $\tilde{\phi}_2$, which is given by

$$\tilde{\phi}_2 = \frac{\tilde{\lambda}^2(r)}{\sinh(r)^4\,(2 + \cosh(2\,r))}\,, \tag{6.72}$$

where

$$\tilde{\lambda}^2 = \frac{9}{4}\int \sinh(r)\;(2+\cosh(2\,r))^{1/4}\left[-3\,\tilde{\xi}_1 + 7\,\tilde{\xi}_3 - \frac{4}{3}\,\frac{\sinh(r)^2}{\cosh(r)}\;(2+\cosh(2\,r))^{3/4}\;\tilde{\phi}_1\right] + Y_2^{IR}\,. \tag{6.73}$$

From this, we obtain an integral expression for $\tilde{\phi}_3$:

$$\tilde{\phi}_3 = \frac{9}{4}\int \frac{\left[\tilde{\xi}_1 - 3\,\tilde{\xi}_3 + \frac{2}{3}\,\frac{\sinh(r)^2}{\cosh(r)}\;(2+\cosh(2\,r))^{3/4}\;\tilde{\phi}_1 + 2\,\frac{\sinh(r)^2\,\cosh(r)^3}{(2+\cosh(2\,r))^{1/4}}\,\tilde{\phi}_2\right]}{\sinh(r)^3\;(2+\cosh(2\,r))^{3/4}} + Y_3^{IR}\,.$$

The fluxes $\left(\tilde{\phi}_5, \tilde{\phi}_6\right) = (f, h)$ are given by

$$\begin{pmatrix} \tilde{\phi}_5 \\ \tilde{\phi}_6 \end{pmatrix} = \begin{pmatrix} \cosh(r)^3\;\tanh(r)^6 & \cosh(r)^3\;\left[2 - 3\;\tanh(r)^2\right] \\ \frac{1}{2}\;\left[\mathrm{sech}(r) - \cosh(r)^3\right] & \frac{1}{2}\;\cosh(r)^3 \end{pmatrix}\begin{pmatrix} \tilde{\lambda}_5 \\ \tilde{\lambda}_6 \end{pmatrix}, \tag{6.74}$$

where the derivatives of $\tilde{\lambda}_5$ and $\tilde{\lambda}_6$ are given by

$$\begin{pmatrix} \tilde{\lambda}_5' \\ \tilde{\lambda}_6' \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\;\cosh(r)\,\coth(r)^2 & \frac{1}{2}\;[\cosh(r) - 2\coth(r)\,\mathrm{csch}(r)] \\ \frac{1}{8}\;[3+\cosh(2\,r)]\;\mathrm{sech}(r) & \frac{1}{2}\;\sinh(r)\;\tanh(r)^3 \end{pmatrix}\begin{pmatrix} b_5 \\ b_6 \end{pmatrix}, \tag{6.75}$$

and b_5, b_6 are the right–hand side of (6.48) and (6.49) respectively. The 2×2 matrix appearing in (6.75) is the inverse of the matrix of homogeneous solutions written in (6.74). We will call Y_5 and Y_6 the constants arising from integrating (6.75), even though the two functions $\tilde{\phi}_5$ and $\tilde{\phi}_6$ depend on both of them.

Finally, relying on the same method, the equation for $\tilde{\phi}_4$ is solved to

$$\tilde{\phi}_4 = e^{-3z_0(r)}\,\tilde{\lambda}_4\,, \qquad \tilde{\lambda}_4 = \int e^{3z_0(r)}\,b_4(r) + Y_4^{IR}\,, \tag{6.76}$$

where $b_4(r)$ is the right–hand side of (6.50) (setting $\tilde{\phi}_4$ to zero).

IR behavior

We now give the IR expansions of the ϕ^i's. We only write the divergent and constant terms since terms which are regular in the IR do not provide any constraint on our solution space. Z_0 is defined in (6.57). The X_i integration constants are those appearing in the exact solutions for the $\tilde{\xi}_i$'s (6.61)–(6.67) :

$$\tilde{\phi}_1 = -\frac{1}{r^2}\left[\frac{27\,X_1 + 30\,X_4 - 16\,\sqrt{3}\,X_5}{4\,3^{3/4}}\right] + \frac{1}{2\,r}\,Y_1^{IR} + \left[\frac{189\,X_1 + \left(498 - 198\,3^{1/4}\,Z_0\,m^2\right)\,X_4 + 80\,\sqrt{3}\,X_5}{12\,3^{3/4}}\right] + \mathcal{O}(r)\,, \tag{6.77}$$

$$\tilde{\phi}_2 = \frac{Y_2^{IR}}{3\,r^4} + \frac{1}{r^2}\left[\frac{9}{4}\,3^{1/4}\,X_1 + \frac{3}{2}\,3^{1/4}\,X_4 - 2\,\sqrt{3}\,3^{1/4}\,X_5 - \frac{4}{9}\,Y_2^{IR}\right] - \frac{1}{2\,r}\,Y_1^{IR} - \left[6\,3^{1/4}\,X_1 + \frac{23}{2}\,3^{1/4}\,X_4 - 6\,\sqrt{3}\,Z_0\,m^2\,X_4 - \frac{1}{3^{1/4}}\,X_5 - \frac{41}{135}\,Y_2^{IR}\right] + \mathcal{O}(r)\,, \tag{6.78}$$

$$\tilde{\phi}_3 = -\frac{Y_2^{IR}}{8\,r^4} - \frac{1}{r^2}\left[\frac{9\,3^{1/4}\,X_1 - 12\,3^{3/4}\,X_5 - 4\,Y_2^{IR}}{24}\right] + \left[Y_3^{IR} + \frac{3^{1/4}}{8}\left(-18\,3^{1/4}\,Z_0\,m^2\,X_4 + 21\,X_1 + 48\,X_4 + 4\sqrt{3}\,X_5\right)\log(r)\right] + \mathcal{O}(r)\,, \tag{6.79}$$

$$\begin{aligned}\tilde{\phi}_4 = &-\frac{1}{r^2}\left[\frac{18\,X_4 - 4\sqrt{3}\,X_5 + Z_0\,m^2\,\left(Y_2^{IR} - 24\sqrt{3}\,Y_6^{IR}\right)}{8\,3^{3/4}}\right] \\ &-\left[\frac{1}{4}\left(Z_0\,m^2\left(\frac{3\sqrt{3}}{2}\,X_4 - X_5\right) - 4\,Z_0\,Y_4^{IR}\right)\right. \\ &+\frac{1}{48}\,Z_0^2\,m^4\left(\sqrt{3}\,Y_2^{IR} - 72\,Y_6^{IR}\right) + \left[\frac{3}{2}\,3^{1/4}\,X_4 - \frac{X_5}{3^{1/4}}\right. \\ &\left.\left.+\frac{1}{36}\,Z_0\,m^2\left(81\sqrt{3}\,X_1 + 78\sqrt{3}\,X_4 - 168\,X_5 + 11\,3^{1/4}\,Y_2^{IR} - 72\,3^{3/4}\,Y_6^{IR}\right)\right]\log(r)\right] \\ &+\mathcal{O}(r)\,,\end{aligned} \tag{6.80}$$

$$\tilde{\phi}_5 = 2Y_6^{IR} + \left[\frac{9}{8}\,3^{3/4}\,X_1 + \frac{3}{4}\,3^{3/4}\,X_4 - 2\,3^{1/4}\,X_5 + \frac{1}{2\,Z_0\,m^2}\left(X_5 + \frac{\sqrt{3}}{2}\,X_4\right)\right]r^2 + \mathcal{O}(r^3), \tag{6.81}$$

$$\begin{aligned}\tilde{\phi}_6 = &\frac{1}{r^2}\,\frac{X_5 + \frac{\sqrt{3}}{2}\,X_4}{6\,Z_0\,m^2} \\ &+\left[\frac{3^{3/4}}{16}\,X_1 - \frac{1}{18}\,\frac{X_5 + \frac{\sqrt{3}}{2}\,X_4}{Z_0\,m^2} - \frac{7}{72}\,3^{3/4}\,X_4 - \frac{5}{18}\,3^{1/4}\,X_5 + \frac{1}{2}\,Y_6^{IR}\right] + \mathcal{O}(r)\,.\end{aligned} \tag{6.82}$$

Note that in the $\tilde{\phi}_5$ expansion we have also displayed the term of order r^2 – this term will be relevant for the singularity analysis in Section 6.6.

UV behavior

We provide the UV asymptotics for all six $\tilde{\phi}_i$'s, incorporating terms which decay not faster than $e^{-13r/2}$. However, as appears in Table 1 below, a few modes have leading behavior in the UV which is even more convergent than this.

$$\begin{aligned}\tilde{\phi}_1 = &\frac{18}{2^{1/4}}\,X_3\,e^{-r/2} + 2\,Y_1^{UV}\,e^{-2r} - 4\,2^{3/4}\left[\frac{27}{2}\,X_1 - 27\,X_3 + 8\sqrt{3}\,(X_5 + X_6)\right]e^{-5r/2} \\ &-\left[\frac{1089}{10\,2^{1/4}}\,X_3 - \frac{128}{5}\,2^{3/4}\sqrt{3}\,(X_5 + X_6)\right]e^{-9r/2} + 2\,Y_1^{UV}\,e^{-6r} \\ &+\mathcal{O}(e^{-13r/2})\,,\end{aligned} \tag{6.83}$$

$$\tilde{\phi}_2 = \frac{21}{5\,2^{1/4}}\,X_3\,e^{3r/2} - \frac{17523}{140\,2^{1/4}}\,e^{-5r/2} X_3 - 12\,Y_1^{UV}\,e^{-4r}$$
$$+ 4\,2^{3/4}\left[99\,X_1 - \frac{1719}{10}\,X_3 + 64\,\sqrt{3}\,(X_5+X_6)\right]e^{-9r/2} + 32\,Y_2^{UV} e^{-6r}$$
$$+ \mathcal{O}(e^{-13r/2})\,, \tag{6.84}$$

$$\tilde{\phi}_3 = -\frac{27}{10\,2^{1/4}}\,X_3\,e^{3r/2} + Y_3^{UV} + \frac{9693}{280\,2^{1/4}}\,X_3\,e^{-5r/2} + \frac{15}{4}\,Y_1^{UV}\,e^{-4r}$$
$$- 2^{3/4}\left[130\,X_1 - \frac{1113}{5}\,X_3 + \frac{256}{\sqrt{3}}\,(X_5+X_6)\right]e^{-9r/2} - 12\,Y_2^{UV}\,e^{-6r}$$
$$+ \mathcal{O}(e^{-13r/2})\,, \tag{6.85}$$

$$\tilde{\phi}_4 = \frac{3}{16\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{9r/2} + \frac{27}{26\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{5r/2} + \frac{9}{5\,2^{1/4}}\,X_3\,e^{3r/2} + \frac{350271}{183872\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{r/2}$$
$$- 2\left[Y_3^{UV} + \sqrt{3}\,\left(Y_5^{UV} - Y_6^{UV}\right)\right] + \frac{216}{325}\,2^{3/4}\,X_3\,e^{-r/2} + \frac{484605}{298792\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{-3r/2}$$
$$+ \frac{144}{13}\,\sqrt{3}\,Y_6^{UV}\,e^{-2r} + \frac{3985953003}{14077700\,2^{1/4}}\,X_3\,e^{-5r/2} + \frac{7978373883}{21130570240\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{-7r/2}$$
$$+ \left[\frac{273}{34}\,Y_1^{UV} + \frac{78912\,\sqrt{3}}{2873}\,Y_6^{UV}\right]e^{-4r}$$
$$- 2^{3/4}\left[4\,\frac{229}{5}\,X_1 - \frac{1707341851}{2691325}\,X_3 + 4\,\frac{256}{3\,\sqrt{3}}\,(X_5+X_6)\right]e^{-9r/2}$$
$$+ \frac{473729599251}{995778122560\,2^{3/4}}\,\frac{Y_4^{UV}}{m^2}\,e^{-11r/2} + \mathcal{O}(e^{-6r})\,, \tag{6.86}$$

$$\tilde{\phi}_5 = \frac{1}{8}\,\left(Y_5^{UV} - Y_6^{UV}\right)\,e^{3r} - \frac{9}{8}\,\left(Y_5^{UV} - Y_6^{UV}\right)\,e^{r} + \frac{1}{8}\,\left(39\,Y_5^{UV} + 9\,Y_6^{UV}\right)\,e^{-r}$$
$$+ 19\,\frac{4\,2^{3/4}}{\sqrt{3}}\,X_3\,e^{-3r/2} + \left[\frac{14}{3\,\sqrt{3}}\,Y_1^{UV} - \frac{1}{8}\,\left(111\,Y_5^{UV} + Y_6^{UV}\right)\right]e^{-3r}$$
$$- 4\,2^{3/4}\left[2\,\frac{279}{65}\,\sqrt{3}\,X_1 + \frac{147}{65}\,\sqrt{3}\,X_3 + 2\,\frac{308}{39}\,(X_5+X_6)\right]e^{-7r/2}$$
$$+ 10\left[-\frac{2}{\sqrt{3}}\,Y_1^{UV} + 3\,Y_5^{UV}\right]e^{-5r}$$
$$+ \frac{56}{1105}\,2^{3/4}\left[3071\,\sqrt{3}\,X_1 - \frac{166409\,\sqrt{3}}{56}\,X_3 + \frac{18716}{3}\,(X_5+X_6)\right]e^{-11r/2}$$
$$+ \mathcal{O}(e^{-13r/2})\,, \tag{6.87}$$

$$
\begin{aligned}
\tilde{\phi}_6 = &-\frac{1}{16}\left(Y_5^{UV}-Y_6^{UV}\right)e^{3r}-\frac{3}{16}\left(Y_5^{UV}-Y_6^{UV}\right)e^{r}+\frac{1}{16}\left(13\,Y_5^{UV}+3\,Y_6^{UV}\right)e^{-r}\\
&+\frac{10}{\sqrt{3}}\,2^{3/4}X_3\,e^{-3r/2}+\left[\frac{1}{3\sqrt{3}}\,Y_1^{UV}-\frac{1}{16}\left(17\,Y_5^{UV}-Y_6^{UV}\right)\right]e^{-3r}\\
&-4\,2^{3/4}\left[\frac{33}{65}\sqrt{3}\,X_1+\frac{9\sqrt{3}}{130}\,X_3+\frac{116}{117}\left(X_5+X_6\right)\right]e^{-7r/2}\\
&-\left[\frac{2}{3\sqrt{3}}\,Y_1^{UV}-Y_5^{UV}\right]e^{-5r}\\
&+\frac{4}{1105\sqrt{3}}\,2^{3/4}\left[3713\,X_1-\frac{30221}{8}\,X_3+2932\sqrt{3}\left(X_5+X_6\right)\right]e^{-11r/2}\\
&+\mathcal{O}(e^{-13r/2})\,.
\end{aligned}
\tag{6.88}
$$

To understand the holographic physics of the $\tilde{\phi}^i$ modes, we tabulate the leading UV behavior coming from each mode. To each local operator $\mathcal{O}_i$ of quantum dimension Δ in the field theory, the holographic dictionary associates two modes in the dual AdS space, one normalizable and one non–normalizable [24, 20]. These two supergravity modes are dual respectively to the vacuum expectation value (VEV) $\langle 0 \mid \mathcal{O}_i \mid 0\rangle$ and the deformation of the action $\delta S \sim \int d^d x \mathcal{O}_i$:

$$
\begin{aligned}
\text{normalizable modes } &\sim \rho_{AdS}^{-\Delta} \leftrightarrow \text{field theory VEV's}\\
\text{non–normalizable modes } &\sim \rho_{AdS}^{\Delta-3} \leftrightarrow \text{field theory deformations of the action}\,.
\end{aligned}
$$

Here we refer to the standard AdS radial coordinate ρ_{AdS}, to be distinguished from the radial coordinate on the cone, ρ. In the UV, we have $\rho \sim e^{3r/4}$ and $\rho_{AdS} \sim \rho^2/m^{1/3}$ with the factor of $m^{1/3}$ taken with respect to the conventions of [175].

In Table 1 we have summarized which integration constants correspond to normalizable and non–normalizable modes. As stated in a previous section, the X_i are integration constants for the ξ_i modes and break supersymmetry, while the Y_i are integration constants for the modes ϕ^i. It is very interesting to note that in all cases a normalizable/non–normalizable pair consists of one BPS mode and one non–BPS mode.

As already mentioned, the mode $\tilde{\xi}_4$, whose integration constant is X_4 and which is the only mode accountable for the force felt by a probe M2–brane in the first–order perturbation to the CGLP background [66], is the most convergent mode in the UV, though this cannot be seen from the expansions we have provided but is apparent at higher order in the asymptotics that we have computed.

dim Δ	non–norm/norm	int. constant
6	$\rho^3_{AdS}/\rho^{-6}_{AdS}$	Y_4^{UV}/X_4
5	$\rho^2_{AdS}/\rho^{-5}_{AdS}$	$Y_5^{UV}-Y_6^{UV}/X_5-X_6$
4	$\rho_{AdS}/\rho^{-4}_{AdS}$	X_3/Y_2^{UV}
3	$\rho^0_{AdS}/\rho^{-3}_{AdS}$	Y_3/X_2
7/3	$\rho^{-2/3}_{AdS}/\rho^{-7/3}_{AdS}$	$Y_5^{UV}+Y_6^{UV}/X_5+X_6$
5/3	$\rho^{-4/3}_{AdS}/\rho^{-5/3}_{AdS}$	Y_1^{UV}/X_1

Table 6.1: The UV behavior of the twelve $SO(5)$–invariant modes in the deformation space of the CGLP solution. As discussed below, only ten of these modes are physical, and the mode of dim. 3 is a gauge artifact.

Taking into account a rescaling which culls Y_3 and the zero energy condition which eliminates X_2, we are left with a total of ten integration constants or five modes. The absence of a physical mode behaving as ρ^0_{AdS} is related to the quantization of the level of the Chern–Simons matter theory. This is unlike in four–dimensional gauge theories, where we expect a dimension–four operator corresponding to the dilaton. Note also that in Table 1 we see explicitly the dimension $\Delta = 7/3$ operator discussed in [175]. We have been somewhat glib in writing $X_5 - X_6$ or $Y_5 + Y_6$. The numerical factors in the combination of those integration constants are actually different, but can be rescaled to the shorthand notation we use.

6.5 Boundary conditions for M2 branes

Within the space of solutions that we have derived in Section 6.4. we now proceed to find the modes which arise from the backreaction of a set of anti–M2 branes smeared on the finite–sized S^4 at the tip of the Stenzel-CGLP solution ($r = 0$). For describing them it is necessary to carefully impose the correct infrared boundary conditions.

The gravity solution for a stack of localized M2–branes in flat space has a warp factor $H(\rho) = 1 + Q/\rho^6$ and as $\rho \to 0$ the full solution is smooth due to the infinite throat. However when these branes are smeared in n–dimensions, the warp factor scales as ρ^{-6+n} as $\rho \to 0$ since it is now the solution to a wave equation in dimension $d = 8 - n$. This is the IR boundary condition that we will impose on the solution.

We must furthermore bring to bear appropriate boundary conditions on the various fluxes. This is rather simple for M2 branes in flat space, where the energy from $G^{(4)}$ is the same as that from the curvature. In the presence of other types of flux, the IR boundary conditions are more intricate. When the background is on–shell, contributions to the stress tensor from all types of flux taken together cancel the energy from the curvature: this is the basic nature of Einstein's equation but this is too wobbly a criterion to signal the presence of M2 branes. Instead, the right set of boundary conditions for M2 branes should enforce that the dominant contribution to the stress–energy tensor comes from the $G^{(4)}$ flux.

6.5.1 BPS M2 branes

The M2 brane charge varies with the radial coordinate r of a section of the Stenzel space [244]:

$$\begin{aligned} \mathcal{Q}_{M2}(r) &= \frac{1}{(2\,\pi\,\ell_p)^6} \int_{\mathcal{M}_7} \star G_4\,, \\ &= -\frac{6\,m^2\,\mathrm{Vol}\,(V_{5,2})}{(2\,\pi\,\ell_p)^6} \left(h_0(r)\,\left(f_0(r) - 2\,h_0(r)\right) - \frac{1}{54}\right)\,, \end{aligned} \tag{6.89}$$

with ℓ_p the Planck length in eleven dimensions, $\mathcal{M}_7$ a constant r section of the transverse Stenzel space of volume $\mathrm{Vol}\,(V_{5,2}) = \frac{27\,\pi^4}{128}$ [40]. The number of units of G_4 flux through the S^4 is

$$\begin{aligned} q(r) &= \frac{1}{(2\,\pi\,\ell_p)^3} \int_{S^4} G_4\,, \\ &= -\frac{16\,\pi^2\,m}{(2\,\pi\,\ell_p)^3}\,h_0(r)\,. \end{aligned} \tag{6.90}$$

In the smooth solution their IR values ($r \to 0$) are

$$\mathcal{Q}^{IR}_{M2} = 0\,, \qquad q^{IR} = \frac{1}{(2\,\pi\,\ell_p)^3}\,\frac{8\,\pi^2 m}{3^{3/2}}\,, \tag{6.91}$$

reflecting the fact that all M2 charge is dissolved in fluxes. One can obtain a BPS solution in which smeared M2 branes are added at the tip of the Stenzel space [244] simply by shifting $\star G_4$

in such a way that $f-4h$ does not change[6]. Under shifts of $f \to f + 2\,N$ and $h \to h + \frac{N}{2}$, the IR M2 brane charge changes to

$$\mathcal{Q}_{M2} \to \mathcal{Q}_{M2} + \Delta\mathcal{Q}_{M2}\,, \tag{6.92}$$

where we define

$$\Delta\mathcal{Q}_{M2} = -\frac{6\,m^2\,\mathrm{Vol}\,(V_{5,2})}{(2\,\pi\,\ell_p)^6}\left(\frac{1}{2}\,N^2 - \frac{2}{3^{3/2}}\,N\right)\,, \tag{6.93}$$

whereas the variation in the units of flux through the S^4 amounts to $\frac{8\,\pi^2\,m\,N}{(2\,\pi\,\ell_p)^3}$. This introduces in the IR a $-\Delta\mathcal{Q}_{M2}/r^2$ singularity in the warp factor

$$H_0(r) = 162\,m^2 \int^r \frac{h_0\,\left(f_0 - 2h_0\right) - \frac{1}{54}}{\sinh(r')^3\,\left(2 + \cosh(2\,r')\right)^{3/4}}\,dr'\,. \tag{6.94}$$

This singularity is to be expected as we have smeared BPS M2 branes (whose harmonic function diverges as $1/r^6$ near the sources) on the S^4 of the transverse space. It is interesting to see how this BPS solution arises in the first–order expansion around the BPS CGLP background [66] in the context of our perturbation apparatus. Given that the ξ^i modes are associated to supersymmetry–breaking, all the X_i must be set to zero :

$$X_i = 0\,. \tag{6.95}$$

Since all the $\tilde{\xi}^i$ are zero,

$$Y_1^{IR} = Y_1^{UV}\,. \tag{6.96}$$

In the IR and the UV, $e^{z_0+2\alpha_0}$, $e^{z_0+2\beta_0}$ and $e^{z_0+2\gamma_0}$ do not blow up but reach constant or vanishing values instead. So we impose

$$Y_1^{IR} = 0\,, \qquad Y_2^{IR} = 0\,, \qquad Y_4^{UV} = 0\,. \tag{6.97}$$

As a result of (6.97) and (6.96), the mode $\tilde{\phi}_1$ is identically zero. This yields $Y_2^{IR} = Y_2^{UV}$, $Y_3^{IR} = Y_3^{UV}$.

Since BPS M2 branes do not change the geometry of the Stenzel space but only the warp factor (much like BPS D3 branes also only change the warp factor and not the transverse geometry [100]) we expect the first–order perturbation to $e^{z+2\beta}$ to vanish both in the UV and in the IR, and thus

$$2\,Y_3 + e^{-3z_0(0)}\,Y_4^{IR} + \frac{3}{2}\,m^4\,e^{-6z_0(0)}\,Y_6^{IR} = 0\,, \qquad Y_5^{UV} = Y_6^{UV}\,. \tag{6.98}$$

The constant Y_4^{IR} is in turn determined by Y_4^{UV}. Furthermore, the fields $\tilde{\phi}_5$, $\tilde{\phi}_6$ now obey the corresponding homogeneous equations and the solution is found by replacing $\tilde{\lambda}_{5,6}$ by $Y_{5,6}$.

The mode $\tilde{\phi}_4$ corresponds to the first–order perturbation of the warp factor. We allow an $1/r^2$ IR divergence, which means that Y_6^{IR} doesn't necessarily need to vanish. We will see in a moment that this mode is related to the number $\Delta\mathcal{Q}_{M2}$ of added M2 branes. But first, we note that this does not give rise to a singularity that would be associated with $\tilde{\phi}_5 - 4\,\tilde{\phi}_6$, the perturbation to the term in F_4 (6.13) with legs on $\nu\wedge\sigma_i\wedge\tilde{\sigma}_j\wedge\tilde{\sigma}_k$. Indeed, the conditions we have imposed render this term harmless and independent of Y_6^{IR}: $\tilde{\phi}_5 - 4\,\tilde{\phi}_6 = 2\,Y_6 - 2\,Y_6 + \mathcal{O}(r) = \mathcal{O}(r)$.

Given that Y_4^{IR} first shows up in the $\mathcal{O}(r^0)$ part of the IR expansion of $\tilde{\phi}_4$ there is no restriction on it. Moreover, Y_5 does not arise in any of the divergent or constant pieces in the $\tilde{\phi}^i$ IR expansions, but requiring no exponentially divergent terms in the UV imposes $Y_5 = Y_6$, in agreement with (6.98).

[6] This combination multiplies a four-form field strength with one leg along ν, one along σ^i and two legs along two of the $\tilde{\sigma}^j$ directions which shrink in the IR ($e^{2\beta_0} \sim r^2$)

As a result, the perturbation corresponding to adding $\Delta\mathcal{Q}_{M2}$ M2 branes at the tip is obtained by just setting $Y_5 = Y_6 \sim -\Delta\mathcal{Q}_{M2}$. This perturbation causes the warp factor to diverge in the infrared as $-\Delta\mathcal{Q}_{M2}/r^2$ while all the other ϕ^i change by sub–leading terms apart from ϕ^5 and ϕ^6 which shift by some N related to $\Delta\mathcal{Q}_{M2}$ through (6.93).

The UV expansion of the new warp factor is

$$\begin{aligned} H &= e^{3z_0}\left(1+3\,\tilde{\phi}_4\right), \\ &= \frac{16}{3}\,2^{3/4}\,m^2\,e^{-9r/2}\,(1-6\,Y_3) + \mathcal{O}(e^{-13r/2})\,, \\ &= \frac{16}{3}\,2^{3/4}\,m^2\,e^{-9r/2}\left(1+3\,e^{-3z_0(0)}\,Y_4^{IR} + \frac{9}{2}\,m^4\,e^{-6z_0(0)}\,Y_6\right) + \mathcal{O}(e^{-13r/2})\,, \end{aligned} \tag{6.99}$$

where in the last line we used (6.98), and one can see that Y_6 multiplies a $1/\rho^6$ term, as expected from the exact solution.

6.6 Constructing the anti–M2 brane solution

In order to construct a first–order backreacted solution sourced by anti–M2 branes at the tip of the CGLP solution, the first necessary condition is that the force a probe M2 brane feels be nonzero, which implies:

$$X_4 \neq 0\,. \tag{6.100}$$

Furthermore, since the infrared is that of a smooth solution perturbed with smeared anti–M2 branes, we require that no other field except those sourced by these anti–M2 branes have a divergent energy density in the infrared.

Requiring no $\frac{1}{r^2}$ or stronger divergences in $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{\phi}_3$ and $\tilde{\phi}_6$ immediately implies:

$$\begin{aligned} X_5 &= -\frac{\sqrt{3}}{2}\,X_4\,, \\ Y_2^{IR} &= 0\,, \\ X_1 &= -2\,X_4\,, \end{aligned} \tag{6.101}$$

Barring any $\frac{1}{r}$ divergence in $\tilde{\phi}_{1,2}$ results in

$$Y_1^{IR} = 0\,. \tag{6.102}$$

The divergence in $\tilde{\phi}_4$ is now

$$\tilde{\phi}_4 = 3^{1/4}\,\frac{\sqrt{3}\,Z_0\,m^2\,Y_6^{IR} - X_4}{r^2} + \mathcal{O}(r^0) \tag{6.103}$$

and this is the proper divergence for the warp factor of anti–M2 branes spread on the S^4 in the infrared. The energy density that one can associate with this physical divergence is

$$\rho(E) \sim \frac{d\tilde{\phi}_4}{dr} \sim \frac{1}{r^6} \tag{6.104}$$

Another more subtle divergence in the infrared comes from the M–theory four–form field strength, which is

$$G_4 \;=\; dK(\tau)\wedge dx^0\wedge dx^1\wedge dx^2 + m\,F_4\;, \tag{6.105}$$

where (6.13)

$$\begin{aligned} F_4 \;=\;& \dot{f}\,d\tau\wedge\tilde{\sigma}_1\wedge\tilde{\sigma}_2\wedge\tilde{\sigma}_3 + \dot{h}\,\epsilon^{ijk}\,d\tau\wedge\sigma_i\wedge\sigma_j\wedge\tilde{\sigma}_k \\ &+\frac{1}{2}\,(4h-f)\,\epsilon^{ijk}\,\nu\wedge\sigma_i\wedge\tilde{\sigma}_j\wedge\tilde{\sigma}_k - 6\,h\,\nu\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\;\;. \end{aligned} \tag{6.106}$$

The unperturbed metric in the IR is regular and is given by

$$ds^2 = Z_0^{2/3}\, ds_4^2 + \frac{1}{3^{3/4}}\, Z_0^{-1/3}\, \left[dr^2 + \nu^2 + \sigma_i^2 + r^2\, \tilde{\sigma}_i^2\right] \,, \tag{6.107}$$

with the constant Z_0 given in (6.57). The vanishing metric components $g_{\tilde{\sigma}\tilde{\sigma}}$ lead to a divergent energy density from the four–form field strength components:

$$F_{\nu\sigma\tilde{\sigma}\tilde{\sigma}}\, F_{\nu\sigma\tilde{\sigma}\tilde{\sigma}}\, g^{\nu\nu}\, g^{\sigma\sigma}\, g^{\tilde{\sigma}\tilde{\sigma}}\, g^{\tilde{\sigma}\tilde{\sigma}} \;=\; \frac{9\,\sqrt{3}\, Z_0^{4/3}\, X_4^2}{r^4} + \mathcal{O}(r^{-2}) \tag{6.108}$$

$$F_{r\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}}\, F_{r\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}}\, g^{rr}\, g^{\tilde{\sigma}\tilde{\sigma}}\, g^{\tilde{\sigma}\tilde{\sigma}}\, g^{\tilde{\sigma}\tilde{\sigma}} \;=\; \frac{81\,\sqrt{3}\, Z_0^{3/4}\, X_4^2}{r^4} + \mathcal{O}(r^{-2}). \tag{6.109}$$

Unlike the analogous computations in IIB [30], when integrating these energy densities the factor of $\sqrt{-G} \sim r^{-3}$ is not strong enough to render the action finite. Hence, this singularity has both a divergent energy density, and a divergent action.

As discussed in the Introduction, if this singularity is physical then the perturbative solution we find corresponds to the first–order backreaction of a set of anti–M2 branes in the Stenzel-CGLP background. If this singularity is not physical, then our analysis indicates that anti–M2 branes cannot be treated as a perturbation of this background, and hints towards the fact that antibranes in backgrounds with positive brane charge dissolved in fluxes do not give rise to metastable vacua.

Chapter 7

On The Inflaton Potential From Antibranes in Warped Throats

We compute the force between a stack of smeared antibranes at the bottom of a warped throat and a stack of smeared branes at some distance up the throat, both for anti–D3 branes and for anti–M2 branes. We perform this calculation in two ways: first, by treating the antibranes as probes in the background sourced by the branes and second, by treating the branes as probes in the candidate background sourced by the antibranes. These two very different calculations yield exactly the same expression for the force, for all values of the brane–antibrane separation. This indicates that the force between a brane and an antibrane is not screened in backgrounds where there is positive charge dissolved in flux, and gives a way to precisely compute the inflaton potential in certain string cosmology scenarios.

7.1 Introduction and motivation

Anti–D3-branes in warped deformed conifold throats are widely used in string theory model building and string cosmology, both to get de Sitter solutions [157], and to construct string theoretic models of inflation using D3 branes moving in such throats [158].

In a previous work [30], the construction of the first–order backreacted supergravity solution for a stack of anti–D3 branes in the Klebanov–Strassler (KS) background [171] was attempted. Such antibranes were conjectured in [156] to give rise to holographic duals to metastable vacua of a strongly–coupled gauge theory, and the supergravity analysis implies that the would–be anti–D3 brane solution must have a certain infrared singularity. A similar result was obtained by investigating anti–M2 branes in a warped Stenzel background [32]. If these singularities have a physical origin, then the solutions found in [30, 32] describe the first–order backreaction of antibranes in these backgrounds. If these singularities are pathological, the analyses of [30, 32] imply that antibranes in backgrounds with positive brane charge dissolved in fluxes cannot be treated in perturbation theory.

In the present work we will work under the assumption that the singularities found in [30, 32] are physical, and that antibranes can be treated as perturbations of their respective backgrounds with charge dissolved in fluxes[1].

In certain string inflation models, the inflaton is the position of a BPS D3 brane in a warped background with anti–D3 branes at its bottom, and the brane–antibrane force gives the derivative of the inflaton potential. There exist two methods to compute this potential. The first, introduced in [158] and widely used in string cosmology constructions, treats the anti–D3 branes as probes in the (easy to find) backreacted solution sourced by BPS D3 branes up the throat.

[1]Note that this does not automatically imply that antibranes give rise to metastable vacua – for this one would have to show also that the antibrane solution does not contain other non–normalizable modes.

This method involves calculating the change in the potential of the anti–D3 branes as the position of the D3 branes is altered. This yields the force felt by these D3 branes in the warped deformed conifold with anti–D3 branes.

The second method to derive the inflaton potential consists in constructing the first–order backreacted solution sourced by anti–D3 branes placed at the bottom of a warped deformed conifold [30] and to compute the force felt by a probe D3 brane in this background. Despite the rather complicated nature of the first–order deformation space, the force on a probe D3 turns out to depend only on one of the fourteen integration constants that parametrize the space of $SU(2) \times SU(2) \times \mathbb{Z}_2$–invariant deformations [30]. Furthermore, the leading large–distance behavior of the inflaton potential agrees with the one computed in [158].

One natural question to ask is whether the two calculations for the inflaton potential agree also beyond leading–order. At first glance, one expects that they should indeed agree, as this ought to be merely a consequence of Newton's third law: the force exerted by the brane on the antibrane is the same as the force exerted by the antibrane on the brane [158].

However, the answer does not appear to be so simple. If in the vacuum the calculations of the force using the bare action of one brane in the background of the other should indeed agree, there is no reason this should happen in a background where the charge/anticharge symmetry is broken by the D3 charge dissolved in flux. Indeed, because of harmonic superposition, the fields of the D3 brane are not screened [100]. Yet, there is no reason why the anti–D3 would not be screened by the D3 charge dissolved in flux. Hence, one would expect to have a screening cloud around the anti–D3 branes, which would affect the potential felt by a bare D3 brane. Note that this is a generic phenomenon in media where positive and negative charges are screened differently: because of the different profiles of the screening clouds, the force computed using the action of a bare negative charge in the background of the screened positive charge needs not agree with the force computed using the action of a bare positive charge around the screened negative charge. In the language of plasma physics, the Debye screening lengths of the positive and of the negative charges need not be equal.

The purpose of this letter is to show that the forces computed in the two approaches outlined above agree not only in leading behavior, but in full functional form, modulo a to–be–determined overall normalization constant. This indicates that this force is not screened by the brane charge dissolved in flux[2]. There are two obvious explanations for this: either anti-D3 branes are not screened by the positive D3 brane charge dissolved in flux, or they are screened, but the screening cloud does not interact with D3 branes. This latter possibility would imply that antibranes change the profile of the cloud of charge dissolved in fluxes, but do not alter its properties, in particular the fact that the local D3 charge density remains equal to the mass density; such a cloud would not interact with probe D3 branes and would not screen the force.

We find no brane–antibrane force screening, both for anti D3–branes at the bottom of the Klebanov–Strassler solution, and for anti–M2 brane at the bottom of a warped Stenzel space with M2 brane charge dissolved in flux [244, 66], and hence we believe this is likely a generic phenomenon in flux compactifications.[3]

Hence, in an optimistic scenario (if the IR singularities found in [30] and [32] are physical, and we can trust perturbation theory), modulo this subtle issue about the overall constant, our calculation yields the exact functional form of the inflaton potential in a brane/antibrane realization of inflation in string theory. It also demonstrates that the force between branes and antibranes is not screened, and therefore the probe antibrane calculation à la KKLMMT [158] of this inflaton potential in other string inflationary models gives the exact functional form of

[2]Our analysis does not formally exclude screening by a delta–function–shaped screening cloud, which would keep the same functional expression of the force while changing the overall normalization constant. However, it is hard to believe this is anything but a formal possibility. We leave the actual computation of this constant to a forthcoming publication [34].

[3]In an upcoming paper [97] we will also show this for anti–D2 branes in backgrounds with D2 brane charge dissolved in fluxes [67].

the potential, not only its leading behavior. This should allow in turn to accurately compute the power spectrum in those models and to compare them with observation.

This work is organized as follows. In Section 7.2. we review the calculation of the brane/antibrane force, treating the smeared antibranes as probes, both for anti–D3 branes in KS and for anti–M2 branes in a warped Stenzel background [244, 66]. In Section 7.3. we use the first–order backreacted solutions of [30] and [32] to compute this force using the action of probe D3 and M2 branes, respectively. As advertised, the two calculations agree.

Note: as this work was being prepared for submission we learnt that Anatoly Dymarsky has independently found some of the analytic results presented here.

7.2 Computing the force using the action of probe antibranes

To establish whether antibranes are screened by charge dissolved in flux in the warped deformed conifold or the Stenzel space, we first smear them at the tip of those two geometries. This way we preserve the symmetries of the solution without antibranes, and render the calculation of the backreaction of the antibranes an achievable task. The force between the smeared antibranes and the BPS branes at some distance $r = r_0$ up the throat will then be the same, by symmetry, as the force between the smeared antibranes and a uniform shell of BPS branes at the same distance.

We demonstrate how to compute the force generated between a stack of antibranes at the bottom of a warped throat and a stack of branes some distance up the throat. This is computed in two ways: either by backreacting the branes while leaving the antibranes as probes ; or from backreacting the antibranes and leaving the branes as probes.

7.2.1 Backreacted D3 branes in the warped deformed conifold

To obtain a fully backreacted solution with BPS D3 branes in the warped deformed conifold one simply needs to add to the warp factor a harmonic function (given by the Green's function on this Calabi-Yau manifold) sourced by these branes [100]. While in general this is a non–trivial task [176, 225], here we are considering smeared branes and as such the Green's function is radially symmetric and the problem is tractable.

The two radially symmetric solutions to the Laplace equation on the deformed conifold are

$$H_1(\tau) = c_1 \,, \tag{7.1}$$

$$H_2(\tau) = c_2 \int_\tau^\infty \frac{d\tau'}{\left(\sinh 2\,\tau' - 2\,\tau'\right)^{2/3}} \,. \tag{7.2}$$

With a shell of D3 branes at $\tau = \tau_0$, the full warp factor is

$$H(\tau) = H_0(\tau) + \delta H(\tau) \ . \tag{7.3}$$

Here $H_0(\tau)$ is the zeroth–order warp factor for the warped deformed conifold:

$$\begin{aligned} H_0 &= e^{-4A_0-4p_0+2x_0} \\ &= h_0 - 32\,P^2 \int_0^\tau \frac{t\,\coth t - 1}{\sinh^2 t} \left(\tfrac{1}{2}\,\sinh(2\,t) - t\right)^{1/3} dt \ , \end{aligned} \tag{7.4}$$

where P is the RR three–form flux through the S^3 of the deformed conifold, and h_0 is a constant[4]. On top of the warp factor for the zeroth–order solution, there is the following contribution from the N D3 branes at $\tau = \tau_0$:

$$\delta H(\tau) = \begin{cases} H_1(\tau)\,, & \tau < \tau_0\,, \\ H_2(\tau)\,, & \tau > \tau_0\ . \end{cases} \tag{7.5}$$

[4] Explicitly, we have $h_0 = 32\,P^2 \int_0^\infty \frac{\tau\,\coth\tau - 1}{\sinh^2\tau} \left(\frac{1}{2}\,\sinh(2\,\tau) - \tau\right)^{1/3} d\tau = 18.2373 P^2$.

The two integration constants (c_1, c_2) are related by matching at the source the solutions in the two domains above:

$$c_1 = H_2(\tau_0)\,. \tag{7.6}$$

To fix the other integration constant in terms of the number of D3 branes, we rely on the standard quantization formula for the five–form field strength:

$$\frac{1}{(4\,\pi^2\,\alpha')^2}\int F_5 = N\,, \tag{7.7}$$

and integrate on the $T^{1,1}$ surfaces right outside and right inside the shell using

$$g_s F_5 = *_{10} dH^{-1} \wedge dx_0 \wedge \ldots \wedge dx_3\,. \tag{7.8}$$

The difference of the two integrals gives the D3 brane charge of the shell and its relation to the coefficient in δH:

$$c_2 = 4\,\pi\left(\frac{2^{1/3}\,\alpha'}{\varepsilon^{4/3}}\right)^2 g_s\,N\,, \tag{7.9}$$

where we use the conventions of [129].

We now compute the potential of probe anti–D3 branes placed at the tip of the cone. Since for a BPS D3 brane the DBI and WZ potentials cancel, for anti–D3 branes these potentials are equal in magnitude and sign:

$$V_{\overline{D3}} = V_{DBI} + V_{WZ} = 2\,V_{WZ}\,. \tag{7.10}$$

Expanding the potential to first–order in the number of D3 branes we find

$$\begin{aligned} V_{\overline{D3}} &= 2\,H^{-1}\,, \\ &= 2\,H_0^{-1}(1 - \frac{\delta H}{H_0}) + \mathcal{O}((N/P)^2). \end{aligned} \tag{7.11}$$

The force exerted by the anti–D3 branes on the D3 branes can then be obtained from the variation of this potential as the source D3 branes are moved [158]

$$\begin{aligned} F_{D3} &= -\frac{\partial V_{\overline{D3}}}{\partial \tau_0}\Big|_{\tau=0} \\ &= -\frac{1}{H_0^2|_{\tau=0}}\,\frac{c_2}{(\sinh 2\,\tau_0 - 2\,\tau_0)^{2/3}}\,. \end{aligned} \tag{7.12}$$

The dependence of this force on N appears through the constant c_2 (7.9).

7.2.2 M–Theory on a warped Stenzel space

The generalization of the probe brane computation of Kachru, Pearson and Verlinde [156] to a warped Stenzel space M–theory background [244, 66] has recently been performed in [175]. Motivated by this analysis, three of the authors have used the technology of [30] to study the backreaction of anti–M2 branes in this space [32]. The probe brane analysis of the previous section can also be performed, and we find that although the Green's function itself is a complicated combination of incomplete elliptic integrals:

$$\begin{aligned} H_1(y) =& d_1\,, \\ H_2(y) =& \frac{2}{45}\,d_2\,\Big[\frac{9\,\sqrt{y^4-1}}{y^5} + 3\,E\,(\arcsin(1/y)\mid -1) - 3\,F\,(\arcsin(1/y)\mid -1) \\ &+ 5\,\sqrt{3}\,\Big(\Pi\left(\sqrt{3}; -\arcsin(1/y)\mid -1\right) - \Pi\left(-\sqrt{3}; -\arcsin(1/y)\mid -1\right)\Big)\Big]\,, \end{aligned} \tag{7.13}$$

with d_i integration constants and k a constant that ensures that H_2 vanishes at large y, the derivative of this Green's function is very simple[5]:

$$H_2'(r) = \frac{3\sqrt{2}\, d_2 \operatorname{csch}^3 r}{(2+\cosh 2\, r)^{3/4}}\,. \tag{7.14}$$

From flux quantization

$$\frac{1}{(2\,\pi\,\ell_p)^6}\int_{V_{5,2}} *_{11} G_4 = N\,, \tag{7.15}$$

with

$$G_4 = dH^{-1}\wedge dx_0 \wedge dx_1 \wedge dx_2\,,$$

we find that the M2 brane charge of the shell, N, is related to the constant in the new warp factor via

$$d_2 = (2\,\pi)^2\,\ell_p^6\,\sqrt{2}\,N\,. \tag{7.16}$$

In addition, there is the matching condition

$$d_1 = H_2(y_0)\,. \tag{7.17}$$

If we now consider the change in the potential of probe antibranes with the position of the source M2 branes in this background, we obtain the force:

$$F_{M2} = -\frac{1}{H_0^2|_{r=0}}\frac{3\sqrt{2}\,d_2 \operatorname{csch}^3 r_0}{(2+\cosh 2\,r_0)^{3/4}}\,. \tag{7.18}$$

7.3 Computing the force on probe branes

7.3.1 Warped deformed conifold

We now use the results from [30] and refer to this work for much of the notation. In that paper three of the authors found that the force felt by a probe D3 brane in the first–order deformed KS background has the remarkably–simple form

$$F_{D3} = \frac{2}{3}\,e^{-2x_0}\,\tilde{\xi}_1\,, \tag{7.19}$$

where $\tilde{\xi}_1$ is one of the sixteen modes parameterizing the deformation space[6] [45] and is given by

$$\tilde{\xi}_1 = \widetilde{X}_1 \exp\Big(\int_0^\tau d\tau' e^{-2x_0}\left[2\,P\,f_0 - F_0\,(f_0-k_0)\right]\Big)\,. \tag{7.20}$$

Here X_1 is an integration constant and

$$\begin{aligned} e^{x_0} &= \frac{1}{4}\,H_0^{1/2}\,(\tfrac{1}{2}\,\sinh(2\,\tau)-\tau)^{1/3}\,,\\ f_0 &= -P\,\frac{(\tau\,\coth\tau-1)(\cosh\tau-1)}{\sinh\tau}\,,\\ k_0 &= -P\,\frac{(\tau\,\coth\tau-1)(\cosh\tau+1)}{\sinh\tau}\,,\\ F_0 &= P\,\frac{(\sinh\tau-\tau)}{\sinh\tau}\,, \end{aligned} \tag{7.21}$$

with H_0 given in (7.4).

[5] The standard coordinate we use is $y^4 = 2+\cosh 2\,r$.

[6] This deformation space has been considered previously in various respects [180, 38, 39, 37, 82].

We make great use of the simple yet elusive observation that this integral can in fact be performed exactly

$$\begin{aligned} \tilde{\xi}_1 &= \widetilde{X}_1 \exp\Big(\int_0^\tau d\tau' \, \frac{H_0'}{H_0}\Big) \\ &= X_1 \, H_0(\tau) \, . \end{aligned} \tag{7.22}$$

The force now takes the form

$$\begin{aligned} F_{D3} &= \frac{2}{3} \, e^{-2x_0} \, X_1 \, H_0(\tau) \\ &= \frac{32}{3} \, \frac{2^{2/3} \, X_1}{(\sinh 2\,\tau - 2\,\tau)^{2/3}} \, . \end{aligned} \tag{7.23}$$

Remarkably enough, this has exactly the same functional form as the force computed in (7.12) using the probe antibrane potential. As mentioned in the Introduction, the fact that the two calculations of the force agree implies that this force is not screened by the positive D3 brane charge dissolved in flux.

As has been explained in [30] the value of X_1 can be determined in terms of the UV and IR boundary conditions, but this requires relating the UV and IR values of all sixteen integration constants involved in the full solution, which can only be done numerically. Once this numerical work is completed, we will be able to compare the coefficient of the force computed in this section with the calculation of section 7.2.1. Whether these two numbers agree or not will help elucidate the physics of anti–D3 branes in the Klebanov–Strassler background. We plan to report on these results soon [34].

7.3.2 M–Theory on a warped Stenzel space

The same steps for M–theory on a Stenzel space have recently been performed in [32] and we merely quote the results and refer to this work for the notation. When considering the candidate backreacted solution corresponding to anti–M2 branes, the force felt by a probe M2 brane is

$$\begin{aligned} F &= -\frac{2}{3} \, e^{-3\,(\alpha_0+\beta_0)(r)} \, e^{-3z_0(0)} \, X_4 \\ &= -\frac{18 \, e^{-3z_0(0)} \, X_4 \, \mathrm{csch}^3 r}{(2+\cosh 2r)^{3/4}} \, . \end{aligned} \tag{7.24}$$

This has again the same functional form as (7.18), up to the determination of the integration constant X_4 in terms of the charges of the system. This demonstrates that, much like in the anti–D3 brane story, the force between anti–M2 branes and M2 branes is not screened by the charge dissolved in flux.

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