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QUELQUES ASPECTS DE LA BRISURE DE SUPERSYMÉTRIE EN THÉORIE DES CORDES DE TYPE IIA: VIDES ET DÉFORMATIONS

ASPECTS OF SUPERSYMMETRY BREAKING IN TYPE IIA STRING THEORY: VACUA AND DEFORMATIONS

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Abstract

This thesis is devoted to the study of non–supersymmetric flux compactifications in type IIA string theory. After a brief review of type II theories, we introduce the mathematical framework of Generalized Complex Geometry, which provides an encompassing geometric interpretation and organizing principle for supersymmetric vacua. We introduce the class of solvmanifolds, which have been extensively used as compactification manifolds, and discuss their mathematical properties, with particular attention to the compactness criteria. We then present our first example of non-supersymmetric compactification, a vacuum which has a de Sitter external space. We solve the equations of motion and in the process we argue about the behavior of D–branes in non–supersymmetric backgrounds; a short analysis of the four dimensional physics is also provided. We speculate about the use of Generalized Geometry for non-supersymmetric vacua too and about the right variables to describe the supposed underlying geometric structure. Motivated by AdS/CFT considerations we investigate a supersymmetry breaking vacuum which is supposed to be the gravity dual to a metastable non–supersymmetric vacuum of a supersymmetric gauge theory. Supersymmetry is here broken by the addition of anti–branes; it is notoriously difficult to take into account their backreaction and we resort to use a perturbative technique. We compute the most general first order deformations of a D2–brane background, discuss the space of solutions of the deformed fields and argue about the nature of the unavoidable singularities which we encounter in the process.

Résumé court

Cette thèse porte sur l'étude des compactifications non–supersymétriques avec flux en théorie des cordes de type IIA. Après une introduction aux théories de type II, nous introduisons le cadre mathématique de la Géométrie Complexe Généralisée, celle ci donne une interprétation géométrique des vides supersymétriques et une principe permettant de les organiser. Nous introduisons la classe des solvmanifolds, qui ont été largement utilisées comme variétés de compactification, et discutons leurs propriétés mathématiques, notament les critères de compacité. Nous présentons ensuite notre premier exemple de compactification non–supersymétrique, un vide qui a pour espace externe un espace de Sitter. Nous résolvons les équations du mouvement et dans le même temp nous discutons le comportement de D–branes dans les fonds non–supersymétriques. Une brève analyse de la physique à quatre dimensions est également fournie. Nous spéculons sur l'utilisation de la Géométrie Généralisée pour les vides non–supersymétriques et sur leur structure géométrique. Motivé par des considérations issues de la dualité AdS/CFT nous analysons un vide non–supersymétrique, censé être le correspondant gravitationnel d'un vide métastable non supersymétrique d'une théorie de jauge supersymétrique. La supersymétrie est ici brisée par l'ajout d' anti–branes, dont il est notoirement difficile de prendre en compte la contre réaction. Ainsi nous avons recours à l'utilisation d'une technique perturbative. Nous calculons les déformations du premier ordre d'un fond D2–brane, discutons l'espace des solutions et argumentons sur la nature des singularités inévitables que nous rencontrons dans le processus.

Résumé détaillé

La théorie des cordes et sa limite de basse énergie (supergravité) ont été l'objet de recherches intenses au cours des dernières décennies. Dans sa version supersymétrique la théorie est définie à dix dimensions, il est donc nécessaire d'expliquer le fait que nous faisons l'expérience de quatre dimensions, et de relier la physique d'une théorie à dix dimensions et celle à quatre dimensions. La façon la plus étudiée et fructueuse pour y parvenir est connue sous le nom de compactification. L'idée est que l'espace dix dimensionnel se décompose en quatre dimensions étendues tandis que les six autres sont enveloppées dans un espace compact. La taille des dimensions compactes est généralement considérée comme très petite pour justifier le fait que nous ne pouvons pas les détecter par nos expériences. Les propriétés de la théorie effective à quatre dimensions sont obtenues à partir d'une solution dix dimensionnelle par intégration sur les degrés de libertés internes, et dépendent donc de la géométrie de l'espace interne. Bien qu'il soit un mécanisme intéressant pour construire des théories à quatre dimensions cette procédure souffre d'un inconvénient immédiat: même si la théorie de départ est unique il y a un grand degré d'indétermination une fois à quatre dimensions. Les exemples les plus simples des compactifications sont basés sur des variétés différentielles plates, et plus particulièrement sur des espaces de Calabi–Yau. Ils présentent un grand nombre de champs scalaires non massifs qui, d'un point de vue à quatre dimensions, ont une valeur moyenne dans le vide indéterminée, ce qui est certainement insatisfaisante pour la phénoménologie. La présence des flux à la fois du secteur NS–NS et RR améliore cette situation, offrant un moyen de fixer cette valeur moyenne dans le vide. Dans cette thèse, nous allons se concentrer sur l'analyse des vides avec flux, en particulier dans la théorie de type IIA, avec quelques autres ingrédients supplémentaires: des D–branes et orientifolds. Ce sont des objets non–perturbatifs qui, en plus d'être nécessaires pour reproduire des extensions du modèle standard en théorie des cordes, sont aussi des sources pour les flux RR.
La structure des vides supersymétriques des théories de type II a été un objet d'étude important, depuis l'article fondateur [32]. Le travail a consisté en la détermination des propriétés géométriques de la variété différentielle interne, et de nombreux progrès mathématiques ont été inspirés par cette question. Cela a résulté en particulier en la Géométrie Complexe Généralisée: il s'agit d'un cadre mathématique qui fournit une compréhension claire de la géométrie, ainsi qu'un critère de classification avec des outils de calcul pour trouver des exemples concrets. Nous allons adopter ce point de vue, mais l'intérêt est vers une situation différente.

C'est une évidence expérimentale que, à l'énergie à laquelle nous sommes capable de sonder, la supersymétrie n'est pas réalisée. Donc, si elle est une symétrie de la théorie, elle est brisée à des échelles d' énergie supérieures, que nous espérons être de l'ordre du régime du LHC. En attendant une éventuelle preuve expérimentale, la compréhension de certaines caractéristiques des vides non–supersymétriques est un problème important et intéressant en lui–même. L'objectif de cette thèse est d'analyser, au travers de deux exemples concrets, certains aspects de cette question.

Le premier exemple est une compactification sur variétés résolubles, qui sont une certaine classe de variétés différentielles qui peuvent admettre une courbure négative. Cette propriété, avec la présence d'un flux de RR de degré zéro, a été prouvé être un ingrédient nécessaire pour une compactification sur un espace extérieur avec constante cosmologique positive: un espace de Sitter. Cette configuration est intéressante pour deux raisons. Tout d'abord, cet espace est, comme nous l'expliquerons dans le texte, intrinsèquement non–supersymétrique en raison de certaines considérations d'ordre général dans les théories de supergravité. La deuxième raison est phénoménologique: il existe désormais des preuves expérimentales qui soutiennent l'affirmation que la constante cosmologique a une valeur positive. Avec notre analyse, nous ne proposons pas de construire un modèle phénoménologique réaliste, tâche très compliquée et au–delà des objectifs de cette thèse. Bien que se baser sur un exemple n'est pas une procédure générale, cela permet toujours d'étudier certaines des propriétés qui devraient caractériser de tels vides. En particulier, nos solutions s'appuient sur une déformation de la supersymétrie des sources (D–branes et orientifolds), et celles–ci sont ensuite responsables de la rupture. Nous allons également compléter l'analyse par quelques observations et spéculations sur un plan plus formel et général.

Le deuxième exemple provient de considérations sur la correspondance AdS/CFT [143, 97, 187]. Cette dualité permet un aperçu du régime de couplage fort des théories de jauge supersymétriques dans des dimensions diverses, et nous nous concentrons ici sur le dual gravitationnel d'une théorie à $2+1$ dimensions. Il y a eu récemment un intérêt croissant pour la brisure dynamique de la supersymétrie vers des états métastables dans la théorie quantique des champs. Une question naturelle est de savoir si cela peut être réalisé dans le cadre de la correspondance AdS/CFT par des déformations non–supersymétriques de solutions de supergravité. Un travail important a été réalisé dans la théorie de type IIB, principalement dans le cas de Klebanov–Strassler [124], d'abord avec l'approximation de sonde [119], puis par une étude linéaire de la "backreaction" [16, 13, 14, 62]. Une analyse similaire a été faite en théorie M [123, 11]. Moins d'attention, à notre connaissance, a été portée à des configurations de type IIA. Nous proposons ici quelques progrès afin de combler cette lacune, et nous étudions l'espace des déformations linéaires non–supersymétriques d'un vide de type IIA, qui décrit des D2–branes fractionnaires régulières. La solution a deux supercharges, elle est donc le dual gravitationnel d'une théorie jauge en $2+1$ dimensions avec $\mathcal{N}=1$. Nous nous concentrons principalement sur le côté gravitationnel de la dualité. Nous ferons une résolution pour l'espace des perturbations linéaires autour du vide supersymétrique [47] utilisant la technique développée dans [24]. Nous montrons que cette configuration présente aussi deux caractéristiques principales communes à d'autres analyses [16, 13, 14, 62, 11]: la force sur une brane sonde dépend d'un seul mode de perturbation qui est lié à la brisure de supersymétrie, et la région IR est affectée par quelques singularités. C'est là une question ouverte de savoir si ces singularités sont admissibles ou non, et ce point n'est à ce jour pas résolu; nous montrons que dans la configuration que nous avons choisi d'analyser, elles sont plus sévères que dans les autres cas.

La thèse est ainsi structurée.

Le *chapitre 2* contient un rappel des propriétés principales de la supergravité de type II: son contenu en champs, son action, ses équations du mouvement et ses variations de supersymétrie.

Nous fournissons notre définition du vide et expliquons le problème géométrique qu'elle pose avec l'exemple de la compactification de Calabi–Yau, principalement pour fournir la logique que nous voulons suivre. Nous introduisons également quelques notions géométriques comme celle de la G–structure.

Le *chapitre 3* est plus technique et de nature mathématique. Il fournit une courte introduction à la Géométrie Complexe Généralisée. Nous introduisons la notion de fibré tangent généralisé, de structure complexe généralisée et nous définissons un crochet qui permet de définir une notion d' intégrabilité. Nous introduisons les spineurs associés au fibré généralisé et les détails de leur relation avec les formes différentielles sur la variété interne. Ce sera un point clé qui permettra d'obtenir un ensemble d'équations différentielles qui reformulent les conditions de supersymétrie. Ces équations agissent sur un couple d'objets donnés par la somme de formes différentielles de degrés différents (polyformes). Nous terminons le chapitre par une brève description des sources (D–branes et orientifolds) dans ce contexte.

Le *chapitre 4* contient une description des solvmanifolds. Il s'agit d'une classe de variétés différentielles à six dimensions qui ont été largement utilisées comme variétés internes pour des compactifications supersymétriques [87]. Nous les considérons ici pour deux raisons: l'une est que nous voulons réinterpréter certains résultats bien établis sur leurs propriétés mathématiques de compacité et l'existence de formes différentielles globalement définies dans le cadre de la transformation de twist développée dans [5], étendant l'analyse aux solvmanifolds qui ne sont pas des nilmanifolds. Nous ne présentons pas de résultats originaux, mais il s'agit néanmoins d'une neuve reformulation utile qui nous permet d'obtenir une solution supersymétrique sur un solvmanifold noté $G_{5.17}^{p,-p,\pm1} \times S^1$, qui sera considéré comme point de départ pour l'analyse non–supersymétrique du chapitre 5.

Le *chapitre 5* contient l'analyse d'une solution non–supersymétrique de type de Sitter. Nous commençons avec quelques considérations sur les caractéristiques de telles solutions, avec une attention particulière sur la description des sources. Nous allons ensuite fournir les détails de la solution qui présente des flux F_2 et H non nuls, avec également un flux non nul F_0 et une courbure négative de la variété interne, comme indiqué nécessaire par certaines analyses générales. Nous fournissons les détails du calcul du tenseur énergie–impulsion pour les sources qui brisent la supersymétrie et expliquons quel genre de déformations nous avons besoin pour résoudre les équations d'Einstein et du dilaton. Nous terminons le chapitre par une analyse de la (méta)–stabilité de notre solution d'un point de vue à quatre dimensions.

Le *chapitre 6* est une tentative de développement plus formel de l'intuition que l'on peut obtenir à partir de l'exemple précédemment analysé. La géométrie a été une idée clé dans l'analyse des configurations supersymétriques: depuis le premier exemple de compactification sur Calabi–Yau jusqu'à la reformulation en termes de Géométrie Complexe Généralisée, les structures géométriques sont intimement liées à la dynamique du problème. Notre avis est que les vides $\mathcal{N} = 0$ (ou au moins un sous–ensemble d'entre eux) peuvent aussi être décrits par des outils géométriques. Dans ce chapitre, nous recueillons certaines observations et les réorganisons à un niveau plus formel. En particulier, nous essayons de trouver un ensemble de variables bispinorielles adaptées à une généralisation aux cas non–supersymétriques du formalisme de premier ordre, via la manipulation directe sur les équations de spineurs purs ou via la T–dualité. Nous essayons également de discuter du problème des branes non–supersymétriques, dont le comportement et la description est a priori différent du cas des configurations supersymétriques. Le but de ce chapitre n'est pas de fournir des résultats établis, mais d'en déduire qu'une certaine structure géométrique peut également être développée pour le problème plus difficile des vides non–supersymétriques.

Le *chapitre 7* est consacré à l'analyse de l'espace des déformations supersymétriques et non–supersymétriques du premier ordre, pour le vide de type IIA découvert dans [47]. La variété à dix dimensions dans ce cas est divisée en trois dimensions qui composent un espace de Minkowski, et une certaine variété non compacte à sept dimensions, asymptotiquement conique et d'holonomie G_2, qui est fait d'une fibre $\mathbb{R}^3$ sur S^4. La Géométrie Complexe Généralisée pourrait être utilisée dans certaines configurations similaires [98], mais nous choisissons d'utiliser une approche différente basée sur la technique développée dans [24] qui est appropriée pour notre analyse et nous allons commencer par l'examiner. Après avoir rappelé les propriétés importantes du vide [47], nous présentons les solutions aux équations qui paramétrisent les déformations linéaires. Nous discutons ensuite les principales caractéristiques de la physique IR et commentons sur les singularités que nous allons rencontrer.

Dans le *chapitre 8* on conclue avec des considérations finales.

Quelques détails techniques sont présentés dans les appendices. L'appendice A résume nos conventions et quelques formules utiles. L'appendice B contient une note sur l'application de la T–dualité aux solvmanifolds. L'appendice C contient les définitions et les théorèmes sur les groupes de Lie qui sont beaucoup utilisés dans la thèse. L'appendice D contient la liste des algèbres et des variétés résolubles qui sont considérées dans la thèse, en particulier elles sont présentées dans la base adaptée pour l'analyse de la compacité.

Contents

Chapter 1

Introduction

The belief in the existence of a unified description of all the known interactions has been one of most fruitful heuristic principles in theoretical physics since ever and in particular in the last century. Our current understanding of the physics at a fundamental level is based, it is by now a many-times-told story, on two rather different theories: on one side there is Standard Model with its microscopic description of electro-weak and strong interactions based on gauge theory and fully compatible with quantum principles, on the other side sits Einstein General Relativity, based on Riemannian geometry and not incorporating the quantum principles, which describes gravitational interactions and whose aim is to describe the intimate dynamics of space-time and the relation between its geometry and the distribution of matter. Both theories have been tested to a high degree of accuracy and revealed incredibly fruitful as well as both are incomplete and clearly only one piece of the puzzle. Standard Model is a rather ad hoc construction where many parameters are not fixed from first principles, there is no explanation of why that particular set of gauge groups and multiplets has been selected by Nature and of course it does not include gravity. On the other hand General Relativity is a theory whose validity, already at the classical level, breaks down because of the singularities met when, for example, attempts of describing "extreme" objects such as black holes or attempts to extrapolate back in time the history of the universe are made. Moreover renormalization, which can be performed with success for the Standard Model, fails when we try to connect quantum field theory and General Relativity. Last but not least in this (partial) list of unanswered questions is the hierarchy problem, some of the parameters are smaller than expected and we lack an explanation for the small ratio $M_{\text{Weak}}/M_{\text{Planck}}$ between the electroweak scale (where electromagnetic and weak interaction have a unified description, at around 246 GeV) and the Planck scale (a first guess on the regime where quantum gravity effects are important, a much higher scale around 10^{19} Gev). This brief account of questions left open by the Standard Model + General Relativity paradigm should be enough to understand the motivations which pushed the community of theoretical physicists to look for some other description that encompasses and unifies the two, with the aim of providing the lacking answers (and maybe pose new questions).

The main outcome of the last forty years' efforts is string theory. Even a far from complete summary of it is hopeless in the restrict space of this Introduction, we will thus present some of the peculiar features which are of most interest in this thesis and refer the reader to more complete works on the subject [165, 91, 9]. String theory arises from a simple, but with far reaching consequences, idea which is to replace point particles, that quantum field theory assumes as fundamental objects of the theory, with one-dimensional objects: strings[1]. It was first developed in late Sixties as an attempt to describe strong interactions, but soon later it was clear that it was better suited to describe gravity at a quantum level because the spectrum presents a massless spin-two particle which can be identified with the graviton. Particles arise as oscillating modes of the fundamental object and its one-dimensional nature is distinguishable at energy scales comparable to its characteristic length ℓ_s, which is usually assumed to be of the order of the Planck length ($\sim 10^{-35}$ m). At lower energies they appear as point-like objects and we can thus recover the quantum field theory limit. It turns out that, in order to have fermions in the spectrum, the theory must be supersymmetric. This, together with anomaly cancellation, restricts quite a lot the possible string models. In fact there are only five know consistent string theories: Type I, Type IIA/IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$. Moreover there is a web of dualities which relate the different string theories to one another and this can be interpreted as a signal of an underlying common theory. In this thesis we will mainly concentrate on Type II theories and more specifically on type IIA.

Supersymmetry is a symmetry between bosons and fermions which arrange themselves in multiplets of equal mass. Symmetry has always been an over comprehensive fundamental concept in physics and a necessary tool to handle the difficulties of the calculations and, as such, supersymmetry has an important role already at the level of theories beyond the Standard Model. It is common knowledge that, among many other advantages, it keeps under control quantum corrections to the Higgs mass, it is a candidate solution to the dark matter problem, and determines the very non trivial fact that the electroweak and strong coupling constants meet at the "Grand Unification" scale, approximately 10^{16} Gev.

The attempt to merge supersymmetry with General Relativity by making it a local symmetry bears the name of supergravity. Despite the first encouraging results it turned out that it is not enough to cure all the divergencies, solving the problem of the quantization of gravity.[2] However it is a remarkable fact that the massless spectrum of each string theory corresponds to the spectrum of some supergravity and that conformal anomaly cancellation at one-loop[3] imposes equations among the fields which correspond to the equations of motion of some supergravity. We can thus conclude that the low energy effective theory of string theory is supergravity. These considerations motivate the conclusion that string theory is, if not the definitive answer, at least an important step towards the understanding of the fundamental physics of our universe.

One of the most striking predictions of string theory is that, again as a consequence of conformal anomaly cancellation, the space-time dimension has to be ten[4]. We thus have to face the fact that we experience only four dimensions and we need to explain the relation between the

[1]One could ask why should one and not higher dimensional objects be the fundamental building blocks. A priori this is not obvious but in the course of analysis it has become clear that this is the right compromise to keep under control the divergencies of quantum field theory and gravity and the one arising by the increasing of the internal degrees of freedom. The theory is defined in two dimensions where the (local) conformal symmetry is richer than in higher dimensions.

[2]This is true for models with a minimal amount of supersymmetry which are the most suitable for phenomenology. For maximal $\mathcal{N} = 8$ supergravity the outcome could be different and the theory finite. It is a very active field in these days, high loop orders have been computed showing rather striking cancellations.

[3]String theory is defined by the conformal two-dimensional world-sheet action which defines a sigma model with target space our space-time. In a path integral based quantization the sum is over the different two-dimensional world-sheet topologies.

[4]This is true for supersymmetric string theories, a similar result was first obtained for the simpler bosonic string where the critical dimension is 26.

physics of a ten dimensional theory and the one in four dimensions. The most studied and fruitful way to achieve this is compactification. The idea is that the ten dimensional space has four extended dimensions while the other six are wrapped on a compact space. The scale of the compact dimensions is usually taken to be very small to justify the fact that we cannot probe them directly[5], nevertheless once we "integrate" the ten dimensional theory over the compact space we are left with an effective four dimensional theory whose properties depend on the geometry of the internal space. Despite being an attractive mechanism to build lower dimensional theories it suffers from an immediate drawback, even if the starting theory is unique there is a large degree of indeterminateness regarding the four dimensional outcome. The simplest examples of compactifications based on Ricci-flat manifolds, most notably the case of Calabi-Yau spaces, exhibit a large number of scalar fields, called moduli, which encode the information about the internal geometry, but which, from a four dimensional point of view, have undetermined vacuum expectation value which is certainly phenomenologically unsatisfactory. The presence of fluxes both from the NS-NS and the R-R sector improve this situation providing a mean of fixing the value of the moduli. In this thesis we will in fact concentrate on the analysis of background with fluxes, especially in type IIA theory, together with some other additional ingredient namely D-branes and O-planes. These are non-perturbative objects which beside being necessary in constructing four-dimensional Standard-like models are also sources for the RR fluxes. We can now give an outline of the thesis.

1.1 Outline of the thesis

The structure of supersymmetric vacua of type II theories has been subject of intensive analysis during the last decades. Since the seminal work [32] the effort has been to determine the geometrical properties of the internal manifold and many mathematical advancements have been inspired by such a problem. The outcome of this is Generalized Complex Geometry, it is an encompassing framework which provides a clear understanding of the geometry and a classifying criterion together with a computational tool to find concrete examples. We will adopt this point of view but the interest is towards a different situation.

It is an experimental evidence that, at the energies we are able to probe, supersymmetry is not realized, thus, if it is a symmetry of the theory, it is broken an some higher energy scale which hopefully will be in the range of the now running LHC experiment. Waiting for a possible experimental evidence, it is an important and interesting problem by itself to understand some of the features of non-supersymmetric vacua. The aim of this thesis is to analyze, by means of two concrete examples, some aspects of the phenomenon.

The first example is a compactification on solvmanifolds, which are a certain class of manifolds that could admit negative curvature. This property, together with the presence of a zero-form flux field, has been proved to be a necessary ingredient for a compactification with an external space with positive cosmological constant: a de Sitter space. This configuration is interesting for two reasons. First of all it is, as we will explain in the text, intrinsically non-supersymmetric because of some general considerations in supergravity theories. The second reason is phenomenology, growing experimental evidence supports the claim that the cosmological constant has a positive value. Our analysis is far from being phenomenological viable and

[5]For models with large extra dimensions an the possibility of detecting them at LHC see [139] and references therein.

being example based it is not a general procedure, but it still provides a way to investigate some of the properties which should characterize such vacua. In particular our solutions rely on a deformation of the supersymmetry of the sources, which are then responsible for the breaking. We will complete the analysis with some observations and speculations at a more formal and general level.

The second example stems from considerations based on the AdS/CFT correspondence [143, 97, 187]. The duality provided deep insights of the strong coupling regime of supersymmetric gauge theories in diverse dimensions and we concentrate here on the gravity dual of a $2+1$ dimensional theory. There has recently been a growing interest on metastable dynamical supersymmetry breaking in quantum field theory and a natural question is if this can be achieved in the holographic set up by non-supersymmetric deformations of supergravity solutions. A great effort has been done in type IIB context, mainly in case of the Klebanov-Strassler background [124], first in probe approximation [119] and then by a study of the linearized backreaction [16, 13, 14, 62]. A similar analysis has been done in M-theory [123, 11]. Less attention, to our knowledge, has been devoted to type IIA configurations; we propose here to do some steps with the aim of filling this gap and we study non-supersymmetric deformations of a type IIA background which describes regular fractional D2–branes. The solution has two supercharges and thus it is dual to an $\mathcal{N}=1$ gauge theory in 2+1 dimension. We will concentrate mainly on the gravity side of the duality, limiting to few comments about the gauge dual which is not fully understood. We will solve for the linearized space of perturbations around the supersymmetric background of [47] using the technique first proposed in [24]. We will show that our configuration exhibits two main features common to the other analysis [16, 13, 14, 62, 11], namely the force on a probe brane depends on only one supersymmetry breaking perturbation mode and the IR region is affected by some singularities. It is an open question whether this singularities are admissible or not and no clear answer is available, we will show that in the configuration we have chosen to analyze they are more severe than in the other cases.

The thesis is organized as follows:

Chapter 2 contains a review of the main properties of Type II supergravity, its field content and action, equations of motion and supersymmetry variations. We provide our definition of vacuum and set the geometric problem it poses together with the example of Calabi-Yau compactification, mainly to provide the guide line we want to follow. We introduce some geometric notions such as the one of G-structure.

Chapter 3 is more technical an mathematical in nature. It provides a short introduction to Generalized Complex Geometry. We will introduce the concept of generalized tangent bundle and generalized complex structure and introduce a bracket which allows for a proper concept of integrability. We will introduce spinors associated to the generalized bundle and detail their relation with differential forms on the manifold. This will be a key point to derive a set of differential equations for a couple of object which are sum of forms of different degree (polyforms) and that reformulate the supersymmetry conditions. We end the Chapter with a brief description of calibrations and sources in this context.

Chapter 4 contains a quick review on solvmanifolds. These are a class of six dimensional manifolds which have been extensively used as internal manifolds in supersymmetric compactifications [87]. We consider them here for two reasons, one is that we want to reinterpret some well established mathematical result about their compactness properties and the existence of globally defined one-forms in the framework of twist construction as developed in [5], extending their analysis to solvmanifolds which are not nilmanifolds. We do not claim any truly original result, nevertheless it is a useful reformulation that allow us to obtain a supersymmetric solution on a certain solvmanifold, denoted as $G_{5.17}^{p,-p,\pm1} \times S^1$, which we will take as a starting point for the non-supersymmetric analysis of Chapter 5.

Chapter 5 contains our analysis of a non-supersymmetric de Sitter solution. We will start with some considerations about the features such solutions have to posses, with particular emphasis on the description of the sources. We will then provide the details of the solution, which exhibits a non zero F_2 and H fluxes together with a non zero F_0 flux and a negative curvature of the internal manifold, as necessary according to some general analysis. We provide the details of the computation of the energy-momentum tensor for the supersymmetry breaking sources and explain which kind of deformations we need in order to solve Einstein and dilaton equations. We end the Chapter by an analysis of the (meta)stability of our solution from a four dimensional point of view.

Chapter 6 is an attempt of a more formal development of the intuition one can get from the previously analyzed example. Geometry has been a guiding principle in the analysis of supersymmetric configurations, from the first example of Calabi-Yau compactifications till the reformulation in terms of Generalized Complex Geometry for arbitrary flux backgrounds, geometrical structures are intimately related to the dynamics of the problem. Our believe is that also $\mathcal{N} = 0$ vacua (or at least some subset) can be described by geometrical tools. In this Chapter we collect some observations and reorganize them at a more formal level. In particular we try to guess a set of bispinorial variables suitable for a generalization of first order formalism to the non-supersymmetric case, via direct manipulation on the pure spinors equations or via T-duality arguments. We also try to discuss the problem of branes in non-supersymmetric background, whose behavior and description is a priori different from the case of supersymmetric configurations. We will argue about a set of equations relating the brane dynamics with the non-supersymmetric bulk dynamics. The aim of the Chapter is not to provide established results, but to infer that a certain geometrical structure can be developed also for the more challenging problem of non-supersymmetric vacua.

Chapter 7 is devoted to the analysis of the space of supersymmetric and non-supersymmetric first order deformations of the type IIA background discovered in [47]. The ten dimensional manifold in this case is split into a three dimensional warped Minkowski space and a certain seven dimensional, non-compact, asymptotically conical G_2 manifold which is an $\mathbb{R}^3$ bundle over S^4. Generalized Complex Geometry could be used is some similar configurations [98][6] but we choose to use a different approach based on the technique developed in [24] which is suitable for our analysis and we will begin by reviewing it. After recalling the salient properties of the background [47], we will proceed by presenting the solution to the equations which parametrize the linearized deformations. We will then discuss the main features of the IR physics and comment about the singularities we will encounter.

[6]The splitting in $3+7$ dimensions does not make it evident because of the odd dimensionality of the internal manifold. However in presence of a preferred direction (as a radial one) one can decompose $7 = 6 + 1$ and apply Generalized Complex Geometry. We thank A. Tomasiello for pointing this out.

Chapter 8 contains our concluding remarks.

Appendix A collects our conventions and a set of useful formulae.

Appendix B is a short note about T-duality applied to solvmanifolds.

Appendix C contains some definitions and theorems about Lie groups which are extensively used in the main text.

Appendix D collects lists of algebras and related solvmanifolds which have been considered in this thesis, in particular we present them in a basis suitable to the compactness analysis as explained in Chapter 4.

This thesis is based on the following publication and preprint:

1. D. Andriot, E. Goi, R. Minasian, M. Petrini, *Supersymmetry breaking branes on solvmanifolds and de Sitter vacua in string theory*, JHEP, **1105**, 2011, 028, [arXiv:1003.3774].

2. G. Giecold, E. Goi, F. Orsi, *Assessing a candidate IIA dual to metastable supersymmetry-breaking*, [arXiv:1108.1789].

Chapter 2

Type II supergravity

This thesis is devoted to the study of compactifications of type II supergravity. We thus start with a description of the main features of these theories and we provide an explanation of the mathematical techniques we want to apply. The material presented here is now a standard topic of many textbooks [165, 91] and review/lecture notes, [31, 83, 180, 127] among many others. We will follow their logic in the exposition.

2.1 Action and equations of motion

Type II supergravity is the low energy effective theory which describes the dynamics of the massless sector of type II superstring theory. As the string theory from which they derive, type II supergravity comes in two manifestations, know as type IIA and type IIB, according to the chirality of their supersymmetry parameters (type IIA is a non–chiral theory while type IIB is chiral). They are the unique, maximally supersymmetric supergravity theories in ten dimensions; they have $\mathcal{N} = 2$ supersymmetry corresponding to 32 supercharges.[1] We will use the so called "democratic" formulation [18], which incorporates the Romans' mass parameter [169] of type IIA theory, because, as we will see, it is the most suitable for our approach based on generalized complex geometry.

The fermionic sector of the theory includes two Majorana–Weyl gravitinos Ψ^a_M, $a = 1, 2$, they are spin 3/2 fields of opposite chirality in type IIA and same chirality in type IIB. There are also two dilatino fields λ^a, again Majorana–Weyl spinors of spin 1/2 and with opposite chirality than the gravitinos. The supersymmetric parameters ϵ^a have the same chirality as the corresponding gravitinos, there are two of them making the theory $\mathcal{N} = 2$.

The bosonic superpartners include:

- NS–NS sector:
  - a scalar field ϕ: the dilaton;
  - a symmetric 2–tensor g_{MN}: the metric;
  - an antisymmetric 2–tensor B_{MN}: gauge potential of the three–form field H.

[1] We refer to Appendix A.2 for our convention regarding spinors in different dimensions.

- R–R sector:
  - type IIA: gauge fields $C_p^{(10)}$ which are forms of odd degree $p = 1, \ldots, 9$;
  - type IIB: gauge fields $C_p^{(10)}$ which are forms of even degree $p = 0, \ldots, 8$.

We denote with $C^{(10)}$ and $F^{(10)}$ the sum of all the RR potentials and field strengths respectively. The field strengths are given by:

$$F^{(10)} = \mathrm{d}C^{(10)} - H \wedge C^{(10)} + e^B F_0^{(10)} \tag{2.1}$$

where we have introduced the Romans' mass parameter $F_0^{(10)}$, e^B has to be intended through its series expansion where the product is the wedge product. Not all the gauge potentials are independent, thus, to have the right number of degrees of freedom, we impose a Hodge duality condition on the corresponding field strength $F_{p+1}^{(10)}$:

$$F_p^{(10)} = (-)^{\lfloor \frac{p}{2} \rfloor} *_{10} F_{10-p}^{(10)}. \tag{2.2}$$

The Bianchi identities for the form fields are:

$$\mathrm{d}H = 0 \qquad\qquad (\mathrm{d} - H\wedge)\, F^{(10)} = \delta. \tag{2.3}$$

We consider the possibility of having RR sources introducing a δ–function which gives their charge density. We will discuss in Section 3.6 with more detail the kind of sources we will allow for; note that we choose to not include NS5–sources, thus for us the Bianchi identity for H will be always as in (2.3).

The dynamics of these fields is determined by the action which we present here in string frame:

$$S_{10} = \frac{1}{2k_{10}^2} \int_{M_{10}} \mathrm{d}^{10}x \sqrt{|g_{10}|} \left[e^{-2\phi} \left(R_{10} + 4|\nabla\phi|^2 - \frac{|H|^2}{2} \right) - \frac{1}{2} \left(|F_0|^2 + |F_2|^2 + |F_4|^2 \right) \right]. \tag{2.4}$$

We have chosen to present the type IIA version because it is the one we will study mainly in this thesis. Mutatis mutandis type IIB action is easily recovered and most of the results we will state in the following are valid for it too. Note that we use conventions of [18, 36], a difference in the definition of the Hodge dual gives us an extra sign depending on the parity of forms[2]. In our notation we have $2k_{10}^2 = (2\pi)^7(\alpha')^4$, $\alpha' = l_s^2$ and

$$F_k \wedge \hat{*} F_k = \mathrm{d}^{10}x \sqrt{|g_{10}|} \frac{(-)^{(10-k)k}}{k!} F_{\mu_1 \ldots \mu_k} F^{\mu_1 \ldots \mu_k} = \mathrm{d}^{10}x (-)^{(10-k)k} |F_k|^2. \tag{2.5}$$

As customary $|g_{10}|$ denotes the determinant of the ten dimensional metric, $\hat{*}$ denotes the ten dimensional Hodge dual. There are other pieces which complete the action for the bosonic sector: a topological Chern–Simons term and an effective action describing the dynamics of the

[2] See Appendix A.1 for conventions. In IIA, the sign is always positive on RR fields, but not on the odd forms, H and $\mathrm{d}\phi$, hence the sign difference with respect to [36] for the corresponding terms in the action. The sign difference is related to the fact we use the Mukai pairing to give the norm: for a real form α_i, we have $\langle *\lambda(\alpha_i), \alpha_i \rangle = |\alpha_i|^2 \times \mathrm{vol}$.

source terms and their coupling with the other fields. The topological part will not be of interest in our analysis[3] before Chapter 7 and thus we will present it there, the action for the sources will be discussed in Section 3.6 and Chapter 5. There is obviously also an action for the fermionic fields. We do not report it here because we will not need it in the following, the reason will be clear in a while when we will discuss the kind of vacua we are looking for.

We are now ready to present the equations of motion. Einstein and dilaton equation of motion are:

$$
\begin{aligned}
R_{MN} - \frac{g_{MN}}{2} R_{10} = 2 g_{MN} (\nabla^2 \phi - 2|\nabla\phi|^2) - 2\nabla_M \nabla_N \phi + \frac{1}{4} H_{MPQ} H_N{}^{PQ} + \frac{e^{2\phi}}{2} F_{2\ MP} F_{2\ N}{}^P \\
- \frac{g_{MN}}{2} \left(-4|\nabla\phi|^2 + \frac{1}{2}|H|^2 + \frac{e^{2\phi}}{2}(|F_0|^2 + |F_2|^2) \right) + e^{\phi}\, \frac{1}{2} T_{MN}\ , \qquad (2.6)
\end{aligned}
$$

$$
8(\nabla^2\phi - |\nabla\phi|^2) + 2R_{10} - |H|^2 = -e^{\phi} \frac{T_0}{p+1}\,. \qquad (2.7)
$$

Here T_{MN} and T_0 are the source energy momentum tensor and its partial trace (see (5.28) and (5.27)). The equation of motion for the H–field and the RR fields are:

$$
\mathrm{d}\left(e^{-2\phi} \hat{*} H \right) + F_0 \wedge \hat{*} F_2 + F_2 \wedge \hat{*} F_4 + \frac{1}{2} F_4 \wedge F_4 = \text{source term} \qquad (2.8)
$$

$$
(\mathrm{d} + H\wedge)\, (\hat{*} F) = 0 \qquad (2.9)
$$

The theory is supersymmetric and thus we should complete this section by providing the behavior of the fields under a supersymmetry variation. We present explicitly only the variation of the fermionic fields, which will involve bosonic ones, we postpone again our motivations to the next section. The supersymmetry variations are:

$$
\begin{aligned}
\delta\Psi^1_M &= \left(\nabla_M + \frac{1}{4} \not{H}_M \right) \epsilon^1 + \frac{e^{\phi}}{16} \not{F}\, \Gamma_M \Gamma \epsilon^2 \\
\delta\Psi^2_M &= \left(\nabla_M - \frac{1}{4} \not{H}_M \right) \epsilon^2 - \frac{e^{\phi}}{16} \not{F}^{\dagger}\, \Gamma_M \Gamma \epsilon^1 \\
\Gamma^M \delta\Psi^1_M - \delta\lambda^1 &= \left(\not{\nabla} - \not{\partial}\phi - \frac{1}{4} \not{H} \right) \epsilon^1 \\
\Gamma^M \delta\Psi^2_M - \delta\lambda^2 &= \left(\not{\nabla} - \not{\partial}\phi + \frac{1}{4} \not{H} \right) \epsilon^2
\end{aligned} \qquad (2.10)
$$

A slash means contraction with a gamma matrix[4].

2.2 Compactification ansatz

We are interested in solutions of the theory such that the ten dimensional space–time M_{10} is fibered over a four dimensional base M_4. We are looking for the most simple configuration:

[3]See Footnote 5 in Chapter 5.

[4]See Appendix A.1 for our conventions.

a vacuum. In our definition it is a solution where there are no particles in four dimensions, we thus require M_4 to admit the maximum possible number of Killing vectors, namely to be maximally symmetric. It is possible to prove [185] that there are only three four–dimensional spaces with this characteristic: AdS_4, Mink_4 and dS_4. To preserve the maximal symmetry of the external space we have to require the fibration to be trivial, as a consequence the most general ten dimensional metric turns our to be a warped product of a fully constrained (by the maximal symmetry) four–dimensional part and an unconstrained six–dimensional part:

$$\mathrm{d}s_{10}^2 = e^{2A} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} \mathrm{d}y^m \mathrm{d}y^n. \tag{2.11}$$

The internal metric g_{mn} and the warping factor A are functions of the internal coordinates y^m only. Gamma matrices and spinors decompose accordingly, we detail more about this point in Appendix A.2. Here we need only the following consideration. The metric we choose in ten dimensions is always of signature $(1, 9)$ and thus under the ansatz (2.11) the spinors reduce as $\mathrm{Spin}(1, 9) \to \mathrm{Spin}(1, 3) \times \mathrm{Spin}(6)$. This reduction does not contain an invariant under $\mathrm{Spin}(1, 3)$, thus if we were to allow for an non zero expectation value of a spinorial field we would have had a breaking of four dimensional maximal symmetry. This forces us to impose the vanishing of all the expectation values of the fermionic fields in the vacuum.

The compactification ansatz imposes constraints also on the other bosonic fields of the theory. First of all, they can have a coordinate dependence with respect to the internal ones only. The H field can have only internal indexes: H_{mnp}. The RR fluxes decompose as follows:

$$F_{(n)}^{(10)} = F_{(n)} + e^{4A} \mathrm{vol}_4 \wedge \tilde{F}_{(n-4)}. \tag{2.12}$$

Here $F_{(n)}$ and $\tilde{F}_{(n-4)}$ denote purely internal forms, vol_4 denotes the unwarped four-dimensional volume and the duality relation (2.2) reads:

$$\tilde{F} = \lambda(*F) \tag{2.13}$$

with $*$ the six dimensional Hodge duality operator. Accordingly also the gauge potentials get decomposed as:

$$C_{(n)}^{(10)} = C_{(n)} + \mathrm{d}x^{0123} \wedge e^{4A} \tilde{C}_{(n-4)}. \tag{2.14}$$

2.3 Supersymmetry and geometry

In the previous Section we have introduced the definition of vacuum we will use in this thesis. In addition we can require that the solution preserves some fraction of the original supersymmetry, that is to say we must impose that the supersymmetry variations (2.10) are zero for at least some of the supersymmetry generators (supercharges). We thus obtain a set of first order equations among the fields of the theory. In principle there is also a set of equations coming from the supersymmetry variation of the bosonic fields, however these will always involve a fermionic field. We have said that, according to our definition of vacuum, we are forced to put to zero all their expectation values and thus the supersymmetry variations for the bosonic fields are automatically zero.

The are many reason to look for supersymmetric solutions, among them there is the fact that, under certain mild assumptions, it is enough to solve the Killing spinor equations (2.10) and

Bianchi identities (2.3) to have a solution that satisfies all the equations of motion. We will discuss this in more detail in Section 3.5. A second reason is phenomenology, considerations related to the hierarchy problem set the susy breaking scale at a much lower energy than the compactification case. Of course non supersymmetric solutions are of extreme importance, both for obvious phenomenological reasons and for the understanding of fundamental properties of the theory and they will be the main interest of this thesis. Needless to say, in order to explore less known configurations it is of extreme heuristic usefulness to start from well known ones and try to deform them. It is also of clear evidence the importance to have a known limit where to check our results. For these reasons we have chosen to discuss the supersymmetry case before addressing non supersymmetric solutions in Chapter 5 and Chapter 7.

We are interested in supersymmetric solutions with $\mathcal{N}=1$ in four dimensions, that is to say with four conserved supercharges. The most general decomposition of the supersymmetry parameters ϵ^a, compatible with $\mathcal{N}=1$ and our definition of a vacuum, is (see [180] for a detailed discussion about this point):

$$\begin{aligned}\epsilon^1 &= \zeta_+ \otimes \eta_+^{(1)} + \zeta_- \otimes \eta_-^{(1)} \\ \epsilon^2 &= \zeta_+ \otimes \eta_\mp^{(2)} + \zeta_- \otimes \eta_\pm^{(2)}\end{aligned} \tag{2.15}$$

where the upper sign is for type IIA and the lower for type IIB. We can now use this decomposition in the equations (2.10), again maximal symmetry of the external space imposes that solutions of (2.10) have to be valid for an arbitrary ζ. We can now restrict to AdS_4 or Mink_4 spaces. There are two reasons, one is that dS_4 is not compatible with supersymmetry (see Footnote 1 in Chapter 5 for a discussion). The other reason is related to the existence of Killing spinors for maximally symmetric spaces [136], for Minkowski space we can find a basis such that $\nabla_\mu \zeta_\pm = 0$ and for anti de Sitter spaces such that $\nabla_\mu \zeta_- = \frac{1}{2}\mu\gamma_\mu\zeta_+$, where μ is related to a negative cosmological constant $\Lambda = -|\mu|^2$, while for de Sitter space there is no such a basis. Taking into account these considerations one can see that equations (2.10) reduce to:

$$\begin{aligned}
&\left(\nabla_m - \frac{1}{4}\not{H}_m\right)\eta_+^{(1)} \mp \frac{e^\phi}{8}\not{F}\gamma_m\eta_\mp^{(2)} = 0\,,\\
&\left(\nabla_m + \frac{1}{4}\not{H}_m\right)\eta_\mp^{(2)} - \frac{e^\phi}{8}\not{F}^\dagger\gamma_m\eta_+^{(1)} = 0\,,\\
&\mu e^{-A}\eta_-^{(1)} + \not{\partial}A\eta_+^{(1)} - \frac{1}{4}e^\phi\not{F}\eta_\mp^{(2)} = 0\,,\\
&\mu e^{-A}\eta_\pm^{(2)} + \not{\partial}A\eta_\mp^{(2)} - \frac{1}{4}e^\phi\not{F}^\dagger\eta_+^{(1)} = 0\,,\\
&2\mu e^{-A}\eta_-^{(1)} + \not{\nabla}\eta_+^{(1)} + \left(\not{\partial}(2A-\phi) + \frac{1}{4}\not{H}\right)\eta_+^{(1)} = 0\,,\\
&2\mu e^{-A}\eta_\pm^{(2)} + \not{\nabla}\eta_\mp^{(2)} + \left(\not{\partial}(2A-\phi) - \frac{1}{4}\not{H}\right)\eta_\mp^{(2)} = 0\,.
\end{aligned} \tag{2.16}$$

These equations involve only quantities which are defined on the internal manifold, they give a set of constraints that will determine the geometry of M_6. The fact that supersymmetry conditions translate to geometric conditions which determine the internal manifold is known since decades, the result is valid also for the other string theories and since the celebrated work

[32] there has been a huge effort to investigate its far reaching consequences. The most fruitful geometrical technique has been the one based on G–structures. A not exhaustive list of references for its application on type II context can be [72, 34, 74, 121, 122, 51, 70, 69, 50, 10] and references therein.

Let us consider the definition of G–structure [43, 71, 116, 171].

Definition. *A G–structure is a principal subbundle of the frame bundle with fiber $G \subset \mathrm{Gl}(d, \mathbb{R})$.*

Stated in another way, when we transform objects from one patch to another, we allow for transition functions which are elements of a subgroup $G \subset \mathrm{Gl}(d, \mathbb{R})$. The logic behind the use of G–structures to describe physical situations is to interpret the field content of a theory as a topological data for the structure and to relate the equations they obey to some integrability condition. Once the correspondence is stated one can exploit representation theory to look for solutions, a problem which is usually easier than to directly address the field equations.

A simple example is provided by a Riemannian metric g, on each patch we can choose frames such that, in frame indexes, g is simply the identity. It defines a reduction to $\mathrm{O}(d, \mathbb{R})$ which is clearly the group of transformations that leave invariant the metric. From this simple example we can see a way to characterize G-structures, namely they are given in terms of globally defined, non–degenerate G–invariant objects, which can be tensors, as we have just seen, or spinors. The simplest application to type II supergravity gives the well know Calabi–Yau geometry which we review here as an example of the approach we want to follow. Let us consider a configuration where there are no fluxes and we set to zero dilaton and warp factor. We look for supersymmetric solutions with $\eta^{(1)} = \eta^{(2)} = \eta$ [5] and an external Minkowski space. We need a metric and thus the structure group is reduced to $\mathrm{O}(6)$. Supersymmetry requires spinors and thus the manifold has to be a spin manifold, in particular it has to be orientable, reducing the structure group to the orientation preserving $\mathrm{SO}(d, \mathbb{R})$[6]. Moreover the spinor η entering the supersymmetry transformations has to be globally defined and nowhere vanishing and thus we can further reduce the structure group to the stabilizer of η in $\mathrm{SO}(6)$ which is $\mathrm{SU}(3)$. This last piece of information can be translated into the existence on M_6 of globally defined non–degenerate two form J_{mn} and three form Ω_{mnp} together with some compatibility condition[7]. They define an almost symplectic structure and an almost complex structure. Up to now we have only topological information. The supersymmetry equations (2.16) reduce to simply:

$$\nabla_m \eta = 0 \tag{2.17}$$

and provide a differential condition which translates into integrability conditions for J and Ω. In fact one can prove that (2.17) is equivalent to

$$\mathrm{d}J = 0 \qquad\qquad \mathrm{d}\Omega = 0 \tag{2.18}$$

We immediately recognize the condition for a symplectic structure to be integrable, namely $\mathrm{d}J = 0$ and after a little bit of work one can see that if $\mathrm{d}\Omega = 0$ then the Nijenhuis tensor of the

[5]This case admits $\mathcal{N} = 2$ non–minimal supersymmetry because in (2.15) we can choose two different ζ's. In the case of $F \neq 0$ we are forced to chose the same ζ's because F mixes ϵ^1 and ϵ^2 in (2.10).

[6]The manifold has to be spin in order to lift $\mathrm{SO}(d, \mathbb{R})$ to its universal cover $\mathrm{Spin}(d, \mathbb{R})$ which admits spinor representations. It is a very well known result in differential geometry that a spin structure exists if and only if the second Stiefel–Whitney class of the tangent bundle vanishes [132].

[7]We refer to Appendix A.3 for their definition and properties.

associated almost complex structure vanishes[8], making it a complex structure. The condition (2.17), which we recall is equivalent to (2.18), is well known, in fact it says that the holonomy of M_6 is contained in SU(3) because the spinor is invariant under parallel transport, which is a way to define a Calabi–Yau manifold [90].

As a last example of the method we can see how supersymmetry implies the equations of motion (see for example [91] vol. 2, ch. 15). In this simple case we are left with the Einstein equation (2.6) which reduces to

$$R_{mn} - \frac{1}{2} g_{mn} R = 0. \tag{2.19}$$

It is an immediate consequence of (2.17) that:

$$0 = [\nabla_m, \nabla_n]\eta_+ = \frac{1}{4} R_{mnpq}\gamma^{pq}\eta_+, \tag{2.20}$$

where R_{mnpq} is the Riemann tensor. Multiplying (2.20) by γ^n and using (A.8) we obtain:

$$R_{mq}\gamma^q\eta_+ + \frac{1}{4} R_{mnpq}\gamma^{npq}\eta_+ = 0. \tag{2.21}$$

The second term in the equation is zero by symmetry properties of the Riemann tensor, we are thus left with $R_{mq} = 0$ which is clearly a solution of (2.19). We recover the well know property of Ricci flatness which is a characteristic of Calabi–Yau manifolds.

We have discussed which kind of geometrical tools we need to find solutions of type II supergravity, the procedure is well defined and in principle one could try to extend it to more complicated configurations in which we consider non vanishing fluxes or SU(2) structures. For example one can look for solutions with, e.g., NS–NS flux only (see [178] in the context of heterotic strings) also known as type A solutions or to type B solutions in type IIB string theory with NS–NS 3-form, RR 5–form and 3–form among which there is the class of conformally Calabi–Yau solutions which is the simplest deformation from the Calabi–Yau configuration. However the supersymmetry equations become rapidly intricate and we soon lose a nice geometrical interpretation. It turns out that a more suitable language is given by generalized complex geometry which we are going to review in the next chapter.

[8] Actually this is not an if and only if condition. The vanishing of the Nijenhuis tensor is equivalent to the existence of a form W_5 such that $\mathrm{d}\Omega = W_5 \wedge \Omega$. Clearly $\mathrm{d}\Omega = 0$ is a subcase of it. In this case the canonical bundle K_I admits a holomorphic section.

Chapter 3

Generalized Complex Geometry

In the previous Chapter we have seen how geometry enters the description of supersymmetric vacua in type II string theory and how supersymmetry conditions translate into geometric conditions for the internal manifold which, in the simplest case of fluxless supersymmetric configurations, has to be a Calabi–Yau.

In this chapter we review the fundamentals of generalized complex geometry and explain why this is a suitable language to address the more complicated problem of compactification with fluxes and sources. The Killing spinor equations (2.10), when fluxes are turned on, are much more complicated to analyze, but, as we will see, generalized complex geometry will give them a clear geometric meaning and will allow us to reformulate the problem into one involving differential forms on the internal manifold which are more tractable mathematical objects than spinors.

Generalized complex geometry has been developed on the mathematical side by Hitchin [106] and collaborators [95, 96], its application to string theory has been almost concurrently, see [85, 86] or the later [115] from a more abstract point of view. Here, without any pretension of completeness or originality in the exposure, we recall the main facts about it and we refer the reader to the previously cited original works or to one of the many available reviews or lecture notes [108, 38, 180, 189, 127] and references therein for a more comprehensive development of the subject.

3.1 Generalized tangent bundle

Let us consider a d–dimensional manifold M; many of the following definitions and results are valid for any even d but the reader should keep in mind that we will soon specialize to the case $d = 6$ which is, as we have seen in Chapter 2, the case for compactification to four dimensions of type II theories.

Generalized complex geometry describes structures on an extension of the tangent bundle TM by the cotangent bundle T^*M. The extension, called generalized tangent bundle, is defined by the short exact sequence:

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0\,. \tag{3.1}$$

Locally its sections (generalized vectors) are written as $\mathbb{X} = X + \xi$ with $X \in TM$ and $\xi \in T^*M$.

Passing from one patch U_α to another U_β the elements of a section are glued as follows:

$$X_{(\alpha)} + \xi_{(\alpha)} = a_{(\alpha\beta)} X_{(\beta)} + \left(a^{-t}_{(\alpha\beta)} \xi_{(\beta)} + \iota_{a_{(\alpha\beta)} X_\beta} \omega_{(\alpha\beta)} \right) \tag{3.2}$$

where $a_{(\alpha\beta)} \in \mathrm{Gl}(d, \mathbb{R})$ gives the usual patching of vector and one forms, $\omega_{(\alpha\beta)}$ is a two–form and $\iota_v \alpha$ denotes the contraction of a form α by the vector v[1]. We further require that $\omega_{(\alpha\beta)} = -\mathrm{d}A_{(\alpha\beta)}$ with $A_{(\alpha\beta)} \in \Omega^1(U_\alpha \cap U_\beta)$ one–forms satisfying:

$$A_{(\alpha\beta)} + A_{(\beta\gamma)} + A_{(\gamma\alpha)} = g^{-1}_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \tag{3.3}$$

on threefold intersections $U_\alpha \cap U_\beta \cap U_\gamma$. The functions $g_{(\alpha\beta\gamma)} = e^{i\alpha}$ are S^1 valued functions which satisfy a cocycle condition on fourfold intersections and define a *gerbe* [105, 107, 26, 79]. This construction is reminiscent of the patching in U(1) bundles but now "the connective structure" of a gerbe given by the one–forms $A_{(\alpha\beta)}$ have the role of "transition" functions. This is the rigorous mathematical structure one needs in order to give the NS–NS two–form B a geometric interpretation. It will play the role which was of the U(1) connection one–form and it has a correspondingly quantized (on compact manifolds) globally defined three–form field strength $H = \mathrm{d}B$. One the overlaps the B–field is patched as $B_{(\alpha)} = B_{(\beta)} - \mathrm{d}A_{(\alpha\beta)}$. The transition functions which define E have the form

$$t_{(\alpha\beta)} = \mathcal{A}_{\alpha\beta} \circ e^{\omega_{(\alpha\beta)}} \tag{3.4}$$

$$\mathcal{A}_{(\alpha\beta)} = \begin{pmatrix} a_{(\alpha\beta)} & 0 \\ 0 & a^{-t}_{(\alpha\beta)} \end{pmatrix} \in \mathrm{Gl}(d)_+ \subset \mathrm{SO}(d,d)_+ \, . \tag{3.5}$$

In the following we will consider a local patch and drop related subscripts in most of the objects we define; one has to keep in mind that by carefully gluing we can extend the results all over the manifold. A local trivialization of the bundle E will look like the direct sum $TM \oplus T^*M$ with elements $\mathbb{X} = X + \xi$. It is endowed with a natural bilinear form of indefinite signature (d, d) given by the pairing of vectors and one–forms:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left(\xi(Y) + \eta(X) \right) \, , \tag{3.6}$$

which is left invariant by the non–compact orthogonal group $\mathrm{O}(d,d)$[2]. If we adopt a two–component notation the metric given by the pairing (3.6) can be written as:

$$\mathcal{I} = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3.7}$$

A generic element $O \in \mathrm{O}(d,d)$ will act on sections $\mathbb{X} = X + \xi$ as:

$$\mathbb{X} = \begin{pmatrix} X \\ \xi \end{pmatrix} \to O\mathbb{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.8}$$

[1]See Appendix A for conventions.

[2]Note that usually we can make a natural choice for the orientation once we identify $\Lambda^{2d}(TM \oplus T^*M) \equiv \mathbb{R}$ which allows to reduce the $\mathrm{O}(d,d)$ group to its special subgroup $\mathrm{SO}(d,d)$, [95].

where the $d \times d$ matrices a, b, c, d are forced by the orthogonal condition to satisfy $a^t c + c^t a = 0$, $b^t d + d^t b = 0$ and $a^t d + c^t b = \mathbb{1}$. The Lie algebra can be decomposed as $\mathfrak{o}(d,d) = \mathrm{End}(TM) \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM$, consequently its elements[3] are of the type:

$$G = \begin{pmatrix} a & \beta \\ B & -a^t \end{pmatrix} \tag{3.9}$$

where $a \in \mathrm{End}(TM)$, $B : TM \to T^*M$ and $\beta : T^*M \to TM$ are skew–symmetric, i.e. they can be interpreted as a two–form and a two–vector respectively. By exponentiation we can obtain the embedding of three subgroups of $\mathrm{O}(d,d)$ namely:

- $\mathrm{Gl}(d)$ subgroup

$$\mathbb{X} \to \mathbb{X}' = \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.10}$$

  whose action on $\mathbb{X}$ is as usual for vectors and one-forms;

- G_B (Abelian) subgroup

$$\mathbb{X} \to \mathbb{X}' = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.11}$$

  where B acts by shearing in the T^*M direction, $\mathbb{X}' = \mathbb{X} + \imath_X B$;

- G_β (Abelian) subgrup

$$\mathbb{X} \to \mathbb{X}' = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.12}$$

  where β acts by shearing in the TM direction, $\mathbb{X}' = \mathbb{X} + \imath_\xi \beta$.

3.2 Generalized (almost) complex structures

Let us consider the complexification[4] of the generalized tangent bundle $(TM \oplus T^*M) \otimes \mathbb{C}$, as we would do in the case of the ordinary tangent bundle. We want to introduce a structure which is the direct generalization of an (almost) complex structure.

Definition. *A generalized almost complex structure is a map $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = -1$ and satisfies an Hermiticity condition, $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$, with respect to the metric $\mathcal{I}$ in* (3.7).

In presence of $\mathcal{J}$ the structure group of $TM \oplus T^*M$ is further reduced to the stabilizer of $\mathcal{J}$ in $\mathrm{O}(d,d)$ which is $\mathrm{U}(d/2, d/2)$[5].

[3] By abuse of notation we use the same symbol for the Lie algebra element or the corresponding group element.

[4] In what follows all the bundles we will meet are complex. We will denote with the same symbol both the real bundle and its complexification.

[5] As usual we consider the real representation of $\mathrm{Gl}(d/2, \mathbb{C})$ into $\mathrm{Gl}(d, \mathbb{R})$ given by:

$$A = \mathrm{Re}(A) + i\,\mathrm{Im}(A) \to \begin{pmatrix} \mathrm{Re}(A) & \mathrm{Im}(A) \\ -\mathrm{Im}(A) & \mathrm{Re}(A) \end{pmatrix} .$$

As for the (almost) complex case, we can always split the complexified generalized tangent bundle into its "$(1,0)$" and "$(0,1)$" components. The $+i$ eigenbundles of $\mathcal{J}$ are given by generalized vectors satisfying the condition $\Pi\,\mathbb{X} = \mathbb{X}$, where the projector Π is defined as:

$$\Pi = \frac{1}{2}(1 - i\mathcal{J}) \tag{3.13}$$

It is easy to see that such subbundle is isotropic (or null) with respect to the metric (3.7) and has dimension d, which is the maximal one for isotropic subspaces in signature (d, d).

We state here some general facts about maximally isotropic subbundles, their proof can be found in [95]. As an example of maximally isotropic subspace consider any subspace $V \subset TM$, it is easy to see that the subspace

$$V \oplus \mathrm{Ann}(V) \subset TM \oplus T^*M\,, \tag{3.14}$$

where $\mathrm{Ann}(V)$ is the annihilator space of V in T^*M, is a maximally isotropic subspace. We can also give the general form of maximally isotropic subspaces. Consider as before any subspace $V \subset TM$ and let $\omega \in \Lambda^2 V^*$, one can prove that every maximally isotropic subspace is of the form:

$$L(V,\omega) = \{X + \xi \in V \oplus T^*M \mid \xi|_{V^*} = \omega(X)\}\,. \tag{3.15}$$

We define the *type* of a maximal isotropic subspace $L(V,\omega)$ as the codimension k of its projection onto TM. We will see later how this is related to the correspondent concept for a pure spinor. Among the subgroups of $\mathrm{O}(d,d)$ previously described the B–transformations do not change the type on the other hand β–transformations do.

3.2.1 The Courant bracket

In the (almost) complex case there is a definition of integrability given in terms of Lie bracket. An almost complex structure is said to be integrable if the holomorphic subbundle $T^{(1,0)}M$ is closed under the Lie bracket, i.e.: $[T^{(1,0)}M, T^{(1,0)}M]_{\mathrm{Lie}} \subset T^{(1,0)}M$.

We want to extend this notion to the generalized case and thus we need a bracket on (smooth) sections of $TM \oplus T^*M$ which will replace the Lie bracket in the definition of an integrability condition. There is no non–trivial bracket satisfying the Jacobi identity on $TM \oplus T^*M$, but it is possible to define one that satisfies it when restricted to isotropic subbundles. It is the so called *Courant bracket*, it has been first introduced in [45, 46] but was also present in the work [58][6]. We will introduce it as a derived bracket (see [131]). As a remark, we point out that the mathematics of the Courant bracket can be understood in a more deep way when considered in the more general framework of Lie algebroid theory [166, 141, 134]; in what follows we will not need such a sophisticate mathematical insight and we will keep a more pedestrian approach. We recall here Cartan formulae relating Lie derivative $\mathcal{L}$, exterior derivative d, and interior product ι_x:

$$\mathcal{L}_x = \{\iota_x, \mathrm{d}\} \qquad \mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y] \qquad [\mathcal{L}_x, \mathrm{d}] = 0 \tag{3.16}$$

In particular we are interested in the definition of the Lie bracket as a derived bracket:

$$[\{\iota_x, \mathrm{d}\}, \iota_y] = \iota_{[x,y]_{\mathrm{Lie}}}. \tag{3.17}$$

[6]Actually the Courant bracket is none but the anti–symmetrization of the Dorfman bracket.

The brackets on the left–hand side are commutators or anti–commutators of operators acting on differential forms, while the bracket on the right–hand side is the Lie bracket. The Courant bracket is defined by analogy:

$$\frac{1}{2}\Big([\{\mathbb{X}\cdot,\mathrm{d}\},\mathbb{Y}\cdot]-[\{\mathbb{Y}\cdot,\mathrm{d}\},\mathbb{X}\cdot]\Big)\equiv[\mathbb{X},\mathbb{Y}]_{\mathrm{Courant}}. \tag{3.18}$$

$\mathbb{X}$ and $\mathbb{Y}$ are sections of $TM\oplus T^*M$ and considered as operators on a differential form ω they act as: $\mathbb{X}\cdot\omega=\iota_X\omega+\xi\wedge\omega$.

The Courant bracket can be expressed as:

$$[X+\xi,Y+\eta]_{\mathrm{Courant}}=[X,Y]_{\mathrm{Lie}}+\mathcal{L}_X\eta-\mathcal{L}_Y\xi-\frac{1}{2}\mathrm{d}(\iota_X\eta-\iota_Y\xi). \tag{3.19}$$

It reduces to the Lie bracket on vector fields (which are a maximally isotropic subbundle of the generalized tangent bundle) on the contrary it vanishes on one–forms (which also are a maximally isotropic subbundle). Both the inner product (3.7) and the Courant bracket are invariant under diffeomorphisms, as well as the Lie bracket is, but they have an additional symmetry with respect to the latter. By simple application of definitions and Cartan formulae one can prove that transformations described in (3.11) are automorphisms of the Courant bracket if and only if $\mathrm{d}B=0$. Moreover it is possible to prove [95] that the semidirect product

$$\mathrm{Gl}(d,\mathbb{R})\rtimes\Omega^2_{closed}(M) \tag{3.20}$$

of diffeomorphisms and B–transformations is the group of orthogonal automorphisms of the Courant bracket. These results are valid for a complex two–form, later we will restrict to real forms when we will identify B with the NS–NS potential of string theory.

We have said that the Courant bracket does not satisfy the Jacobi identity on $TM\oplus T^*M$; an expression for the Jacobiator (which quantifies that failure) is given in [95] (see also [134]):

$$\begin{aligned}\mathrm{Jac}(\mathbb{X},\mathbb{Y},\mathbb{Z})&=[[\mathbb{X},\mathbb{Y}],\mathbb{Z}]+[[\mathbb{Y},\mathbb{Z}],\mathbb{X}]+[[\mathbb{Z},\mathbb{X}],\mathbb{Y}]\\&=\mathrm{d}(\mathrm{Nij}(\mathbb{X},\mathbb{Y},\mathbb{Z})).\end{aligned}$$

Thus the Jacobi identity is satisfied up to an exact term given by the Nijenhuis operator:

$$\mathrm{Nij}(\mathbb{X},\mathbb{Y},\mathbb{Z})=\frac{1}{3}\left(\langle[\mathbb{X},\mathbb{Y}],\mathbb{Z}\rangle+\langle[\mathbb{Y},\mathbb{Z}],\mathbb{Z}\rangle+\langle[\mathbb{Z},\mathbb{X}],\mathbb{Y}\rangle\right). \tag{3.21}$$

From this expression it is clear that the Jacobiator is zero on isotropic subbundles.

It is possible to twist the Courant bracket by a closed 3–form H [174, 95]. Let us define the new bracket:

$$[X+\xi,Y+\eta]_H=[X+\xi,Y+\eta]+\iota_X\iota_YH. \tag{3.22}$$

One can compute again the Nijenhuis tensor and the Jacobiator:

$$\mathrm{Nij}_H(\mathbb{X},\mathbb{Y},\mathbb{Z})=\mathrm{Nij}(\mathbb{X},\mathbb{Y},\mathbb{Z})+H(X,Y,Z)$$

$$\mathrm{Jac}_H(\mathbb{X},\mathbb{Y},\mathbb{Z})=\mathrm{d}(\mathrm{Nij}(\mathbb{X},\mathbb{Y},\mathbb{Z}))+\iota_X\iota_Y\iota_Z\mathrm{d}H\,.$$

If $\mathrm{d}H=0$ then the twisted bracket has the same properties as the untwisted one: the Jacobi identity vanish up to exact terms and its Jacobiator is zero on isotropic subbundles. As before B–transforms are a symmetry of the bracket only if $\mathrm{d}B=0$, in fact:

$$[e^B(\mathbb{X}),e^B(\mathbb{Y})]_H=e^B[\mathbb{X},\mathbb{Y}]_{H+\mathrm{d}B}. \tag{3.23}$$

We will not enter in more detail the relation of the twisted bracket and the gerbe construction sketched at the beginning of this chapter, we readdress the interested reader to [95]. It is worth noticing that the structure group of the "twisted" generalized tangent bundle E, as we have introduced it at the beginning of Section 3.1, is actually the group of symmetries of the Courant bracket and not the whole $\mathrm{O}(d,d)$ group which is the symmetry group of the metric (3.7). In particular β transformations introduced in (3.12) are not in this subgroup. It has been suggested that they are related to T–duality and non–geometric constructions [5, 88, 89, 110, 111, 64, 99].

After this detour on the Courant bracket and its properties it is easy to generalize the notion of integrability:

Definition. *A generalized almost complex structure $\mathcal{J}$ is said to be integrable if its $+i$ eigenbundle $L_{\mathcal{J}}$ is closed under the Courant bracket*

$$\bar{\Pi}[\Pi(X+\xi),\Pi(Y+\eta)]_{\text{Courant}}=0, \tag{3.24}$$

where the projector Π *has been defined in* (3.13).

As an example we provide the embedding of the usual complex and symplectic structures in a generalized one. Generalized geometry thus encompasses both complex and symplectic geometries but contains more structures which can be thought as "interpolating" between the two. Using the two–component notation we consider the following generalized complex structures:

$$\mathcal{J}_I=\begin{pmatrix} I & 0\\ 0 & -I^t\end{pmatrix} \qquad\qquad \mathcal{J}_\omega=\begin{pmatrix} 0 & -\omega^{-1}\\ \omega & 0\end{pmatrix}, \tag{3.25}$$

where I is an almost complex structure on TM and ω is a non–degenerate two–form. It is immediate to check that they square to minus the identity and it is also possible to show that requiring the integrability condition (3.24) forces I to be a complex structure and ω to be closed and thus a symplectic form.

Examples of manifolds admitting generalized complex structures but not complex or symplectic ones have been first found on nilmanifolds (the next chapter will be dedicated to the class of solvmanifolds which contains the one of nilmanifolds, we refer to it for more details) in the work [37] by Cavalcanti and Gualtieri. In [95] there is also an example, based on a hyperkähler manifold, of a generalized complex structure which depends on a real parameter and interpolates between a complex and a symplectic structure.

3.3 Spinors and differential forms

In this section we introduce the spin bundle associated to the generalized tangent bundle, provide its relation with the vector space of differential forms $\Lambda^\bullet(T^*M)$ on the manifold M together with some properties. We will see how pure spinors are related to generalized complex structures and how the integrability condition described in the previous section has a translation in terms of a differential condition on forms.

Let us consider a vector space V and a quadratic form Q on it. From this data we can construct the associated Clifford algebra defined by the relation:

$$v\cdot u=(v,u)\,1 \qquad \forall v,u\in V, \tag{3.26}$$

where $(,)$ is the symmetric bilinear form associated to the quadratic form Q[7] and 1 is the unit element of the algebra. We call spinors the elements of the Clifford module, that is to say the elements of the irreducible representation space of the Clifford algebra (see [42, 132]).

As we have seen the generalized tangent bundle is equipped with the symmetric bilinear form (3.7) and thus we can considered the associated Clifford algebra $\mathrm{Cliff}(TM \oplus T^*M)$. There is a natural representation on the space $\Lambda^\bullet(T^*M)$ where the Clifford action is defined as:

$$\mathbb{X} \cdot \phi = \iota_X \phi + \xi \wedge \phi \tag{3.27}$$

for $\mathbb{X} = X + \xi \in TM \oplus T^*M$ and $\phi \in \Lambda^\bullet(T^*M)$. By easy manipulation of wedge product and contraction on forms we can convince ourself that this is actually an algebra representation. This is the standard spin representation and due to the signature (d, d) the spinor bundle S splits into positive and negative helicity, which correspond to even respectively odd degree differential forms:

$$S = S^+ \oplus S^- = \Lambda^{\mathrm{even}}(T^*M) \oplus \Lambda^{\mathrm{odd}}(T^*M). \tag{3.28}$$

This splitting is preserved by the spin group $\mathrm{Spin}(d, d)$ sitting in the Clifford algebra[8] and $S^\pm$ are two Weyl irreducible representations. Actually we need to be more careful. The Lie algebras $\mathfrak{spin}(d, d)$ and $\mathfrak{so}(d, d)$ are isomorphic and thus we can ask how the transformations (3.10), (3.11), (3.12) act on the spin representation [106, 95, 115, 186]. It is rather straightforward to compute the action of B and β transformations:

$$e^{-B} \cdot \phi = \left(1 - B\wedge + \frac{1}{2} B \wedge B \wedge + \dots\right)\phi \qquad e^{\beta} \cdot \phi = \left(1 + \iota_\beta + \frac{1}{2}\iota_\beta^2 + \dots\right)\phi \tag{3.30}$$

For $a \in \mathrm{Gl}_+(d)$ a careful computation gives:

$$a \cdot \phi = \sqrt{\det a}\, e^{-a^m{}_n \,\mathrm{d}x^n \wedge \iota_m} \phi \tag{3.31}$$

Thus as a $\mathrm{Gl}_+(d)$-module we are led to identify the following isomorphism:

$$S^\pm \cong \Lambda^{even/odd} T^*M \otimes \sqrt{\Lambda^d TM}. \tag{3.32}$$

Thus spinors are isomorphic to the tensor product of differential forms and the square root of the d–vectors line bundle, this will be important later when we will explain the connection with the NS–NS massless sector of type IIA/B superstrings.

We can state all of this in a more familiar language, the gamma matrices acting on spinors (here polyforms, i.e. sum of forms of different degree) are given by vectors acting by contraction and one–forms acting by wedge products. If we consider as a basis ι_m and $\mathrm{d}x^m\wedge$ then the usual commutation relations for the metric (3.7) are easily recovered:

$$\{\mathrm{d}x^m\wedge, \mathrm{d}x^n\wedge\} = 0 \qquad \{\mathrm{d}x^m, \iota_n\} = \delta^m_n \qquad \{\iota_m, \iota_n\} = 0. \tag{3.33}$$

[7] For $u, v \in V$, given the quadratic form Q, the symmetric bilinear form $(,)$ is defined as $(u, v) \equiv Q(u + v) - Q(u) - Q(v)$.

[8] The spin group $\mathrm{Spin}(d, d)$ is given by the even degree elements of the Clifford algebra:

$$\mathrm{Spin}(d, d) = \{\mathbb{X}_1 \cdot \ldots \mathbb{X}_{2l} \,|\, \langle \mathbb{X}_i, \mathbb{X}_i\rangle = \pm 1\}. \tag{3.29}$$

There is a standard way to define a $\mathrm{Spin}(d,d)_+$–invariant inner product on $\Gamma(S)$ (see [42]) which for the case of the generalized tangent bundle coincides with the Mukai pairing[9] for differential forms [156]:

$$\langle \phi, \sigma \rangle = (\phi \wedge \lambda(\sigma))_d \,, \tag{3.34}$$

where ϕ and σ are differential forms and λ is an operator which act reversing all the indexes of a form $\lambda(\sigma_p) = (-)^{\mathrm{Int}[p/2]}\sigma_p$ (p is the degree of the form). For $d = 6$ the Mukai pairing is antisymmetric. We define the norm of a spinor $\Phi \in S$ as follows:

$$\langle \Phi, \bar{\Phi} \rangle = -i||\Phi||^2 \mathrm{vol} \tag{3.35}$$

thus once we choose a volume form vol on M we can define the norm of a spinor as the proportionality constant in (3.35). Another definition we need is that of null or annihilator space $L_\Phi \subset TM \oplus T^*M$ of a spinor:

$$L_\Phi = \{\mathbb{X} = X + \xi \in TM \oplus T^*M \,|\, (X + \xi) \cdot \Phi = 0\} \,. \tag{3.36}$$

From the definition of the Clifford action it is straightforward to see that this is an isotropic subspace and if it is of maximal dimension d then Φ is called pure spinor.

We are now ready to state the correspondence between generalized complex structures $\mathcal{J}$ and pure spinors Φ. In Section 3.2 we said that the $+i$–eigenbundle of $\mathcal{J}$ is a maximally isotropic subspace and we have just stated that, by definition, the null space of a pure spinor has the same property. We can thus identify the two:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_\Phi. \tag{3.37}$$

To be precise the correspondence is between $\mathcal{J}$ and a line bundle of pure spinors, because a rescaling of Φ does not change its annihilator space. If the line bundle admits a global section (it does not need to be the case in general) the structure group of $TM \oplus T^*M$ is further reduced from $\mathrm{U}(d/2, d/2)$ to $\mathrm{SU}(d/2, d/2)$. In the following we will also require that the spinor has non–vanishing norm[10], this will allow us to have an explicit formula for $\mathcal{J}$ which is independent on the rescaling:

$$\mathcal{J}^\Lambda{}_\Sigma = -4\frac{\langle \mathrm{Re}\,\Phi, \Gamma^\Lambda{}_\Sigma \,\mathrm{Re}\,\Phi \rangle}{i\langle \Phi, \bar{\Phi} \rangle} \tag{3.38}$$

where Λ, Σ are generalized tangent bundle indexes, $\Gamma^\Lambda{}_\Sigma$ is the antisymmetrized product of gamma matrices and indexes are raised and lowered with the metric (3.7).

In the previous section we have defined the notion of integrability of a generalized complex structure as a requirement on the $+i$–eigenbundle: it has to be closed under the Courant bracket. Given the correspondence with pure spinors one can imagine that there is a correspondent condition given in terms of some requirement on the spinors. We recall that the Courant bracket contains a differential which is one of the most important operators in the theory of differential forms, thus we can expect a condition involving it. If $\mathbb{X}, \mathbb{Y} \in L_{\mathcal{J}}$ and $[\mathbb{X}, \mathbb{Y}]_C \in L_{\mathcal{J}}$ than the structure is integrable. From the definition of the Courant bracket (3.18) and the identification (3.37) we have:

$$0 = [\mathbb{X}, \mathbb{Y}]_C \cdot \Phi = \frac{1}{4}(\mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X}) \cdot \mathrm{d}\Phi. \tag{3.39}$$

[9]See Appendix A.4 for more details and a list of useful properties.

[10]An example of pure spinor with zero norm is the zeroth order differential form 1. Its wedge with itself has clearly non top–form part however it is a pure spinor whose annihilator space is TM which has clearly maximal dimension.

If we require the spinor to be closed, $\mathrm{d}\Phi = 0$, then the condition is clearly satisfied, i.e. $[\mathbb{X}, \mathbb{Y}]_C \in L_\Phi = L_\mathcal{J}$ and the generalized complex structure is integrable. The closure condition is actually too restrictive and it can be relaxed to

$$\mathrm{d}\Phi = (\iota_X + \xi\wedge)\Phi, \tag{3.40}$$

the structure is integrable if and only if there are X and ξ such that this condition is satisfied. Note that the condition is actually true for line bundles of pure spinors because it does not depend on an overall rescaling (as it was the case for the null space of the pure spinor). In fact if we change Φ as $\Phi' = f\,\Phi$ where f is an arbitrary no–where vanishing (smooth) function than the integrability condition (3.40) is easily recovered if we take $\mathbb{X}' = \mathbb{X} + \mathrm{d}f$.

Definition. *A Generalized Calabi–Yau is a manifold that admits a closed pure spinor with non–vanishing norm*[11].

In general a manifold admits more than one closed pure spinor, we recall that B–transformations with a closed B are a symmetry of the Courant bracket, thus it is not unexpected that we can transform Φ to another closed pure spinor Φ_B:

$$\Phi_B = e^B \wedge \Phi \quad \leftrightarrow \quad \mathcal{J}_B = \mathcal{B}\mathcal{J}\mathcal{B}^{-1}, \qquad \begin{pmatrix} 1 & 0 \\ \mathcal{B} & 1 \end{pmatrix}. \tag{3.41}$$

It is possible to extend the formalism even to non–closed two–forms, we have seen how the Courant bracket can be twisted by a three–form H and we can express (3.22) as a derived bracket provided we find a suitable differential. It is easy to see that $\mathrm{d} - H\wedge$, with H the curvature of B, is the operator we look for provided $\mathrm{d}H = 0$. It gives the right expression for the twisted bracket and $e^B\Phi$ is now $\mathrm{d} - H\wedge$ closed. From the point of view of string theory the requirement $\mathrm{d}H = 0$ means considering situations without NS–fivebranes. The condition (3.40) is straightforwardly generalized to:

$$(\mathrm{d} - H\wedge)\Phi = (\iota_X + \xi\wedge)\Phi\,. \tag{3.42}$$

It is possible to prove [95] that every complex pure spinor can be written as:

$$\Phi = \Omega_k \wedge e^{i\omega + B} \tag{3.43}$$

with ω and B real two–forms and Ω complex k–form which is decomposable, i.e. $\Omega = \theta_1 \wedge \ldots \wedge \theta_k$ and $\theta_{j=1\ldots k}$ linearly independent complex one–forms. The maximally isotropic subspace it defines is said to be of real index zero if:

$$\langle \Phi, \bar{\Phi} \rangle = (-)^{k(k-1)/2} \frac{2^{d/2-k}}{(d/2-k)!} \Omega_k \wedge \bar{\Omega}_k \wedge \omega^{d/2-k} \neq 0 \tag{3.44}$$

which corresponds to the non–zero norm condition we were imposing before. We say that the spinor is of type k, this is of course related to the type of the maximally isotropic subspaces as defined around (3.15).

[11] The norm requirement is to avoid a trivial definition otherwise every manifold would be generalized Calabi–Yau, cfr. Footnote 10.

3.4 $\mathrm{SU}(d/2) \times \mathrm{SU}(d/2)$ structures

In this section we introduce on the manifold M a second generalized complex structure and explain how this allows us to recast in this language the content of the NS–NS massless sector of type II string theories. Let us suppose it is possible to define two generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. They are said to be compatible if $[\mathcal{J}_1, \mathcal{J}_2] = 0$ and $\mathcal{H} \equiv -\mathcal{I}\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$. The operator $G = \mathcal{I}^{-1}\mathcal{H}$ is such that:

$$G^2 = \mathbb{1}_{d+d} \qquad\qquad \mathcal{I}G = G^t\mathcal{I}\,, \tag{3.45}$$

that is to say it is an hermitian projector. Its $+1$–eigenbundle C_+ together with the orthogonal complement C_- will give a decomposition of $TM \oplus T^*M$ such that G is given as:

$$G(\cdot,\cdot) = \mathcal{I}(\cdot,\cdot)|_{C_+} - \mathcal{I}(\cdot,\cdot)|_{C_-}. \tag{3.46}$$

The two structures $\mathcal{J}_1$ and $\mathcal{J}_2$ commute thus we can simultaneously diagonalize them, splitting $TM \oplus T^*M$ into four subbundles:

$$L_{++} = L_{\mathcal{J}_1} \cap L_{\mathcal{J}_2} \qquad\qquad L_{+-} = L_{\mathcal{J}_1} \cap \bar{L}_{\mathcal{J}_2} \tag{3.47}$$

and their complex conjugates. The subbundles $C_\pm$ are then given by $C_\pm = L_{+\pm} \oplus L_{-\mp}$ and we can observe that $C_\pm$ have both rank d while $L_{\pm\pm}$ have rank $d/2$. Such a choice of decomposition (with the condition of positivity) is equivalent to the reduction of the structure group $\mathrm{U}(d/2, d/2)$ to its maximal compact subgroup $\mathrm{U}(d/2) \times \mathrm{U}(d/2)$ which clearly preserves the splitting. Note that in terms of the exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\;i\;} E \xrightarrow{\;\pi\;} TM \longrightarrow 0\,, \tag{3.48}$$

the choice of C_+ bundle implies it is split. In fact identifying T^*M with its image in E under the natural inclusion i then $T^*M \cap C_+ = \{0\}$ because T^*M is isotropic. Thus the projection π restricted to C_+ is injective and this defines a splitting of the exact sequence (3.48).

It is possible to prove [95] that the most general form of $\mathcal{H}$ is:

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{3.49}$$

where g is symmetric and non–degenerate while B is skew–symmetric. This shows that given two compatible generalized complex structure we can extract from them a Riemannian metric and a B–field. It is also useful to notice that (3.49) has already been introduced in the T–duality context [82].

In terms of pairs of line bundles of pure spinors the compatibility condition corresponds to ask that the metric constructed from the associated generalized complex structures is positive definite; the commutativity condition is equivalent to [179, 84]:

$$\langle \Phi_1, \mathbb{X} \cdot \Phi_2 \rangle = 0 = \langle \Phi_1, \mathbb{X} \cdot \bar{\Phi}_2 \rangle \qquad \forall \mathbb{X} \in TM \oplus T^*M. \tag{3.50}$$

As a last step we consider the case in which the two line bundles of pure spinors Φ_1 and Φ_2 admit global sections, in that case the structure group can be further reduced to $\mathrm{SU}(d/2) \times \mathrm{SU}(d/2)$ which will be the case of interest in the following discussion (with $d = 6$). We will also require the normalization condition:

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle \neq 0. \tag{3.51}$$

3.5 Generalized Complex Geometry and Type II theory

After the exposition of the mathematical background we can now explain its relation with type IIA/B string theory. In Chapter 2 we have described the kind of problem we want to address and we have seen that, motivated by the simplest case of fluxless compactifications and the Calabi–Yau case, there is an underlying geometric description. In this section we will complete the "geometrization" for the more general case of flux compactifications.

Let us start by the vector space isomorphism between Cliff(d,d) spinors (which are polyforms in our language) and bispinors. In the literature this is known as Clifford ("/") map. We remind here that bispinors are elements of the tensor product of two Cliff(d) spinor bundles and that can be written as:

$$\not{C} = \sum_{i,j} C_{ij}\eta^{(i)} \otimes \eta^{(j)} \qquad \text{or in components} \qquad \not{C}_{\alpha\beta} = \sum_{i,j} C_{ij}\eta^{(i)}_\alpha \eta^{(j)*}_\beta \tag{3.52}$$

with η^i a basis for Cliff(d) spinors. The map is[12]:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k}_{\alpha\beta} , \tag{3.53}$$

where $\gamma^{i_1 \ldots i_k}$ are the completely antisymmetrized products of gamma matrices which furnish a basis for the bispinor space.

To link generalized complex geometry and type II theory we start by constructing a pair of compatible pure spinors from the spinors which enter the supersymmetry variations (2.10). From now on we will restrict to $d = 6$. According to our definition of $\mathcal{N} = 1$ compactification the susy spinors decompose as in (2.15). In particular η^1_+ and η^2_+ are a pair of six–dimensional chiral spinors which are globally defined and nowhere vanishing. From them we can construct a pair of compatible Cliff$(6,6)$ pure spinors as:

$$\Phi_+ = \eta^{(1)}_+ \otimes \eta^{(2)\dagger}_+ , \qquad \qquad \Phi_- = \eta^{(1)}_+ \otimes \eta^{(2)\dagger}_- . \tag{3.54}$$

To have an expression for $\Phi_\pm$ it is useful to first expand them in the basis for the bispinors and then use the Clifford map to get back a differential form, by Fierz identity[13] we have:

$$\eta^{(1)}_+ \otimes \eta^{(2)\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \left(\eta^{(2)\dagger}_\pm \gamma_{m_k \ldots m_1} \eta^{(1)}_+ \right) \gamma^{m_1 \ldots m_k} . \tag{3.55}$$

The pair $\Phi_\pm$ is pure by construction. Purity can be defined for spinors in arbitrary dimension and signature, a spinor is said to be pure if it is annihilated by $d/2$ linear combinations of gamma matrices. It is known that the $d/2$ annihilators of a Cliff(d) pure spinor are the holomorphic gamma matrices and that for $d \le 6$ all the Cliff(d) spinors are pure (see for example [31]). If we now restrict to six dimensions and we consider bispinors like in (3.54) they have six annihilators, the three of $\eta^{(1)}$ acting from the left and the three of $\eta^{(2)}$ acting from the right, and we can conclude that they are pure.

[12] We refer the reader to Appendix A.4 for more details about the Clifford map and its relation with the Mukai pairing.

[13] As usual the scalar product between two bispinors seen as matrices is given by the trace of their product. The coefficients of the expansion on the bispinor basis are given by $\frac{1}{8k!}\mathrm{tr}(\Phi_\pm \gamma^{m_k \ldots m_1}) = \eta^{(2)\dagger}_\pm \gamma^{m_k \ldots m_1} \eta^{(1)+}$. The factor $\frac{1}{8k!}$ assures that the basis given by the antisymmetrized gamma matrices is orthonormal.

To conclude that $\Phi_\pm$ are a compatible pair we need to check that the generalized complex structures $\mathcal{J}_\pm$ they define are commuting. The two pure spinors clearly share three annihilator, the three gamma matrices acting from the left on $\eta_+^{(1)}$ which give the subbundle $L_{++} \subset TM \oplus T^*M$. With the same reasoning we can see that Φ_+ and $\bar{\Phi}_-$ share three annihilators (this time the three from the right) and give the subbundle $L_{+-} \subset TM \oplus T^*M$. This way we can construct four subbundles $L_{\pm\pm}$ on which we define $\mathcal{J}_\pm$ to be $+i$ or $-i$ as before.

If the pure spinors $\Phi_\pm$ are globally defined we have a reduction of the structure group of the generalized tangent bundle to SU(3) × SU(3). The most general relation between $\eta_+^{(1)}$ and $\eta_-^{(2)}$ is[14]:

$$\eta_+^{(2)} = c\eta_+^{(1)} + V_i \gamma^i \eta_-^{(1)} \, , \tag{3.56}$$

with c and V_i a complex constant and vector respectively. In terms of the structure group of the manifold we can distinguish the following situations [86, 4]:

- Strict SU(3) structure: $V^i = 0$ everywhere and the pure spinor $\Phi_\pm$ are of type $(0, 3)$.
- Static SU(2) structure: $c = 0$ everywhere and the pure spinor $\Phi_\pm$ are of type $(2, 1)$.
- Intermediate SU(2) structure: c and $|V|^2$ are everywhere non–vanishing and the pure spinor $\Phi_\pm$ are of type $(0, 1)$.
- Type changing dynamic SU(3) × SU(3) structure: generically c and $|V|^2$ are non zero but there are loci on the manifold where $c = 0$ or $|V|^2 = 0$ so that there is a type change in the pure spinors.

For static and intermediate SU(2) structure there are two everywhere linear independent spinors and thus the structure group of the manifold is reduced to SU(2). In Chapter 5 we will consider the first of these possibilities where the structure group of the manifold is reduced to SU(3) and the two spinors are everywhere parallel.

We are now ready to present the reformulation of the supersymmetry Killing conditions (2.16) in terms of differential equations on a pair of pure spinors. The result has been first achieved in [86], see also Appendix A of [87] for more details. The equations are:

$$\mathrm{d}_H \left(e^{2A-\phi} \Phi_1 \right) = -2\mu e^{A-\phi} \, \mathrm{Re}\, \Phi_2 \tag{3.57}$$

$$\mathrm{d}_H \left(e^{A-\phi} \, \mathrm{Re}\, \Phi_2 \right) = \frac{c_-}{16} F \tag{3.58}$$

$$\mathrm{d}_H \left(e^{3A-\phi} \, \mathrm{Im}\, \Phi_2 \right) = -3 e^{2A-\phi} \, \mathrm{Im}\, (\bar{\mu} \Phi_1) + \frac{c_+}{16} e^{4A} * \lambda(F) \tag{3.59}$$

$\Phi_1 = \Phi_\pm$ and $\Phi_2 = \Phi_\mp$ for type IIA/IIB respectively, A is the warp factor, ϕ is the dilaton, H is the NS–NS three–form potential, $\mathrm{d}_H = (\mathrm{d} - H\wedge)$ is a differential operator, μ is related to the cosmological constant as $\Lambda = -|\mu|^2$, F denotes the sum of the RR–fluxes on the internal manifold:

$$F = F_0 + F_2 + F_4 + F_6 \qquad \text{for type IIA}$$
$$F = F_1 + F_3 + F_5 \qquad \text{for type IIB.}$$

[14]See Appendix A.2 for our conventions on spinors and gamma matrices.

The two constants c_+ and c_- are expressed in terms of the norms $|a|^2 = |\eta_+^{(1)}|^2$ and $|b|^2 = |\eta_+^{(2)}|^2$ of the $d = 6$ spinors and the warp factor A as follows:

$$c_+ e^A = |a|^2 + |b|^2 \qquad\qquad c_- e^A = |a|^2 - |b|^2. \tag{3.60}$$

This relation allows also to fix a possible invariance of (3.54) which are clearly left unaltered if $\eta^1 \to f\eta^1$ and $\eta^2 \to f^{-1}\eta^2$ for any real nowhere vanishing function f. The norm of the pure spinors are determined by:

$$||\Phi_{1(2)}||^2 = \frac{1}{8}|\eta^{(1)}|^2|\eta^{(2)}|^2 = \frac{1}{32}\left(c_+^2 e^{2A} - c_-^2 e^{-2A}\right). \tag{3.61}$$

It is worth noticing that for the AdS case, where $\mu \neq 0$, taking equations (3.57) and (3.58) into account we can conclude that $c_- = 0$.[15] For Minkowski vacua ($\mu = 0$) the argument is slightly less direct. One has to distinguish between compact and non compact case. General arguments [75, 56, 144, 87] tell us that supersymmetric compact Minkowski vacua require sources with an overall negative charge, thus one needs at least one orientifold. Equation (3.58) tell us that $c_-(\mathrm{d} - H\wedge)F = 0$ but this cannot be satisfied in presence of sources where the Bianchi identity for F reads $(\mathrm{d} - H\wedge)F = \delta$ unless $c_- = 0$.
It is a common fact for supersymmetric theories that imposing invariance under supersymmetry together with Bianchi identities guarantees that (some of) the equations of motion are satisfied. This is the case also for type IIA/B supergravity. For compactifications on Minkowski one can prove (the general proof has been achieved by steps, see [73, 140, 130]) that equations (3.57), (3.58) and (3.59) together with Bianchi identities[16] (2.3) imply that the Einstein equation, the equation of motion for the RR fluxes (2.9), H–field and dilaton are satisfied also in presence of calibrated (see next Section) magnetic sources[17]. In the reformulation we presented the equations of motion for the RR fluxes can be derived straightforwardly, it is sufficient to act by d_H on equation (3.59) and make use of (A.38) and (A.39).

For compactification to AdS the result is similar but, in presence of sources, a modification of the calibration form is needed [129].

3.5.1 Spinors on E and the dilaton field: a brief note

By extending the results of Section 3.1 about the patching on E it is possible to see [88, 186] that the E–spinor bundle is defined by the following transition functions:

$$\tilde{t}_{(\alpha\beta)} = \tilde{\mathcal{A}}_{(\alpha\beta)} \circ e^{\omega_{(\alpha\beta)}} \tag{3.62}$$

Here $\tilde{\mathcal{A}}$ is as in (3.5) but we consider $\mathrm{Gl}(d)_+$ as a subgroup of $\mathrm{Spin}(d,d)_+$ instead of $\mathrm{SO}(d,d)_+$ and the exponentiation of $\omega_{(\alpha\beta)}$ is in $\mathrm{Spin}(d,d)_+$

Thus an E–spinor is defined by a collection of maps $\Phi_{(\alpha)} : U_{(\alpha)} \to S(E)_\pm$ together with their transition functions. We have already remarked that taking into account the action of $\mathrm{Gl}(d)_+$ on spinors we have the isomorphism (3.32). We can notice that it does not transform as a differential form under diffeomorphisms and, even if $S_\pm$ and even/odd forms are isomorphic

[15] F is assumed to be different from zero.

[16] We recall that in our formulation we do not consider NS5–branes, thus the Bianchi identity for H is always $\mathrm{d}H = 0$.

[17] Electric sources would look like points in M_4 and thus break the maximal symmetry which is one of our requirements in the definition of vacuum.

as a vector spaces, we cannot define the exterior derivative. Fortunately there is an easy way out. We can choose a nowhere vanishing globally defined n–vector $\nu = e^{2\phi}\nu_0$, that is to say a set of smooths maps $\nu_{(\alpha)} = e^{2\phi_{(\alpha)}}\nu_0 : U_{(\alpha)} \to \Lambda^n\mathbb{R}^{n*} \cong \mathbb{R}$, which we can assume to be strictly positive definite. The coefficients $e^{2\phi_{(\alpha)}}$ will transform as $e^{2\phi_{(\alpha)}} = (\det a_{(\alpha\beta)})e^{2\phi_{(\beta)}}$ because the vector ν is globally defined. We can thus define an isomorphism:

$$\begin{aligned} \mathfrak{F}^{\phi} &: \Gamma\left(S(E)_{\pm}\right) \longrightarrow \Lambda^{\text{even/odd}}T^*M \\ \mathfrak{F}^{\phi}_{(\alpha)} &: \Phi_{(\alpha)} \longmapsto e^{-B_{(\alpha)}} \wedge e^{-\phi_{(\alpha)}}\Phi_{(\alpha)} \end{aligned} \tag{3.63}$$

It can be shown that it transforms in the right way: $\mathfrak{F}^{\phi}_{(\alpha)} \circ \tilde{t}_{(\alpha\beta)} = a_{(\alpha\beta)} \cdot \circ \mathfrak{F}^{\phi}_{(\beta)}$ where $a_{(\alpha\beta)}\cdot$ is the induced action of $\mathrm{Gl}(d)_+$ on forms (that is to say the exponential term (without the determinant) in (3.31)). It can be shown that there exist a parity reversing map $\mathrm{d}_\nu : \Gamma(S(E)_{\pm}) \to \Gamma(S(E)_{\mp})$ that descends to the H–twisted differential

$$\phi\left(\mathrm{d}_\nu\Phi\right) = \mathrm{d}_H\mathfrak{F}^{\phi}\left(\Phi\right) \tag{3.64}$$

Choosing ν is the same as choosing the dilaton field and thus it determines the isomorphism between the spinor bundle $S(E)_{\pm}$ and differential forms on M. Moreover this allows to extract a scalar from the Mukai paring (3.34), in fact in presence of a Riemannian metric there is a canonical choice for ν. We can choose $\nu = e^{2\phi}\nu_g$ where ν_g is the dual n–vector of the canonical volume form. Thus we can define the Mukai pairing as:

$$\langle\tau, \sigma\rangle = \nu\left([\mathfrak{F}^{\phi}(\tau), \lambda(\mathfrak{F}^{\phi}(\sigma))]_d\right) \tag{3.65}$$

We have thus seen how the fields of the NS–NS sector of the theory, namely g, B and ϕ, are incorporated in the geometry of the generalized tangent bundle.

3.6 Sources, calibrations and generalized complex geometry

In the search of supersymmetric vacua and their non–supersymmetric deformations we need to take into account the possibility of having sources namely D–branes or O–planes. There are various reasons to consider them, among others D–branes are charged objects under the gauge RR–fields [164], thus their presence it is not unexpected in flux backgrounds moreover, under some mild assumptions (like absence of higher derivative corrections), there are no–go theorems [56, 144, 77] which state that in case of flux compactifications on Minkowski space we must require the presence of negatively charged objects. They are not truly supergravity objects but despite their stringy nature they have a description in terms of an effective action [133, 59]:

$$S_s = -T_p\int_\Sigma \mathrm{d}^{p+1}\xi e^{-\phi}\sqrt{|\det(\iota^*[g_{10}] + \mathcal{F})|} + T_p\int_\Sigma \iota^*[C] \wedge e^{\mathcal{F}} \tag{3.66}$$

Here $T_p = \frac{\pi}{k_{10}^2}(4\pi^2\alpha')^{3-p}$ is the brane tension. Σ is the world–volume of the source and it will be a certain submanifold of M_{10}, ι^* denote the pullback on the world–volume of bulk tensors. g_{10} denotes as usual the ten dimensional metric and C are the RR–gauge potentials. With $\mathcal{F} = \iota^*[B] + 2\pi\alpha' F$ we indicate the gauge invariant combination of the field strength F of the world–volume gauge field and the pullback of B. The action of an O–plane is easily recovered from (3.66) substituting T_p with $T_{Op} = -2^{p-5}T_p$ and setting $\mathcal{F}$ to zero.

The first term in the action is known as Dirac–Born–Infeld action and it describes the interaction of the source with the NS–NS sector of the theory. It will thus contribute to the energy–momentum tensor in the Einstein equation and to the dilaton equation. The second term is know as Wess–Zumino (or Chern–Simons) term, it described the coupling of the source with the RR–gauge potentials and it will contribute to the equations of motion and Bianchi identities for the RR–fields.

Here we want to present a description based on the concept of (generalized) calibration, detail its connection with generalized complex geometry and κ–symmetry. This point of view will be useful in the following when we will describe deviations from the supersymmetric case. The concept of calibration has been introduced in [101] as a tool to describe minimal surfaces in curved spaces.

Definition. *Let (M, g) be a Riemaniann manifold, a p–form φ is said to be a calibration if:*

- *in every point $q \in M$ and for every oriented tangent p–plane ξ it satisfies the algebraic condition $\varphi|_\xi \leq \mathrm{vol}|_\xi$;*
- *it satisfies the differential condition $\mathrm{d}\varphi = 0$.*

It is then possible to prove that a p–dimensional submanifold Σ such that at every of its points the previous inequality is saturated is volume minimizing in its homology class. Well known examples of calibrated submanifolds are the $2l$ dimensional complex submanifolds whose calibration form is $\varphi = \frac{1}{l!}\omega^l$, where ω is the Kähler form or the special Lagrangian submanifolds whose calibration form is $\varphi = \mathrm{Re}(e^{i\theta}\Omega)$, where Ω is the holomorphic $(d/2, 0)$–form. Before to discuss the extension of these concepts to the generalized case we need to specify the concept of generalized complex submanifold [95]. The guiding principle is invariance under the symmetries of the theory. Let us consider a submanifold $\Sigma \subset M$ together with its tangent space $T\Sigma$, seen as a natural subbundle of TM, and we construct the subbundle

$$T\Sigma \oplus \mathrm{Ann}(T\Sigma) \subset (TM \oplus T^*M)|_\Sigma. \tag{3.67}$$

We could ask it to be stable under $\mathcal{J}$; this, at a first glance, is a good definition because it reduces to the right ordinary submanifolds if we consider $\mathcal{J}$ as in (3.25) namely complex and Lagrangian respectively. However this definition does not provide a subbundle which is invariant under B–transformations, which are supposed to be a symmetry of the theory, in fact it will change $\mathcal{J}$ leaving (3.67) unchanged. It turns out that to define a suitable notion of generalized complex submanifold we need to supply Σ with some extra information, this is enclosed in a two–form $\mathcal{F}$ defined over Σ such that $\mathrm{d}\mathcal{F} = \iota^*[H]$ where H is a closed three–form on M. This is part of the data which enter the effective action (3.66), it is then clear the connection with sources. The generalized tangent bundle of a generalized submanifold $(\Sigma, \mathcal{F})$

$$T_{(\Sigma,\mathcal{F})} = \{X + \xi \in T\Sigma \oplus T^*M|_\Sigma \,|\, \iota^*(\xi) = \iota_X\mathcal{F}\} \tag{3.68}$$

is real and maximally isotropic, moreover if it is stable under $\mathcal{J}$ the submanifold is said generalized complex.
We want to use calibration forms to describe sources which extremize their energy rather then their volume, we seek thus for a suitable generalization. In compactification with fluxes it first appeared in [161, 162, 35] and then extended in [126, 149] to take into account non–trivial $\mathcal{F}$.

To proceed we introduce the concept of generalized real current j [128], formally it is a functional defined over the space of smooth polyforms and it can be seen as a polyform j such that it acts on ϕ as:

$$j(\phi) = \int_M \langle \phi, j \rangle. \tag{3.69}$$

To every pair $(\Sigma, \mathcal{F})$ we can associate a real current $j_{(\Sigma,\mathcal{F})}$ such that:

$$j_{(\Sigma,\mathcal{F})}(\phi) = \int_M \langle \phi, j_{(\Sigma,\mathcal{F})} \rangle = \int_\Sigma \iota^*(\phi) \wedge e^{\mathcal{F}}. \tag{3.70}$$

Using (A.29) one can prove that $\mathrm{d}_H j_{(\Sigma,\mathcal{F})}$ is a current whose action is given by:

$$\mathrm{d}_H j_{(\Sigma,\mathcal{F})}(\phi) = \int_M \langle \phi, \mathrm{d}_H j_{(\Sigma,\mathcal{F})} \rangle = (-)^d \int_{\partial\Sigma} \iota^*(\phi) \wedge e^{\mathcal{F}|_{\partial\Sigma}}; \tag{3.71}$$

if we then consider a cycle Σ, i.e. $\partial\Sigma = 0$, then $\mathrm{d}_H j_{(\Sigma,\mathcal{F})} = 0$ and we call $(\Sigma, \mathcal{F})$ a generalized cycle. The current $j_{(\Sigma,\mathcal{F})}$ defines a generalized cocycle in H–twisted cohomology and it can be thought as a real pure spinor whose annihilator space is the generalized tangent subbubdle (3.68). Thus to every single source we can associate a localized pure spinor and the generalized current looks like $e^{-\mathcal{F}} \wedge \delta^{(d-p)}(\Sigma)$ where $\delta^{(d-p)}(\Sigma)$ is the Poincaré dual of Σ. The generalization to the smeared case is immediate.
From now on we consider cases which fit in our definition of a vacuum (see Section 2.2) and sources which fill the external space–time and wrap an internal cycle which, by abuse of notation, we denote Σ. We make the further assumptions that the sources are static and that there are no electric world–volume gauge fields. The energy density (3.66) is thus proportional to:

$$\mathcal{E}(\Sigma, \mathcal{F}) = e^{4A-\phi} \sqrt{|\det(\iota^*[g_{10}] + \mathcal{F})|}\, \mathrm{d}\xi^1 \ldots \mathrm{d}\xi^{p-3} - e^{4A} \left(\sum_k \iota^*[\tilde{C}_k] \wedge e^{\mathcal{F}} \right)_{(p-3)}, \tag{3.72}$$

where ξ^i are world–volume coordinates and $\tilde{C}_k$ are the dual potentials as in (2.14). We are now ready to give the definition of a generalized calibration form φ.

Definition. *A polyform φ is said to be a generalized calibration if:*

- $\iota^*(\varphi) \wedge e^{\mathcal{F}} \leq \mathcal{E}(\Sigma, \mathcal{F})$,
- $\mathrm{d}_H \varphi = 0$,

If a source $(\Sigma, \mathcal{F})$ saturates the bound it is said to be generalized calibrated and it is possible to show that it is energy minimizing in its generalized homology class.

We are now left with the problem of finding a calibration form φ for a space–time filling source. In the case of supersymmetric vacua with sources this is dictated by supersymmetry itself. The sources will be energy minimizing and thus stable. We will sketch the derivation here and refer the reader to the previously cited original works for more details. A source preserves the background supersymmetry generated by ϵ^1 and ϵ^2 if it satisfies the so called κ–symmetry condition [40, 41, 2, 3, 19]

$$\Gamma_{Dp} \epsilon^2 = \epsilon^1, \tag{3.73}$$

where Γ_{Dp} is the world-volume chiral operator

$$\Gamma_{Dp} = \frac{1}{\sqrt{|\det(\iota^*[g_{10}] + \mathcal{F})|}} \sum_{2l+s=p+1} \frac{\varepsilon^{\alpha_1 \ldots \alpha_{2l} \beta_1 \ldots \beta_s}}{l! s! 2^l} \mathcal{F}_{\alpha_1 \alpha_2} \ldots \mathcal{F}_{\alpha_{2l-1} \alpha_{2l}} \Gamma_{\beta_1 \ldots \beta_s}\,. \tag{3.74}$$

If we consider the metric ansatz (2.11) and the consequent decomposition of the spinors ϵ^a and operator Γ_{Dp}, a careful analysis shows that (3.73) reduce to:

$$i\gamma^{Dp}_{(p-3)}\eta^{(2)}_{+} = \eta^{(1)}_{\mp} \qquad (\mp \text{ refers to type IIA/B}) \tag{3.75}$$

where $\gamma^{Dp}_{(p-3)}$ is the internal part of Γ_{Dp} and $p-3$ is the dimension (odd/even for type IIA/B) of the internal cycle wrapped by the source. Expanding (3.75) in the spinorial basis defined by $\eta^{(1)}$ gives for type IIA/B:

$$\left(\mathrm{Re}\left(\iota^*\left[-i\Phi_{\mp}\right]\right)\wedge e^{\mathcal{F}}\right)_{(p-3)} = \frac{|a|^2}{8}\sqrt{\det(\iota^*[g]+\mathcal{F})}\,\mathrm{d}\xi^1\ldots\mathrm{d}\xi^{p-3}, \tag{3.76}$$

where $\Phi_{\mp}$ are the image of the bispinors (3.54) under the Clifford map. Note that $\gamma^{Dp}_{(p-3)}$ is unitary and thus from (3.75) we can easily infer that the norms of $\eta^{(1)}$ and $\eta^{(2)}$ must be equal, which is in agreement with the discussion at the end of Section 3.5.

At this point we are on the right way to get the calibration form φ. Confronting (3.72) it is easy to see that

$$\varphi = \mathrm{Re}\left(-i\frac{8}{|a|^2}e^{4A-\phi}\,\Phi_{\mp}\right) - \sum_{\text{k odd/even}} e^{4A}\tilde{C}_k \tag{3.77}$$

is a form that saturates the bound in Definition 3.6 when we take into account supersymmetric sources. To match all the requirements of a generalized calibration form we need to show that it actually satisfies the inequality and that it is d_H closed. The Schwarz inequality

$$|i\gamma^{Dp}_{(p-3)}\eta^{(2)}_{+} + \eta^{(1)}_{\mp}| \leq |i\gamma^{Dp}_{(p-3)}\eta^{(2)}_{+}| + |\eta^{(1)}_{\mp}| \tag{3.78}$$

gives the following general inequalities (again $\mp$ refers to type IIA/B theory)

$$\mathrm{Re}\left(i\eta^{(1)\dagger}_{\mp}\gamma^{Dp}_{(p-3)}\eta^{(2)}_{+}\right) \leq |a|^2. \tag{3.79}$$

They are equivalent to

$$\left(\mathrm{Re}\left(\iota^*\left[-i\Phi_{\mp}\right]\right)\wedge e^{\mathcal{F}}\right)_{(p-3)} \leq \frac{|a|^2}{8}\sqrt{\det(\iota^*[g]+\mathcal{F})}\,\mathrm{d}\xi^1\ldots\mathrm{d}\xi^{p-3}, \tag{3.80}$$

which allows to conclude that (3.77) fulfills the algebraic requirement. The d_H closure is a direct consequence of (3.59). We can thus conclude that φ in (3.77) is a generalized calibration form. Beside to furnish a clear description of sources in a well established mathematical framework this approach provides an immediate technical advantage which will be important in our analysis of Chapter 5. It is usually very difficult to take the variation of DBI action with respect to the bulk fields g_{mn} and B_{mn} but considering equations (3.70) and (3.76) we can rewrite the DBI action as [130]

$$S_{DBI} = -T_p\int_{\Sigma}\iota^*[\Upsilon]\wedge e^{\mathcal{F}} + \mathcal{O}(\mathrm{cal}^2) = -T_p\int_M\langle\Upsilon, j_{(\Sigma,\mathcal{F})}\rangle + \mathcal{O}(\mathrm{cal}^2), \tag{3.81}$$

where

$$\Upsilon = \frac{8}{|a|^2}e^{4A-\phi}\,\mathrm{Im}(\Phi_{\mp})\,. \tag{3.82}$$

The corrections are thus quadratic in the calibration condition (3.75). We can now take the variation using the right–hand side of (3.81) and we obtain:

$$\frac{\delta S_{DBI}}{\delta g^{mn}} = -\frac{T_p}{2}\langle g_{p(m}\mathrm{d}x^p \otimes \iota_{n)}\Upsilon, j_{(\Sigma,\mathcal{F})}\rangle \tag{3.83}$$

$$\frac{\delta S_{DBI}}{\delta \phi} = T_p\langle \Upsilon, j_{(\Sigma,\mathcal{F})}\rangle\,. \tag{3.84}$$

The current $j_{(\Sigma,\mathcal{F})}$ can be read out of the Bianchi identities for the RR–fluxes. Note that in (3.83) there could be corrections coming from the $\mathcal{O}(\mathrm{cal}^2)$ piece but this would be linear in the calibration condition and thus they vanish in the supersymmetric case.

A number of remarks are needed. We have presented here the case of space–time filling branes and we have assumed our ansatz (2.11) for the metric but there are a number of generalizations one can consider. In [149] the authors show that also sources which span two space–time (domain walls) or one space–time (string like) dimensions are compatible with κ–symmetry, they provide the corresponding generalized calibration forms which are related to $\Phi_\pm$ and to $\mathrm{Re}(\Phi_\mp)$ respectively. Clearly these configurations break four dimensional Poincaré symmetry and we will not consider them in the following. In [147] it was shown that the calibration conditions are related to F–flatness and to D–flatness conditions of the four dimensional effective theory on the D–brane. The study of calibrations in type II context for different splittings of the ten dimensional space–time has been done in [138]. Many results we presented here have been derived under the assumption of a Minkowski external space, in order to extend them to the AdS case one needs to modify the concept of calibration because of the presence of a boundary, e.g. in AdS the energy does not have to be positive definite to have absence of tachions due to the Breitenlohner–Freedman bound, thus a modification of the calibration condition is expected, we refer to [129] for the analysis about this point.

We conclude with few words about the constraints on the various fields in presence of an orientifold. The orientifold action is a composition of a reflection, Ω_{WS}, on the world–sheet and a target–space involution, σ, acting on the internal manifold (sometimes there is an extra sign depending on the fermion number of the left–movers). A complete analysis of its action requires a distinction according to the dimensionality of the O–plane and we refer, for example, to [130] for it, here we recall only the action of an O6 because we will need it in Chapter 5. Under the action of an O6–plane we have:

$$\begin{aligned}
\sigma^*(\phi) &= \phi & \sigma^*(g_{mn}) &= g_{mn}\\
\sigma^*(H) &= -H & \sigma^*(F) &= \lambda(F)\\
\sigma^*(\epsilon^1) &= \epsilon^2 & \sigma^*(\epsilon^2) &= \epsilon^1\\
\sigma^*(\eta^{(1)}_\pm) &= \eta^{(2)}_\mp & \sigma^*(\eta^{(2)}_\pm) &= \eta^{(1)}_\mp\\
\sigma^*(\Phi_+) &= \lambda(\Phi_+) & \sigma^*(\Phi_-) &= \lambda(\bar{\Phi}_-)
\end{aligned} \tag{3.85}$$

The compatibility of the orientifold involution with SU(3)×SU(3) structures has been studied first in [87] based on previous works [93, 94, 17] and completed in [130].

Chapter 4

Solvmanifolds

As presented in Chapter 2 one usually looks for compact solutions in which the internal space is six dimensional and compact. We have also described the general geometric requirements that such a space has to satisfy. A complete classification of six dimensional manifolds is not available and thus one starts his searching from a restricted class whose properties are (well) understood. In this chapter we will describe in detail six dimensional nilmanifolds and solvmanifolds which have been extensively used in type II compactifications, both to four dimensional Minkowski or Anti de Sitter [120, 87, 4] and appear to be good candidates for possible de Sitter vacua as well. Indeed their geometry is pretty well understood (for instance all nilmanifolds are generalized Calabi–Yau [37]) and, in particular, they can have negative curvature and therefore support internal fluxes (as well as D–branes and O–plane sources).

The results we present here are not original, in particular the ones regarding the compactness criterion. Nevertheless we reformulate them in a slightly different way with respect to the mathematical literature following more the spirit of twist construction [5]. In Section 4.3.1 we will apply the results of [5] (there authors' focus was on nilmanifolds) to construct a supersymmetric solution on the solvmanifold $G_{5.17}^{p,-p,\pm 1} \times S^1$ from a known solution on $G_{5.17}^{0,0,\pm 1} \times S^1$ [29, 87, 4]. It will be our starting point for the analysis presented in Chapter 5.

4.1 Definition and classification

Nil– and solvmanifolds are compact homogeneous spaces constructed from nilpotent or solvable Lie groups G via a quotient G/Γ, where Γ is a lattice in G, i.e. a discrete co–compact subgroup [6, 159, 23]. We provide here an important remark. This definition of solvmanifold is not the most general: one could consider cases where a d–dimensional solvmanifold is a compact quotient of a higher dimensional group with respect to a closed continuous subgroup. A famous example of this kind is the Klein bottle. With the general definition the number of six dimensional solvmanifolds is very high and there are no complete classifications. In the rest of the manuscript we will consider only solvmanifolds according to the more restricting definition, where a full classification of solvable Lie algebras up to dimension six is available. The dimension of the resulting manifold will be thus the same as that of the group G. We refer to Appendix C for definitions and details about Lie groups.

As usual in Lie theory many properties of the groups and their classification (up to global issues) are inferred from Lie algebras. According to Levi's decomposition, any real finite dimensional Lie algebra is the semidirect sum of its largest solvable ideal called the radical, and a semi–simple subalgebra. Therefore solvable and nilpotent algebras do not enter the usual Cartan classification. Solvable algebras $\mathfrak{g}$ are classified with respect to the dimension of their nilradical $\mathfrak{n}$. One can show [152, 23] that dim $\mathfrak{n} \geq \frac{1}{2}$dim $\mathfrak{g}$. Since we are interested in six dimensional manifolds we will consider dim $\mathfrak{n} = 3, \ldots, 6$. If dim $\mathfrak{n} = 6$, $\mathfrak{n} = \mathfrak{g}$ and the algebra is nilpotent (they clearly are a subset of the solvable ones). There are 34 (isomorphism) classes of six–dimensional nilpotent algebras (see for instance [87, 172] for a list), among which 24 are indecomposable. Among the 10 decomposable algebras, there is of course the abelian one, $\mathbb{R}^6$. There are 100 indecomposable solvable algebras with dim $\mathfrak{n} = 5$ (99 were found in [153], and [181] added 1, see [30] for a complete and corrected list), and 40 indecomposable solvable algebras with dim $\mathfrak{n} = 4$ [181]. Finally, those with dim $\mathfrak{n} = 3$ are decomposable into sums of two solvable algebras. There are only 2 of them, see Corollary 1 of [154]. In total, there are 164 indecomposable six–dimensional solvable algebras. For a list of six–dimensional indecomposable unimodular solvable algebras, see [23].

Most of the solvable groups are semidirect products (see Appendix C.2 for definitions and details). For G a solvable group and N its nilradical, we consider the following definitions:

- If $G = \mathbb{R} \ltimes_\mu N$, G is called almost nilpotent. All three– and four–dimensional solvable groups are of this kind [23].
- If furthermore, the nilradical is abelian (i.e. $N = \mathbb{R}^k$), G is called almost abelian.

Since N has codimension 1 in G, we can consider μ as a one–parameter group $\mathbb{R} \to \mathrm{Aut}(N)$. N is a normal subgroup of G and we can thus apply some of the results presented in Appendix C.1. Let us label the $\mathbb{R}$ direction with a parameter t, which we can take as a coordinate, with the corresponding algebra element being ∂_t. According to (C.4), we then have

$$\mu(t) = exp^N \circ Ad_{e^{t\partial_t}}(\mathfrak{n}) \circ log^N \ , \ Ad_{e^{t\partial_t}}(\mathfrak{n}) = e^{ad_{t\partial_t}(\mathfrak{n})} = e^{t \ ad_{\partial_t}(\mathfrak{n})} \ . \tag{4.1}$$

Furthermore, for the almost abelian case, we can identify N and $\mathfrak{n}$, so the *exp* and *log* correspond to the identity. Then, we obtain the simpler formula

$$\mu(t) = Ad_{e^{t\partial_t}}(\mathfrak{n}) = e^{t \ ad_{\partial_t}(\mathfrak{n})} \ . \tag{4.2}$$

We will mainly focus on solvable algebras with dim $\mathfrak{n} = 5$ (to which correspond almost nilpotent solvable groups) because, as we will discuss further, the compactness question is simpler to deal with.

4.2 The geometry of solvmanifolds

4.2.1 Fibration structure

Given a d–dimensional Lie algebra $\mathfrak{g}$ expressed in some vector basis $\{E_1, \ \ldots, \ E_d\}$ as

$$[E_b, E_c] = f^a{}_{bc} E_a \ , \tag{4.3}$$

where $f^a{}_{bc}$ are the structure constants, we can define the dual space of one–forms $\mathfrak{g}^*$ with basis $\{e^1, \ \ldots, \ e^d\}$. They satisfy the Maurer–Cartan equation

$$\mathrm{d}e^a = -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c = -\sum_{b<c} f^a{}_{bc} \ e^b \wedge e^c \ , \tag{4.4}$$

where d is the exterior derivative. Since $\mathfrak{g}^* \approx T_e G^*$, the set $\{e^1, \ldots, e^d\}$ provide, by left invariance, a basis for the cotangent space $T_x G^*$ at every point $x \in G$ and consequently its elements are globally defined one–forms on the manifold. When we consider the quotient of G by a lattice Γ, the one–forms will have non trivial identification through the lattice action[1]. The less general definition of nil– and solvmanifolds we use in this thesis allows to prove that they are always parallelizable (see [7] for an example of non parallelizable solvmanifold) and hence orientable (as opposed to the Klein bottle example which is a non–orientable surface), even if they are not necessarily Lie groups [159].

The Maurer–Cartan equations reflect the topological structure of the corresponding manifolds. For example, nilmanifolds all consist of iterated fibrations of circles over tori, where the iterated structure is related to the descending or ascending series of the algebra (see [37, 23, 168]). This can be easily seen on a very simple example, the nilmanifold obtained from the three–dimensional Heisenberg algebra

$$[E_2, E_3] = E_1 \qquad \Leftrightarrow \qquad \mathrm{d}e^1 = -e^2 \wedge e^3 \,. \tag{4.5}$$

The Maurer–Cartan equation is solved by the one–forms

$$e^1 = \mathrm{d}x^1 - x^2 \mathrm{d}x^3 \ , \ e^2 = \mathrm{d}x^2, \ e^3 = \mathrm{d}x^3 \ . \tag{4.6}$$

From the connection form, $-x^2\mathrm{d}x^3$, one can read the topology of the nilmanifold in question, which is a non–trivial fibration of the circle in direction 1 on the two–torus in directions $2, 3$:

$$\begin{array}{ccc} S^1_{\{1\}} & \hookrightarrow & H/\Gamma_1 \\ & & \downarrow \\ & & T^2_{\{23\}} \end{array} \tag{4.7}$$

As we have said solvmanifolds are classified according to the dimension of the nilradical $\mathfrak{n}$ (the largest nilpotent ideal) of the corresponding algebra; at the level of the group[2] we have that, if $\dim N < 6$, then G contains an abelian subgroup of dimension k [6, 30]. This means we have $G/N = \mathbb{R}^k$. If the group admits a lattice Γ (see section 4.2.2 for a detailed discussion about compactness), one can show that $\Gamma_N = \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is a closed subgroup of G, and so $G/(N\Gamma) = T^k$ is a torus. The solvmanifold is a non–trivial fibration of a nilmanifold over the torus T^k

$$\begin{array}{ccc} N/\Gamma_N = (N\Gamma)/\Gamma & \hookrightarrow & G/\Gamma \\ & & \downarrow \\ & & T^k = G/(N\Gamma) \end{array} \tag{4.8}$$

This construction, called the Mostow bundle [151], is one of the main results in the theory of solvmanifolds. As we shall see, the corresponding fibration can be more complicated than in

[1] In general there is a natural inclusion $(\Lambda\mathfrak{g}^*, \delta) \to (\Lambda(G/\Gamma), d)$ between the Chevalley–Eilenberg complex on G and the de Rham complex of differential forms on G/Γ. This inclusion induces an injection map between cohomology groups $H^*(\mathfrak{g}) \to H^*_{dR}(G/\Gamma)$ which turns out to be an isomorphism for completely solvable groups. We recall that a Lie group G with Lie algebra $\mathfrak{g}$ is said to be completely solvable if the linear map $ad_X : \mathfrak{g} \to \mathfrak{g}$ has only real roots $\forall X \in \mathfrak{g}$. By root of a linear map we mean a root of its characteristic polynomial. Note that all nilmanifolds are completely solvable and thus the injection is an isomorphism (Nomizu's theorem [157]), the extension to non–nilpotent completely solvable groups being the so called Hattori theorem [159]. For more details and for a list of Betti numbers of solvmanifolds up to dimension six see [23].

[2] We denote by $\mathfrak{n}$ the ideal in the algebra and with N the corresponding subgroup.

the nilmanifold case. In general the Mostow bundle is not principal, nevertheless, under certain hypotheses, it is possible to find sufficient conditions for it to be a principal bundle (Theorem 3.6 in [23]).

The iterated structure of fibrations for a nilmanifold is related to the descending serie of $\mathfrak{n}$:

$$\mathfrak{n}^{k=0\dots p} \text{ with } \mathfrak{n}^0 = \mathfrak{n} \ , \ \mathfrak{n}^p = \{0\} \ .$$

Every $\mathfrak{n}^k$ is an ideal of $\mathfrak{g}$, so $\forall k \geq 1$, $\mathfrak{n}^k = [\mathfrak{n}, \mathfrak{n}^{k-1}] \subset [\mathfrak{g}, \mathfrak{n}^{k-1}] \subset \mathfrak{n}^{k-1}$. Let us now define another serie:

$$\text{For } 1 \leq k \leq p, \ s^k = \{E \in \mathfrak{n}^{k-1} \text{ with } E \notin \mathfrak{n}^k\} \ . \tag{4.9}$$

Let us prove some property of this serie. Assume that $\exists X \in s^p \bigcap s^q$, $p > q$ with $X \neq 0$. Then $X \in \mathfrak{n}^{p-1} \subset \mathfrak{n}^{p-2} \subset \cdots \subset \mathfrak{n}^q \subset \mathfrak{n}^{q-1}$. So $X \in \mathfrak{n}^{q-1}$ and $X \in \mathfrak{n}^q$, so $X \notin s^q$, which is a contradiction. So $s^p \bigcap s^q = \{0\}$ for $p \neq q$. Furthermore, we always have $s^p = \mathfrak{n}^{p-1}$. So $s^{p-1} \bigcup s^p = s^{p-1} \bigcup \mathfrak{n}^{p-1} = \mathfrak{n}^{p-2} \bigcup \mathfrak{n}^{p-1} = \mathfrak{n}^{p-2}$. Assume that $s^k \bigcup s^{k+1} \bigcup \dots \bigcup s^{p-1} \bigcup s^p = \mathfrak{n}^{k-1}$. Then $s^{k-1} \bigcup s^k \bigcup \dots \bigcup s^{p-1} \bigcup s^p = s^{k-1} \bigcup \mathfrak{n}^{k-1} = \mathfrak{n}^{k-2} \bigcup \mathfrak{n}^{k-1} = \mathfrak{n}^{k-2}$. So by recurrence, we get that $\bigcup_{k=1\dots p} s^k = \mathfrak{n}$. In other words, each element of $\mathfrak{n}$ appears in one and only one element of the serie $s^{\{k\}}$.

In this thesis we will label the algebras according to their Maurer–Cartan equations. For example, the previously introduced Heisenberg algebra, is denoted with the following concise form: $(-23, 0, 0)$, where each entry i gives the result of $\mathrm{d}e^i$. Let us give an example based on the five–dimensional solvable algebra $(0, 31, -21, 23, 24)$. We have

$$\begin{aligned} \mathfrak{g} = \{1,2,3,4,5\} \quad , \quad \mathfrak{n} = \{2,3,4,5\} \ , \ \mathfrak{n}^1 = \{4,5\} \ , \ \mathfrak{n}^2 = \{5\} \ , \ \mathfrak{n}^3 = \{0\} \\ s^1 = \{2,3\} \ , \ s^2 = \{4\} \ , \ s^3 = \{5\} \ . \end{aligned}$$

The descending serie of $\mathfrak{n}$ is known to be related to the fibration structure of the nilpotent group: each element gives a further fibration. Now we understand that the serie $s^{\{k\}}$ gives us which directions are fibered at each step. The correspondence between basis, fibers and series for a general iteration is given in the following diagram (of course it should be understood in terms of group elements instead of algebra elements as given here, see [168]):

$$\begin{array}{ccc} \mathcal{F}^{p-1} = s^p & \hookrightarrow & \mathcal{M}^{p-1} = \mathfrak{n} \\ & & \downarrow \\ \mathcal{F}^{p-2} = s^{p-1} & \hookrightarrow & \mathcal{M}^{p-2} = \mathcal{B}^{p-1} \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \mathcal{F}^2 = s^3 & \hookrightarrow & \mathcal{M}^2 = \mathcal{B}^3 \\ & & \downarrow \\ \mathcal{F}^1 = s^2 & \hookrightarrow & \mathcal{M}^1 = \mathcal{B}^2 \\ & & \downarrow \\ & & \mathcal{B}^1 = s^1 \end{array}$$

We see the unique decomposition of $\mathfrak{n}$ into the serie $s^{\{k\}}$. We have $\mathcal{B}^i = \bigcup_{k=1\dots i} s^k$ and $\mathcal{F}^i = s^{i+1}$. The case for a solvmanifold which is not a nilmanifold is more complicated and the structure is given in two steps. According to (4.8) the Mostow bundle is given by a nilmanifold which is fibered over a torus. The structure of the nil–fiber can be described as before looking at the nilradical of the solvable algebra, what is left is the information related to the fibration over the torus. We will discuss later in more detail the case of almost nilpotent groups which is the simplest one.

4.2.2 Compactness

Given an algebra (or the corresponding group) it is far from obvious that a lattice, and hence the corresponding solvmanifold, exists. A theorem by Malcev [142] states that a connected and simply–connected nilpotent Lie group G admits a lattice if and only if there exists a basis for the Lie algebra $\mathfrak{g}$ such that the structure constants are rational numbers. This condition is always satisfied for all the 34 classes of nilpotent six dimensional algebras. The theory of non–nilpotent solvmanifolds is less developed with respect to the nilmanifolds and their construction is much more difficult, mainly due to the lack of an easy criterion for the existence of a lattice. Several criteria have been proposed. The first is due to Auslander [6]; despite its generality the criterion is difficult to use in concrete situations and we will not refer to it in this manuscript. Details about it can be found in the original paper [6] and in [23]. Another criterion, which is closer to the one we use in this thesis, is due to Saitô [170]. It is less general than Auslander's because it applies to solvable groups that are algebraic subgroups of $\mathrm{Gl}(n,\mathbb{R})$ for some n. The criterion deals with the adjoint action of the group G over the nilradical $\mathfrak{n}$ of its algebra $\mathfrak{g}$. For an illustration, see [87].

It is possible to prove that a necessary condition for the existence of a lattice is unimodularity of the algebra [150]. A Lie algebra $\mathfrak{g}$ is unimodular if $\forall X \in \mathfrak{g}$, $\mathrm{tr}(ad_X) = 0$, with the use of (C.2) it is easy to see that it is equivalent to:

$$\sum_a f^a{}_{ba} = 0, \forall b\,. \tag{4.10}$$

We will present a criterion for the compactness which is valid for almost abelian solvmanifolds, for which the Mostow bundle is particularly simple. Remember that for this class the group action μ which gives the semidirect product of $\mathbb{R}$ and the codimension one nilradical N is given by:

$$\mu(t) = Ad_{e^{t\partial_t}}(\mathfrak{n}) = e^{t\ ad_{\partial_t}(\mathfrak{n})}\ . \tag{4.11}$$

From a geometrical point of view, $\mu(t)$ encodes the fibration of the Mostow bundle. For almost abelian solvable groups the criterion to determine whether the associated solvmanifolds exist is rather simple: the group admits a lattice if and only if there exists a $t_0 \neq 0$ for which $\mu(t_0)$ can be conjugated to an integer matrix. As an example, we can consider two three–dimensional almost abelian solvable algebras

$$\begin{aligned} \varepsilon_2 \quad : \quad & [E_2, E_3] = E_1 && \Leftrightarrow && \mathrm{d}e^1 = -e^2 \wedge e^3 \\ & [E_1, E_3] = -E_2 && \Leftrightarrow && \mathrm{d}e^2 = e^1 \wedge e^3 \end{aligned} \tag{4.12}$$

$$\begin{aligned} \varepsilon_{1,1} \quad : \quad & [E_1, E_3] = E_1 && \Leftrightarrow && \mathrm{d}e^1 = -e^1 \wedge e^3 \\ & [E_2, E_3] = -E_2 && \Leftrightarrow && \mathrm{d}e^2 = e^2 \wedge e^3\,. \end{aligned} \tag{4.13}$$

For the algebra $\varepsilon_2 : (-23, 13, 0)$, the nilradical is given by $\mathfrak{n} = \{E_1, E_2\}$ and $\partial_t = E_3$. In this basis, the restriction of the adjoint representation to the nilradical is

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{4.14}$$

which gives a μ matrix of the form

$$\mu(t) = e^{t\ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} . \tag{4.15}$$

It is easy to see that, for $t_0 = n\frac{\pi}{2}$, with $n \in \mathbb{Z}^*$, $\mu(t_0)$ is an integer matrix and hence the corresponding manifold is compact.

For the algebra $\varepsilon_{1,1} : (-13, 23, 0)$ the analysis is less straightforward. The nilradical is $\mathfrak{n} = \{E_1, E_2\}$ and again $\partial_t = E_3$. Then, in the (E_1, E_2) basis,

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \ \mu(t) = e^{t\ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} , \tag{4.16}$$

and it is clearly not possible to find a $t_0 \neq 0$ such that $\mu(t_0)$ is an integer. To see whether the group admits a lattice, we then have to go to another basis. In other words, $\mu(t_0)$ will be conjugated to an integer matrix. Let us consider the particular change of basis given by

$$P = \begin{pmatrix} 1 & c \\ 1 & \frac{1}{c} \end{pmatrix} , \ P^{-1} = \frac{1}{c - \frac{1}{c}} \begin{pmatrix} -\frac{1}{c} & c \\ 1 & -1 \end{pmatrix} , \tag{4.17}$$

where $c = e^{-t_1}$ and $t_1 \neq 0$. Then:

$$\hat{\mu}(t) = P^{-1} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} P = \begin{pmatrix} \frac{\sinh(t_1 - t)}{s_1} & -\frac{\sinh(t)}{s_1} \\ \frac{\sinh(t)}{s_1} & \cosh(t) + c_1 \frac{\sinh(t)}{s_1} \end{pmatrix} , \tag{4.18}$$

with $s_1 = \sinh(t_1)$ and $c_1 = \cosh(t_1)$. For $t = t_1$, we get

$$\hat{\mu}(t = t_1) = \begin{pmatrix} 0 & -1 \\ 1 & 2c_1 \end{pmatrix} . \tag{4.19}$$

The conjugated matrix $\hat{\mu}(t)$ can have integers entries for some non–zero $t = t_1$ when $2\cosh(t_1)$ is integer. In [23], $2\cosh(t_1) = 3$.

Another change of basis follows [100] and it leads to conjugate matrices which are similar to the ones we obtain for the six dimensional solvmanifolds considered later for de Sitter purposes. The basis in given by:

$$E_1 \rightarrow \sqrt{\frac{q_2}{q_1}} \frac{E_1 - E_2}{\sqrt{2}} , \ E_2 \rightarrow \frac{E_1 + E_2}{\sqrt{2}} , \ E_3 \rightarrow \sqrt{q_1 q_2}\, E_3 , \tag{4.20}$$

with q_1, q_2 strictly positive constants, such that the algebra reads

$$[E_1, E_3] = q_2 E_2 \qquad [E_2, E_3] = q_1 E_1 . \tag{4.21}$$

In this new basis

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} 0 & -q_1 \\ -q_2 & 0 \end{pmatrix} \ , \ \mu(t) = \begin{pmatrix} \cosh(\sqrt{q_1 q_2} t) & -\sqrt{\frac{q_1}{q_2}} \sinh(\sqrt{q_1 q_2} t) \\ -\sqrt{\frac{q_2}{q_1}} \sinh(\sqrt{q_1 q_2} t) & \cosh(\sqrt{q_1 q_2} t) \end{pmatrix} \ , \qquad (4.22)$$

so that $\mu(t)$ can be made integer by the choice of parameters

$$t_0 \neq 0 \ , \ \cosh(\sqrt{q_1 q_2} t_0) = n_1 \ , \ \frac{q_1}{q_2} = \frac{n_2}{n_3} \ , \ n_2 n_3 = n_1^2 - 1 \ , \ n_{1,2,3} \in \mathbb{Z}^* \ . \qquad (4.23)$$

Thus also the algebra $\varepsilon_{1,1}$ can be used to construct compact solvmanifolds. Notice that the values $q_1 = q_2 = 1$ are not allowed by the integer condition (4.23).

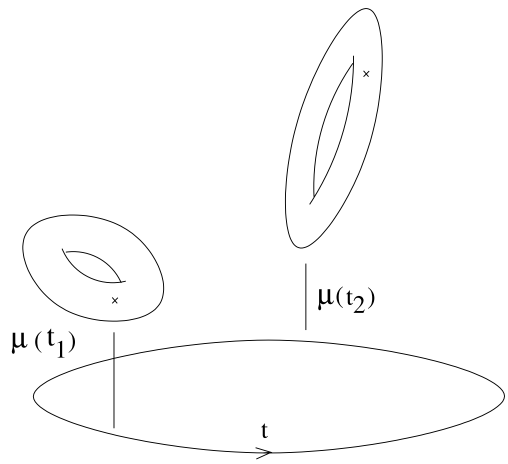

Figure 4.1: Mostow bundle for the solvmanifolds ϵ_2 and $\epsilon_{1,1}$. The base is the circle in the t direction, and due to the nilradical being abelian the fiber is T^2. The fibration is encoded in $\mu(t)$ which is either a rotation or a "hyperbolic rotation" twisting the T^2 moving along the base.

4.2.3 Twist construction of (almost abelian) solvmanifolds

The knowledge that different supersymmetric backgrounds are related by a web of dualities and their connection to geometric transitions have been one of the most intriguing and fruitful developments of string theory [82]. In [5] a transformation called twist has been developed, which was suitable to relate certain backgrounds on nilmanifolds which were previously not connected by any simple direct duality. As we will see in the next section such a transformation has a clear embedding in generalized complex geometry, in this section we are interested on the fact that it allows for a topology change.

We have shown how to obtain explicitly, at least for almost abelian solvmanifolds, the operator $\mu(t)$ giving the structure of the Mostow bundle and what condition it has to satisfy in order for the manifold to be compact.

We focus here on six–dimensional almost abelian algebras and the corresponding compact solvmanifolds, and, following [5], we discuss how to use the adjoint action $\mu(t)$ to construct the globally defined one–forms of the solvmanifolds from those on T^6.

Let us first discard global issues related to the compactness of the manifolds. Then, given an almost abelian solvable group G, we want to relate one–forms on $T^*\mathbb{R}^6$ to those of $T^*G = \mathfrak{g}^*$

$$A \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^6 \end{pmatrix} = \begin{pmatrix} e^1 \\ \vdots \\ e^6 \end{pmatrix} . \tag{4.24}$$

Here A is a local matrix that should contain the bundle structure of G. The one–forms in (4.24) must satisfy the corresponding[3] Maurer–Cartan equation. The matrix A should reproduce the different fibrations of the solvable group (the bundle structure is manifest in the Maurer–Cartan equations). Given the general form of solvable groups (a nilradical subgroup N and an abelian left over subgroup $G/N = \mathbb{R}^{\dim\ G - \dim\ N}$), we will consider A to be a product of two pieces:

$$A = \left(\begin{array}{c|c} A_N & 0 \\ \hline 0 & \mathbb{I}_{6-\dim\ N} \end{array}\right) \left(\begin{array}{c|c} A_M & 0 \\ \hline 0 & \mathbb{I}_{6-\dim\ N} \end{array}\right) , \tag{4.25}$$

where we take A_M and A_N to be $\dim N \times \dim N$ matrices, and we put the abelian directions of $\mathbb{R}^{\dim\ G - \dim\ N}$ in the last entries. A_M will provide the non–trivial fibration of N over $\mathbb{R}^{\dim\ G - \dim\ N}$, the Mostow bundle fibration of the solvmanifold for the compact case. In turn, A_N will provide fibrations inside N, the fibrations within the nilmanifold piece for the compact case. If the solvable group is nilpotent, then we take A_M to be the identity. To explicitly construct the matrices A_M and A_N we will now restrict ourselves to $G = N$ (nilpotent) or $G = \mathbb{R} \ltimes_\mu N$ (almost nilpotent).

4.2.4 Mostow bundle structure: A_M

From the Mostow bundle, (4.8), it is natural to identify the direction x^6 with the $\mathbb{R}$ subalgebra. We thus take a coordinate t parametrizing the $\mathbb{R}$ subalgebra with basis $\partial_t = \partial_6$ and the corresponding one–form $\mathrm{d}x^6 = \mathrm{d}t$. We define

$$A_M = Ad_{e^{-t\partial_t}}(\mathfrak{n}) = e^{-t\ ad_{\partial_t}(\mathfrak{n})} , \tag{4.26}$$

and

$$e^i = (A_M)^i{}_k\, \mathrm{d}x^k . \tag{4.27}$$

Let us prove that this action will give forms which do verify the Maurer–Cartan equations. Consider first the simpler case of an almost abelian group, i.e. with $N = \mathbb{R}^5$, which has $A_N = \mathbb{I}_N$. Then

$$\begin{aligned} \mathrm{d}e^i &= \mathrm{d}(e^{-t\ ad_{\partial_t}})^i{}_k \wedge \mathrm{d}x^k \\ &= -\mathrm{d}t \wedge (ad_{\partial_t} e^{-t\ ad_{\partial_t}})^i{}_k \mathrm{d}x^k \\ &= -\mathrm{d}t \wedge (ad_{\partial_t})^i{}_j (e^{-t\ ad_{\partial_t}})^j{}_k \mathrm{d}x^k \\ &= -\mathrm{d}t \wedge (ad_{\partial_t})^i{}_j e^j \\ \mathrm{d}e^i &= -f^i{}_{tj}\ \mathrm{d}t \wedge e^j . \end{aligned} \tag{4.28}$$

[3]Whether the exterior derivative is defined on these new forms will not be treated (see Footnote 1): we will just define it as the exterior derivative of $\mathbb{R}^6$ acting on the left–hand side of (4.24).

The fact that we used the adjoint action allows to easily verify the Maurer–Cartan equations.

Expression (4.26) for the matrix A_M holds also for the more general case of almost nilpotent algebras. In this case the Maurer–Cartan equations have component in direction $\mathrm{d}t$ and also in the directions of the nilradical. The t dependence is always determined by A_M and hence it is not modified by the presence of a non–trivial nilradical.

4.2.5 Nilmanifold fibration structure: A_N

In section 4.2.1 we have presented how the structure of successive fibrations is related to the properties of the algebra, with special attention for the nilradical part. We show here how to construct the corresponding twist matrix. The matrix giving a single fibration was worked out in [5], we recover this result here. In the general case of an iteration, we consider a product of several operators, each of them giving one fibration of the iteration:

$$A_N = A_{p-1} \dots A_1 \ , \ A_i = e^{-\frac{1}{2}f_i} \qquad (\text{for } p=1,\ \mathfrak{n}=\mathbb{R}^5 \text{ and } A_N = 1) \ ,$$

with $f_i \in \mathrm{End}(\mathfrak{n})$:

$$\begin{aligned} \text{For } i = 1 \dots p-1 \ , \ f_i : \mathfrak{n} \ &\to \ \mathfrak{n} \\ X \ &\mapsto \ Y = ad_{\mathcal{B}^i}(X) \text{ if } X \in \mathcal{B}^i \text{ and } ad_{\mathcal{B}^i}(X) \in \mathcal{F}^i \ , \\ & \quad\ \ Y = 0 \text{ otherwise} \ . \end{aligned} \tag{4.29}$$

We choose to give a basis of $\mathfrak{n}$ in the order given by $s^1, s^2, \dots, s^p$, and in each s^k we can choose some order for the elements. Then in that basis, f_i, as a matrix, is an off-diagonal block with lines corresponding to $\mathcal{F}^i = s^{i+1}$ and columns to $\mathcal{B}^i = \bigcup_{k=1\dots i} s^k$. Then A_i is the same plus the identity. Furthermore, the block depends on parameters a^j of a generic element $a^j E_j$ of $\mathcal{B}^i$, and we have $ad_{a^j E_j \in \mathcal{B}^i} = a^j ad_{E_j \in \mathcal{B}^i}$. So for instance for the algebra $(0, 31, -21, 23, 24)$ we have previously taken as an example, we get:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}a^3 & -\frac{1}{2}a^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}a^4 & 0 & -\frac{1}{2}a^2 & 1 \end{pmatrix} \ . \tag{4.30}$$

The parameters a^j can be understood as a coordinate along E_j, so they are such that $\mathrm{d}a^j = e^j$, dual of E_j.

Let us prove that the operator A_r gives the fibration of directions of $\mathcal{F}^r$ over a base $\mathcal{B}^r$, and the correct corresponding Maurer–Cartan equation. As explained, an element of A_r is given by:

$$(A_r)^i{}_k = \delta^i_k - \frac{1}{2} \sum_{j \in \mathcal{B}^r} a^j (ad_{E_j})^i{}_k \ \Theta(i \in \mathcal{F}^r)\Theta(k \in \mathcal{B}^r) = \delta^i_k - \frac{1}{2} \sum_{j,k \in \mathcal{B}^r} a^j f^i{}_{jk} \ \Theta(i \in \mathcal{F}^r) \ . \tag{4.31}$$

The forms on which we act with A_r at the step r of the iteration are labelled e^k, and they become after the operation $\tilde{e}^i$:

$$\tilde{e}^i = (A_r)^i{}_k \, e^k \ . \tag{4.32}$$

The directions we fiber with A_r are initially not fibered, so $e^{k\in\mathcal{F}^r} = \mathrm{d}x^k$. All the other directions are not modified by A_r, so in particular $\tilde{e}^{i\in\mathcal{B}^r} = e^{i\in\mathcal{B}^r}$. Thus the Maurer–Cartan equations of the forms not in $\mathcal{F}^r$ are not modified at this step. Their equation is then only modified at the step when they are fibered, so we don't have to consider it here. For the directions $\mathcal{F}^r$, we get:

$$\begin{aligned}\tilde{e}^{i\in\mathcal{F}^r} &= e^{i\in\mathcal{F}^r} - \frac{1}{2}\sum_{j,k\in\mathcal{B}^r} a^j f^i{}_{jk}\, e^k \\ &= \mathrm{d}x^i - \frac{1}{2}\sum_{j,k} a^j f^i{}_{jk}\, e^k \ ,\end{aligned}$$

where we dropped the restriction $j,k \in \mathcal{B}^r$ because due to the iterated structure, for $i \in \mathcal{F}^r$, $f^i{}_{jk} = 0$ if k or $j \notin \mathcal{B}^r$. This operation then gives the fibration structure, since we can read the connection. We can verify that we have the correct Maurer–Cartan equation:

$$\begin{aligned}\mathrm{d}\tilde{e}^{i\in\mathcal{F}^r} &= -\frac{1}{2} f^i{}_{jk}\, \mathrm{d}a^j \wedge e^k \\ &= -\frac{1}{2} f^i{}_{jk}\, e^j \wedge e^k \\ \mathrm{d}\tilde{e}^i &= -\frac{1}{2} f^i{}_{jk}\, \tilde{e}^j \wedge \tilde{e}^k \ .\end{aligned}$$

4.2.6 Is this construction consistent?

We now come back to the consistency of this construction and the question of compactness. To this end we need to investigate the monodromy properties of the matrix A_M and the related one–forms under a complete turn around the base circle (we restrict to the case of almost nilpotent groups).

Let us consider the following identification: $t \sim t + t_0$ where t_0 is the periodicity of the base circle. To obtain a consistent construction (having globally defined one–forms) we must preserve the structure of the torus we are fibering over the t direction. This amounts to asking that an arbitrary point of the torus is sent to an equivalent one after we come back to the point t from which we started. The monodromies of the fiber are fixed, thus the only allowed shifts are given by their integer multiples. The way points in the torus are transformed when we go around the base circle is encoded in a matrix $M_{\mathcal{F}}$ which has to be integer valued. The identification along the t direction is given by

$$T_6 : \begin{cases} t \to t + t_0 \\ x^i \to (M_{\mathcal{F}})^i{}_j x^j \end{cases} \qquad i,j = 1,\ldots,5\,, \tag{4.33}$$

while those along the remaining directions are trivial

$$T_i : \begin{cases} x^i \to x^i + 1 \\ x^j \to x^j \\ t \to t \end{cases} \qquad i,j = 1,\ldots,5\,;\ i \neq j\,. \tag{4.34}$$

Let us now consider the one–forms (4.24) we have constructed via the twist A_M. It is straightforward to see that (4.24) are invariant under the trivial identifications, while under the non–trivial T_6, we have for $i,j = 1,\ldots,5$

$$\tilde{e}^i = A_M(t+t_0)^i{}_j \mathrm{d}\tilde{x}^j = [A_M(t)A_M(t_0)M_{\mathcal{F}}]^i{}_j \mathrm{d}x^j \ . \tag{4.35}$$

The one–forms are globally defined if they are invariant under this identification:

$$\tilde{e}^i = e^i = A_M(t)^i{}_j \mathrm{d}x^j \,. \tag{4.36}$$

Therefore, in the construction, we have to satisfy the following condition:

$$A_M(t_0) M_{\mathcal{F}} = \mathbb{1}_5 \Leftrightarrow M_{\mathcal{F}} = A_M^{-1}(t_0) = A_M(-t_0) \,. \tag{4.37}$$

Consistency requires the matrix A_M to be such that $A_M(-t_0)$ is integer valued for at least one $t_0 \neq 0$. This will impose a quantization condition on the period of the base circle, which can take only a discrete set of values (in general it will be a numerable set, as we will see in the examples). Once we fix t_0, the integer entries of $A_M(-t_0)$ will provide the set of identifications.

It is worth stressing that being able to give the correct identifications of the one–forms of the manifold is the same as having a lattice: the identifications (4.37) express the lattice action, and give globally defined one–forms only if $A_M(-t_0) = \mu(t_0)$ is integer valued for some t_0. As already discussed, this is the condition to have a lattice (as stated in [23]). Let us emphasize that the one–forms (4.24), constructed via the twist, are globally defined only if we start from a basis of the Lie algebra where $A_M(t)$ is integer valued for some value of t. We give a list of algebras in such a basis in Appendix D.2.

Note that obtaining a set of globally defined one–forms is an expected result, since we are transforming a six–torus into a solvmanifold, which we know to be parallelizable. Moreover, we also know that, with a consistent twist, we are not leaving the geometrical framework.

As an example, we write the explicit form of the twist matrix for the two almost abelian six–dimensional algebras we will need in the following chapters[4]. In the basis where the one–forms are globally defined the two algebras are

$$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R} \quad : \quad (q_1 25, q_2 15, q_2 45, q_1 35, 0, 0) \,, \tag{4.38}$$

$$\mathfrak{g}_{5.17}^{p,-p,\pm 1} \oplus \mathbb{R} \quad : \quad (q_1(p25 + 35), q_2(p15 + 45), q_2(p45 - 15), q_1(p35 - 25), 0, 0) \,. \tag{4.39}$$

In both cases the parameters q_1 and q_2 are strictly positive. This is not the most general form of these algebras, which in general[5] contain some free parameters p, q and r. Here we wrote the values of the parameters for which we were able to find a lattice: $p = -q = -r = 1$ for the first algebra and $r = \pm 1$ for the second.

In the rest of the thesis, by abuse of notation, we will denote the algebra and the corresponding solvmanifold with the same name.

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[4] We use the same notation as in the standard classification of solvable algebras [152, 181, 23]: the number 5 indicates the dimension of the (indecomposable) algebra, while the second simply gives its position in the list of indecomposable algebras of dimension 5.

[5] The general form for $\mathfrak{g}_{5.7}^{p,q,r}$ is

$$\frac{1}{2}\Big(-\beta(1+r)15 + q_1(1-r)25, -\beta(1+r)25 + q_2(1-r)15, -\beta(q+p)35 + q_2(p-q)45, -\beta(q+p)45 + q_1(p-q)35, 0 \Big) \,,$$

where we set $\beta = \sqrt{q_1 q_2}$. Similarly, for $\mathfrak{g}_{5.17}^{p,-p,r}$ we have

$$\Big(q_1 p25 + \frac{1}{2}[q_1(r^2+1)35 + \beta(r^2-1)45], q_2 p15 + \frac{1}{2}[q_2(r^2+1)45 + \beta(r^2-1)35], q_2(-15 + p45), q_1(-25 + p35), 0 \Big) \,.$$

For (4.38), a type IIA solution with O6 planes was found in [29]. The algebra being a direct product of a trivial direction and a five–dimensional indecomposable algebra, the adjoint matrix $ad_{\partial_{x^5}}(\mathfrak{n})$ is block–diagonal, with the non–trivial blocks given by $-ad_{\partial_t}(\mathfrak{n})$ in (4.22) and its transpose. Then the twist matrix is

$$A = \begin{pmatrix} A_M & \\ & \mathbb{I}_2 \end{pmatrix} \qquad A_M = \left(\begin{array}{cc|cc} \alpha & -\beta & & \\ -\gamma & \alpha & & \\ \hline & & \alpha & -\gamma \\ & & -\beta & \alpha \end{array}\right) , \tag{4.40}$$

where, not to clutter notation, we defined

$$\begin{aligned} \alpha &= \cosh(\sqrt{q_1 q_2}x^5) , \\ \beta &= \sqrt{\frac{q_1}{q_2}} \sinh(\sqrt{q_1 q_2}x^5) , \\ \gamma &= \sqrt{\frac{q_2}{q_1}} \sinh(\sqrt{q_1 q_2}x^5) . \end{aligned} \tag{4.41}$$

The forms obtained by the twist (4.40) are globally defined [100]. Indeed they are invariant under constant shifts of each x^i for $i = 1, 2, 3, 4$ and 6, with the other variables fixed, and the following non–trivial identification under shifts for x^5

$$(x^1, \dots, x^6) = (\alpha x^1 + \beta x^2, \gamma x^1 + \alpha x^2, \alpha x^3 + \gamma x^4, \beta x^3 + \alpha x^4, x^5 + l, x^6) , \tag{4.42}$$

where in α, β, γ we took $x^5 = l$. For the above identifications to be discrete [100] α, β, and γ must be all integers. This is equivalent to having the matrix $\mu(x^5 = l)$ integer and, hence, it is the same as the compactness criterion. The existence of a lattice for the solution in [29] was also discussed in [87]. In that case the parameters α, β and γ were set to $\alpha = 2$, $\beta = 3$, $\gamma = 1$.

For the second algebra, $\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$, we will consider separately the cases $p = 0$ and $p \neq 0$. For $p = 0$ it reduces to $(q_1 35, q_2 45, -q_2 15, -q_1 25, 0, 0)$ with $r^2 = 1$. This algebra and the associated manifold have been already considered in [87], where it was called s 2.5. For $p \neq 0$ the algebra can be seen as the direct sum

$$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R} \approx s\ 2.5 + p\ (\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}) . \tag{4.43}$$

The twist matrix is given by

$$A = \begin{pmatrix} A_1 A_2 & \\ & \mathbb{I}_2 \end{pmatrix} . \tag{4.44}$$

The two matrices A_1 and A_2 commute and give the two parts of the algebra

$$A_1 = \left(\begin{array}{cc|cc} \mathrm{ch} & -\eta\,\mathrm{sh} & & \\ -\frac{1}{\eta}\mathrm{sh} & \mathrm{ch} & & \\ \hline & & \mathrm{ch} & -\frac{1}{\eta}\mathrm{sh} \\ & & -\eta\,\mathrm{sh} & \mathrm{ch} \end{array}\right) \qquad A_2 = \left(\begin{array}{cc|cc} \mathrm{c} & & -\eta \mathrm{s} & \\ & \mathrm{c} & & -\frac{1}{\eta}\mathrm{s} \\ \hline \frac{1}{\eta}\mathrm{s} & & \mathrm{c} & \\ & \eta \mathrm{s} & & \mathrm{c} \end{array}\right) , \tag{4.45}$$

where now we define $\eta = \sqrt{\frac{q_1}{q_2}}$ and

$$\begin{aligned} \mathrm{ch} &= \cosh(p\sqrt{q_1 q_2}x^5) \qquad & \mathrm{c} &= \cos(\sqrt{q_1 q_2}x^5) \\ \mathrm{sh} &= \sinh(p\sqrt{q_1 q_2}x^5) \qquad & \mathrm{s} &= \sin(\sqrt{q_1 q_2}x^5) . \end{aligned}$$

In this case, imposing that the forms given by the twist (4.44) are globally defined under discrete identifications fixes the parameters in the twist to (with $x^5 = l$)

$$\begin{aligned} &\text{ch c} = n_1 \ , \ \eta \text{ sh c} = n_2 \ , \ \frac{1}{\eta}\text{sh c} = n_3 \\ &\text{sh s} = n_4 \ , \ \eta \text{ ch s} = n_5 \ , \ \frac{1}{\eta}\text{ch s} = n_6 \ , \quad n_i \in \mathbb{Z} \ . \end{aligned} \tag{4.46}$$

The equations above have no solutions if the integers n_i are all non–zero. The only possibilities are either $n_1 = n_2 = n_3 = 0$ or $n_4 = n_5 = n_6 = 0$ (plus the case where all are zero, which is of no interest here). If one also imposes that the constraints must be solved both for $p = 0$ and $p \neq 0$, the first option, $n_1 = n_2 = n_3 = 0$, has to be discarded and the only solution is

$$\begin{aligned} &n_4 = n_5 = n_6 = 0 \ , \ s = 0 \ , \ l = \frac{k\ \pi}{\sqrt{q_1 q_2}} \ , \ c = (-1)^k \ , \ \tilde{n}_1 = (-1)^k n_1 > 0 \ , \ k \in \mathbb{Z} \\ &\text{ch} = \tilde{n}_1 \ , \ \text{sh}^2 = n_2 n_3 \ , \ n_3 \eta^2 = n_2 \ , \ n_2 n_3 = \tilde{n}_1^2 - 1 \ , \ p = \frac{\cosh^{-1}(\tilde{n}_1)}{k\ \pi} \ . \end{aligned} \tag{4.47}$$

p is quantized by two integers, but one can show that it can be as close as we want to any real value (the ensemble is dense in $\mathbb{R}$).

4.3 Twist transformations in generalized geometry

The twist defined in the previous section has a natural embedding in generalized geometry. We briefly review here its main properties and we refer the reader to the original paper [5] for more details. In Chapter 3 we have introduced the generalized tangent bundle E and discussed many of its properties, in the following we will consider background such that $B = 0$ thus we can identify $E = TM \oplus T^*M$. Recall that E is endowed with two metrics

$$\mathcal{I} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad\qquad \mathcal{H} = \begin{pmatrix} g - Bg^{-1}g & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} , \tag{4.48}$$

where $\mathcal{I}$ is the natural metric on E (which is used to derive the Clifford algebra) while the generalized metric $\mathcal{H}$ encodes the information about the metric and the B–field of the background.

On E one can define generalized vielbeine $\mathcal{E}$, such that

$$\mathcal{I} = \mathcal{E}^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathcal{E} \qquad\qquad \mathcal{H} = \mathcal{E}^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \mathcal{E} \, . \tag{4.49}$$

Explicitly, the generalized vielbeine can be put in the form

$$\mathcal{E}^A_{\ M} = \begin{pmatrix} e^a_{\ m} & 0 \\ -(\hat{e}B)_{am} & \hat{e}_a^{\ m} \end{pmatrix} , \tag{4.50}$$

where $e^a_{\ m}$ are the vielbeine on M, $\hat{e} = (e^T)^{-1}$, and B is the B–field. Comparing the $\mathrm{O}(d,d)$ action on $\mathcal{E}$

$$\mathcal{E} \mapsto \mathcal{E}' = \mathcal{E}O = \begin{pmatrix} e^a_{\ m} & 0 \\ -(\hat{e}B)_{am} & \hat{e}_a^{\ m} \end{pmatrix} \begin{pmatrix} A^m_{\ n} & B^{mn} \\ C_{mn} & D_m^{\ n} \end{pmatrix} , \tag{4.51}$$

with (4.24), it is natural to embed the twist transformation as

$$O_{\rm tw} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} . \tag{4.52}$$

We focus on manifolds of dimension six and construct O(6, 6) bispinors from the Killing spinors on M, $\eta^{1,2}$ as in (3.54):

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\,\dagger}_\pm . \tag{4.53}$$

Here we will consider the SU(3) structure manifolds, which admit a single globally defined spinor η_+ of unitary norm. Hence

$$\eta^1_+ = |a|\, e^{i\alpha}\eta_+ \ , \ \eta^2_+ = |b|\, e^{i\beta}\eta_+ \, ,$$

where $|a|$ and $|b|$ are, as before, the norms of $\eta^{1,2}$. The corresponding pure spinors $\Psi_\pm$ on E (see discussion in Section 3.5.1) are

$$\begin{aligned} \Psi_+ &= e^{-\phi}e^{-B}\frac{\sqrt{8}}{||\Phi_+||}\Phi_+ \, , \\ \Psi_- &= e^{-\phi}e^{-B}\frac{\sqrt{8}}{||\Phi_-||}\Phi_- \, , \end{aligned}$$

with $\sqrt{8}||\Phi_\pm|| = |a|^2 = |b|^2$. The phases of the two pure spinor are $\theta_+ = \alpha - \beta$ and $\theta_- = \alpha + \beta$[6].

The O(d, d) action on pure spinors was first introduced in Section 3.3; it is given by its spinorial representation

$$O \cdot \Psi = e^{-\frac{1}{4}\Theta_{MN}[\Gamma^M,\Gamma^N]} \cdot \Psi \, , \tag{4.54}$$

where Γ^M are the Cliff(d, d) gamma matrices ($\Gamma^m = {\rm d}x^m$ and $\Gamma_m = \iota_m$) and Θ_{MN} are the O(d, d) parameters

$$\Theta_{MN} = \begin{pmatrix} a^m{}_n & \beta^{mn} \\ b_{mn} & -a_m{}^n \end{pmatrix} . \tag{4.55}$$

Here $a^m{}_n$, b_{mn} and β^{mn} parametrise the Gl(d) transformations, B–transforms and β–transform, respectively. Then the twist action (4.26) on the spinor reads [5]

$$O_{\rm tw} \cdot \Psi = \frac{1}{\sqrt{\det A}}\ e^{-t\ [ad_{\partial_t}(\mathfrak{n})]^m{}_n e^n \wedge \iota_m} \cdot \Psi \, , \tag{4.56}$$

where e^m is a given basis of one–forms on M, and ι_m the associated contraction.

4.3.1 Type IIA supersymmetric solutions from twist transformations

Type IIA supersymmetric compactifications to four–dimensional Minkowski where the internal manifold is the solvmanifold $\mathfrak{g}^{0,0,\pm1}_{5.17} \times S^1$ were found in [29, 87, 4]. As shown in Section 4.2.6, this manifold is related by twist to the more general manifold $\mathfrak{g}^{p,-p,\pm1}_{5.17} \times S^1$. It is then natural to ask what is the effect of twisting the solutions in [87, 4].

[6]Actually what matters is the relative phase between the two spinors $\eta^{1,2}$; we can express everything in terms of only one phase that is, in general, fixed by the orientifold projection. We already know that supersymmetry implies $|a| = |b|$, we can impose $a = \bar{b}$ and define the phase $e^{i\theta} = b/a$.

We will take as starting point Model 3 of [87]. This is an SU(3) structure solution with smeared D6–branes and O6–planes in the directions (146) and (236). For SU(3) structure, the two pure spinors are[7]

$$\Phi_+ = |a|^2 \frac{e^{i\theta}}{8}\, e^{-iJ} \qquad\qquad \Phi_- = -|a|^2 \frac{i}{8}\, \Omega\,. \tag{4.57}$$

The phase in Φ_+ is, in general, determined by the orientifold projection. For O6 planes θ is actually free and we set it to zero. We take

$$\Omega = \sqrt{t_1 t_2 t_3}\, \chi^1 \wedge \chi^2 \wedge \chi^3 \qquad\qquad J = \frac{i}{2} \sum_k t_k \chi^k \wedge \overline{\chi}^k\ , \tag{4.58}$$

with complex structure[8]

$$\begin{aligned}
\chi^1 &= e^1 + i\,\lambda \frac{\tau_3}{\tau_4}\ e^2\,,\\
\chi^2 &= \tau_3\, e^3 + i\tau_4\ e^4\,,\\
\chi^3 &= e^5 - i\tau_6\ e^6\ .
\end{aligned} \tag{4.60}$$

For simplicity, we introduce $\lambda = \frac{t_2 \tau_4^2}{t_1}$. e^i are globally defined one–forms, obtained as in (4.24)

$$e^m = (A_2)^m_{\ n} \mathrm{d}x^n\,, \tag{4.61}$$

with A_2 given by (4.45). With this choice the metric is diagonal

$$g = \mathrm{diag}\left(t_1, \lambda\, t_2\, \tau_3^2, t_2\, \tau_3^2, \lambda t_1, t_3, t_3 \tau_6^2\right)\ . \tag{4.62}$$

Positivity of the volume imposes the following constraints on the complex structure and Kähler moduli

$$\tau_6 > 0\ ,\ t_1,\ t_2,\ t_3 > 0\,. \tag{4.63}$$

Due to the presence of intersecting sources, the warp factor A is set to zero and the dilaton to a constant. By splitting the pure spinor equations (3.57) - (3.59) into forms of fixed degree, it is easy to verify that supersymmetry implies

$$\mathrm{d}(\mathrm{Im}\,\Omega) = 0\,, \tag{4.64}$$
$$\mathrm{d}J = 0\,, \tag{4.65}$$
$$\mathrm{d}(\mathrm{Re}\,\Omega) = g_s * F_2\,, \tag{4.66}$$
$$F_6 = F_4 = F_0 = H = 0\,. \tag{4.67}$$

The only non–zero RR flux reads

$$g_s F_2 = \frac{\sqrt{\lambda}\,(q_1 t_1 - q_2 t_2 \tau_3^2)}{\sqrt{t_3}} (e^3 \wedge e^4 - e^1 \wedge e^2)\,, \tag{4.68}$$

---

[7] See Appendix A.3 for our conventions on SU(3) structures.

[8] Ω and J are normalised as

$$\frac{4}{3} J^3 = i\Omega \wedge \overline{\Omega} = -8\ \mathrm{vol}_{(6)} = -8\ \sqrt{|g|}\ e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \tag{4.59}$$

where $\mathrm{vol}_{(6)}$ is the internal volume form.

and it is straightforward to check that its Bianchi identity is satisfied. Let us also recall [87, 130] the transformation the forms should satisfy under the O6–plane involution σ:

$$\sigma(J) = -J \ , \ \sigma(\Omega) = \overline{\Omega} \ , \ \sigma(H) = -H \ , \ \sigma(F) = \lambda(F) \ . \tag{4.69}$$

Given the directions of the sources here, these orientifold projection conditions are clearly verified by the solution.

Given the solution above, we want to use the twist action to produce solutions, still with O6–planes and D6–branes, on $\mathfrak{g}_{5.17}^{p,-p,\pm1} \times S^1$. The manifolds $\mathfrak{g}_{5.17}^{p,-p,\pm1} \times S^1$ and $\mathfrak{g}_{5.17}^{0,0,\pm1} \times S^1$ are related by the twist matrix A_1 in (4.45), whose adjoint matrix is

$$ad_{\partial_5}(\mathfrak{n})|_p = \begin{pmatrix} a_{12} & \\ & a_{34} \end{pmatrix} \qquad\qquad a_{12} = a_{34}^T = \begin{pmatrix} 0 & pq_1 \\ pq_2 & 0 \end{pmatrix} \ . \tag{4.70}$$

The sixth direction being a trivial circle, we identify $t = x^5$. Then the twist action on pure spinors,

$$\Phi_\pm \mapsto \Phi'_\pm = O_{\rm tw} \Phi_\pm \ , \tag{4.71}$$

can be rewritten as

$$\begin{aligned} O_{\rm tw} &= e^{-px^5(q_2 e^1 \wedge \iota_2 + q_1 e^2 \wedge \iota_1)}\, e^{-px^5(q_1 e^3 \wedge \iota_4 + q_2 e^4 \wedge \iota_3)} \\ &= O_{12} O_{34} \ , \end{aligned} \tag{4.72}$$

with

$$\begin{aligned} O_{12} &= \mathbb{1} + [\cosh(p\sqrt{q_1 q_2} x^5) - 1](e^1 \wedge \iota_1 + e^2 \wedge \iota_2 + 2e^1 \wedge e^2 \wedge \iota_1 \wedge \iota_2) \\ &\quad - \frac{1}{\sqrt{q_1 q_2}} \sinh(p\sqrt{q_1 q_2} x^5)(q_2 e^1 \wedge \iota_2 + q_1 e^2 \wedge \iota_1) \ , \end{aligned} \tag{4.73}$$

$$\begin{aligned} O_{34} &= \mathbb{1} + [\cosh(p\sqrt{q_1 q_2} x^5) - 1](e^3 \wedge \iota_3 + e^4 \wedge \iota_4 + 2e^3 \wedge e^4 \wedge \iota_3 \wedge \iota_4) \\ &\quad - \frac{1}{\sqrt{q_1 q_2}} \sinh(p\sqrt{q_1 q_2} x^5)(q_1 e^3 \wedge \iota_4 + q_2 e^4 \wedge \iota_3) \ . \end{aligned} \tag{4.74}$$

Note that unimodularity of the algebra implies $\det(A) = 1$. In comparison to the procedure described in [5], here we do not introduce a phase in the twist operator, since we do not modify the nature of the fluxes and sources.

It is straightforward to check that the transformed pure spinors have formally the same expression as in (4.57) - (4.60) but with the one-forms e^i now given by

$$e^m = (A_1 A_2)^m{}_n {\rm d}x^n \ . \tag{4.75}$$

Also the metric, which is completely specified by the pure spinors, has the same form as for the initial solution, but in the new e^i basis

$$g = {\rm diag}\left(t_1, \lambda t_2 \tau_3^2, t_2 \tau_3^2, \lambda t_1, t_3, t_3 \tau_6^2\right) \ . \tag{4.76}$$

In order for the twist transformation to produce new solutions, the transformed pure spinors should again satisfy the supersymmetry equations

$$\begin{aligned} {\rm d}_{H'}(\Phi'_+) &= 0 \ , \\ {\rm d}_{H'}({\rm Re}\Phi'_-) &= 0 \ , \\ {\rm d}_{H'}({\rm Im}\Phi'_-) &= g_s \ R' \ , \end{aligned} \tag{4.77}$$

where R' is the new RR field $R = \frac{1}{8} * \lambda(F)$. The conditions

$$H' = 0 \qquad\qquad \mathrm{d}J' = 0 \tag{4.78}$$

are automatically satisfied, so that the first two equations in (4.77) reduce to[9]

$$0 = \mathrm{d}(\mathrm{Im}\,\Omega') = -p(\lambda - 1)\ \tau_3\tau_6\sqrt{t_1 t_2 t_3}\ (q_2\ e^1 \wedge e^4 \wedge e^5 + q_1\ e^2 \wedge e^3 \wedge e^5) \wedge e^6 \,. \tag{4.80}$$

From this we see that, in addition to $p = 0$ case, supersymmetric solutions exist for $p \neq 0$ provided $\lambda = 1$.

The last equation in (4.77) defines the transformed RR field

$$g_s R' = g_s O_{\mathrm{tw}} \cdot R + \mathrm{d}_{H'}(O_{\mathrm{tw}}) \cdot \mathrm{Im}\,\Phi_- \ . \tag{4.81}$$

Since the twist operator does not change the degree of forms, it follows from (4.81) that no new RR fluxes have been generated

$$F_0 = F_4 = F_6 = 0\,, \tag{4.82}$$

and (we have already set $\lambda = 1$)

$$g_s F_2 = \frac{q_1 t_1 - q_2 t_2 \tau_3^2}{\sqrt{t_3}}(e^3 \wedge e^4 - e^1 \wedge e^2) + \frac{p(q_1 t_1 + q_2 t_2 \tau_3^2)}{\sqrt{t_3}}(e^2 \wedge e^4 + e^1 \wedge e^3)\,. \tag{4.83}$$

The Bianchi identity for F_2 is satisfied

$$g_s \mathrm{d}F_2 = c_1 v^1 + c_2 v^2 \ , \tag{4.84}$$

with $v^1 = t_1\sqrt{t_3}\ e^1 \wedge e^4 \wedge e^5$ and $v^2 = t_2\tau_3^2\sqrt{t_3}\ e^2 \wedge e^3 \wedge e^5$ being the covolumes of the sources in (236) and (146). Let us note that the orientifold projection conditions (4.69) are again satisfied with such sources. The sign of the charges

$$\begin{aligned} c_1 &= \frac{2q_2}{t_3 t_1}\left[t_1 q_1(1-p^2) - (1+p^2) t_2 q_2 \tau_3^2\right] \\ c_2 &= \frac{2q_1}{t_3 t_2 \tau_3^2}\left[\tau_3^2 t_2 q_2 (1-p^2) - (1+p^2) t_1 q_1\right] \end{aligned} \tag{4.85}$$

depends on the parameters, but the sum of the two charges is clearly negative. This guarantees that the transformed background with $p \neq 0$ and $\lambda = 1$ is indeed a solution of the full set of ten–dimensional equations of motion. In Chapter 5 we will use the non–supersymmetric version, with $\lambda \neq 1$, as starting point for our search for de Sitter solution.

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[9] Note that a slightly more general solution given by $\chi^1 = e^1 + i\left(\frac{\tau_3}{\tau_4}\lambda\ e^2 - \frac{\tau_2}{\tau_4}\ e^3\right)$, $\chi^2 = \tau_2\ e^2 + \tau_3\ e^3 + i\tau_4\ e^4$ and the same χ^3 leads to the same $\mathrm{d}(\mathrm{Im}\,\Omega')$ and to

$$\mathrm{d}(J') = -p(\lambda - 1)\ \tau_2\sqrt{\frac{t_1 t_2}{\lambda}}\ (q_2\ e^1 \wedge e^4 \wedge e^5 + q_1\ e^2 \wedge e^3 \wedge e^5)\ . \tag{4.79}$$

A supersymmetric solution, requiring $\mathrm{d}(\mathrm{Im}\Omega) = \mathrm{d}J = 0$, needs $\lambda = 1$. For $\tau_2 = 0$ we can have non–supersymmetric configurations with a closed J'.

In the literature, de Sitter backgrounds are given in terms of SU(3) structure torsions,

$$\begin{aligned} \mathrm{d}J &= \frac{3}{2}\,\mathrm{Im}(\bar{W}_1\Omega) + W_4\wedge J + W_3 \\ \mathrm{d}\Omega &= W_1 J^2 + W_2\wedge J + \bar{W}_5\wedge\Omega\,, \end{aligned} \tag{4.86}$$

where W_1 is a complex scalar, W_2 is a complex primitive $(1,1)$ form, W_3 is a real primitive $(2,1)+(1,2)$ form, W_4 is a real vector and W_5 is a complex $(1,0)$ form. For the more general SU(3) structure solution ($p\neq 0$, $\lambda\neq 1$, $\tau_2\neq 0$) mentioned in Footnote 9, we obtain

$$\begin{aligned} W_1 &= \frac{p\,\tau_2\,(A+B)(1-\lambda)}{6(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2t_3}} \\ W_2 &= \frac{1}{6(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2t_3}}\Big[-it_1\Big(p\tau_2\,(A+B)(\lambda+2)+3\lambda\tau_3(A-B)\Big)\chi^1\wedge\bar{\chi}^1 + \\ &\quad +3\sqrt{\lambda t_1t_2}\Big(\tau_2(B-A)+p\tau_3(\lambda A+B)\Big)\chi^1\wedge\bar{\chi}^2 - 3\sqrt{\lambda t_1t_2}\Big(\tau_2(B-A)+p\tau_3(A+\lambda B)\Big)\chi^2\wedge\bar{\chi}^1 + \\ &\quad +it_2\Big(p\tau_2(A+B)(1+2\lambda)+3\lambda\tau_3(A-B)\Big)\chi^2\wedge\bar{\chi}^2 - ip\tau_2t_3(A+B)(\lambda-1)\chi^3\wedge\bar{\chi}^3\Big] \\ W_3 &= \frac{ip\tau_2(\lambda-1)}{8(\tau_2^2+\lambda\tau_3^2)}\Big[(A+B)\chi^1\wedge\chi^2\wedge\bar{\chi}^3 - (A+B)\chi^3\wedge\bar{\chi}^1\wedge\bar{\chi}^2 + \\ &\quad -(A-B)(\chi^1\wedge\chi^3\wedge\bar{\chi}^2 - \chi^1\wedge\bar{\chi}^2\wedge\bar{\chi}^3 + \chi^2\wedge\chi^3\wedge\bar{\chi}^1 - \chi^2\wedge\bar{\chi}^1\wedge\bar{\chi}^3)\Big] \\ W_4 &= 0 \\ W_5 &= \frac{ip\sqrt{\lambda}\tau_3(A+B)(\lambda-1)}{4(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2}}\,\chi^3\,, \end{aligned} \tag{4.87}$$

with $A=q_1t_1$, $B=q_2t_2(\tau_3^2+\frac{\tau_2^2}{\lambda})$.

Localizing the sources and warping

The supersymmetric solution discussed in the previous section is global, the warp factor and the dilaton being constant. It is an interesting question to see whether localised solutions also exist (see e.g. [61] for a discussion about the importance of warping or [20] for general considerations about smearing vs. localization). The strategy for finding localized solutions used in [87] was first to look for a smeared solution at large volume and then localize it by scaling the vielbeine, longitudinal and transverse with respect to the source, with e^A and e^{-A}, respectively. This procedure works in a number of cases, provided only parallel sources are present. Unfortunately this is not the case for the supersymmetric solution we took as a departure point for our construction - the intersecting O6/D6 solution on $s\,2.5$.

It is however possible to find a completely localised solution on $s\,2.5$ with O6 planes. The solution has a simpler form in a basis where the algebra is $(25,-15,r45,-r35,0,0)$, $r^2=1$. In this basis the O6–plane is along the directions (345).

The SU(3) structure is constructed as in (4.58) with

$$\begin{aligned}\chi^1 &= e^{-A}e^1 + ie^A(\tau_3 e^3 + \tau_4 e^4)\,,\\ \chi^2 &= e^{-A}e^2 + ie^A r(-\tau_4 e^3 + \tau_3 e^4)\,,\\ \chi^3 &= e^A e^5 + ie^{-A} r\tau_6 e^6\,,\\ \tau_6 &> 0\,,\ t_1 = t_2,\ t_3 > 0\,,\end{aligned} \tag{4.88}$$

where the non–trivial warp factor, A, depends on x^1, x^2, x^6. The metric is diagonal

$$g = \text{diag}\left(t_1 e^{-2A}, t_1 e^{-2A}, t_1(\tau_3^2+\tau_4^2)e^{2A}, t_1(\tau_3^2+\tau_4^2)e^{2A}, t_3 e^{2A}, t_3\tau_6^2 e^{-2A}\right)\,, \tag{4.89}$$

and the only non–zero flux is the RR two–form

$$g_s F_2 = -r\Big[\tau_6\sqrt{t_3}\partial_1(e^{-4A})\,\mathrm{d}x^2\wedge e^6 - \tau_6\sqrt{t_3}\partial_2(e^{-4A})\,\mathrm{d}x^1\wedge e^6 + \frac{1}{\tau_6}\sqrt{\frac{t_1^2}{t_3}}\partial_6(e^{-4A})\,\mathrm{d}x^1\wedge \mathrm{d}x^2\Big]\,. \tag{4.90}$$

Setting the parameters $t_1 = t_2$ in the Kähler form (4.58) allows to have a single source term in the F_2 Bianchi identity

$$g_s \mathrm{d}F_2 \sim e^{-A}\Delta(e^{-4A})e^1\wedge e^2\wedge e^6\,, \tag{4.91}$$

where Δ is the laplacian with unwarped metric.

As $A \to 0$ this solution becomes fluxless ($s\,2.5$ can indeed support such solutions), hence it cannot be found following the strategy of localizing the large volume smeared solutions. Unfortunately this solution does not satisfy the twist to $p \neq 0$, (4.39), since for $p \neq 0$ the action of the involution of an O6–plane with a component along direction 5 is not compatible with the algebra.

4.3.2 Twist, non–compactness and non–geometric backgrounds

We want to come back to the question of consistency of the twist transformation. As explained already, the transformation is obstructed unless the matrix A is conjugated to an integer–valued matrix. In many cases, the twist can result in a topology change similar to what is achieved by T–duality. The latter also can be obstructed, and yet these obstructions do not stop us from performing the duality transformation. So what about the obstructed twist?

To keep things simple, let us consider again an almost abelian algebra and the gluing under $t \to t + t_0$. We should have in general

$$T_6 : \begin{cases} t \to t + t_0 \\ x^i \to \tilde{A}_M(-t_0)^i{}_j x^j \end{cases} \qquad i,j = 1,\ldots,5\,, \tag{4.92}$$

where $\tilde{A}_M(-t_0)$ is necessarily an integer–valued matrix for $t_0 \neq 0$. In the case of compact solvmanifolds this matrix is given by (4.25). For the algebras that do not admit an action of a lattice, $\tilde{A}_M(-t_0)$ has nothing to do with the algebra. Then the one forms $e^i = A(t)^i{}_j \mathrm{d}x^j$ ($\mathrm{d}x^6 = \mathrm{d}t$) are defined only locally and have discontinuities under $t \to t + t_0$. These kinds of discontinuity are actually familiar from the situations when an obstructed T–duality is performed, and are commonly referred to as non–geometric backgrounds. One way to see this is to work on the generalized tangent bundle and use local $\mathrm{O}(6) \times \mathrm{O}(6)$ transformations (for six–dimensional internal manifolds) to bring the generalized vielbeine to the canonical lower diagonal form (4.50). In geometric backgrounds, this is a good transformation, while in the non–geometric case it involves non–single valued functions [88].

As an example, let us consider the manifold $\mathfrak{g}_{4.2}^{-p} \times T^2$, where the algebra $\mathfrak{g}_{4.2}^{-p}$ is:

$$[E_1, E_4] = -pE_1 \ , \ [E_2, E_4] = E_2 \ , \ [E_3, E_4] = E_2 + E_3 \ , \ p \neq 0 \ . \tag{4.93}$$

or in the shorter notation $\mathfrak{g}_{4.2}^{-p} \times T^2 = (p14, -24 - 34, -34, 0, 0, 0)$.

The corresponding group does not admit a lattice. For generic p this is very easy to see since the group is not unimodular. For $p = 2$, the group is unimodular but there still is no lattice.

We have $\mathfrak{n} = \{E_1, E_2, E_3\}$ and $\partial_t = E_4$ (the algebra is almost abelian). Then, in the (E_1, E_2, E_3) basis,

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} p & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \ , \ \mu(t) = e^{t \ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} e^{pt} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & -te^{-t} & e^{-t} \end{pmatrix} . \tag{4.94}$$

Following [23], we are going to prove that this matrix cannot be conjugated to an integer matrix[10] except for $t = 0$. A way to verify if the matrix $\mu(t)$ can be conjugated to an integer one is to look at the coefficients of its characteristic polynomial $P(\lambda)$. This is independent of the basis in which it is computed, and hence, for the criterion to be satisfied it should have integer coefficients. Here we have:

$$P(\lambda) = (\lambda - e^{2t})(\lambda - e^{-t})^2 = \lambda^3 - \lambda^2(2e^{-t} + e^{2t}) + \lambda(e^{-2t} + 2e^t) - 1 \ . \tag{4.95}$$

The coefficients are given by sums and products of roots. We can use Lemma (2.2) in [102]. Let

$$P(\lambda) = \lambda^3 - k\lambda^2 + l\lambda - 1 \in \mathbb{Z}[\lambda] \, . \tag{4.96}$$

Then $P(\lambda)$ has a double root $\lambda_0 \in \mathbb{R}$ if and only if $\lambda_0 = +1$ or $\lambda_0 = -1$ for which $P(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1$ or $P(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1$ respectively.

In our case, we find the double root e^{-t}. This means the only way to have this polynomial with integer coefficients is to set $t = 0$ and therefore there is an obstruction to the existence of a lattice.

If we now consider the algebra together with its dual, i.e. examine the existence of a lattice on the generalized tangent bundle, we should study the 6×6 matrix $M(t) = \mathrm{diag}(\mu(t), \mu(-t)^T)$ instead of the matrix $\mu(t)$. One has

$$M(t) = \begin{pmatrix} e^{pt} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 \\ 0 & -te^{-t} & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-pt} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^t & te^t \\ 0 & 0 & 0 & 0 & 0 & e^t \end{pmatrix} . \tag{4.97}$$

[10] A naïve reason one could think of would be that it is due to the off–diagonal piece, but as we are going to show, this piece actually does not contribute.

For $t_0 = \ln(\frac{3+\sqrt{5}}{2})$ and $p \in \mathbb{N}^*$, $M(t = t_0)$ is conjugated to an integer matrix, $P^{-1}M(t_0)P = N$, where N is an integer matrix (Theorem 8.3.2 in [23]):

$$P = \begin{pmatrix} 1 & 0 & 0 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{2(2+\sqrt{5})}{3+\sqrt{5}} \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & 0 & \frac{2(2+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} & 0 \\ 1 & 0 & 0 & \frac{2}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & 0 & -\frac{(1+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} & 0 \\ 0 & -1 & 0 & 0 & 0 & \frac{1+\sqrt{5}}{3+\sqrt{5}} \end{pmatrix}, \tag{4.98}$$

$$N = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ a_{41} & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.99}$$

The piece

$$N_4 = \begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}^p \tag{4.100}$$

comes from the entries e^{pt} and the result can be obtained[11] from (4.19). We see that on the generalized tangent bundle the basic obstruction to the existence of a lattice is easily removed. Moreover it is not hard to see that, due to putting together the algebra and its dual, even the requirement of unimodularity can be dropped.

On the generalized tangent bundle we can therefore obtain a lattice. For non–geometry, one may ask for more: the integer matrix N being in $\mathrm{O}(3,3)$. This question can be decomposed into $N_4 \in \mathrm{O}(1,1)$ and the 4×4 integer matrix in $\mathrm{O}(2,2)$. Actually, the latter is true[12]. But $N_4 \notin \mathrm{O}(1,1)$. Moreover, one can prove that $\mathrm{diag}(e^{pt}, e^{-pt})$ can only be conjugated to an integer $\mathrm{O}(1,1)$ matrix for $t = 0$. Indeed, the eigenvalues of an integer $\mathrm{O}(1,1)$ matrices are ± 1, and those are not changed by conjugation.

This is reminiscent of the twist construction of the IIB background $n\,3.14$ discussed in [5]. The internal manifold is a circle fibration over a five manifold M_5, which itself is a bundle with a two–torus fiber, but the only obvious duality seen there is the $\mathrm{O}(2,2)$ associated with the two–torus. The solution on $M_5 \times S^1$ is obtained from IIB solution on $\mathbb{T}^6$ with a self–dual three–form flux, but not $n\,3.14$ itself [87].

By taking $p = 0$ in (4.97), we obtain a different topology. In $M(t)$ the corresponding direction becomes trivial, and we can forget about it. Up to an $\mathrm{O}(1,1)$ action, the non–trivial part of $M(t)$ can still be thought of as corresponding to the algebra on $T(\varepsilon_{1,1}) \oplus T^*(\varepsilon_{1,1})$. Indeed, $\varepsilon_{1,1}$ has two *local* isometries, and T–duality (the $\mathrm{O}(1,1)$ in question) with respect to any of them

[11] Another possible conjugation is given in (4.22). The other part of N, the 4×4 integer matrix, can also be different, see the change of basis in Proposition 7.2.9 in [23].

[12] Note it is not true for the one given in Proposition 7.2.9 of [23].

will yield a non–geometric background. This can be inferred by simply noticing that the result of the duality in (any direction) is not unimodular; more detailed discussion of T–duality on $\varepsilon_{1,1}$ can be found in Appendix B.

Chapter 5

Supersymmetry breaking sources and de Sitter vacua

In most of the vacua which have been found so far the external space is Minkowski or Anti de Sitter. One reason is due to the fact that, once solutions of the supersymmetry conditions (2.10) (which are first order) and of the Bianchi identities are found, one is guaranteed that the equations of motion are satisfied (or a subset of them, see discussion at the end of Section 3.5). Another reasons can be find in the interest for AdS vacua as gravity duals of gauge theories in the framework of AdS/CFT duality.

The status for vacua with a de Sitter space as external manifold is rather different, less attention has been devoted to them; they are not compatible with unbroken supersymmetry[1] and thus all the technical advantages are lost and one has to face the full set of equations which could be rather complicated to solve. Moreover it is seems that purely supergravity backgrounds solutions with a positive cosmological constant $\Lambda > 0$ require non trivial fine tuning of the geometric parameters and fluxes. Nevertheless in the last decade there has been a renewed interest in de Sitter backgrounds, mainly because, beyond being an interesting (and challenging) problem on their own [188, 66, 8], recent observations support the claim of a positive cosmological constant, [167, 163]. In this chapter we will investigate the problem in the context of type IIA theory, our analysis is carried with the aim of understanding some general properties of the issue trough the study of a concrete example and we advance no pretension of being general or to provide any phenomenological prediction. Since the famous [117], all known examples of (meta)stable de Sitter vacua require some additional non–perturbative ingredients such as KK monopoles and Wilson lines [176], non–geometric fluxes [54, 55], or α' corrections and probe branes [27, 160]. We would like to find a de Sitter solution directly in ten dimensions focusing only on simple conservative compactifications (i.e "geometric" set–up) and investigate a way of breaking supersymmetry by relaxing some conditions imposed on the sources. We would like also to make some first steps towards determining a set of first order equations for configurations where four–dimensional supersymmetry is broken.

[1] For $\mathcal{N} = 1$ vacua in four dimensions the potential has the following general form:

$$V = e^K \left(|DW|^2 - 3|W|^2\right) + D^2 \tag{5.1}$$

where W is the superpotential, K is the kähler potential and D are D–terms. Supersymmetry implies $D = 0$, $DW = 0$ and thus it is compatible only with negative (or zero) cosmological constant.

5.1 Setting the stage

Several no–go theorems and ways of circumventing them have been proposed [144, 112, 103, 176, 100, 52, 39, 67, 53]. In particular, in presence of O6/D6 sources, a minimal requirement to evade the no–go theorem [103] is to have a negatively curved internal manifold and a non–zero F_0 (Romans mass parameter) [144, 100, 39]. Therefore, we will focus on type IIA configurations with non–zero NS three–form and RR zero and two–forms. Moreover, we assume that all the sources (there may be intersecting ones) are space–time filling and are of the same dimension $p = 6$.

Tracing the four–dimensional part of Einstein equation and using the dilaton equation of motion, one can show that the four–dimensional curvature and the "source term" can be written as

$$R_4 = \frac{2}{3}\left[-R_6 - \frac{g_s^2}{2}|F_2|^2 + \frac{1}{2}(|H|^2 - g_s^2|F_0|^2)\right] , \tag{5.2}$$

$$g_s\frac{T_0}{p+1} = \frac{1}{3}\left[-2R_6 + |H|^2 + 2g_s^2(|F_0|^2 + |F_2|^2)\right] , \tag{5.3}$$

where, for simplicity, we have taken constant dilaton, $e^{\phi} = g_s$, and no warping[2].

Further simplifications are possible when one assumes that the sources preserve the supersymmetry of the bulk; as we have seen in Section 3.6 this condition is usually expressed in terms of an equation involving the bulk supersymmetry parameters and the world–volume chiral operator entering the κ–symmetry transformations. Up to terms quadratic in the κ–symmetry condition, one can always rewrite the brane world–volume action as the pullback of the non–integrable pure spinor

$$\left(i^*[\mathrm{Im}\,\Phi_2] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8}\sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^{\Sigma}x , \tag{5.6}$$

where i denotes the embedding of the world–volume into the internal manifold M, g is the internal metric and $\mathcal{F}$ is the gauge invariant combination of the field strength of the world–volume gauge field and the pullback of B. For sources preserving the supersymmetry of the bulk, one can then replace the DBI action by the left–hand side of (5.6). The equations of motion derived from both actions are the same, since the corrections would be linear in the κ–symmetry condition, and then vanishing in the supersymmetric case. In particular, one can show that the world–volume equations of motion are then automatically implied by equation (3.59). So the condition (5.6) together with equation (3.59) give (generalized) calibrated sources, i.e. their energy density is minimized [126, 149, 128, 130].

[2] In general, with non–trivial dilaton and a ten dimensional metric of the form

$$ds^2 = e^{2A(y)}g_{\mu\nu}(x)\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n , \tag{5.4}$$

equation (5.2) becomes

$$\begin{aligned} e^{-2A}R_4 &= \frac{2}{3}[-R_6 - \frac{e^{2\phi}}{2}|F_2|^2 + \frac{1}{2}(|H|^2 - e^{2\phi}|F_0|^2)] \\ &\quad -8\nabla^2 A + 20|\partial_m A|^2 - \frac{8}{3}\nabla^2\phi + \frac{8}{3}|\partial_m\phi|^2 - \frac{32}{3}g^{mn}\partial_m A\partial_n\phi . \end{aligned} \tag{5.5}$$

All derivatives are taken with respect to coordinates on M.

For such supersymmetric configurations, the four– and six–dimensional traces of the source energy–momentum tensor and the source term in the dilaton equation are all proportional to each other, and one arrives at

$$R_4 = \frac{2}{3}(g_s^2|F_0|^2 - |H|^2 - g_s^2|F_4|^2) \ , \tag{5.7}$$

$$R_6 + \frac{1}{2}g_s^2|F_2|^2 + \frac{3}{2}(g_s^2|F_0|^2 - |H|^2) - \frac{1}{2}g_s^2|F_4|^2 = 0 \, , \tag{5.8}$$

together with eq. (5.3). The last equation is just a constraint on internal quantities, while the two others fix R_4 and the source term T_0. From these two equations we recover the minimal requirement of having $F_0 \neq 0$ and $R_6 < 0$. In practice, however, this is not enough to find a de Sitter vacuum. In particular, we can see that F_0 alone can give a positive value to the cosmological constant, and adding more fluxes, F_4 and F_6, does not help since they give negative contributions.

Since we are interested in non–supersymmetric backgrounds, there is a priori no reason to impose that the sources preserve the bulk supersymmetry. The condition (5.6) could therefore be violated. To do so, we make the following proposal: we replace (5.6) by

$$\left(i^*[\operatorname{Im} X_-] \wedge e^{\mathcal{F}}\right) = \sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^\Sigma x \, , \tag{5.9}$$

where X_- is an odd polyform given by a general expansion in the generalized Hodge diamond[3]

For supersymmetric configurations, X_- reduces to $8\Phi_-$, but in general, it is no longer a pure spinor.

An advantage of replacing DBI by the pullback of a form from the bulk is that it is actually easier to take the variation with respect to the various fields, in particular the bulk ones. Moreover the variation of the left–hand side of (5.9) with respect to the metric will lead to interesting consequences for de Sitter solutions: new terms are generated in the energy momentum tensor which help to lift the cosmological constant to positive values as we will see when we will investigate the problem from a $4d$ perspective. This is the main motivation for using this proposal and we do not aim in this thesis to provide a full understanding. One possible interpretation is that such sources could be thought as standard D–branes or O–planes but their embedding into space–time (here into M) is modified. While for supersymmetric configurations the geometry of the subspace wrapped by the source is encoded in $\operatorname{Im}\Phi_2$, here it would be encoded in the more general expansion $\operatorname{Im} X_-$, of which $\operatorname{Im}\Phi_2$ is only one possible term. Therefore, the breaking of bulk supersymmetry seems to come from allowing more general geometries for the wrapped subspaces, and the new terms in the energy momentum tensor could come from the non standard embedding, in particular a dependence of the embedding functions on the metric moduli.

Since the bulk supersymmetry is broken, we could as well modify (3.58) and (3.59) thus, in view of (5.9), we propose here the following generalization of the first order conditions:

$$\mathrm{d}_H(e^{2A-\phi}\operatorname{Re} X_-) = 0 \, , \tag{5.10}$$

$$\mathrm{d}_H(e^{4A-\phi}\operatorname{Im} X_-) = c_0 e^{4A} * \lambda(F) \, , \tag{5.11}$$

where c_0 is a positive constant fixed by the parameters of the solution. Hence the introduction of X_- allows, as for the supersymmetric case, to trade the RR equations of motion for first order

[3]See Appendix A.5.

equations (clearly the equations of motion for the RR fluxes follow by differentiating (5.11) in the same way as for the susy case, see discussion at the end of Section 3.5), while, in addition, it helps via (5.9) to solve the internal Einstein equation. This is a first step towards developing a more systematic procedure to find non–supersymmetric backgrounds.

The idea of investigate which non–supersymmetric configurations can still be derived from a first order set of equations is certainly not original of this work. Recently a procedure has been proposed in [137] that generalizes to non-supersymmetric backgrounds the first order pure spinor equations. The idea of [137] is to decompose the supersymmetry breaking terms in the Spin(6,6) basis constructed from the pure spinors. For instance, for Minkowski compactifications, the modified first order equations are

$$\begin{aligned}
&\mathrm{d}_H(e^{2A-\phi}\Phi_1) = \Upsilon\,,\\
&\mathrm{d}_H(e^{A-\phi}\,\mathrm{Re}\,\Phi_2) = \mathrm{Re}\,\Xi\,,\\
&\mathrm{d}_H(e^{3A-\phi}\,\mathrm{Im}\,\Phi_2) - \frac{|a|^2}{8}e^{3A} * \lambda(F) = \mathrm{Im}\,\Xi\,,
\end{aligned} \tag{5.12}$$

where schematically

$$\begin{aligned}
\Upsilon &= a_0\Phi_2 + \widetilde{a}_0\overline{\Phi}_2 + a^1_m\gamma^m\Phi_1 + a^2_m\Phi_1\gamma^m + \widetilde{a}^1_m\gamma^m\overline{\Phi}_1 + \widetilde{a}^2_m\overline{\Phi}_1\gamma^m\\
&\quad + a_{mn}\gamma^m\Phi_2\gamma^n + \widetilde{a}_{mn}\gamma^n\overline{\Phi}_2\gamma^m\,, \qquad (5.13)\\
\Xi &= b_0\,\Phi_1 + \widetilde{b}_0\,\overline{\Phi}_1 + b^1_m\gamma^m\Phi_2 + b^2_m\Phi_2\gamma^m + b_{mn}\gamma^m\Phi_1\gamma^n + \widetilde{b}_{mn}\gamma^n\overline{\Phi}_1\gamma^m\,. \qquad (5.14)
\end{aligned}$$

In the particular case of an SU(3) structure, this decomposition is equivalent to the expansion of (3.57), (3.58) and (3.59) in the SU(3) torsion classes.

Equations (5.12) rely on the assumption that the four–dimensional space–time admits Killing spinors and that the supersymmetry breaking is due to the internal spinors only. This applies of course to Minkowski and Anti de Sitter backgrounds, but not for de Sitter solutions or cases when supersymmetry is broken in four–dimensions.

As discussed, replacing the source action as in (5.6) for sources preserving bulk supersymmetry is correct up to quadratic terms in the κ–symmetry condition, and corrections to the equations of motion derived from it will vanish linearly if the condition holds. In our case the structure of the corrections is not explicit, and we cannot conclude that the equations of motion derived from left–hand side of (5.9) are the same as those derived from DBI. We will thus proceed as follows: we first find solutions using the equations derived from the left–hand side of (5.9) and then we will check whether these are solutions to the equations derived from the standard DBI action.

For the NS–NS fields, we will check explicitly that our solution is a solution to the equations of motion derived from DBI, making use of a dependence of the embedding functions on the metric moduli. What remains are the world–volume fields (note that $\mathcal{F}$ will be trivial for us). Let us comment on their equations. As mentioned previously, for sources preserving bulk supersymmetry, a world–volume equation of motion, obtained by varying (5.6) augmented by the WZ terms, turns out to follow simply from a partial pullback of the bulk pure spinor equation. Then the minimization of the world–volume energy is automatic [126, 149]. The equations of motion derived from the left–hand side of (5.9) should also be compared with the partial pullback of (5.11). We shall denote the transverse differentiation by ∂_α and a flux with all but one index pulled back to the world–volume by $i^*[F]_\alpha$. Neglecting the world–volume gauge fields, we can write the resulting equation as

$$\partial_\alpha \left(i^*[e^{4A-\phi}\,\mathrm{Im}(e^{-B}X_-)] \right) - i^*[e^{4A}e^{-B} * \lambda(F)]_\alpha = 0 \ . \tag{5.15}$$

Comparison with the components of (5.11) gives

$$(c_0 - 1)\ i^*[e^{-B} * \lambda(F)]_\alpha = 0 \ . \tag{5.16}$$

In the supersymmetric case, where we replace X_- by Φ_-, $c_0 = 1$ and the equation is automatically satisfied. Here we will consider solutions with a vanishing partial pullback $i^*[e^{-B} * \lambda(F)]_\alpha$, so the world–volume equations derived from the left–hand side of (5.9) will be satisfied, making the energy of our sources extremized. We will also check that our solution satisfies the equations of motion obtained by the variation of the standard DBI+WZ action.

The strategy to find a non–supersymmetric solution to our proposed action is the following. We start with a particular solution to the pure spinor equations on a solvmanifold defined by the algebra $(q_1(p25+35), q_2(p15+45), q_2(p45-15), q_1(p35-25), 0, 0)$, which is the supersymmetric compactification of IIA discussed in Section 4.3.1. There we found that the pure spinor equations can be satisfied for $p \neq 0$ provided a certain combination of moduli, which we call λ, takes value 1. In other words, for generic p and $\lambda = 1$ we find supersymmetric solutions (corresponding to a vanishing four–dimensional curvature). For generic λ, the pure spinor equations are not satisfied and supersymmetry is broken. It is certainly of great practical importance to have a convenient limit in which our construction can be tested. The solution involves intersecting O6 planes (and possibly D6 branes - depending on the choice of parameters). Due to the general problems in constructing localized intersecting branes, the sources are smeared, and hence the model would suffer from general criticism [61, 21]. It does have some convenient features though, and it serves as a good illustration to the method we would like to propose.

The proposed source action (5.9) allows to rewrite (5.7) for the four dimensional Ricci tensor as

$$R_4 = \frac{2}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2 |F_0|^2 - |H|^2 \right) \ , \tag{5.17}$$

where the source term T_0 is different from the trace of the energy momentum tensor T. As can be seen from (5.3), T_0 gives a positive contribution to R_4 and in our case, it turns out that $T_0 - T$ is also positive. Thus, with our proposal (5.9) we are indeed able to find a ten–dimensional de Sitter solution. Checking that it also satisfies the equations of motion derived from the standard source action (with a dependence of the embedding functions on the metric moduli) will make it a solution of type IIA supergravity.

In order to derive the ten–dimensional equations of motion, we shall need source terms, and to this end let us consider the DBI action of only one Dp–brane in string frame

$$S_s = -T_p \int \mathrm{d}^{p+1}x\ e^{-\phi}\sqrt{|i^*[g_{10}] + \mathcal{F}|} \ , \qquad T_p^2 = \frac{\pi}{\kappa^2}(4\pi^2\alpha')^{3-p} \ .$$

Here T_p is the tension of the brane; for an O–plane, one has to replace T_p by $-2^{p-5}T_p$. The open string excitations will not be important for our solution, and we shall discard the $\mathcal{F}$ contribution from now on (note as well that the B–field will pull back to zero along the sources).

To derive the equations of motion, a priori, we should take a full variation of the DBI action with respect to the bulk metric. For supersymmetry preserving (calibrated) sources, there exists a convenient way of dealing with this. As discussed before one can think of an expansion of the DBI action around the supersymmetric configuration and, to leading order, replace the DBI action by a pullback of the calibration form. As shown in [130], this allows to prove that, for Minkowski compactifications, the equations of motion follow from the first order pure spinor equations, and the flux Bianchi identities. A similar treatment of space–time filling sources is also possible for non–supersymmetric Minkowski and AdS_4 configurations [137]. It is worth stressing that, even in these cases, the sources continue being (generalized) calibrated and are not responsible for the supersymmetry breaking. However convenient, as we shall see, these kinds of source are not going to be helpful in our search for a dS vacuum.

At this point we shall consider an important assumption: inspired by the supersymmetric case just described, we make a proposal for sources breaking the bulk supersymmetry. The latter can be applied in the case of an internal space with SU(3) structure, and the triviality of the canonical bundle is going to be important. We shall assume that, in analogy with the supersymmetric case, the DBI action can be replaced to leading order by the pullback of a (poly)form X in the bulk, as discussed around (5.9). The bulk does have invariant forms and hence pure spinors can be constructed, but X cannot be pure, otherwise the source would preserve bulk supersymmetry. The form X is expandable in the Hodge diamond defined by the pure spinors. This amounts to consider forms that are equivalent not to simply the invariant spinor η_+ (defining the SU(3) structure) but to a full spinorial basis, η_+, η_-, $\gamma^{\bar\imath}\eta_+$ and $\gamma^i\eta_-$, where $i, \bar\imath = 1, ...3$ are the internal holomorphic and antiholomorphic indices[4]. To be concrete we shall consider a generic odd form

$$\begin{aligned}
X &= \sqrt{|g_4|}\ \mathrm{d}^4x \wedge X_- = \sqrt{|g_4|}\ \mathrm{d}^4x \wedge (\mathrm{Re}\, X_- + i\, \mathrm{Im}\, X_-)\,,\\
X_- &= \mathrm{Re}\, X_- + i\, \mathrm{Im}\, X_- = \frac{8}{||\Phi_-||}\Big(\alpha_0\Phi_- + \tilde\alpha_0\overline{\Phi}_- + \alpha_{mn}\gamma^m\Phi_-\gamma^n + \tilde\alpha_{mn}\gamma^m\overline{\Phi}_-\gamma^n\\
&\qquad +\alpha^L_m\gamma^m\Phi_+ + \tilde\alpha^L_m\gamma^m\overline{\Phi}_+ + \alpha^R_n\Phi_+\gamma^n + \tilde\alpha^R_n\overline{\Phi}_+\gamma^n\Big)\,, \qquad (5.18)
\end{aligned}$$

where $\Phi_\pm$ are given in (4.57) and the γ's act on even and odd forms via contractions and wedges

$$\gamma^m\Phi_\pm = (g^{mn}\imath_n + \mathrm{d}x^m)\,\Phi_\pm\,, \qquad \text{and} \qquad \Phi_\pm\gamma^m = \mp(g^{mn}\imath_n - \mathrm{d}x^m)\,\Phi_\pm\,. \qquad (5.19)$$

The action for a single source term becomes

$$\begin{aligned}
S_s &= -T_p\int_\Sigma \mathrm{d}^{p+1}x\ e^{-\phi}\sqrt{|i^*[g_{10}]|}\\
&= -T_p\int_\Sigma\ e^{-\phi}i^*[\mathrm{Im}\, X]\\
&= -T_p\int_{M_{10}}\ e^{-\phi}\langle j_p, \mathrm{Im}\, X\rangle\\
&= T_p\int_{M_{10}}\mathrm{d}^{10}x\sqrt{|g_{10}|}\ e^{-\phi}\hat{*}\langle j_p, \mathrm{Im}\, X\rangle\ , \qquad (5.20)
\end{aligned}$$

[4]The covariant derivative on the invariant spinor contains the same information as the intrinsic torsions. For the explicit dictionary for SU(3) structure see [65]. In the supersymmetric backgrounds the (H–twisted) derivative on the spinor cancels against the RR contribution [85], and the entire content of that cancellation is captured by first order equation on the pure spinors (3.57)-(3.59). For the non–supersymmetric backgrounds, the unbalance between the NS and RR contributions results in the presence of terms that need to be expanded in the full basis (see e.g. [137]).

where $i : \Sigma \hookrightarrow M_{10}$ is the embedding of the subspace Σ wrapped by the source in the bulk and $j_p = \delta(\Sigma \hookrightarrow M_{10})$ is the dimensionless Poincaré dual of Σ. The change of sign between the last two lines is due to the Lorentzian signature which gives a minus when taking the Hodge star. For the sum of all sources we then take the action

$$S_s = T_p \int_{M_{10}} \mathrm{d}^{10}x \sqrt{|g_{10}|}\ e^{-\phi} \hat{*} \langle j, \mathrm{Im}\, X \rangle \ , \qquad j = \sum_{Dp} j_p - \sum_{Op} 2^{p-5} j_p \, . \tag{5.21}$$

As stated before, our interpretation is that sources remain standard D–branes or O–planes, but their embedding into M, in particular the form which describes the subspace wrapped by them, is modified from $\mathrm{Im}\, \Phi_-$ to the more general $\mathrm{Im}\, X_-$. The difference with the supersymmetric case is that we are not sure anymore that the equations of motion derived from both actions are the same. Our procedure will consist in finding solutions to the equations derived from the proposed source action, which are much easier to deal with. We will then argue that these solutions are also solutions of the equations derived from the standard source action. Until this is done in Section 5.2.3, we mean by solution a solution to the equations of motion derived with our proposed source action.

In the following, we will consider solutions where the only non–trivial fluxes are H, F_0 and F_2 on the internal manifold, and the RR magnetic sources are $D6$'s and $O6$'s. The sources will be smeared, so we take $\delta \to 1$ and the warp factor $e^{2A} = 1$. The relevant part of the action[5], in string frame, is then

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{|g_{10}|}\ [e^{-2\phi}(R_{10} + 4|\nabla\phi|^2 - \frac{1}{2}|H|^2) - \frac{1}{2}(|F_0|^2 + |F_2|^2) + 2\kappa^2 T_p\ e^{-\phi} \hat{*} \langle j, \mathrm{Im}\, X \rangle] \ , \tag{5.22}$$

where $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$.

With the flux ansatz (2.12), the flux equations of motion and Bianchi identities reduce to the six–dimensional equations

$$\begin{aligned}
&\mathrm{d}H = 0 \, , \\
&\mathrm{d}F_0 = 0 \ , \\
&\mathrm{d}F_2 - H \wedge F_0 = 2\kappa^2 T_p\ j \ , \\
&H \wedge F_2 = 0 \ , \\
&\\
&\mathrm{d}(e^{-2\phi} * H) = -F_0 \wedge *F_2 - e^{-\phi}\ 4\kappa^2 T_p\ j \wedge \mathrm{Im}\, X_1 \, , \\
&\mathrm{d}(*F_2) = 0 \, ,
\end{aligned}$$

[5]By relevant we mean the parts of the bulk and source actions that give non–trivial contributions to the Einstein and dilaton equations of motion and to the derivation of the four–dimensional effective potential of Section 5.3. We do not write down the Chern–Simons terms of the bulk action and the Wess–Zumino part of the source action. Indeed they do not have any metric nor dilaton dependence and, since we do not allow for non–zero values of RR gauge potentials in the background, they will not contribute to the vacuum value of the four–dimensional potential either. However, both terms contribute the flux e.o.m. and Bianchi identities (in particular, see [118, 57, 36] for a discussion of the Chern–Simons terms in the presence of non–trivial background fluxes).

where $\mathrm{Im}\, X_1$ is the one–form part of $\mathrm{Im}\, X_-$ in (5.18)[6].

The ten–dimensional Einstein and dilaton equations in string frame now become

$$R_{MN} - \frac{g_{MN}}{2} R_{10} = 2g_{MN}(\nabla^2\phi - 2|\nabla\phi|^2) - 2\nabla_M\nabla_N\phi + \frac{1}{4} H_{MPQ} H_N{}^{PQ} + \frac{e^{2\phi}}{2} F_{2\ MP} F_{2\ N}{}^P$$
$$- \frac{g_{MN}}{2}\left(-4|\nabla\phi|^2 + \frac{1}{2}|H|^2 + \frac{e^{2\phi}}{2}(|F_0|^2 + |F_2|^2)\right) + e^{\phi}\ \frac{1}{2} T_{MN}\ , \quad (5.23)$$

$$8(\nabla^2\phi - |\nabla\phi|^2) + 2R_{10} - |H|^2 = -e^{\phi}\frac{T_0}{p+1}\ . \quad (5.24)$$

Here T_{MN} and T_0 are the source energy momentum tensor and its partial trace, respectively[7]

$$T_{MN} = 2\kappa^2 T_p\, \hat{*}\langle j, g_{P(M} dx^P \otimes \iota_{N)}\, \mathrm{Im}\, X\ - \delta^m_{(M} g_{N)n} C_m{}^n\rangle\ , \quad (5.26)$$
$$T_0 = 2\kappa^2 T_p\, \hat{*}\langle j, dx^N \otimes \iota_N\, \mathrm{Im}\, X\rangle = (p+1)\ 2\kappa^2 T_p \hat{*}\langle j, \mathrm{Im}\, X\rangle\ , \quad (5.27)$$
$$T = g^{MN} T_{MN} = T_0\ - 2\kappa^2 T_p\, \hat{*}\langle j, C_m{}^m\rangle\ . \quad (5.28)$$

m, n are real internal indices, $C_m{}^n = \sqrt{|g_4|}\ \mathrm{d}^4x \wedge c_m{}^n$ and

$$c_m^n = \frac{8}{||\Phi_-||}\, \mathrm{Im}\left(\alpha^L_m \gamma^n \Phi_+ + \tilde{\alpha}^L_m \gamma^n \overline{\Phi}_+ + \alpha^R_m \Phi_+ \gamma^n + \tilde{\alpha}^R_m \overline{\Phi}_+ \gamma^n\right.$$
$$\left. + \alpha_{pm}\gamma^p \Phi_- \gamma^n + \alpha_{mp}\gamma^n \Phi_- \gamma^p + \tilde{\alpha}_{pm}\gamma^p \overline{\Phi}_- \gamma^n + \tilde{\alpha}_{mp}\gamma^n \overline{\Phi}_- \gamma^p\right)\ . \quad (5.29)$$

For supersymmetric configurations, $\mathrm{Im}\, X_- = 8\, \mathrm{Im}\Phi_-$, $c_m{}^n = 0$, T_0 reduces to the full trace of the source energy–momentum tensor, $T = T_0$ and one recovers the formulae in [130].

We can now split (5.23) into its four and six–dimensional components. Since for maximally symmetric spaces, $R_{\mu\nu} = \Lambda g_{\mu\nu} = (R_4/4) g_{\mu\nu}$, for constant dilaton, $e^{\phi} = g_s$, the four–dimensional Einstein equation has only one component and reduces to

$$R_4 = -2R_6 + |H|^2 + g_s^2(|F_0|^2 + |F_2|^2) - 2g_s \tilde{T}_0 = 4\Lambda\ . \quad (5.30)$$

Not to clutter equations, in the rest of the thesis we set $\tilde{T}_0 = T_0/(p+1)$.

This equation defines the cosmological constant, Λ. Using the dilaton equation (5.24), the source contribution can be eliminated and we obtain

$$R_4 = \frac{2}{3}[-R_6 - \frac{g_s^2}{2}|F_2|^2 + \frac{1}{2}(|H|^2 - g_s^2|F_0|^2)]\ , \quad (5.31)$$
$$R_{10} = \frac{1}{3}[R_6 + |H|^2 - g_s^2(|F_0|^2 + |F_2|^2)]\ . \quad (5.32)$$

[6] We refer to [130] for a discussion of the last term in the H equation of motion.

[7] In our conventions

$$\frac{1}{\sqrt{|g_{10}|}}\frac{\delta S_s}{\delta\phi} = -\frac{e^{-\phi}}{2\kappa^2}\frac{T_0}{p+1}\ , \qquad \frac{1}{\sqrt{|g_{10}|}}\frac{\delta S_s}{\delta g^{MN}} = -\frac{e^{-\phi}}{4\kappa^2} T_{MN}\ . \quad (5.25)$$

To derive (5.26), we considered the fact that each γ_m matrix in the bispinors $\Phi_\pm$ carries one vielbein. To derive C_m^n the metric dependence of the full Hodge decomposition (5.18) must be taken into account. For supersymmetric cases, the operator $g_{P(M} dx^P \otimes \iota_{N)}$ in T_{MN} is the projector on the cycle wrapped by the source [77].

We are left with the internal Einstein equation,

$$R_{mn} - \frac{1}{4}H_{mpq}H_n{}^{pq} - \frac{g_s^2}{2}F_{2\ mp}F_{2\ n}{}^p - \frac{g_{mn}}{6}[R_6 - \frac{1}{2}|H|^2 - \frac{5}{2}g_s^2(|F_0|^2 + |F_2|^2)] = \frac{g_s}{2}T_{mn}\ , \quad (5.33)$$

and the dilaton equation

$$g_s\tilde{T}_0 = \frac{1}{3}[-2R_6 + |H|^2 + 2g_s^2(|F_0|^2 + |F_2|^2)]\,. \quad (5.34)$$

Provided the flux equations of motion and Bianchi identities are satisfied, solving the Einstein and dilaton equations becomes equivalent to finding the correct energy–momentum tensor for the sources. We shall now consider an explicit example and see how the non–supersymmetric modifications to the energy momentum tensor help in looking for de Sitter solutions. In the process we shall establish some properties of the form $\mathrm{Im}\, X_-$.

5.2 Solvable de Sitter

Our starting point is the solution described in Section 4.3.1, based on the algebra

$$(q_1(p25 + 35), q_2(p15 + 45), q_2(p45 - 15), q_1(p35 - 25), 0, 0)\ . \quad (5.35)$$

Among the different O6 projections compatible with the algebra for $p = 0$, only those along 146 or 236 are still compatible with the full algebra with $p \neq 0$. In Section 4.3.1 we showed that, acting with a twist transformation on the supersymmetric solution with $p = 0$ and the right O6 planes, one finds a family of backgrounds characterised by the SU(3) structure

$$\Omega = \sqrt{t_1t_2t_3}(e^1 + i\lambda\frac{\tau_3}{\tau_4}e^2) \wedge (\tau_3\ e^3 + i\tau_4\ e^4) \wedge (e^5 - i\tau_6\ e^6)\,, \quad (5.36)$$

$$J = t_1\lambda\frac{\tau_3}{\tau_4}e^1 \wedge e^2 + t_2\tau_3\tau_4e^3 \wedge e^4 - t_3\tau_6e^5 \wedge e^6\,, \quad (5.37)$$

which satisfy the supersymmetry equations (3.57) - (3.59) only when the parameter $\lambda = \frac{t_2\tau_4^2}{t_1}$ is equal to one. One motivation to consider what happens when supersymmetry is violated comes from the form of the Ricci scalar for this class of backgrounds[8]

$$R_6 = -\frac{1}{t_1 t_2 t_3 \tau_3^2}\left[(A-B)^2 + p^2\left(\frac{(\lambda-1)^2}{2\lambda}(A^2+B^2) + (A+B)^2\right)\right] \,, \tag{5.41}$$

where we introduced the following quantities

$$A = q_1 t_1 \qquad B = q_2 t_2 \tau_3^2 \,. \tag{5.42}$$

Indeed, R_6 gets more negative when the SUSY breaking parameters p and $|\lambda - 1|$ leave their SUSY value 0. Therefore, the value R_4 as given in (5.31) is lifted by SUSY breaking and this is a priori promising for a de Sitter vacuum.

The rest of this section is devoted to the search of de Sitter solutions on the class of backgrounds discussed above. We will take the same SU(3) structure as in (5.36) and metric

$$g = \text{diag}\left(t_1, \lambda t_2 \tau_3^2, t_2 \tau_3^2, \lambda t_1, t_3, t_3 \tau_6^2\right) \tag{5.43}$$

in the basis of e^m given in (4.75). Dilaton and warp factor are still constant: $e^{\phi} = g_s$ and $e^{2A} = 1$. For the fluxes, beside the RR two–form, we will allow for non–trivial RR zero–form and NS three–form

$$\begin{aligned} H &= h\,(t_1\sqrt{t_3\lambda}\, e^1\wedge e^4\wedge e^5 + t_2\tau_3^2\sqrt{t_3\lambda}\; e^2\wedge e^3\wedge e^5)\,, && (5.44)\\ g_s F_2 &= \gamma\sqrt{\frac{\lambda}{t_3}}\left[(A-B)(e^3\wedge e^4 - e^1\wedge e^2) + \frac{p}{\lambda}\,(A+B)(\lambda^2\; e^2\wedge e^4 + e^1\wedge e^3)\right]\,, && (5.45)\\ g_s F_0 &= \frac{h}{\gamma}\,. && (5.46)\end{aligned}$$

We have introduced here another parameter $\gamma > 0$ which is given by the ratio of NS and RR zero–form fluxes. We consider again D6 or O6 sources along (236) and (146), and one can check that the chosen SU(3) structure forms and fluxes satisfy the orientifold projection

[8] The Ricci tensor of a group manifold is easily computed in frame indices (where the metric is the unit one) in terms of the group structure constants

$$R_{ad} = \frac{1}{2}\left(\frac{1}{2} f_a{}^{bc} f_{dbc} - f^c{}_{db} f_{ca}{}^b - f^b{}_{ac} f^c{}_{db}\right)\,. \tag{5.38}$$

In our case, with the appropriate rescaling of the one–forms e^a and of the structure constants, we find that the only non–zero components of the Ricci tensor are

$$\begin{aligned} R_{11} = -R_{22} &= \frac{1}{2t_1t_2t_3\tau_3^2}\left[A^2 - B^2 + \frac{p^2}{\lambda}(A^2-\lambda^2B^2)\right]\,,\\ R_{33} = -R_{44} &= \frac{1}{2t_1t_2t_3\tau_3^2}\left[B^2 - A^2 + \frac{p^2}{\lambda}(B^2-\lambda^2A^2)\right]\,,\\ R_{55} &= -\frac{1}{t_1t_2t_3\tau_3^2}\left[(A-B)^2 + p^2\left(\frac{1+\lambda^2}{2\lambda}(A^2+B^2) + 2AB\right)\right]\,, && (5.39)\\ R_{14} = R_{23} &= \frac{1}{2t_1t_2t_3\tau_3^2}\frac{p}{\sqrt{\lambda}}(\lambda-1)(A^2-B^2)\,. && (5.40)\end{aligned}$$

Notice that the curvature only receives contributions from R_{55}.

conditions (4.69). Note that the NS flux has component along the covolumes[9] of the sources, $v^1 = t_1\sqrt{t_3\lambda}\, e^1 \wedge e^4 \wedge e^5$ and $v^2 = t_2\tau_3^2\sqrt{t_3\lambda}\; e^2 \wedge e^3 \wedge e^5$.

The SUSY solutions are recovered setting

$$\lambda = 1 \text{ or } p = 0,\ \gamma = 1\ , F_0 = h = 0\ . \tag{5.48}$$

5.2.1 The solution

We will first consider the four–dimensional Einstein equation (5.31). Using the ansatz for the fluxes we obtain

$$\begin{aligned} g_s^2|F_2|^2 &= \frac{2\gamma^2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2 + p^2(A+B)^2\left(\frac{(\lambda-1)^2}{2\lambda}+1\right)\right]\ , \\ |H|^2 &= 2h^2\ . \end{aligned} \tag{5.49}$$

Notice that

$$g_s^2|F_2|^2 = 2\gamma^2\left[-R_6 + p^2\frac{(\lambda-1)^2}{\lambda}\frac{q_1q_2}{t_3}\right]\ . \tag{5.50}$$

This allows to write the four dimensional Ricci scalar as

$$R_4 = \frac{2}{3}\left[(1-2\gamma^2)(-R_6 - \frac{1}{2}g_s^2|F_0|^2) + \gamma^2\left(-R_6 - \frac{q_1q_2}{t_3}\ p^2\frac{(\lambda-1)^2}{\lambda}\right)\right]\ . \tag{5.51}$$

Since the second bracket is positive (see (5.41)), we see that de Sitter solutions are possible, for instance, for $\gamma^2 \leq \frac{1}{2}$ and small F_0. Note also that R_4 clearly vanishes in the supersymmetric solution where $\lambda = 1$, $\gamma = 1$ and $F_0 = 0$.

To solve the dilaton and internal Einstein equations it is more convenient to go to frame indices and take a unit metric. As already discussed in Footnote 8, this choice makes the computation of the Ricci tensor very simple. To simplify notations we introduce the constant

$$C = -\frac{1}{6}\left(R_6 - \frac{1}{2}|H|^2 - \frac{5}{2}g_s^2(|F_0|^2 + |F_2|^2)\right)\ . \tag{5.52}$$

Then the dilaton equation becomes

$$g_s\tilde{T}_0 = 4C - \frac{h^2}{\gamma^2} - \frac{2\gamma^2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2 + p^2(A+B)^2\left(\frac{(\lambda-1)^2}{2\lambda}+1\right)\right]\ . \tag{5.53}$$

[9] In order not to clutter the notations we did not divide v^i by $\sqrt{2}$ (and recalibrate the cycles accordingly) with an unfortunate consequence that H comes out as even–quantized, and γ is rational up to multiplication by $\sqrt{2}$. This is due to our choice of normalization of the RR kinetic terms which differ by a factor of 2 with respect to [176]. For a k–flux α through a k–cycle Σ (with embedding i into the bulk manifold M), we have

$$\frac{1}{(2\pi\sqrt{\alpha'})^{k-1}}\frac{1}{\mathrm{vol}_M}\int_\Sigma i^*\alpha = \frac{1}{(2\pi\sqrt{\alpha'})^{k-1}}\frac{1}{\mathrm{vol}_M}\int_M \langle\delta(\Sigma \hookrightarrow M), \alpha\rangle = n\,, \tag{5.47}$$

where n is an integer.

For the internal Einstein equations, only some components are non–trivial

$$
\begin{aligned}
g_sT_{14} &= \frac{1}{t_1t_2t_3\tau_3^2}\frac{p}{\sqrt{\lambda}}(A^2-B^2)(\lambda-1)(1-\gamma^2)\,,\\
g_sT_{23} &= \frac{1}{t_1t_2t_3\tau_3^2}\frac{p}{\sqrt{\lambda}}(A^2-B^2)(\lambda-1)(1-\gamma^2)\,,\\
g_sT_{11} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[A^2-B^2+\frac{p^2}{\lambda}(A^2-B^2\lambda^2)-\gamma^2((A-B)^2+\frac{p^2}{\lambda}(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{22} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[B^2-A^2+\frac{p^2}{\lambda}(B^2\lambda^2-A^2)-\gamma^2((A-B)^2+p^2\lambda(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{33} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[B^2-A^2+\frac{p^2}{\lambda}(B^2-A^2\lambda^2)-\gamma^2((A-B)^2+\frac{p^2}{\lambda}(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{44} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[A^2-B^2+\frac{p^2}{\lambda}(A^2\lambda^2-B^2)-\gamma^2\left((A-B)^2+p^2\lambda(A+B)^2\right)\right]-h^2+2C\,,\\
g_sT_{55} &= -\frac{2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2+p^2\left(\frac{(\lambda^2+1)}{2\lambda}(A^2+B^2)+2AB\right)\right]-2h^2+2C\,,\\
g_sT_{66} &= 2C\,.
\end{aligned}
\tag{5.54}
$$

The remaining components set to zero the corresponding source term $T_{ab}=0$.

To solve these equations we need the explicit expressions for the source energy–momentum tensor, (5.26). In six–dimensional frame indices we have

$$
\begin{aligned}
T_{ab} &= 2\kappa^2T_p\hat{*}\langle j,\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X\ -\delta^c_{(a}\delta_{b)d}C^d_c\rangle\\
&= 2\kappa^2T_p\hat{*}\left(\sqrt{|g_4|}\,\mathrm{d}^4x\wedge\langle j,\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_-\ -\delta^c_{(a}\delta_{b)d}c^d_c\rangle\right)\\
&= 2\kappa^2T_p\frac{1}{\sqrt{|g_6|}}\ \left[j\wedge\left(\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_3\ -\delta^c_{(a}\delta_{b)d}c^d_c|_3\ \right)\right]_{1\ldots6}\\
&= \frac{1}{\sqrt{|g_6|}}\ \left[(\mathrm{d}F_2-HF_0)\wedge\left(\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_3\ -\delta^c_{(a}\delta_{b)d}c^d_c|_3\ \right)\right]_{1\ldots6}\ .
\end{aligned}
\tag{5.55}
$$

Since, in our case, the source j is a three–form,

$$2\kappa^2T_p\ j=\mathrm{d}F_2-HF_0\,, \tag{5.56}$$

only the three–form parts $\mathrm{Im}\,X_3$ and $c^d_c|_3$ of $\mathrm{Im}\,X_-$ and c^d_c contribute to the equations.

In the same way, we obtain

$$g_s\tilde{T}_0=g_s\ 2\kappa^2T_p\,\hat{*}\langle j,\mathrm{Im}\,X\rangle=\frac{1}{\sqrt{|g_6|}}\ \left[g_s\,(\mathrm{d}F_2-HF_0)\wedge\mathrm{Im}\,X_3\right]_{1\ldots6}\ . \tag{5.57}$$

Combining (5.18) and the explicit expression for SU(3) pure spinors, it is easy to see that $\mathrm{Im}\,X_-$ decomposes into a one–form, a three–form and a five–form piece

$$\mathrm{Im}\,X_-=\mathrm{Im}\,X_1+\mathrm{Im}\,X_3+\mathrm{Im}\,X_5\,, \tag{5.58}$$

where[10]

$$
\begin{aligned}
\operatorname{Im} X_1 &= (a_k^{i\,L} + a_k^{i\,R})\mathrm{d}x^k - (a_k^{r\,L} - a_k^{r\,R}) g^{kj}\iota_j J + (g^{km} g^{jl}\iota_m \iota_l)[-a^r_{kj}\operatorname{Re}\Omega + a^i_{kj}\operatorname{Im}\Omega]\,,\\
\operatorname{Im} X_3 &= -(a_k^{r\,L} + a_k^{r\,R})\mathrm{d}x^k\wedge J - (a_k^{i\,L} - a_k^{i\,R})\, g^{kj}\iota_j J\wedge J\\
&\quad -[a^r_0 - a^r_{kj}\,(g^{kj} - (g^{kl}\mathrm{d}x^j + g^{jl}\mathrm{d}x^k)\iota_l)]\operatorname{Re}\Omega\\
&\quad +[a^i_0 - a^i_{kj}(g^{kj} - (g^{kl}\mathrm{d}x^j + g^{jl}\mathrm{d}x^k)\iota_l)]\operatorname{Im}\Omega\,,\\
\operatorname{Im} X_5 &= -\frac{1}{2}[(a_k^{i\,L} + a_k^{i\,R}))\mathrm{d}x^k - (a_k^{r\,L} - a_k^{r\,R}))g^{kj}\iota_j J]\wedge J^2\\
&\quad -\mathrm{d}x^k\wedge\mathrm{d}x^j\wedge[-a^r_{kj}\operatorname{Re}\Omega + a^i_{kj}\operatorname{Im}\Omega]\,.
\end{aligned}
\tag{5.59}
$$

The superscripts r and i indicate real and imaginary parts:

$$
\begin{aligned}
a^r_0 &= \operatorname{Re}(\alpha_0 - \widetilde{\alpha}_0)\,, & a^r_{jk} &= \operatorname{Re}(\alpha_{jk} - \widetilde{\alpha}_{jk})\,,\\
a^i_0 &= \operatorname{Im}(\alpha_0 + \widetilde{\alpha}_0)\,, & a^i_{jk} &= \operatorname{Im}(\alpha_{jk} + \widetilde{\alpha}_{jk})\,.
\end{aligned}
\tag{5.60}
$$

and

$$
\begin{aligned}
a_k^{r\,L} &= \operatorname{Re}(\alpha^L_k - \widetilde{\alpha}^L_k)\,, & a_k^{r\,R} &= \operatorname{Re}(\alpha^R_k - \widetilde{\alpha}^R_k)\,,\\
a_k^{i\,L} &= \operatorname{Im}(\alpha^L_k + \widetilde{\alpha}^L_k)\,, & a_k^{i\,R} &= \operatorname{Im}(\alpha^R_k + \widetilde{\alpha}^R_k)\,.
\end{aligned}
\tag{5.61}
$$

As already discussed, only the three–form parts of $\operatorname{Im} X_-$ and c^d_c contribute to the equations. Then, for simplicity, we choose to set to zero $\operatorname{Im} X_1$ and $\operatorname{Im} X_5$. This amounts to setting

$$
a_k^{r\,L} = a_k^{i\,L} = a_k^{r\,R} = a_k^{i\,R} = 0\,,
\tag{5.62}
$$

and choosing a^r_{jk} and a^i_{jk} symmetric. Then, in frame indices, $\operatorname{Im} X_3$ becomes

$$
\begin{aligned}
\operatorname{Im} X_3 &= [a^i_0 - Tr(a^i_{bc}) + a^i_{bc}(\delta^{bd}e^c + \delta^{cd}e^b)\iota_d]\operatorname{Im}\Omega\\
&\quad -[a^r_0 - Tr(a^r_{bc}) + a^r_{bc}(\delta^{bd}e^c + \delta^{cd}e^b)\iota_d]\operatorname{Re}\Omega\,.
\end{aligned}
\tag{5.63}
$$

Similarly, we find that the three–form part of c^b_a is given by

$$
\begin{aligned}
c^b_a|_3 &= 2a^i_{ac}[-\delta^{bc} + (\delta^{cd}e^b + \delta^{bd}e^c)\iota_d]\operatorname{Im}\Omega\\
&\quad -2a^r_{ac}[-\delta^{bc} + (\delta^{cd}e^b + \delta^{bd}e^c)\iota_d]\operatorname{Re}\Omega\,.
\end{aligned}
\tag{5.64}
$$

The coefficients in $\operatorname{Im} X_3$ are free parameters which should be fixed by solving the dilaton and internal Einstein equations.

The equations $T_{mn} = 0$ are satisfied by choosing[11]

$$
\begin{aligned}
a^i_0 &= 0 && a = 1,\ldots,6\,,\\
a^i_{bc} &= 0 && b,c = 1,\ldots,6\,,\\
a^r_{bc} &= 0 && (bc)\notin\{(bb),\ (14),\ (23)\}\,.
\end{aligned}
\tag{5.65}
$$

[10]We have not imposed (5.20) yet, and shall return to it later.

[11]The parameters a^i_{12}, a^i_{13}, a^i_{24}, a^i_{34}, a^i_{56} are not fixed by any equation. For simplicity, we decide to put them to zero.

The Einstein and dilaton equations, (5.54) and (5.53) fix the other parameters

$$
\begin{aligned}
a^r_0 &= -g_s \, \frac{\tilde{T}_0 + T_{55} + T_{66} - x_0}{2(c_1+c_2)} \,, \\
a^r_{14} &= g_s \, \frac{T_{14}}{2(c_2-c_1)} \,, \\
a^r_{23} &= g_s \, \frac{T_{23}}{2(c_1-c_2)} \,, \\
a^r_{11} &= g_s \, \frac{1}{2(c_2-c_1)} \left[T_{11} - \frac{c_2 \tilde{T}_0}{c_1+c_2} + \frac{x_0 c_1 c_2}{(c_1^2-c_2^2)} \right] \,, \\
a^r_{22} &= g_s \, \frac{1}{2(c_1-c_2)} \left[T_{22} - \frac{c_1 \tilde{T}_0}{c_1+c_2} + \frac{x_0 c_1 c_2}{(c_2^2-c_1^2)} \right] \,, \\
a^r_{33} &= g_s \, \frac{1}{2(c_1-c_2)} \left[T_{33} - \frac{c_1 \tilde{T}_0}{c_1+c_2} + \frac{x_0 c_1 c_2}{(c_2^2-c_1^2)} \right] \,, \\
a^r_{44} &= g_s \, \frac{1}{2(c_2-c_1)} \left[T_{44} - \frac{c_2 \tilde{T}_0}{c_1+c_2} + \frac{x_0 c_1 c_2}{(c_1^2-c_2^2)} \right] \,, \\
a^r_{55} &= -g_s \, \frac{T_{55}}{2(c_1+c_2)} \,, \\
a^r_{66} &= g_s \, \frac{T_{66} - \tilde{T}_0}{2(c_1+c_2)} \,,
\end{aligned}
\tag{5.66}
$$

where $x_0 = 2\tilde{T}_0 - (T_{11} + T_{22} + T_{33} + T_{44})$ and T_{ab} are given by (5.54). The coefficients c_1 and c_2 appear in the source term of the Bianchi identity for F_2

$$
g_s(\mathrm{d}F_2 - HF_0) = c_1 \; v^1 + c_2 \; v^2 \,, \tag{5.67}
$$

where v^1 and v^2 are covolumes of sources in the directions (146) and (236) and

$$
\begin{aligned}
c_1 &= -\frac{h^2}{\gamma} + \frac{q_1 q_2}{A t_3} \gamma \left[2(A-B) - p^2 \frac{\lambda^2+1}{\lambda} (A+B) \right] \,, \\
c_2 &= -\frac{h^2}{\gamma} + \frac{q_1 q_2}{B t_3} \gamma \left[2(B-A) - p^2 \frac{\lambda^2+1}{\lambda} (A+B) \right] \,.
\end{aligned}
\tag{5.68}
$$

In agreement with our quantization conventions (see Footnote 9), we impose that $(c_1 + c_2)$ is an integer. We emphasize once more, that the overall tension of the intersecting sources is always negative (and so is $c_1 + c_2$), but depending on the parameters of the solution the individual sources may be either O6 planes or D6 branes.

So far, we have solved the external and internal Einstein equations, the dilaton equation of motion, and checked that the Bianchi identity for F_2 is satisfied. As far as the bulk fields are concerned, we should also solve the equations of motion and the remaining Bianchi identities for the fluxes. These are actually automatically satisfied by our ansatz for the fluxes, provided $j \wedge \mathrm{Im}\, X_1 = 0$. As a matter of fact, our choice of the parameters a in (5.65) already sets $\mathrm{Im}\, X_1$ to zero, so we are done with the bulk fields.

As a last step in the construction of a de Sitter solution (we recall we mean here a solution to the equations derived from our proposed action for the sources), we need to check the source fields equations of motion. One should vary our source action with respect to the world–volume coordinates and the gauge fields. The latter is trivially satisfied, since we do not consider any gauge field here, and the pullback of the B–field giving (5.44) vanishes. For the world–volume coordinates, from our action $-T_p \int_\Sigma e^{-\phi} i^*[\mathrm{Im}\, X]$ and WZ, one can derive an equation of motion of the form

$$\partial_{[i_1}(e^{-\phi}\,\mathrm{Im}\, X_3)_{i_2 i_3]\alpha} \sim (*F_2)_{[i_1 i_2 i_3]\alpha} \ , \tag{5.69}$$

where i_k label world–volume directions, and α is orthogonal. One can check that pulling back any three indices of the four–form $*F_2$ to the world–volume gives zero, as discussed after (5.16). The left–hand side also vanishes (see (5.71)), and so we conclude that the world–volume equations of motion are satisfied.

This concludes our resolution of all equations of motion derived from the action (5.22) which contains our proposal for sources breaking bulk supersymmetry. Provided one chooses the free parameters as discussed below (5.51), one can obtain a de Sitter solution. In the next Section, we come back to the question of generalizing first order differential equations to the non–supersymmetric case. This will fix for us the free parameters to values which indeed give a de Sitter solution. In Section 5.2.3 we will argue that the solution we found here is also a solution to the equations derived with the standard source action.

5.2.2 More on the polyform X

In this section, we will try to provide further justification for our choice of polyform X_-.

In supersymmetric compactifications, the imaginary part of the non–closed pure spinor, Φ_- in type IIA, on one side, defines the calibration for the sources and, on the other, gives the bulk RR fields in the supersymmetry equations (3.59). We will show that, for our de Sitter solution, the polyform X_- satisfies the same equations Φ_- satisfies in the supersymmetric case

$$\begin{aligned} (\mathrm{d}-H)\,\mathrm{Re}\, X_- &= 0\,, \\ (\mathrm{d}-H)\,\mathrm{Im}\, X_- &= c_0\ g_s * \lambda(F)\,, \end{aligned} \tag{5.70}$$

where the constant c_0 can a priori be different from 1.

Keeping only the parameters a that are non-zero in the de Sitter solution (5.66), it is easy to compute

$$\begin{aligned} \mathrm{d}(\mathrm{Im}\, X_-) \;=\; & [(a_0^r + a_{66}^r - a_{55}^r)[p(q_1+q_2)(e^1\wedge e^3 + e^2\wedge e^4) \\ & -(q_1-q_2)(e^1\wedge e^2 - e^3\wedge e^4)]\wedge e^5\wedge e^6 \\ & -(a_{11}^r + a_{44}^r - a_{22}^r - a_{33}^r)[p(q_1-q_2)(e^1\wedge e^3 + e^2\wedge e^4) \\ & -(q_1+q_2)(e^1\wedge e^2 - e^3\wedge e^4)]\wedge e^5\wedge e^6\,, \end{aligned} \tag{5.71}$$

and

$$H\wedge \mathrm{Im}\, X_- = -2h\ (a_0^r + a_{66}^r - a_{55}^r)\ e^1\wedge e^2\wedge e^3\wedge e^4\wedge e^5\wedge e^6\,. \tag{5.72}$$

In order to have $\mathrm{d}(\mathrm{Im}\, X_-)$ proportional to $g_s * F_2$, one must impose the relation

$$a_{11}^r + a_{44}^r - a_{22}^r - a_{33}^r = 0\ . \tag{5.73}$$

Then, one has

$$\begin{aligned} \mathrm{d}(\mathrm{Im}\, X_-) &= -c_0\ g_s * F_2\ , \\ H \wedge \mathrm{Im}\, X_- &= -2\gamma^2\ c_0\ g_s * F_0\ , \end{aligned} \tag{5.74}$$

with

$$c_0 = \frac{a_0^r + a_{66}^r - a_{55}^r}{\gamma} = -g_s \frac{\tilde{T}_0}{\gamma(c_1+c_2)}\ . \tag{5.75}$$

To obtain the second equality, we used the explicit expression (5.66), (5.54) for the parameters a, while c_1 and c_2 are defined in (5.68). Also, using (5.66), it is easy to show that the constraint (5.73) reduces to

$$x_0 = 2\tilde{T}_0 - (T_{11} + T_{22} + T_{33} + T_{44}) = 0 \qquad \Leftrightarrow \qquad (2\gamma^2 - 1)\ h^2 = 0\ . \tag{5.76}$$

Therefore, for[12]

$$\gamma^2 = \frac{1}{2} \tag{5.77}$$

we can write a differential equation for $\mathrm{Im}\, X_-$

$$(\mathrm{d} - H)\, \mathrm{Im}\, X_- = c_0\ g_s * \lambda(F)\, , \tag{5.78}$$

which is the analogue of the supersymmetry equations[13] for $\mathrm{Im}\, \Phi_-$. In addition, fixing the value $\gamma^2 = 1/2$ gives a de Sitter solution, according to the condition (5.51).

The value of the constant c_0 is also fixed by the solution. Indeed, in order for X_- to reproduce the correct Born–Infeld action (5.20) on–shell, we get from our solution that a combination of coefficients of X_- has to be one: $a_0^r + a_{66}^r - a_{55}^r = 1$. Out of (5.75), we deduce that we have to impose $c_0\, \gamma = 1$. This relation is automatically satisfied for supersymmetric backgrounds, where $c_0 = \gamma = 1$ and the pullback of $\mathrm{Re}\, \Omega$ agrees with the DBI action on the solution. In our non–supersymmetric solution, the condition $c_0\, \gamma = 1$ fixes the value of the constant, $c_0 = \sqrt{2}$.

More generally, requiring the two actions being equal on–shell can be formulated as $-g_s \tilde{T}_0 = c_1 + c_2$, where the right–hand side is given by the sum of the source charges. Indeed, as we can see in (5.57), if $\mathrm{Im}\, X$ gives the sum of the source volume forms on–shell, and j or $\mathrm{d}_H F$ gives the sum of the charges times the covolumes (Bianchi identity), then $\tilde{T}_0$ should be given by the sum of the charges; this sum is negative, hence the minus sign. We can verify that this condition is equivalent for our solution to the condition $c_0\, \gamma = 1$, given the second equality in (5.75). Finally, let us note that such a relation would fix one of the three parameters h, γ, λ in terms of the others and the moduli. In particular, for $\lambda = 1$, one gets

$$h^2 = \frac{(A-B)^2 + p^2(A+B)^2}{t_1 t_2 t_3 \tau_3^2}\ \frac{(\gamma-1)(1-2\gamma)\gamma^2}{\gamma^2 - 3\gamma + 1}\ . \tag{5.79}$$

[12]Clearly also $h = 0$ (no NS flux) is a solution to this constraint. It would be interesting to explore the possibility of having de Sitter or non–supersymmetric Minkowski solution with $h = 0$. Notice that, in this case, the condition of having $F_0 \neq 0$ [100], necessary to avoid de Sitter no–go theorems [103], is not required.

[13]Notice that from the equation for $\mathrm{Im}\, X_-$ we recover the condition $T_0 > 0$ (5.34). Indeed, as in [87], starting from (5.57) we have

$$\frac{T_0}{p+1} \int_M \mathrm{vol}_{(6)} = -\int_M \langle \mathrm{d}_H F, \mathrm{Im}\, X_- \rangle = -\int_M \langle F, \mathrm{d}_H\, \mathrm{Im}\, X_- \rangle = c_0\ g_s \int_M \langle *\lambda(F), F \rangle > 0\ .$$

Note one clearly recovers the supersymmetric case with $\gamma = 1$. For our de Sitter solution, one should impose instead $\gamma = \frac{1}{\sqrt{2}}$, and then $h \neq 0$.

We can now show that d_H - closure can be imposed on $\mathrm{Re}\, X_-$. Indeed, the three–form part of $\mathrm{Re}\, X_-$ can be written as

$$\begin{aligned}
\mathrm{Re}\, X_3 &= -[b^r_0 - Tr(b^r_{kj}) + b^r_{kj}(g^{kl}dx^j + g^{jl}dx^k)\iota_l]\,\mathrm{Re}\,\Omega \\
&\quad +[b^i_0 - Tr(b^i_{kj}) + b^i_{kj}(g^{kl}dx^j + g^{jl}dx^k)\iota_l]\,\mathrm{Im}\,\Omega \\
&\quad +[(b^{i\,R}_k - b^{i\,L}_k)\; dx^k + g^{kl}(b^{rR}_k - b^{rL}_k)\; \iota_l J] \wedge J \ ,
\end{aligned} \tag{5.80}$$

where, as for $\mathrm{Im}\, X_3$, we have defined

$$\begin{aligned}
b^r_0 &= \mathrm{Im}(\tilde{\alpha}_0 - \alpha_0) & b^r_{kj} &= \mathrm{Im}(\tilde{\alpha}_{kj} - \alpha_{kj}) \ , \\
b^i_0 &= \mathrm{Re}(\tilde{\alpha}_0 + \alpha_0) & b^i_{kj} &= \mathrm{Re}(\tilde{\alpha}_{kj} + \alpha_{kj}) \ , \\
b^{r\,L}_k &= \mathrm{Re}(\tilde{\alpha}^L_k + \alpha^L_k) & b^{r\,R}_k &= \mathrm{Re}(\alpha^R_k + \tilde{\alpha}^R_k) \ , \\
b^{i\,L}_k &= \mathrm{Im}(\tilde{\alpha}^L_k - \alpha^L_k) & b^{i\,R}_k &= \mathrm{Im}(\alpha^R_k - \tilde{\alpha}^R_k) \ .
\end{aligned} \tag{5.81}$$

Consistently with (5.65), we can choose

$$\begin{aligned}
& b^r_0 = 0\,, \\
& b^{r\,L}_k = b^{r\,L}_k = b^{i\,L}_k = b^{i\,R}_k = 0 \qquad \forall\ k = 1, \ldots 6 \\
& b^r_{jk} = 0 \qquad\qquad\qquad\qquad\qquad\quad \forall\ j,k = 1, \ldots 6 \\
& b^i_{jk} = 0 \qquad \text{for } (kj) \notin \{(kk),\ (14),\ (23),\ (41),\ (32)\}\,.
\end{aligned} \tag{5.82}$$

Furthermore, choosing

$$\frac{b^i_{14}}{t_1} = -\frac{b^i_{23}}{t_2\tau_3^2} \quad , \quad \frac{b^i_{11}}{t_1} + \frac{b^i_{33}}{t_2\tau_3^2} - \frac{b^i_{22}}{t_2\tau_3^2\lambda} - \frac{b^i_{44}}{t_1\lambda} = 0 \ , \tag{5.83}$$

we obtain

$$\mathrm{d}_H(\mathrm{Re}\, X_3) = \sqrt{t_1 t_2 t_3}\ \tau_3\tau_6\ p(1-\lambda)\ \left(b^i_0 + \frac{b^i_{66}}{t_3\tau_6^2} - \frac{b^i_{55}}{t_3}\right)(q_2\ e^1\wedge e^4 + q_1\ e^2\wedge e^3)\wedge e^5\wedge e^6 \ , \tag{5.84}$$

which is zero either in the SUSY solution, or by further setting

$$b^i_0 = -\frac{b^i_{66}}{t_3\tau_6^2} + \frac{b^i_{55}}{t_3}\,. \tag{5.85}$$

While these equations are derived in the vanishing warp factor and constant dilaton limit, their extension to the general case is natural[14]

$$\begin{aligned}
& \mathrm{d}_H(e^{2A-\phi}\,\mathrm{Re}\, X_-) = 0\,, \\
& \mathrm{d}_H(e^{4A-\phi}\,\mathrm{Im}\, X_-) = c_0 e^{4A} * \lambda(F)\,.
\end{aligned} \tag{5.86}$$

[14] Just like Φ_-, X_- is globally defined, and both B–field and the dilaton are needed in order to define an isomorphism between such forms and the positive and negative helicity spin bundles $S^{\pm}(E)$, see discussion in Section 3.5.1. The dilaton assures the correct transformation under Gl(6), making the (non–pure) spinor $e^{-\phi}e^{-B}\,X_-$ the natural variable for the first order equations (5.86).

In general the odd form X_- should receive contribution from both pure spinors, but in our solution we have chosen to "decouple" the even pure spinor completely. Note that any two objects in the trio of the even and odd compatible pure spinors and the metric determine the third. Here we have worked with the almost complex structure and the metric. In the supersymmetric backgrounds it is clearly more convenient to solve the first order equations for the pure spinors rather than the Einstein equation for the metric. Hence it is natural to ask if and when it might be possible to find an even–form counterpart to (5.86), X_+ , so that X_- and X_+ (together with flux Bianchi identities) imply the solution to the Einstein equations. However it is not yet clear to us what the correct generalization of the notion of compatibility is, and what algebraic properties X_+ should satisfy. Hoping for a symmetry with the supersymmetric solutions (and the possibility of having a solution to some variational problem) one may construct X_+ satisfying

$$\mathrm{d}_H(e^{3A-\phi}X_+) = 0\,. \tag{5.87}$$

Assuming X_+ has an expansion similar to that of X_-, which does not receive contributions from Ω, this amounts to finding a closed two–form on $\mathfrak{g}_{5.17}^{p,-p,\pm 1} \times S^1$. It is indeed not hard to construct such a form for our solution, since the symplectic form itself is closed, provided $\tau_2 = 0$ (even if $\lambda \neq 1$, see (4.78)). Even if we do not take $\tau_2 = 0$, finding a conformally closed X_+ of this form is always possible, since the manifold is symplectic. We will return on this issues in the next Chapter.

5.2.3 A solution for the standard source action?

In this Chapter we made a proposal of an action for sources breaking bulk supersymmetry. As discussed in Section 5.1, we cannot conclude (as one would do in the supersymmetric case) whether the equations of motion derived from the action (5.22) are the same as those derived from the standard source action DBI + WZ. Our proposal is to be considered as an assumption with interesting consequences, we are not able to prove such an equivalence. What can be done is to verify that the solution found in our example is indeed a solution to the equations of motion derived from the standard source action. Let us discuss now in practice what should be checked, starting with the world–volume equations of motion.

There are two equations to consider, coming from the variation of DBI + WZ action with respect to the world–volume coordinates and the gauge fields (for a general form of these equations see [177]). The latter is easier, and we shall consider it first. In our solution the dilaton is constant and the world–volume gauge fields vanish. Moreover we recall that the pullback of the B–field computed from (5.44) also vanishes. Then the equation reads

$$\partial_i \left(e^{-\phi} \sqrt{|i^*[g]|}\; (i^*[g])^{[ij]} \right) \sim \epsilon^{jkl}\, (i^*[*F_4])_{kl} \ \ , \tag{5.88}$$

where i, j, k, l are indices along the brane world–volume. Since our solution has no RR four–form flux, both sides vanish trivially. The variation of the world–volume action with respect to the world–volume coordinates (again, in presence of constant dilaton and vanishing pullback of B) connects the trace of the second fundamental form $\mathcal{S}^\alpha_{ij}$ to the RR fluxes (α spans normal directions). It reads

$$e^{-\phi}(i^*[g])^{ij}\mathcal{S}^\alpha_{ij} \sim \epsilon^{jkl}(*F_2)^\alpha_{jkl}\ . \tag{5.89}$$

One can check[15] that pulling back any three indices of the four–form $*F_2$ to the world–volume gives zero. For our intersecting configuration, we need to worry only about $\alpha = 5$, and may use the relation of the second fundamental form with the (components of) the spin connection $\omega_i^\alpha = \mathcal{S}_{ij}^\alpha e^j$. We can check that while the second fundamental form does not vanish (the embedding is not geodesic), it has no diagonal element. However the metric (5.43) in the basis (4.75) is diagonal, and $(i^*[g])^{ij}\mathcal{S}_{ij}^\alpha$ vanishes. Thus the world–volume equations of motion are satisfied.

Let us now consider the bulk field equations of motion. As mentioned at the end of Section 5.2.1, the ansatz chosen for the fluxes guarantees that their equations of motion and Bianchi identities are satisfied. Let us also emphasize the following details: first we do not have any B–field along the sources and therefore a correction term due to the source in its equation of motion could be discarded; second the proposed generalization of the first order equations (5.86), satisfied by our solution, guarantees that the RR equations of motion are satisfied. Therefore, for the bulk fields, only the internal Einstein equation and the dilaton equation of motion remain to be checked.

The dependence of the dilaton equation on the source action is simply through $\tilde{T}_0$ (see for instance (5.34)), which is proportional to the source action on–shell. Therefore, as long as the standard source action and our proposed action match on–shell, the dilaton equations of motion are the same. As discussed in the previous section, this equality amounts in general to the condition $-g_s\tilde{T}_0 = c_1 + c_2$, which for our solution is equivalent to $c_0\gamma = 1$. This fixes one of the three parameters h, γ, λ in terms of the others and the moduli (see for instance (5.79)). Provided this condition is enforced, the dilaton equation of motion derived from DBI is therefore satisfied by our solution.

We are now left with the internal Einstein equation. An explicit check can be done for the family given by:

$$\lambda = 1 \qquad F_0 \neq 0\ ,\ h \neq 0\ ,\ \text{given by (5.79)}\ , \tag{5.90}$$

with particular interest in the non–supersymmetric value $\gamma = \frac{1}{\sqrt{2}}$ giving our de Sitter solution. Solving the Einstein equation amounts to match the values of the energy–momentum tensor T_{ab} given by (5.54). In the supersymmetric case, one can derive from the standard source action that the non–zero components of T_{ab} of one source are the diagonal ones along the source directions, and are all equal. We recover this situation in the family we consider by simply taking $\gamma = 1$. For our non–supersymmetric solution, the supersymmetry breaking will manifest itself as $T_{55} \neq 0$ and $T_{66} \neq T_{11} + T_{22}$. Then, in order to match the results, one needs to consider a non–trivial dependence of the embedding functions on the metric moduli. The computation is rather involved and not particularly enlightening, thus we will not present it here. However, let us note that this non standard embedding corresponds to our interpretation of the proposed action, as discussed in the Introduction. We can also obtain a perturbative solution (the perturbation parameter is $\epsilon = \lambda - 1$) where the deviation from the SUSY solution is more severe due to T_{14} and T_{23} not being zero as opposed to their supersymmetric value.

Let us end this section by adding few words about the stability of our solution. Solving all the equations of motion of course means extremizing the energy density of the bulk plus brane system, but we cannot be sure that the solution is a minimum for arbitrary values of

[15] This check is analoguous to that of the corresponding equation of motion derived from our proposed source action, as discussed at the end of Section 5.2.1.

the parameters. The problem is currently under study. For the time being we can try to give some heuristic justification of the fact we believe our non–supersymmetric solution is stable. For $\lambda = 1$ and $\gamma = 1$ the manifold admits the supersymmetric solution described in Section 4.3.1. By keeping $\lambda = 1$ and setting $\gamma = 1/\sqrt{2}$ we obtain a non–supersymmetric solution with the same internal geometry as in the SUSY case, meaning the metric is not changed and the directions wrapped are the same. The pullback of $\mathrm{Im}\, X_-$ does coincide with the pullback of the (generalized) calibrating form $\mathrm{Re}\,\Omega$. In a sense the brane is still wrapping a minimal volume cycle (even if this is done with a different embedding), and we can imagine the parameters, other than γ, can be chosen in such a way to have small contributions to the potential from the supersymmetry breaking term, and the energy density of combined bulk and brane system at the minimum.

5.3 Four–dimensional analysis

In this section we do a partial study of the stability of our solution by analyzing the four dimensional effective potential with respect to two moduli.

The search for de Sitter vacua, or for no–go theorems against their existence, has generally been performed from a four–dimensional point of view [112, 103, 176, 100, 52, 39, 67, 54, 55], analysing the behaviour of the four–dimensional effective potential with respect to its moduli dependence. In this section, we want to make contact with this approach and show that our solution has the good behaviour one expects to find for de Sitter vacua, as far as the volume and the dilaton are concerned. We use in this section the ten–dimensional action (5.22) which contains our proposal for sources breaking bulk supersymmetry. We will show that this proposal gives rise to interesting new terms in the potential.

5.3.1 Moduli and $4d$ Einstein frame

Let us consider the ten–dimensional action (5.22). By Kaluza–Klein reduction on the internal manifold, we obtain a four–dimensional effective action for the moduli. In particular, in addition to the kinetic terms, the four–dimensional action will contain a potential for the moduli fields. Their number and the way they enter the potential will depend on the peculiar features of the single model.

A de Sitter solution of the four–dimensional effective action will correspond to a positive valued minimum of the potential. Determining the minima of the potential is in general rather difficult, since, a priori one should extremize along all the directions in the moduli space. This complicated problem is generally solved only by numerical analysis, because of the large number of variables. However, some information can be extracted by restricting the analysis to a subset of the moduli fields.

For whatever choice of the manifold on which the compactification is performed, we are always able to isolate two universal moduli: the internal volume and the four–dimensional dilaton. Their appearance in the effective potential at tree–level is also universal. We will then only focus on these two moduli. We define the internal volume as

$$\int_M \mathrm{d}^6 x \sqrt{|g_6|} = \frac{L^6}{2} = \frac{L_0^6}{2}\rho^3 \ , \tag{5.91}$$

where the factor of $\frac{1}{2}$ is due to the orientifold and the vacuum value is $\rho = 1$. Defining the ten–dimensional dilaton fluctuation as $e^{-\tilde{\phi}} = g_s e^{-\phi}$, the four–dimensional dilaton is given by

$$\sigma = \rho^{\frac{3}{2}} \, e^{-\tilde{\phi}} \, . \tag{5.92}$$

Then reducing the action (5.22), we obtain the four–dimensional effective action for gravity, $4d$ dilaton and volume modulus in the string frame

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \sqrt{|g_4|} \left[\frac{L^6}{2} e^{-2\phi} (R_4 + 4|\nabla \phi|^2) - 2\kappa^2 U \right] , \tag{5.93}$$

with $U(\rho, \sigma)$ the four–dimensional potential. To derive the explicit form of the potential, we need to determine how the internal Ricci scalar, fluxes and source terms scale with the volume. For R_6 and the fluxes this is easily computed

$$R_6 \to \rho^{-1} \, R_6 \, , \quad |H|^2 \to \rho^{-3} \, |H|^2 \, , \quad |F_k|^2 \to \rho^{-k} \, |F_k|^2 \, . \tag{5.94}$$

The source term requires some more attention. As shown in (5.57),

$$2\kappa^2 T_p \, \hat{*} \langle j, \mathrm{Im}\, X \rangle = \frac{[(\mathrm{d}F_2 - H F_0) \wedge \mathrm{Im}\, X_3]_{1\ldots 6}}{\sqrt{|g_6|}} \ . \tag{5.95}$$

The terms in $\mathrm{Im}\, X_3$ in (5.59) appearing with a_0, a_{jk} and $a_k^{(L,R)}$ scale differently with the volume. Let us denote them by X_0, X_Ω and X_J, respectively

$$\mathrm{Im}\, X_3 = X_0 + X_\Omega + X_J \ . \tag{5.96}$$

Their ρ dependence is determined by the scaling of the forms J and Ω

$$J \to \rho J \ , \ \Omega \to \rho^{\frac{3}{2}} \Omega \ , \tag{5.97}$$

and by the metric factors in the gamma matrices of (5.59)

$$X_0 \to \rho^{\frac{3}{2}} X_0 \ , \ X_\Omega \to \rho^{\frac{1}{2}} X_\Omega \ , \ X_J \to \rho X_J \ . \tag{5.98}$$

Then, the source term scales as

$$\frac{[(\mathrm{d}F_2 - H F_0) \wedge \mathrm{Im}\, X_3]_{1\ldots 6}}{\sqrt{|g_6|}} \to \rho^{-\frac{3}{2}} \, \left(b_0 + b_1 \, \rho^{-1} + b_2 \, \rho^{-\frac{1}{2}} \right) \ , \tag{5.99}$$

where

$$\begin{aligned} b_0 &= \frac{[(\mathrm{d}F_2 - H F_0) \wedge X_0]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \\ b_1 &= \frac{[(\mathrm{d}F_2 - H F_0) \wedge X_\Omega]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \\ b_2 &= \frac{[(\mathrm{d}F_2 - H F_0) \wedge X_J]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \end{aligned} \tag{5.100}$$

are vacuum values. Then the four–dimensional potential for ρ and σ becomes

$$\begin{aligned} U &= \frac{1}{2\kappa^2}\int_M \mathrm{d}^6x\sqrt{|g_6|}[e^{-2\phi}(-R_6+\frac{1}{2}|H|^2)+\frac{1}{2}(|F_0|^2+|F_2|^2)-2\kappa^2 T_p\, e^{-\phi}\,\hat{*}\langle j,\mathrm{Im}\,X\rangle] \\ &= \frac{L_0^6}{4g_s^2\kappa^2}\,\sigma^2\,[(-\frac{R_6}{\rho}+\frac{|H|^2}{2\rho^3})-\frac{g_s}{\sigma}\,(b_0+\frac{b_1}{\rho}+\frac{b_2}{\sqrt{\rho}})+\frac{g_s^2\,\rho^3}{2\sigma^2}(|F_0|^2+\frac{|F_2|^2}{\rho^2})]\,. \end{aligned} \tag{5.101}$$

Note that the terms in b_1 and b_2 are purely non–supersymmetric contributions of the source. They are due to the new metric dependence of the source action with respect to the supersymmetric case.

In order to correctly identify the cosmological constant, but also to perform the study of the moduli dependence, we need to go to the four–dimensional Einstein frame

$$g_{\mu\nu\ E} = \sigma^2\ g_{\mu\nu}\,. \tag{5.102}$$

The four–dimensional Einstein–Hilbert term transforms as[16]

$$\begin{aligned} \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{|g_4|}\frac{L^6}{2}e^{-2\phi}R_4 &= \frac{L_0^6}{2g_s^2\ 2\kappa^2}\int \mathrm{d}^4x\sqrt{|g_4|}\ \sigma^2 R_4 \\ &= M_4^2\int \mathrm{d}^4x\sqrt{|g_{4E}|}R_{4E}\,, \end{aligned}$$

where we denote Einstein frame quantities by E, and we introduced $M_4^2=\frac{L_0^6}{2g_s^2\ 2\kappa^2}$, the squared four–dimensional Planck mass. Similarly, the four–dimensional potential in the Einstein frame becomes

$$U_E = \sigma^{-4}\ U = 4\kappa^4 M_4^4\frac{e^{4\phi}}{(\frac{L^6}{2})^2}\ U\ , \tag{5.104}$$

and we can write the Einstein frame action as

$$S = M_4^2\int \mathrm{d}^4x\sqrt{|g_{4E}|}\left(R_{4E}+\mathrm{kin}-\frac{1}{M_4^2}U_E\right)\,. \tag{5.105}$$

The cosmological constant, (5.30), is then related to the vacuum value of the potential

$$\Lambda = \frac{1}{2M_4^2}U_E|_0\ . \tag{5.106}$$

Extremization and stability

In order to find a solution, one should determine the minima of the potential. For our choice of moduli, ρ and σ, one has

$$\frac{\partial U_E}{\partial\sigma} = -\frac{M_4^2}{\sigma^5}[2g_s^2\,(|F_0|^2\rho^3+|F_2|^2\rho)+2\sigma^2\ (-\frac{R_6}{\rho}+\frac{|H|^2}{2\rho^3})-3\sigma\ g_s(b_0+\frac{b_1}{\rho}+\frac{b_2}{\sqrt{\rho}})]\ , \tag{5.107}$$

$$\frac{\partial U_E}{\partial\rho} = \frac{M_4^2}{\sigma^2}\ [(\frac{R_6}{\rho^2}-\frac{3|H|^2}{2\rho^4})+\frac{g_s}{\sigma}\ (\frac{b_1}{\rho^2}+\frac{b_2}{2\sqrt{\rho^3}})+\frac{g_s^2}{2\sigma^2}(3|F_0|^2\rho^2+|F_2|^2)]\,. \tag{5.108}$$

[16]Under a conformal rescaling of the four dimensional metric we have

$$g_{\mu\nu}\to e^{2\lambda}g_{\mu\nu}\qquad\Rightarrow\qquad \sqrt{|g_4|}\to e^{4\lambda}\sqrt{|g_4|}\ ,\ R_4\to e^{-2\lambda}R_4\ . \tag{5.103}$$

In our conventions, the extremization conditions are

$$\frac{\partial U_E}{\partial \sigma}|_{\sigma=\rho=1} = 0 \quad , \quad \frac{\partial U_E}{\partial \rho}|_{\sigma=\rho=1} = 0 \ , \tag{5.109}$$

where $\sigma = \rho = 1$ are the values of the moduli on the vacuum. Actually, the conditions (5.109) are equivalent to the ten–dimensional dilaton e.o.m. and the trace of internal Einstein equation. Combining the dilaton equation (5.34) and the trace of the internal Einstein equation, (5.33), we can write the six–dimensional Ricci scalar as

$$R_6 = \frac{3}{2}|H|^2 - \frac{g_s^2}{2}(3|F_0|^2 + |F_2|^2) - \frac{g_s}{2}(T_0 - T) \, , \tag{5.110}$$

where

$$\begin{aligned} T_0 - T &= 2\kappa^2 T_p \hat{*}\langle j, C_m^m \rangle = \frac{[(\mathrm{d}F_2 - HF_0)\wedge(X_J + 2X_\Omega)]_{1\ldots 6}}{\sqrt{|g_6|}} \\ &= 2b_1 + b_2 \, . \end{aligned} \tag{5.111}$$

In the last line we used (5.100). With this expression for $T_0 - T$, it is immediate to verify that (5.110) is indeed equal to the $\partial_\rho U_E$ in (5.109). Similarly, one can see that using (5.108), (5.99), (5.57) and (5.109), the dilaton equation (5.34) reduces to $\partial_\sigma U_E$ in (5.109).

From the equivalence of the ten–dimensional equations and (5.109) we see that the ten–dimensional solution discussed in the previous sections does indeed satisfy the extremization conditions (5.109). The next step is to see whether such extremum correspond to a minimum of the potential and whether, furthermore, it is stable.

Let us consider (5.108) and discuss the ρ dependence of the potential. It is convenient to define the function

$$P(\rho^2) = \frac{\partial U_E}{\partial \rho} \frac{\sigma^2 \rho^4}{M_4^2} \ . \tag{5.112}$$

It is easy to check that $P(\rho^2)$ is negative for $\rho = 0$ and positive for $\rho \to \infty$. Hence there must be a real positive root and this is a minimum of U_E. A priori, $P(\rho^2)$ could have other zeros. Let us focus only on the situation in which $b_2 = 0$, which, in particular, is the case for our ten–dimensional solution. In that case, $P(\rho^2)$ has two other roots which are either complex conjugate[17], or real and negative, according to the value of the parameters. Indeed, studying $\partial_{\rho^2} P$, one can show that $P(\rho^2)$ can be 0 only once. Therefore, at least for $b_2 = 0$, there is only one extremum of U_E in ρ and it is a minimum. So satisfying the extremization in ρ is enough for the stability.

Let us now analyze the σ dependence of (5.104). It is easy to see that the potential admits an extremum for

$$\sigma_\pm = \frac{1}{4a}\left(3b \pm \sqrt{8b^2\left(\frac{9}{8} - \frac{4ac}{b^2}\right)}\right) \qquad \frac{4ac}{b^2} < \frac{9}{8} \ , \tag{5.113}$$

[17]Since the polynomial is real, they come in conjugate pairs.

where for simplicity we introduced

$$
\begin{aligned}
a &= -R_6\rho^{-1} + \frac{1}{2}|H|^2\rho^{-3}\,, \\
b &= g_s(b_0 + b_1\rho^{-1} + b_2\rho^{-\frac{1}{2}})\,, \\
c &= \frac{g_s^2}{2}\rho^3(|F_0|^2 + |F_2|^2\rho^{-2})\,.
\end{aligned} \tag{5.114}
$$

In our case, asking for $\sigma = 1$ and using the extremization in σ in (5.109), which can be written as $2a - 3b + 4c = 0$, we find that the minimum in σ_- corresponds to

$$
a - 2c < 0\ . \tag{5.115}
$$

This condition is satisfied by our solution choosing $\gamma^2 = \frac{1}{2}$, as we can see from (5.50). Therefore, our solution is at the minimum in σ, and it is then stable both in the volume and the dilaton moduli.

It is easy to see that the four–dimensional potential takes a positive value at the minimum, and, hence, the minimum corresponds to a de Sitter vacuum. In [176], it has been shown that the potential has a strictly positive minimum in σ for

$$
1 < \frac{4ac}{b^2} < \frac{9}{8}\ , \tag{5.116}
$$

where the lower bound comes from asking the potential to be never vanishing (strictly positive). This condition is satisfied by our solution.

In addition, we can actually compute the value of the potential at $\sigma = \rho = 1$. Starting from (5.104) and using the two equations of (5.109), we obtain

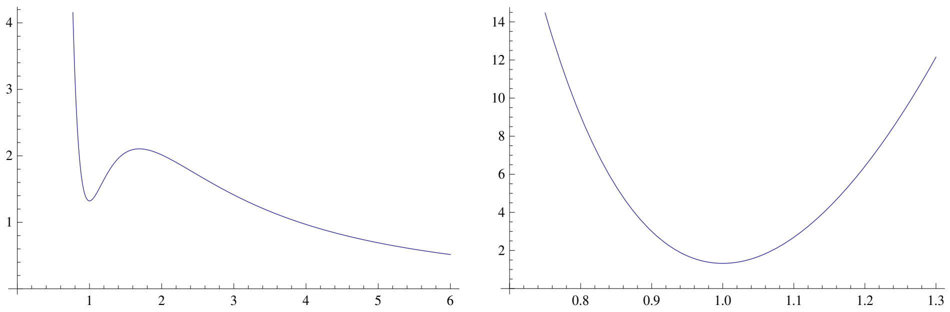
$$
\frac{U_E}{M_4^2} = \frac{1}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2|F_0|^2 - |H|^2\right)\ . \tag{5.117}
$$

Using (5.31) and (5.110), one can show that the four–dimensional Ricci scalar is proportional to (5.117), $R_4 = 2U_E/M_4^2$. For $\gamma^2 = 1/2$, R_4 is positive (see the discussion below (5.51)), and hence so is the value of the potential at the minimum.

Note also that, for $\gamma^2 = 1/2$, the last two terms in (5.117) cancel each other and the entire contribution to the cosmological constant comes from sources, $(T_0 - T)$. For supersymmetry breaking branes, this contribution is never vanishing but, for generic situations, we do not know what its sign is. It would be nice to have a model independent argument to determine whether, for this mechanism of supersymmetry breaking, the resulting four–dimensional space is always de Sitter.

As a further check of the existence of a de Sitter minimum for our solution, we can plot the four–dimensional potential U_E as a function of σ and ρ for some values of the parameters

$$
\begin{aligned}
& t_1 = t_2 = t_3 = \tau_3 = \tau_6 = 1\,, \\
& q_1 = 1\ ,\ q_2 = 3\ ,\ p = \frac{\cosh^{-1}(2)}{\pi}\,, \\
& \lambda = 5\ ,\ \gamma = \frac{1}{\sqrt{2}}\ ,\ h = 4\ .
\end{aligned} \tag{5.118}
$$



$\frac{1}{M_4^2} U_E(\sigma, \rho = 1)$ $\frac{1}{M_4^2} U_E(\sigma = 1, \rho)$

Figure 5.1: Dependence of the potential on dilaton and volume moduli.

Chapter 6

Formal developments

The analysis of non–supersymmetric vacua is complicated by the fact that we are lacking a systematic and comprehensive characterization and we are forced to to face the full problem of second order differential equations of motion. It would be of extreme usefulness to find a modification of the first order conditions which can account for at least some class of non–supersymmetric vacua. The generalized complex geometry formalism, as we have seen, is a good organizing principle to describe $\mathcal{N} = 1$ vacua and it is a natural starting point for a description of non–supersymmetric ones. The authors of [137] have parameterize a quite generic supersymmetry breaking

$$
\begin{aligned}
\delta\psi_\mu^{(i)} &= \frac{1}{2} e^A \gamma_\mu \zeta \otimes \mathcal{V}_i + \text{c.c.}\,, \\
\delta\psi_m^{(i)} &= \zeta \otimes \mathcal{U}_m^{(i)} + \text{c.c.}\,, \\
\Gamma^M \delta\psi_M^{(i)} - \delta\lambda^{(i)} &= \zeta \otimes \mathcal{S}_i + \text{c.c.}\,,
\end{aligned}
$$

where $i = 1, 2$ and the internal spinors parametrizing the breaking are

$$
\begin{aligned}
\mathcal{V}_i &= r_i \eta_i^* + s_m^i \gamma^m \eta_i\,, \\
\mathcal{S}_i &= t_i \eta_i^* + u_m^i \gamma^m \eta_i\,, \\
\mathcal{U}_m^i &= p_m^i \eta_i + q_{mn}^i \gamma^n \eta_i^*\,.
\end{aligned}
$$

The previous equations can be reformulated as a modification of the pure spinor equations (3.57)-(3.59) which, as we have introduced in the previous Chapter, now have additional pieces[1]

$$
e^{-2A+\phi}(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_-) = \mathrm{d}A \wedge \bar{\Phi}_- + \frac{i}{8} e^{A+\phi} * \lambda(F) + \Upsilon\,, \tag{6.1}
$$

$$
e^{-2A+\phi}(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_+) = \Xi\,, \tag{6.2}
$$

[1] We consider here the case of Minkowski external space; the AdS case is easily obtained by adding the terms proportional to the cosmological constant. As elsewhere in the thesis we present the type IIA analysis but it can be straightforwardly extended to the type IIB case.

where

$$\Upsilon = \frac{1}{2}\Big[(\bar{r}_1 + \bar{t}_2)\Phi_+ - (t_1 + r_2)\bar{\Phi}_+ - \bar{s}^1_m\gamma^m\bar{\Phi}_- - s^2_m\bar{\Phi}_-\gamma^m + (u^1_m + \bar{p}^2_m)\gamma^m\Phi_- \\ + (\bar{u}^2_m + p^1_m)\Phi_-\gamma^m - q^1_{mn}\gamma^n\bar{\Phi}_+\gamma^m + \bar{q}^2_{mn}\gamma^m\Phi_+\gamma^n\Big]\,,$$

$$\Xi = \frac{1}{2}\Big[t_2\Phi_- + t_1\bar{\Phi}_- + (u^1_m + p^2_m)\gamma^m\Phi_+ - (u^2_m + p^1_m)\Phi_+\gamma^m - q^1_{mn}\gamma^n\bar{\Phi}_-\gamma^m - q^2_{mn}\gamma^m\Phi_-\gamma^n\Big]\,.$$

However, without any constraint this rewriting provides little information. In [137] an analysis of integrability conditions is carried out, in analogy with the supersymmetric case, they reformulate a certain linear combination of the second order bosonic equations of motion as spinorial equations involving a product of two first order differential operators. They prove that, under certain restrictions, provided the Bianchi identities are imposed and a set of first order equations involving $\mathcal{V}_i$, $\mathcal{U}^i_m$, $\mathcal{S}_i$ and the background fields is satisfied then the second order equations of motion for the bosonic fields are satisfied. Solutions of the spinorial equations in the variables $\mathcal{V}_i$, $\mathcal{U}^i_m$, $\mathcal{S}_i$ select the breaking that lead to solutions of the equations of motion. However, there is no clear geometric interpretation or nice rewriting in terms of bispinors and their analysis is rather involved even in simple sub–cases. Moreover, their analysis is valid for $\Upsilon = 0$ which is equivalent to the assumptions that the background admits certain generalized calibrations which can be used to describe stable sources. Despite the careful analysis of [137] a full understanding of $\mathcal{N} = 0$ vacua in the language of generalized complex geometry is far from being available.

In this short Chapter we collect some observations which could be of some usefulness for a description of supersymmetry breaking backgrounds at a more formal level than the example based configuration we have described before. The nature of the Chapter is speculative and no concrete results are established, nevertheless a glimpse of an underlying structure can still be inferred which suggests that generalized geometry could be a suitable language also for $\mathcal{N} = 0$ vacua.

6.1 Supersymmetry breaking T–duality

Since its inception, T–duality has been of paramount importance in the study of string backgrounds. At the level of supergravity theory there is a corresponding local transformation on the fields, governed by Buscher rules [28], which maps solutions to solutions. The action of T–duality on generalized structures is simply an $\mathrm{O}(d, d)$ gauge transformation on the bundle E and a detailed discussion can be found in [88] where they also establish the necessary condition for the transformation to map $\mathrm{SU}(3) \times \mathrm{SU}(3)$ supersymmetric backgrounds to $\mathrm{SU}(3) \times \mathrm{SU}(3)$ supersymmetric backgrounds. In particular, it turns out that the Lie derivative of the two pure spinors defining the structure along the T–dual direction has to vanish. In this Section we want to relax this condition and argue about its consequences, we find a set of first order equations that should describe T–dual non supersymmetric backgrounds obtained from supersymmetric ones.

We start by a short review of T–duality in the generalized complex geometry, referring the reader to [88] for the details. Buscher rules are valid for backgrounds that admit a Killing direction v satisfying

$$\mathcal{L}_v g = 0 \qquad\qquad \mathcal{L}_v H = 0 \qquad\qquad \mathcal{L}_v F = 0\,.$$

The condition on H allows to find a gauge transformation on $\hat{B} = B + d\hat{\xi}$ such that $\mathcal{L}_v \hat{B} = 0$ and it is on this gauge equivalent background that the Buscher rules are applied. Thus one requires

$$\mathcal{L}_v B = d\xi\,, \qquad\qquad \xi = -\iota_v d\hat{\xi} + df\,,$$

namely the T–duality action is specified by the pair (v, ξ) which can be arranged in a generalized vector $\mathbb{V} = v + \xi$. The freedom given by the shift of ξ by df with f an arbitrary function allows to normalize the generalized vector $\mathbb{V}$ such that

$$\mathcal{I}(\mathbb{V}, \mathbb{V}) = 1 \tag{6.3}$$

where $\mathcal{I}$ is the metric in (3.7). The vector $\mathbb{V}$ takes the general form

$$\mathbb{V} = \frac{\partial}{\partial t} + \left(dt - \iota_{\partial/\partial t} d\hat{\xi}\right)\,, \tag{6.4}$$

where we have introduced the coordinate t such that $v = \partial/\partial t$ and set $f = t$. One can then define an $\mathrm{O}(d,d)$ element

$$T_{\mathbb{V}} = \mathbb{1} - 2\mathbb{V}\mathbb{V}^t\mathcal{I}\,. \tag{6.5}$$

The T–dual transform of generalized metric $\mathcal{H}$ (cfr. (3.49)) is given by $\tilde{\mathcal{H}} = T^t_{\mathbb{V}} \mathcal{H} T_{\mathbb{V}}$ and the pure spinors transform by Clifford action

$$\tilde{\Phi} = T_{\mathbb{V}} \Phi = \iota_{\partial/\partial t}\Phi + \xi \wedge \Phi\,,$$

where $\xi = dt - \iota_{\partial/\partial t}\hat{\xi}$.

Let us start from a supersymmetric type IIB background and without loss of generality we can choose B such that $\mathcal{L}_v B = 0$ and thus $\mathbb{V}_0 = v + dt = \partial/\partial_t + dt$; the background will satisfy the pure spinor equations[2]:

$$\begin{aligned}
\mathrm{d}\left(e^{2A}\Psi_+\right) &= \mathrm{d}A \wedge e^{2A}\bar{\Psi}_+ + \frac{i}{8} e^{3A} *_E (G) \\
\mathrm{d}\left(e^{2A}\Psi_-\right) &= 0
\end{aligned}$$

Let us begin by applying T–duality to the previous equations, we obtain

$$\mathrm{d}\left(e^{2A}\tilde{\Psi}_+\right) = \mathrm{d}A \wedge e^{2A}\bar{\tilde{\Psi}}_+ + \frac{i}{8} e^{3A} \tilde{*}_E(\tilde{G}) + \mathcal{L}_v\left(e^{2A}\Psi_+\right) - (\iota_v \mathrm{d}A) e^{2A}\bar{\Psi}_+\,, \tag{6.6}$$

$$\mathrm{d}\left(e^{2A}\tilde{\Psi}_-\right) = \mathcal{L}_v\left(e^{2A}\Psi_-\right)\,. \tag{6.7}$$

The tilded objects are the result of T–duality, in particular:

$$\tilde{\Psi}_\pm = T_{\mathbb{V}_0}(\Psi_\pm) \qquad\qquad \tilde{*}_E = T_{\mathbb{V}_0} *_E T_{\mathbb{V}_0} \qquad\qquad \tilde{G} = -T_{\mathbb{V}_0}(G)$$

[2] In this Section we prefer to use the variables $\Psi_\pm = e^{-B-\phi}\Phi_\pm$ which have the correct transformation properties on E, the RR fluxes are $G = e^{-B}F$, with Bianchi identity $\mathrm{d}G = 0$ in absence of sources and the Hodge operator on E is related to the usual Hodge dual by $*_E = e^{-B} * \lambda e^B$.

The new background will satisfy pure spinor equations in the transformed variables and it will be supersymmetric if $\mathcal{L}_v\Psi_\pm = 0$ and $\mathcal{L}_v \mathrm{d}A = \iota_v \mathrm{d}A = 0$. If we do not require this condition we can see that the equations are modified by supersymmetry breaking terms which could in principle be rewritten in terms of an expansion in the generalized Hodge diamond constructed from $\tilde{\Psi}_\pm$ in order to recover a form similar to (6.1) and (6.2). Let us consider the following identities

$$\mathrm{d}t \wedge \Psi_\pm = \mathrm{d}t \wedge \iota_v \tilde{\Psi}_\pm \qquad\qquad \iota_v \Psi_\pm = \tilde{\Psi}_\pm - \mathrm{d}t \wedge \iota_v \tilde{\Psi}_\pm$$

We are going to recast the equations (6.6) and (6.7) in a form which is reminiscent of the pure spinor equations but identifying a different set of variables; in the following we will make use only of the supersymmetry equations for the background we started from. Equation (6.7) is easily reformulated as

$$\mathrm{d}\left(\mathrm{d}t \wedge \iota_v \left(e^{2A}\tilde{\Psi}_+\right)\right) = 0\,.$$

Note that if $\tilde{\Psi}$ is pure $\mathrm{d}t \wedge \iota_v \tilde{\Psi}$ is not. Equation (6.6) requires more work but the result can be expressed as

$$\begin{aligned}\mathrm{d}\left[e^{2A}\mathrm{d}t \wedge \iota_v(\tilde{\Psi}_+)\right] &= e^{2A}\mathrm{d}A \wedge (\mathrm{d}t \wedge \iota_v(\bar{\tilde{\Psi}}_-)) + \frac{i}{8}e^{3A}\mathrm{d}t \wedge \iota_v[\tilde{*}_E(\tilde{G})] \\ &= e^{2A}\mathrm{d}A \wedge (\mathrm{d}t \wedge \iota_v(\bar{\tilde{\Psi}}_-)) + \frac{i}{8}e^{3A}\iota_v[*_E(G)] + \frac{i}{8}e^{3A}\tilde{*}_E(\tilde{G})\,. \end{aligned} \qquad (6.8)$$

In the first equality everything is expressed in terms of transformed quantities while the second form it is easier to observe that the equations of motion for the RR fluxes follow if $\iota_v \mathrm{d}A = 0$. It is straightforward to see that

$$\mathrm{d}\left[e^{4A}\tilde{*}_E(\tilde{G})\right] = -\mathcal{L}_v\left[e^{4A} *_E (G)\right] = 0\,.$$

We can now define the following variables

$$\begin{aligned} X_+ &\equiv \mathrm{d}t \wedge \iota_v \tilde{\Psi}_-\,, \\ Y_- &\equiv \mathrm{d}t \wedge \iota_v \tilde{\Psi}_+\,, \end{aligned}$$

and reformulate the equations in a more familiar looking way

$$\begin{aligned} \mathrm{d}\left(e^{A}\,\mathrm{Re}\,Y_-\right) &= 0\,, \\ \mathrm{d}\left(e^{3A}\,\mathrm{Im}\,Y_-\right) &= -\frac{1}{8}e^{4A}\mathrm{d}t \wedge \iota_v\left[\tilde{*}_E(\tilde{G})\right] = -\frac{1}{8}e^{4A}\mathrm{d}t \wedge [*_E(G)]\,, \\ \mathrm{d}\left(e^{2A}X_+\right) &= 0\,. \end{aligned} \qquad (6.9)$$

Note that despite X_+ and Y_- are compatible they are no longer pure.

6.2 A formal system

Let us consider a type IIA configuration where we take into consideration the supersymmetry breaking terms as in [137]

$$\mathrm{d}\left(e^{2A}\Psi_-\right) = \mathrm{d}A\wedge e^{2A}\bar{\Psi}_- + \frac{i}{8}e^{3A} *_E (G) + e^{2A}\Upsilon\,, \tag{6.10}$$

$$\mathrm{d}\left(e^{2A}\Psi_+\right) = e^{2A}\Xi\,. \tag{6.11}$$

Equation (6.11) tells us that we can locally write

$$e^{2A}\Xi = \mathrm{d}\left(e^{2A}\Psi_+ + e^{2A}U_+\right)\,, \tag{6.12}$$

for some conformally closed polyform U_+.

By taking the external differential of equation (6.10) we can obtain a consistency condition for Υ:

$$\mathrm{d}(e^{2A}\Upsilon) - \mathrm{d}A\wedge e^{2A}\bar{\Upsilon} = -\frac{i}{8}e^{-A}\mathrm{d}\left[e^{4A} *_E (G)\right] \tag{6.13}$$

Note that if we demand that RR eom is satisfied, the rhs is zero. Once more, we can solve for the real part of (6.10) by using the closure of the right–hand side and taking

$$e^{A}\,\mathrm{Re}\,\Upsilon = \mathrm{d}\left(e^{A}\,\mathrm{Re}\,\Psi_- + e^{A}\beta_-\right) \tag{6.14}$$

for some real β, which must satisfy $\mathrm{d}(e^{A}\beta_-) = 0$. We would like of course to relate full Υ and Ψ_-, and should look at the imaginary part of equation (6.10). Provided that RR equations of motion are satisfied (i.e. right-hand side of (6.13) vanishes), we may take

$$e^{3A}\,\mathrm{Im}\,\Upsilon = \mathrm{d}\left(e^{3A}\,\mathrm{Im}\,\Psi_- + e^{3A}\gamma_-\right) \tag{6.15}$$

for some real γ_- such that $\mathrm{d}(e^{3A}\gamma_-) = -\frac{1}{8}\left[e^{4A} *_E (G)\right]$. Recall that the ten–dimensional fluxes are written in term of the potentials as: $F_{(n)} = \mathrm{d}C_{(n-1)} - H\wedge C_{(n-3)}$; under the $4+6$ splitting we get:

$$F_{(n)} = \hat{F}_{(n)} + \mathrm{Vol}_4\wedge\tilde{F}_{(n-4)}\,, \tag{6.16}$$

together with the duality relation $\tilde{F}_{(2n-4)} = \lambda(*_6\hat{F}_{(10-2n)})$ (here $n = 2,3,4,5$ for type IIA). According to the split, also the potentials are rewritten as:

$$C_{(n-1)} = \hat{C}_{(n-1)} + \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\wedge e^{4A}\tilde{C}_{(n-4)}\,. \tag{6.17}$$

We thus have:

$$\tilde{F} = e^{B-4A}\mathrm{d}(e^{-B+4A}\tilde{C}) \tag{6.18}$$

where as usual $\tilde{F} = \sum_k \tilde{F}_{(k)}$. The fluxes that appear in the pure spinor equations[3] are the dual ones and we can write them in terms of the dual potentials:

$$\mathrm{d}(e^{2A}\Psi_-) = \mathrm{d}A\wedge e^{2A}\bar{\Psi}_- + \frac{i}{8}e^{-A}\mathrm{d}(e^{4A-B}\tilde{C}) + e^{2A}\Upsilon \tag{6.19}$$

Let us consider the imaginary part of the previous equation:

[3] This is the case if $|a|^2 = |b|^2$ which is equivalent to $c_- = 0$ in (6.1). See also discussion in Section 3.5.

$$\mathrm{d}(e^{3A}\,\mathrm{Im}\,\Psi_-) + \frac{1}{8}\mathrm{d}(e^{4A-B}\tilde{C}) = e^{3A}\,\mathrm{Im}\,\Upsilon\,. \tag{6.20}$$

Thus we can write:

$$e^{3A}\,\mathrm{Im}\,\Upsilon = \mathrm{d}(e^{3A}(\mathrm{Im}\,\Psi_- + \gamma_- + \sigma_-))\,, \tag{6.21}$$

where

$$\gamma_- = \frac{1}{8}e^{A-B}\tilde{C}\,, \qquad\qquad \mathrm{d}(e^{3A}\sigma_-) = 0. \tag{6.22}$$

We can now define $V_- = \beta_- + i\gamma_-$ and solve for Υ

$$e^{A}\Upsilon = \mathrm{d}\left(e^{A}(\Phi_- + V_-)\right) + 2i\mathrm{d}(e^{A}) \wedge \mathrm{Im}(\Phi_- + Y_-)\,, \tag{6.23}$$

and collect equations for U_+ and V_- into familiar looking system:

$$\begin{aligned} \mathrm{d}(e^{2A}U_+) &= 0\,,\\ \mathrm{d}(e^{A}\,\mathrm{Re}\,V_-) &= 0\,,\\ \mathrm{d}(e^{3A}\,\mathrm{Im}\,V_-) &= -\frac{1}{8}e^{4A} *_E (G)\,. \end{aligned} \tag{6.24}$$

The formal rewritings we have tried to sketch in this and the previous section suggests that generalized complex geometry could still be a suitable language to describe $\mathcal{N}=0$ vacua. It is clear to the reader that we are still quite far from a definitive answer which is beyond the scope of this speculative Chapter, in particular the analysis should be completed by a more deep understanding of the properties of the new variables we have introduced. Moreover it is not clear which constraints could be inferred by the equations of motion for the bosonic fields (in the previous analysis we have used only the RR equations of motion).

6.3 Branes in non–supersymmetric backgrounds

In this section we want to further discuss the κ–symmetry condition for sources which has been briefly mentioned in Section 3.6 and speculate about a way to describe branes in supersymmetry breaking backgrounds. The action (3.66) contains only bosonic degrees of freedom and in order to obtain a supersymmetric formulation in a general background one need to use superspace formalism. Elegant and complete formulations have been developed in [40, 19, 2], but couplings between physical fields are not explicit and the fermionic world–volume sector is somehow hidden. Fortunately the authors of [145, 146, 148] have derived an explicit and compact expression for the source action up to terms quadratic in the fermions. We will consider this formulation, in which the interplay of κ–symmetry and supersymmetry is more clear, as a starting point for our considerations about supersymmetry breaking. Let us thus consider the fermionic action (here and in the following we will consider type IIA theory, similar arguments with minor modifications can be developed for type IIB)

$$S^{(F)}_{Dp} = \frac{T_p}{2}\int d^{p+1}\xi e^{-\phi}\sqrt{-\det(\iota^*[g]+\mathcal{F})}\,\bar{\theta}(1-\Gamma_{Dp})\left[(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\right]\theta\,. \tag{6.25}$$

Some detail is needed:

- $\theta = \theta_1 + \theta_2$ denotes a 32–component Majorana spinor such that $\Gamma_{(10)}\theta_1 = \theta_1$ and $\Gamma_{(10)}\theta_2 = -\theta_2$ which eventually can be arranged in a doublet;
- Γ_{Dp} denotes the chiral world–volume operator which we have already encountered in equation (3.74);
- the operator D_α (here α denotes the pullback on the world–volume) and Δ are the operators entering the supersymmetry variations of the gravitino, $\delta_\epsilon \Psi_m = D_\alpha(\epsilon_1 + \epsilon_2)$, and of the dilatino, $\delta_\epsilon \lambda = \Delta(\epsilon_1 + \epsilon_2)$, fields which we have introduced in Section 2.1;
- the matrix $\tilde{M}$ is defined as follows $\tilde{M}_{\alpha\beta} = g_{\alpha\beta} + \Gamma_{(10)}\mathcal{F}_{\alpha\beta}$, where as before, greek letters denote indexes on the world–volume.

The fermionic action together with the bosonic counterpart (3.66) has world–volume diffeomorphism symmetry as well as an additional local fermionic symmetry called κ–symmetry. This last symmetry is necessary to match the number of bosonic and fermionic degrees of freedom. In fact for an arbitrary p–brane the bosonic degrees of freedom are $10 - p - 1$ due to the scalars X^M which define the embedding (after taking into account world–volume diffeomorphism invariance) and $p - 1$ are from the massless gauge vector living on the world–volume, in total 8 degrees of freedom. The Majorana spinor θ has 32 real components that have to be reduced to 8. Half of them are cut due to κ–symmetry and another half of the remaining is cut due to Dirac–like equation, the kinetic terms for θ are linear in time derivative, leaving us with the correct number of 8 fermionic degrees of freedom. We are interested in bosonic backgrounds where, as we have discussed in Section 2.2, we put to zero all the vacuum expectation values of the fermionic fields. The action (6.25) is quadratic in the fermionic field θ and thus it will not give contributions to the equations of motion for the bosonic background fields. We will not provide the details of the κ–symmetry transformation for all the fields and we refer the reader to [41, 40, 19, 2, 3, 145, 146, 148]. For the sake of our exposition it is enough to recall that the supersymmetry and κ–symmetry transformations for θ have the following form

$$\begin{aligned}\delta_\epsilon \theta &= \epsilon \\ \delta_\kappa \theta &= (1 + \Gamma_{Dp})\,\kappa\,.\end{aligned}$$

In the analysis of the supersymmetry of the action one has to take into account a supersymmetry variation which respects some covariant gauge choice. In fact the operator Γ_{Dp} is such that $\Gamma_{Dp}(\mathcal{F})^{-1} = (-)^{\lfloor \frac{p+3}{2} \rfloor}\Gamma_{Dp}(-\mathcal{F})$ and κ–symmetry transformations can actually be written in terms of an irreducible 16–components spinor κ. In general we need to compensate the generic supersymmetry transformation with a κ–symmetry transformation and we thus have[4]

$$\delta_\epsilon \theta = (1 + \Gamma_{Dp})\kappa + \epsilon\,. \tag{6.26}$$

Supersymmetry is clearly preserved if

[4]Here we limit to supersymmetry transformation at the lowest leading order for fermions, namely such that they do not involve fermions fields for the transformation of the fermions and linear in the fermions for the transformation of the bosons. The complete analysis would require higher order terms, nevertheless it is enough for our purposes which considers only variation linear in fermions. If we consider the action truncated to quadratic order in both fermions and bosons around a classical configuration these are exact supersymmetries.

$$\delta_\epsilon \theta = 0\,. \tag{6.27}$$

We discussed in Section 3.6 how this condition can be reformulated in terms of the existence of certain calibration forms and their relation with the bulk supersymmetry. Here we would like to address the possibility of describing branes in non supersymmetric backgrounds. If the background is no longer supersymmetric there is a priori no reason the brane should still satisfy an equation like (6.27) and we expect that it does not wrap one of the cycles calibrated by supersymmetry. Since the possibility of using first order equations is not evident once we leave supersymmetry, we have to face the problem at the level of the equations of motion. The question of the existence of stable branes is a legitimate one also in non supersymmetric backgrounds. We could proceed as follows. As in the previous chapters we want to stay in the class of solutions characterized by an SU(3) × SU(3) structure for which generalized complex geometry is a natural language, we thus choose a certain pair of spinors ϵ_1 and ϵ_2 which will characterize the structure but such that they are not Killing spinors, in particular the gravitino and dilatino supersymmetry variation will no longer be zero but acquire additional pieces whose translation in bispinor language is given by equations (6.1) and (6.2). Let us parametrize the deviation from the supersymmetry in (6.27) by a certain spinor χ:

$$\delta_\epsilon \theta = (1 + \Gamma_{Dp})\kappa + \epsilon = \chi\,. \tag{6.28}$$

If we apply the orthogonal projector $(1 - \Gamma_{Dp})$ we get $(1 - \Gamma_{Dp})\epsilon = (1 - \Gamma_{Dp})\chi$ which, if we impose a covariant gauge like $\Gamma_{(10)}\theta = \theta$ boils down to

$$\Gamma_{Dp}\tilde{\epsilon}_2 = \tilde{\epsilon}_1\,, \tag{6.29}$$

where $\tilde{\epsilon}_{1(2)} = \epsilon_{1(2)} - \chi_{1(2)}$.

This condition looks formally the same as the one that leads to calibrated branes for supersymmetric configurations, but the spinor $\tilde{\epsilon}$ is not the Killing spinor of the background and thus, despite the source seems to preserve some supercharges, these are not "compatible" with the ones that would be associated to the supersymmetric background we are deforming.

Let us abandon the general case by considering our usual 4+6 split and supposing that the deviation of the spinor $\tilde{\epsilon}$ from the Killing one is due to the internal part

$$\begin{aligned} \tilde{\epsilon}^1_+ &= \zeta_+ \otimes \tilde{\eta}^1_+ + \text{c.c.} \qquad & \tilde{\eta}^1_+ &= b_1 \eta^1_+ + c^1_m \gamma^m \eta^1_- \\ \tilde{\epsilon}^2_- &= \zeta_+ \otimes \tilde{\eta}^2_- + \text{c.c.} \qquad & \tilde{\eta}^2_+ &= b_2 \eta^1_+ + c^2_m \gamma^m \eta^1_-\,, \end{aligned}$$

where $b_{1(2)}$ and $c^{1(2)}_m$ are functions on the internal manifold. By similar reasoning involving the chiral world–volume operator (3.74) we can translate the algebraic condition (6.29) into a relation between the DBI Lagrangian and the pullback of a certain form from the bulk[5]:

$$\frac{i}{8}||\tilde{\eta}^1_+||^2 \sqrt{|\det(\iota^*[g] + \mathcal{F})|}\mathrm{d}\xi^1 \ldots \mathrm{d}\xi^{p-3} = \left(\iota^*[X_-] \wedge e^{\mathcal{F}}\right)_{(p-3)}\,. \tag{6.30}$$

The form X_- is defined as

$$X_- \equiv \tilde{\eta}^1_+ \otimes \tilde{\eta}^2_- = b_1 b_2 \Phi_- - b_1 c^2_m \Phi_+ \gamma^m + b_2 c^1_m \gamma^m \bar{\Phi}_+ + c^1_m c^2_n \gamma^m \bar{\Phi}_- \gamma^n\,, \tag{6.31}$$

[5]Note that the unitarity properties of the world–volume chiral operator imply that $||\tilde{\eta}^1_+||^2 = ||\tilde{\eta}^2_+||^2$.

and we can observe that it can be expressed in terms of the pure spinors related to the underlying SU(3) × SU(3) structure.

The energy of the brane is given by the Lagrangian (3.72) and using equation (6.30) we can build a differential form in the bulk

$$\omega_X \equiv \frac{8}{||\tilde{\eta}^1_+||^2}\, e^{4A-\phi}\, \mathrm{Im}\, X_- - e^{4A} \sum_k \tilde{C}_{(k)} \,, \tag{6.32}$$

such that

$$\iota^*[\omega_X] \wedge e^{\mathcal{F}} \leq \mathcal{E}(\Sigma, \mathcal{F}) \,, \tag{6.33}$$

for any brane characterized by Σ and $\mathcal{F}$. The inequality is guaranteed by a Schwarz inequality as in (3.78) and the branes which saturate the bound are the ones which satisfy (6.30). Let us consider a deformation to a different brane configuration $(\Sigma', \mathcal{F}')$ within the same generalized cohomology class, that is to say we can take a chain $\mathcal{C}$ and a field strength $\hat{\mathcal{F}}$ on it such that $\partial\mathcal{C} = \Sigma - \Sigma'$ and the restriction of $\hat{\mathcal{F}}$ to Σ and Σ' gives $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then by Stokes' theorem[126, 149, 130, 128]:

$$\begin{aligned} E(\Sigma, \mathcal{F}) &= \int \mathcal{E}(\Sigma, \mathcal{F}) = \int_\Sigma \iota^*[\omega_X] \wedge e^{\mathcal{F}} = \\ &= \int_{\mathcal{C}} \iota^* \left[\mathrm{d}_H(\omega_X)\right] \wedge e^{\hat{\mathcal{F}}} + \int_{\Sigma'} \iota^*[\omega_X] \wedge e^{\mathcal{F}'} \end{aligned}$$

To conclude that this is a minimal energy condition (i.e. the brane is stable and satisfies its equations of motion) we have to impose the d_H closure of ω_X similarly to the supersymmetric case. If this is true

$$\begin{aligned} E(\Sigma, \mathcal{F}) &= \int_{\mathcal{C}} \iota^* \left[\mathrm{d}_H(\omega_X)\right] \wedge e^{\hat{\mathcal{F}}} + \int_{\Sigma'} \iota^*[\omega_X] \wedge e^{\mathcal{F}'} = \\ &= \int_{\Sigma'} \iota^*[\omega_X] \wedge e^{\mathcal{F}'} \leq \int \mathcal{E}(\Sigma', \mathcal{F}') = E(\Sigma', \mathcal{F}') \,. \end{aligned}$$

The closure condition provides a relation between the deviation from the supersymmetric case in (6.28) and the supersymmetry breaking of the bulk configuration which is encoded in Υ and Ξ. A complete analysis is beyond the scope of the Chapter and we will present a simple situation for illustrative purposes. Let us make the following choice

$$b_1 \,, b_2 \,, c^1_m \in \mathbb{R} \qquad\qquad c^2_m = 0 \,,$$

and let us suppose we have normalized $\tilde{\eta}_+$ such that $||\tilde{\eta}_+||^2 = e^A$. The form ω_X reduces to

$$\omega_X = 8 e^{3A-\phi} \left(b_1 b_2\, \mathrm{Im}\, \Phi_- - b_2\, c^1_m \gamma^m\, \mathrm{Im}\, \Phi_+ \right) - e^{4A} \sum_k \tilde{C}_{(k)} \,.$$

Acting on it with d_H, after some algebra and with the information provided by equations (6.1) and (6.2) we obtain the following expression

$$\begin{aligned} \mathrm{d}_H(\omega_X) = {} & (b_1 b_2 - 1) e^{4A} * \lambda(F) + 8 b_1 b_2 e^{3A-\phi}\, \mathrm{Im}\, \Upsilon + 8\mathrm{d}(b_1 b_2) \wedge e^{3A-\phi}\, \mathrm{Im}\, \Phi_- \\ & - 8\mathrm{d}(b_2\, c^1_{(1)}) \wedge e^{3A-\phi}\, \mathrm{Im}\, \Phi_+ + 8 b_2\, \iota_{c^1} \left[e^{3A-\phi}\, \mathrm{Im}\, \Xi + e^{3A-\phi} \mathrm{d}A \wedge \mathrm{Im}\, \Phi_+ \right] \\ & - 8 b_2\, \mathcal{L}_{c^1} \left(e^{3A-\phi}\, \mathrm{Im}\, \Phi_+ \right) - 8\mathrm{d}b_2 \wedge \iota_{c^1} \left(e^{3A-\phi}\, \mathrm{Im}\, \Phi_+ \right) + 8 b_2 \left(\iota_{c^1} H \right) \wedge e^{3A-\phi}\, \mathrm{Im}\, \Phi_+ \\ & + 8 b_2\, c^1_{(1)} \wedge \left[e^{3A-\phi}\, \mathrm{Im}\, \Xi + e^{3A-\phi} \mathrm{d}A \wedge \mathrm{Im}\, \Phi_+ \right] \,, \end{aligned} \tag{6.34}$$

where we denote by c^1 and $c^1_{(1)}$ the vector $c_1^m \partial_m = g^{mn} c^1_n \partial_m$ and the one–form $c^1_m \mathrm{d}x^m$ respectively. If we ask ω_X to be d_H closed we obtain an equation which relates the parameters b_i and c^i_m, which describe how the brane arranges in a non–supersymmetric background, and the parameters describing the deviation of the bulk from the supersymmetric configuration which enter through Υ and Ξ. The underlying idea is that in a non–supersymmetric background stable sources should not in general be described by the same calibration form as for the supersymmetric case and that they wrap cycles such that the combined bulk+brane action is extremized. The deviation from the supersymmetric configuration is parametrized by X_-, Υ and Ξ which, if not constrained, do not correspond to vacua (meaning the equations of motion are not solved). The constraints should come from the equations of motion of both the bulk and the brane; equations like (6.34), which correspond to minimization of the brane action, should take into account the brane equations of motion whose content is here reformulated in bispinor language suitable for the generalized complex geometry formalism. Another set of constraints should come from the analysis of the equations of motion of the bulk fields, unfortunately, as we have remarked before, there is no reformulation in terms of bispinors and the spinorial equations obtained in [137] are valid only in some restricted situation. Note that we expect the parameters defining X_- to enter such equations, in fact one can prove that[6], provided the algebraic condition (6.29) is satisfied the contribution of the source to the Einstein and dilaton equation can be expressed as

$$\begin{aligned} \frac{\delta S_{DBI}}{\delta \phi} &\propto \langle \mathrm{Im}\, X_-, j_{(\Sigma,\mathcal{F})} \rangle \,, \\ \frac{\delta S_{DBI}}{\delta g^{mn}} &\propto \langle g_{q(m} \mathrm{d}x^q \otimes \iota_{n)} \, \mathrm{Im}\, X_-, j_{(\Sigma,\mathcal{F})} \rangle \,, \end{aligned}$$

where $j_{(\Sigma,\mathcal{F})}$ is the generalized current discussed in Section 3.6.

We do not know if there is a unique solution to (6.34) and the stability of the brane should not be intended as absolute, in the sense that it is more likely that the configurations one could find via this approach correspond to metastable non–supersymmetric states. For a reasonably limited interval of energy we could apply local arguments; the possible decay process towards other minima and its dynamics is far beyond the scope of the present analysis. If this was the case it would not be too surprising, in fact many non–supersymmetric solutions are metastable (though long–lived) as, for example, the configuration we will present in the next Chapter or the ones found in [119, 123].

[6]The reasoning is similar to the supersymmetric case and we refer to [130] for details or the the discussion in Section 3.6.

Chapter 7

The backreaction of Anti–D2 branes on a cone over $\mathbb{CP}^3$

Metastable supersymmetry–breaking is an attractive mechanism from a phenomenological point of view [114]. Furthermore, theories for which a metastable supersymmetry breaking state can be realized — such as $\mathcal{N}=1$, $SU(N_c)$ SQCD in the free magnetic phase with massive flavours — are relatively simple and generic enough, unlike the comparatively baroque ingredients involved in other approaches to dynamical SUSY–breaking (see for instance [175, 113] for a review).

Attempts have been made to embed the proposal of Intriligator, Seiberg and Shih into string theory (see for instance [158, 68]), via brane engineering of the electric and magnetic phases [63]. Nevertheless, in view of the obstruction that seems to arise upon turning on the string coupling $g_s \neq 0$ [15] or the alternative view [81] that involves string tachyons corrections to argue that the brane configuration still describes the ISS state, it is of interest to try and find an alternative stringy embedding and search for would–be supergravity duals to metastable supersymmetry–breaking states.

The dual supergravity solution should be a locally stable non–supersymmetric solution which is usually obtained by a deformation of a supersymmetric one. The most renowned examples are the construction by Kachru, Pearson and Verlinde [119] for D3–branes on the conifold and by Klebanov and Pufu for M2–branes on Stenzel space [123]. The starting supersymmetric background has already less than maximal supersymmetry because it is obtained by putting branes at conical singularities in the transverse space, the remaining supersymmetries are then broken by adding a certain amount of anti–branes which are attracted towards the bottom of the throat. There (part of) the anti–branes can annihilate (via polarization due to the Myers effect) with the positive brane–charge dissolved in flux, a process which is argued to correspond to the decay of the metastable vacuum in the dual field theory description.

The drawback of the previous analysis is due to the fact that they are carried out in probe approximation. Despite this can signal the presence of the metastable vacuum, it is not enough to establish it. The backreacted solution can differ significantly from the probe one, in particular if the backreacted anti–branes source non–normalizable modes, which are not visible in the probe approximation, than the metastable configuration is not dual to a non–supersymmetric background of a supersymmetric theory but rather to a non–supersymmetric background of a non–supersymmetric theory.

Backreacting a solution is a far from trivial task and, with the current status of technology, it can be performed only under certain simplifying conditions. Fortunately, the previous backgrounds are such that a method developed by Borokhov and Gubser in [24] can be applied and one can solve for the backreaction in a series expansion around the supersymmetric vacuum. Recently, an intense analysis has been dedicated to the case of Klebanov–Strassler background [16, 13, 14, 62] and to the case of M2–branes on a warped Stenzel space [11]. Both analysis, despite their different setups and quite different calculations, have pointed out two interesting features. Among the modes describing the perturbations which preserve the symmetries of the original background (14 in the type IIB case and 10 in the M–theory case), only one mode enters the expression for the force that a probe–brane should feel in the perturbed background and, since anti–branes attract probe branes, this mode must be present in order to have a meaningful backreaction. The other feature is the presence of certain singularities in the infrared region which are unavoidable if the mode related to the force is present. In the Klebanov–Strassler case the singularity has finite action[1] while in the M–theory analysis it turns out to be more severe because also the action is not well behaved in the IR. It is of a certain usefulness to analyze different configurations to infer whether the presence of singularities is a feature of a specific background or has a different origin, maybe related to the perturbative nature of the approach or to the problem of backreaction itself. In this Chapter we enlarge the number of examples by investigating the deformation of the type IIA D2–brane background found in [47], which will be reviewed in the Section 7.2. The analysis is not as complete as in the case of Klebanov–Strassler background and it will focus on the IR region of the perturbed solution, nevertheless it is enough to elucidate its features and we will see that such singularities are present also in this setup and they are even more severe than in the other cases, being not sub–leading compared to the kind of singularities that are allowed as a physically sensible ones, that is those stemming from the effect of anti–D2 branes.

Whereas for the backreaction of anti–D3's on the Klebanov–Strassler solution one could have expected, with hindsight, a singularity to arise in analogy with the IIA brane engineering of four–dimensional gauge theories, a similar argument does not hold for string theory constructions of 2+1–dimensional gauge theories.

Indeed, the profile of the NS5–branes featured in those brane engineering constructions is generally not rigid but is instead sourced by the stack of Dp branes in–between (see [80] for pointers to the literature and much more on the physics of those brane constructions). For four–dimensional field theories living on D4–branes between two NS5's, the profile determined upon solving a Laplace equation is logarithmically running. This corresponds to the log–running of the gauge coupling for asymptotically free theories.

On the other hand, for three–dimensional field theories living on D3–branes between two NS5's, the profile decays as $1/r$ away from the location of the D3's on the NS5. Such a mode does not have the potential ability to enhance small IR fluctuations into log–running ones, an ability to which one might roughly ascribe the singularities encountered in the holographic approach to realizing metastable states in string theory, if those singularities are deemed as truly pathological.

So, proceeding in analogy with brane engineering constructions, for 2+1–dimensional IR perturbations should be expected not to affect the UV asymptotics of the background. As we shall see as an outcome of our linearized deformation analysis, this is not quite the case for the candidate supergravity dual to a 2+1–dimensional metastable state. The IR singularities we find are affecting the UV behavior, in the sense that they cannot be completely tamed without

[1] The finite action does not automatically guarantee that the singularity is acceptable, as the negative mass Schwarzschild counterexample shows [109].

switching off at the same time the force felt by a probe D2–brane in the UV.

Besides, having their legs in the wrong directions, those IR divergences cannot be identified as the remnant signature of an NS5 instanton through which the metastable state is been argued to decay in the probe approximation [119, 123].

Such singularities cannot be identified either with those characterizing fractional branes on Ricci–flat transverse geometries before the resolution or deformation of those manifolds (solutions of the Klebanov–Tseytlin [125] type, whose singularities get resolved in the Klebanov–Strassler solution).

The situation is quite puzzling and it might well be that those singularities are an artifact of having to smear anti–branes in order to make the problem tractable. On the other hand, in view of some recent results [20, 21], one might argue that a localization procedure is bound to make things worse, rather than alleviating them. We also comment on another possibility: basically, the issue of those singularities in the smearing approximation should be settled by considering 2nd–order expansions for the deformation modes of a BPS background, a task which has not been attempted so far.

The alternative viewpoint is that those Coulomb–like singularities are of physical significance and could be used to discriminate among solutions of the string theory landscape.

Indeed, consider the following analogy. In QCD, there are free quarks in the linearized approximation. Their "backreaction" results in a Coulomb–like singularity. We know that this is an indication that quarks are not good approximations at all to finite–energy states from the spectrum of QCD, which instead consists of confined, colorless states.

It is beyond the scope of the present work to offer more credence to vindicate or dispel this possibility but it is very tempting to imagine that the IR singularities we keep on finding upon backreacting the effect of antibranes on some BPS background are similarly a hint that some of the constructions which have been proposed has duals to metastable SUSY–breaking might instead belong to some "swampland" [182] once the backreaction of the SUSY–breaking ingredients is duly taken into account.

7.1 The Borokhov–Gubser method

In this Section we review the method proposed by Borokhov and Gubser in [24] to find perturbative solutions of the equations of motion. A fundamental assumption of the method is that the symmetries of the problem are powerful enough to impose that all the fields depend on a single radial coordinate. The idea behind the technique is to trade the n second order equations for n fields ϕ^a for $2n$ first order equations for the fields ϕ^a and their "canonical conjugate variables" ξ^a.

Let us consider the bosonic part of the type IIA supergravity action in Einstein frame[2]

$$S_{IIA} = \frac{1}{2k^2}\int \mathrm{d}^{10}x\sqrt{|g_{10}|}R_{10} - \frac{1}{4k^2}\int \Big[\mathrm{d}\Phi\wedge\hat{*}\,\mathrm{d}\Phi + g_s e^{-\Phi}\,H_3\wedge\hat{*}\,H_3 \\ + g_s^{1/2}e^{3\Phi/2}\,F_2\wedge\hat{*}\,F_2 + g_s^{3/2}e^{\Phi/2}\,\tilde{F}_4\wedge\hat{*}\,\tilde{F}_4 + g_s^2\,B\wedge F_4\;F_4\Big]\,, \qquad (7.1)$$

[2]In this Chapter we use the action in the Einstein frame, it can be obtained from the action in (2.4) by a Weyl transformation of the metric: $g^S_{MN}\to g_s^{-1/2}g^E_{MN}e^{\Phi/2}$. Note that here we will consider also the Chern–Simons term $B\wedge F_4\wedge F_4$.

where

$$\tilde{F}_4 = F_4 - C_1 \wedge H_3\,, \qquad F_4 = \mathrm{d}C_3\,, \qquad H_3 = \mathrm{d}B\,, \qquad F_2 = \mathrm{d}C_1\,. \tag{7.2}$$

If the dependence of the fields is on the radial coordinate r only we can thus reduce (7.1) to a one–dimensional sigma model

$$S_{IIA} \propto \int \mathrm{d}r \mathcal{L}\,, \tag{7.3}$$

where

$$\mathcal{L} = -\frac{1}{2} G(\phi)_{ab} \frac{\mathrm{d}\phi^a}{\mathrm{d}r}\frac{\mathrm{d}\phi^b}{\mathrm{d}r} - V(\phi) = T - V\,. \tag{7.4}$$

We also assume that there exist a superpotential W such that we can write $\mathcal{L}$ as

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\mathrm{d}\phi^a}{\mathrm{d}r} - \frac{1}{2}G^{ac}\frac{\partial W}{\partial \phi^c}\right)\left(\frac{\mathrm{d}\phi^b}{\mathrm{d}r} - \frac{1}{2}G^{bd}\frac{\partial W}{\partial \phi^d}\right) - \frac{1}{2}\frac{\mathrm{d}W}{\mathrm{d}r} \tag{7.5}$$

and $V(\phi)$ as

$$V(\phi) = \frac{1}{8} G^{ab} \frac{\partial W}{\partial \phi^a}\frac{\partial W}{\partial \phi^b}\,. \tag{7.6}$$

The equations of motion derived from $\mathcal{L}$ can be written as

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta \mathcal{L}}{\delta \phi'^a}\right) + \frac{\delta \mathcal{L}}{\delta \phi^a} = &\frac{1}{2}\Big(\partial_a\partial_b W - (\partial_a G_{bc})\, G^{cd}\partial_d W\Big)\left(\phi'^b - \frac{1}{2}G^{be}\partial_e W\right) \\
&- \frac{1}{2}(\partial_a G_{bc})\left(\phi'^b - \frac{1}{2}G^{bd}\partial_d W\right)\left(\phi'^c - \frac{1}{2}G^{ce}\partial_e W\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}r}\left(G_{ab}\left(\phi'^b - \frac{1}{2}G^{bc}\partial_c W\right)\right) = 0\,,
\end{aligned} \tag{7.7}$$

where a prime means derivative with respect to r. The gradient flow equations are

$$\frac{\mathrm{d}\phi^a}{\mathrm{d}r} = \frac{1}{2} G^{ab}\frac{\partial W}{\partial \phi^b}\,, \tag{7.8}$$

and the "zero–energy" condition coming from the G_{rr} Einstein equation is:

$$-\frac{1}{2}G_{ab}\frac{\mathrm{d}\phi^a}{\mathrm{d}r}\frac{\mathrm{d}\phi^b}{\mathrm{d}r} + V(\phi) = 0\,. \tag{7.9}$$

It is immediate to see that solutions of (7.8) are also solutions of the equations of motion (7.7) and satisfy the constraint (7.9) so that one can recover standard results. The idea of [24] is to use the superpotential to find perturbations to a solution of (7.8) that satisfy the equations of motion but not necessarily (7.8) itself. Let us consider an expansion of the fields ϕ^a around their supersymmetric value ϕ_0^a

$$\phi^a = \phi_0^a + \phi_1^a(\alpha) + \mathcal{O}(\alpha^2) \tag{7.10}$$

for some set of small parameters α. Let us introduce the following notation

$$\xi_a = G_{ab}(\phi_0)\left(\frac{\mathrm{d}\phi_1^b}{\mathrm{d}r} - N^b{}_d(\phi_0)\phi_1^d\right) \qquad \text{where} \qquad N^b{}_a(\phi_0) = \frac{1}{2}\frac{\partial}{\partial \phi^a}\left(G^{bc}\frac{\partial W}{\partial \phi^c}\right)\,. \tag{7.11}$$

If we now plug the expansion (7.10) in the equations of motion (7.7) and we keep terms up to the linear order we obtain

$$\frac{\mathrm{d}\xi_a}{\mathrm{d}r} + \xi_b\, N^b{}_a(\phi_0) = 0\,, \tag{7.12}$$

$$\frac{\mathrm{d}\phi_1^a}{\mathrm{d}r} - N^a{}_b(\phi_0)\phi_1^b = G^{ab}(\phi_0)\xi_b\,, \tag{7.13}$$

while the constraint (7.9) can be written as

$$\xi_a \frac{\mathrm{d}\phi_0^a}{\mathrm{d}r} = 0\,. \tag{7.14}$$

The functions ξ_a are deformations of the gradient flow equations (7.8), that is to say if all the ξ_a vanish then the deformation is supersymmetric. The obvious advantage of this method is that one can solve separately for the first order subsystem (7.12) and then solve for (7.13) which are again first order.

7.2 The CGLP background

We review here the supersymmetric type IIA background found by M. Cvetič, G.W. Gibbons, H. Lü and C. N. Pope in [47]. It describes regular deformed D2–branes with fractional D2–branes realized as wrapped D4–branes. The solution is related to the one for a stack of D2–branes placed in flat ten dimensional space, but one can choose the Chern–Simons term

$$\mathrm{d}\left(e^{\Phi/2}\hat{*}F_4\right) = -g_s^{1/2} F_4 \wedge H_3\,, \tag{7.15}$$

to be non zero and thus to contribute to the equation of motion for F_4. This solution can be seen as a deformation of the standard D2–brane solution where additional flux is turned on. To get an everywhere regular solution, it is necessary to replace the transverse flat–space of the original solution with a complete Ricci–flat space which admits square integrable harmonic three–form; one can then choose H_3 in (7.15) to be proportional to such harmonic form and, if it has a non–vanishing integral at infinity, this is interpreted as the magnetic charge of the additional fractional D4–branes. The additional flux enters the equation for the warp factor

$$\Box H = -\frac{1}{6}|H_3|^2\,, \tag{7.16}$$

where the Laplacian as well as the magnitude $|\ldots|^2$ are taken with respect to the seven dimensional transverse metric, giving a smooth solution. The resolution of the singularities enhances the breaking of the original supersymmetries giving rise to a solution that preserves 1/16 of the maximal supersymmetry, namely two supercharges, and it will thus be dual to an $\mathcal{N}=1$ three–dimensional supersymmetric gauge theory.

The Ricci–flat metric is [49]:

$$ds_{10}^2 = e^{-5z(r)}\eta_{\mu\nu}dx^\mu dx^\nu + \ell^2 e^{3z(r)} ds_7^2. \tag{7.17}$$

It is a warped product of a 2+1 dimensional Minkowski space and a complete, Ricci–flat, of G_2 holonomy and asymptotically conical seven dimensional space. These kind of seven dimensional spaces have been obtained in [25, 76] and correspond to $\mathbb{R}^3$ bundles over a quaternionic Kähler

Einstein base. We are interested in the case where the base is S^4; the manifold will be of co-homogeneity one, with level surfaces that are a S^2 bundle over S^4, namely $\mathbb{CP}^3$. The seven dimensional metric in our notation (closely related to [104] but not the same) is:

$$ds_7^2 = h(r)^2 dr^2 + e^{2u(r)} (D\mu^a)^2 + e^{2v(r)} d\Omega_4^2. \tag{7.18}$$

Here μ^a are coordinates on $\mathbb{R}^3$ subject to $\mu^a \mu_a = 1$, $a = 1, 2, 3$, which clearly parametrize S^2; its fibration over S^4 is given by

$$D\mu^a = d\mu^a + \epsilon_{abc} A^b \mu^c \tag{7.19}$$

while $d\Omega_4^2$ is the metric on the unit 4–sphere. The quantities A^a are self–dual SU(2) instanton potentials on S^4 , whose field strengths

$$J^a = dA^a + \frac{1}{2}\epsilon_{abc} A^b \wedge A^c \tag{7.20}$$

satisfy the algebra of the unit quaternions

$$J^a_{\alpha\gamma} J^b_{\gamma\beta} = -\delta_{ab}\delta_{\alpha\beta} + \epsilon_{abc} J^c_{\alpha\beta} \,. \tag{7.21}$$

The functions h, u and v are:

$$h(r) = \left(1 - \frac{1}{r^4}\right)^{-1/2} \qquad e^{2u(r)} = \frac{1}{4} r^2 \left(1 - \frac{1}{r^4}\right) \qquad e^{2v(r)} = \frac{1}{2} r^2 \,. \tag{7.22}$$

The radial coordinate r runs from one to infinity. We can notice that at the tip of the cone $r = 1$ the $\mathbb{R}^3$ directions vanish and the metric approaches

$$ds_7^2 \to \frac{\ell^2}{2} \mathrm{d}\Omega_4^2, \tag{7.23}$$

while for $r \to \infty$ the metric is

$$ds_7^2 = \ell^2 \left[dr^2 + r^2 ds^2_{\mathbb{CP}^3} \right] \,. \tag{7.24}$$

The metric on the asymptotic $\mathbb{CP}^3$ is not the usual Fubini–Study but a squashed version of it[3].

As we said before the solution presents an F_4 and an H_3 flux. The F_4 has the following structure

$$\begin{aligned} g_s F_4 &= K(r) d^3x \wedge dr + m G_4 \\ G_4 &= 2(g_1(r) + c_2) J_2 \wedge J_2 + 2(g_1(r) + c_3) U_2 \wedge J_2 + g_1'(r) \varepsilon_{abc} \mu^a dr \wedge D\mu^b \wedge J^c \,. \end{aligned} \tag{7.25}$$

The first term is related to the electric flux of the ordinary D2–branes in the Minkowski directions, while G_4 is related to the magnetic flux of the fractional D2–branes. These are D4–branes which wrap a vanishing 2–cycle in the internal space. The standard D2–brane solution clearly corresponds to $m = 0$. A non zero m requires a NS–NS three–form flux

$$H_3 = m G_3, \tag{7.26}$$

[3]It is an element of a family of Einstein metrics on $\mathbb{CP}^3$, $ds^2_{\mathbb{CP}^3} = \lambda^2 (D\mu^a)^2 + d\Omega_4^2$ where $\lambda = 1$ corresponds to the usual Fubini–Study metric and $\lambda = 1/2$ to the squashed one [76]. The squashed metric is Hermitian but not Kähler, it is in fact nearly Kähler [1].

where G_3 is an harmonic three–form in the internal space and it is related to G_4 by Hodge duality: $G_4 = *_7 G_3$. The trace of the Einstein equation relates the magnitude of G_3 with the Laplacian of the warp factor H

$$\Box H = -\frac{1}{6}m^2|G_3|^2 = -\frac{1}{6}m^2|G_4|^2 \,. \tag{7.27}$$

The three–form H_3 can be obtained from the following potential

$$\ell B = m\left[g_2(r)U_2 + g_3(r)J_2\right]. \tag{7.28}$$

The Bianchi identities are satisfied provided the following definitions and identities are taken into account

$$\begin{aligned} U_2 &\equiv \frac{1}{2}\varepsilon_{abc}\mu^a D\mu^b \wedge D\mu^c \,, \qquad & J_2 &\equiv \mu^a J^a \qquad & U_3 &\equiv D\mu^a \wedge J^a \\ dU_2 &= U_3 & dJ_2 &= U_3 & dU_3 &= 0\,. \end{aligned} \tag{7.29}$$

The supersymmetric value of the CGLP background functions is

$$\begin{aligned} g_0^1(r) &= \int_1^r f_1(y)dy\,, \quad f_1(r) = e^{u_0(r)+2v_0(r)}u_1\,, \quad u_1(r) = \frac{1}{4r^4(r^4-1)} - \frac{(3r^4-1)\mathcal{P}(r)}{4r^5(r^4-1)^{3/2}} \\ g_0^2(r) &= \int_1^r f_2(y)dy\,, \quad f_2(r) = h_0(r)e^{2u_0(r)}u_2\,, \quad u_2(r) = \frac{1}{r^4} + \frac{\mathcal{P}(r)}{r^5(r^4-1)^{1/2}} \\ g_0^3(r) &= \int_1^r f_3(y)dy\,, \quad f_3(r) = h_0(r)e^{2v_0(r)}u_3\,, \quad u_3(r) = -\frac{1}{2(r^4-1)} + \frac{\mathcal{P}(r)}{r(r^4-1)^{3/2}}\,, \end{aligned} \tag{7.30}$$

where we denote by $\mathcal{P}(r)$ the following function

$$\mathcal{P}(r) = \int_1^r \frac{dy}{\sqrt{y^4-1}} = K(-1) - F\left(\arcsin(1/r)|-1\right). \tag{7.31}$$

By $F(\phi|k)$ we denote the incomplete elliptic integral of the first kind and $K(k) = F(\pi/2|k)$. From now on we will denote $F\left(\arcsin(1/r)|-1\right)$ simply by $\mathcal{F}(r)$. As for the constants c_2 and c_3 in the ansatz (7.25), the background only specifies their difference

$$c_2 - c_3 = \frac{3}{32}\,. \tag{7.32}$$

As for the standard D2–brane solution the dilaton is nonzero and it is related to the function H as $e^{\Phi} = g_s H^{1/4}$, whose value in the supersymmetric background is

$$H_0 \equiv e^{8z_0} = \frac{m^2}{\ell^2}\int_r^\infty y^5\left[u_3(y) - u_2(y)\right]u_1(y)dy\,. \tag{7.33}$$

7.2.1 Few words about the dual field theory

The dual field theory of this supergravity background is not well understood, the authors of [47] argue that the gauge group should be of unitary type with charge changed with respect to the D2–brane case by the additional flux from the wrapped D4–branes. However the transverse space is not $\mathbb{R}^7$ but a cone over $\mathbb{CP}^3$, the more careful analysis of [135] suggest that the dual theory is an $\mathcal{N}=1$ three–dimensional field theory whose classical moduli space of vacua matches such a transverse space. For no fractional branes the gauge group is argued to be [135] $\mathrm{U}(N)\times\mathrm{U}(N)$ with field content an $\mathcal{N}=1$ vector multiplet and four $\mathcal{N}=2$ chiral superfields, one pair transforming in the $(N,\bar{N})$ representation and the other pair in the conjugate. In analogy with the Klebanov–Witten case, adding M fractional branes should change the gauge group to $\mathrm{U}(N)\times\mathrm{U}(N+M)$ [78].

7.3 Reduction and Superpotential

We can now reduce the action (7.1) to a one dimensional sigma model. Inserting the ansatz for the fields and metric we presented in the previous Section one can obtain

$$\mathcal{S}_{\mathrm{IIA}}=\frac{\ell^5\,\mathrm{Vol}\,(M_{1,2})\;\mathrm{Vol}\,(M_6)}{2\,\kappa^2}\int dr\,\mathcal{L} \tag{7.34}$$

where $\mathcal{L}=T-V$ and $\mathrm{M}_{1,2}$, M_6 denote the 2+1 dimensional Minkowski space and the level surfaces of the seven–dimensional G_2 holonomy manifold, respectively. The reduction was first performed in [104], the result is

$$\begin{aligned}T=\frac{e^{2u+4v}}{h}\Big[&-30\,z'^2+2\,u'^2+12\,v'^2+16\,u'\,v'-2\,g_s^{-1/2}\,\frac{m^2}{\ell^6}\,e^{-9z+\Phi/2-2u-4v}\,g_1'^2\\&-\frac{g_s}{2}\,\frac{m^2}{\ell^6}\,e^{-6z-\Phi}\,\left(g_2'^2\,e^{-4u}+2\,g_3'^2\,e^{-4v}\right)-\frac{1}{2}\Phi'^2\Big]\,.\end{aligned} \tag{7.35}$$

The potential will contain terms proportional to $K(r)$ which appears in F_4. Its equation of motion

$$d\left(e^{\Phi/2}\hat{*}F_4\right)=-g_s^{1/2}F_3\wedge H_3 \tag{7.36}$$

turns out to be integrable so that from

$$\hat{*}\left(e^{\Phi/2}K(r)d^3x\wedge dr\right)=-g_s^{1/2}mG_4\wedge B|_{M_6} \tag{7.37}$$

one can obtain an expression for the non–dynamical K

$$K(r)=\frac{4m^2}{\ell^6}g_s^{1/2}he^{-\frac{\Phi}{2}-15z-2u-4v}\left[g_2(g_1+c_2)+g_3(g_1+c_3)\right]. \tag{7.38}$$

Upon evaluation of the Lagrangian at the minimum value (7.38) for K, the potential becomes

$$\begin{aligned}V=&-2\,h\,e^{-2u-4v}\left[e^{2u+8v}-e^{6u+4v}+6\,e^{4u+6v}\right]+2\,g_s\,h\,\frac{m^2}{\ell^6}\,e^{-6z-\Phi}\left[g_2+g_3\right]^2\\&+4\,g_s^{-1/2}\,\frac{m^2}{\ell^6}\,e^{-9z+\Phi/2+2u}\,h\left[2\,\left(g_1+c_2\right)^2\,e^{-4v}+\left(g_1+c_3\right)^2\,e^{-4u}\right]\\&+8\,g_s^{1/2}\,\frac{m^4}{\ell^{12}}\,e^{-15z-\Phi/2-2u-4v}\,h\left[g_1\,\left(g_2+g_3\right)+g_2\,c_2+g_3\,c_3\right]^2\,.\end{aligned} \tag{7.39}$$

In the notation of (7.5) we choose to denote the functions ϕ^a, $a=1,\ldots,7$ in the following order

$$\phi^a=(u,v,z,\Phi,g_1,g_2,g_3)\,, \tag{7.40}$$

and one can find the following superpotential [104]

$$W=-8\left[e^{u+4v}+e^{3u+2v}\right]+8\,\frac{m^2}{\ell^6}\,g_s^{1/4}\,e^{-\frac{15}{2}z-\frac{\Phi}{4}}\left[g_1\,\left(g_2+g_3\right)+g_2\,c_2+g_3\,c_3\right]. \tag{7.41}$$

7.4 ξ functions. Equations and solutions

As explained in Section 7.1 the advantage of the Borokhov–Gubser method is that one can solve separately the systems of first order equations. First one need to solve the system (7.12) for the ξ functions. We do a change of basis in order to have the most "decoupled" system possible

$$\tilde{\xi}_a = (\xi_1, \xi_1 - \xi_2, \xi_3 + 2\,\xi_4, \xi_4, \xi_5, \xi_6, -\xi_6 + \xi_7)\ . \tag{7.42}$$

In this basis the system of equations is

$$\tilde{\xi}_3' = -4\frac{m^2 g_s^{1/4}}{l^6} h_0 e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\phi^0}{4}}\Big[c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0)\Big]\tilde{\xi}_3 \tag{7.43}$$

$$\tilde{\xi}_7' = -\frac{m^2 g_s^{1/4}}{2l^6} h_0 e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\phi^0}{4}}(c_2-c_3)\tilde{\xi}_3 \tag{7.44}$$

$$\begin{aligned}\tilde{\xi}_5' = &-\frac{1}{2g_s^{3/4}l^6} h_0 e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\phi^0}{4}}\Big[4l^6 e^{4v^0+6z^0+\phi^0}(\tilde{\xi}_6+\tilde{\xi}_7) + 8l^6 e^{4u^0+6z^0+\phi^0}\tilde{\xi}_6 \\ &- g_s m^2(g_2^0+g_3^0)\tilde{\xi}_3\Big]\end{aligned} \tag{7.45}$$

$$\tilde{\xi}_6' = \frac{g_s^{1/4}}{2l^6} h_0 e^{-2u^0-4v^0-\frac{3}{4}(10z^0+\phi^0)}\Big[-2g_s^{1/2}l^6 e^{2u^0+4v^0+9z^0}\tilde{\xi}_5 + e^{\frac{\phi^0}{2}}m^2(c_2+g_1^0)\tilde{\xi}_3\Big]$$

$$\begin{aligned}\tilde{\xi}_4' = &\frac{h_0}{8g_s^{3/4}} e^{-\frac{3}{4}(10z^0+\phi^0)}\Big[-24e^{2u^0-4v^0+6z^0+\frac{3}{2}\phi^0}(c_2+g_1^0)\tilde{\xi}_6 - 12e^{-2u^0+6z^0+\frac{3}{2}\phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) \\ &+ 6e^{9z^0}g_s^{3/2}(g_2^0+g_3^0)\tilde{\xi}_5 - \frac{m^2 g_s}{l^6}e^{-2u_0-4v_0+\frac{\phi^0}{2}}(c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\end{aligned} \tag{7.46}$$

$$\begin{aligned}\tilde{\xi}_1' = &\frac{1}{g_s^{3/4}l^6} h_0 e^{-2u^0-4v^0-\frac{15}{2}z^0-\frac{\phi^0}{4}}\Big[g_s^{3/4}l^6 e^{u^0+4v^0+\frac{15}{2}z^0+\frac{\phi^0}{4}}\tilde{\xi}_1 + g_s^{3/4}l^6 e^{\frac{1}{4}(12u^0+8v^0+30z^0+\phi^0)}\tilde{\xi}_2 \\ &- 8l^6 e^{4u^0+6z^0+\phi^0}(c_2+g_1^0)\tilde{\xi}_6 + 4l^6 e^{4v^0+6z^0+\phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) \\ &- g_s m^2(c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\end{aligned} \tag{7.47}$$

$$\begin{aligned}\tilde{\xi}_2' = &\frac{1}{g_s^{3/4}l^6} h_0 e^{-2u^0-4v^0-\frac{15}{2}z^0-\frac{\phi^0}{4}}\Big[g_s^{3/4}l^6 e^{u^0+4v^0+\frac{15}{2}z^0+\frac{\phi^0}{4}}\tilde{\xi}_1 + 3g_s^{3/4}l^6 e^{\frac{1}{4}(12u^0+8v^0+30z^0+\phi^0)}\tilde{\xi}_2 \\ &- 24l^6 e^{4u^0+6z^0+\phi^0}(c_2+g_1^0)\tilde{\xi}_6 + 4l^6 e^{4v^0+6z^0+\phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) \\ &+ g_s m^2(c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\ .\end{aligned} \tag{7.48}$$

We are able to fully integrate[4] the equations and we present their solution in the order in which they have to be solved.

7.4.1 $\tilde{\xi}_3$

After some manipulation the equation for $\tilde{\xi}_3$ can be expressed as

$$\tilde{\xi}_3' = \frac{H_0'}{H_0}\tilde{\xi}_3 \tag{7.49}$$

[4] It is important to have a solution expressed in terms of the least possible number of nested integrals. Usually [16, 13, 14] it is not possible to find a fully integrated solution and one solves in series expansion, if the number of nested integrals is high it could be computationally heavy. In the counting of nested integrals we do not take into account that which enter the definition of the elliptic functions.

whose solution solution is:

$$\tilde{\xi}_3(r) = X_3 H_0(r) e^{-8z_0(1)}\,. \tag{7.50}$$

We define the constant B_1, which we will use in the the following, as

$$B_1 = \frac{m^2}{l^6} X_3 e^{-8z_0(1)}\,. \tag{7.51}$$

We can explicitly do the integration which defines the function H_0. We give some details of the procedure we followed because it is paradigmatic of the way we used to obtain explicit solutions of the $\tilde{\xi}$ equations. Recall that:

$$H_0 = \frac{m^2}{\ell^6} \int_r^\infty y^5 \left[u_3(y) - u_1(y)\right] u_1(y) dy \tag{7.52}$$

The integrand has the following structure:

$$y^5 \left[u_3(y) - u_1(y)\right] u_1(y) = \alpha_2(y)\mathcal{F}(y)^2 + \alpha_1(y)\mathcal{F}(y) + \alpha_0(y) \tag{7.53}$$

Where α_i are some functions which do not contain $\mathcal{F}$. We use simple integration by parts

$$\begin{aligned}\int \alpha_2\mathcal{F}^2 + \alpha_1\mathcal{F} + \alpha_0 &= A_2\mathcal{F}^2 + \int (\alpha_1 - 2\mathcal{F}' A_2) + \int \alpha_0 \\ &= A_2\mathcal{F}^2 + A_3\mathcal{F} + \int (\alpha_0 - \mathcal{F}' A_3) \\ &= A_2\mathcal{F}^2 + A_3\mathcal{F} + A_4\,,\end{aligned} \tag{7.54}$$

where the notation is the following

$$\begin{aligned}&\mathcal{F}' = \frac{\mathrm{d}}{\mathrm{d}y} F(\arcsin(1/y)| - 1) = -\frac{1}{\sqrt{y^4 - 1}}\,, \\ &\alpha_3 = \alpha_1 - 2\mathcal{F}' A_2\,, \\ &\alpha_4 = \alpha_0 - \mathcal{F}' A_3\,, \\ &A_i = \int \alpha_i\,.\end{aligned} \tag{7.55}$$

Once we have a primitive we have just to evaluate it at the two extrema of integration and get:

$$\begin{aligned}H_0(r) = &\frac{m^2}{2l^6}\mathcal{F}(r)^2 \left(\frac{3}{32} - \frac{1}{8r^4(r^4-1)^2}\right) \\ &- \frac{m^2}{2l^6}\mathcal{F}(r)\left(\frac{3r^8 + 3r^4 - 4}{16r^3(r^4-1)^{3/2}} + \frac{K(-1)}{16}\left(3 - \frac{4}{r^4(r^4-1)^2}\right)\right) \\ &+ \frac{m^2}{2l^6}\left(\frac{3r^4 - 4}{32r^2(r^4-1)} + \frac{3r^8 + 3r^4 - 4}{16r^3(r^4-1)^{3/2}} K(-1) - \frac{K(-1)^2}{8r^4(r^4-1)^2}\right)\end{aligned} \tag{7.56}$$

7.4.2 $\tilde{\xi}_7$

The equations for $\tilde{\xi}_7$ is:

$$\tilde{\xi}_7' = -\frac{3}{64}\frac{m^2}{l^6} h^0 e^{-2u^0-4v^0} H_0^{-1}\tilde{\xi}_3 = -\frac{3}{4}\frac{m^2}{l^6} X_3 e^{-8z_0(1)} \frac{1}{(r^4-1)^{3/2}} \tag{7.57}$$

which has the following solution[5]:

$$\tilde{\xi}_7(r) = X_7 + \frac{3}{8} B_1 \left[\frac{r}{\sqrt{r^4-1}} - \mathcal{F}(r)\right] \tag{7.58}$$

7.4.3 $\tilde{\xi}_5$ and $\tilde{\xi}_6$

The functions $\tilde{\xi}_5$ and $\tilde{\xi}_6$ are coupled and the system of equations is:

$$\tilde{\xi}_5' = -2h^0(2e^{2u^0-4v^0} + e^{-2u^0})\,\tilde{\xi}_6 - 2h^0 e^{-2u^0}\,\tilde{\xi}_7 - \frac{32}{3} f_1\,\tilde{\xi}_7' \tag{7.59}$$

$$\tilde{\xi}_6' = -h^0\,\tilde{\xi}_5 - \frac{8}{3}\frac{1}{h^0} e^{-2u+4v} f_2\,\tilde{\xi}_7' \tag{7.60}$$

We can solve for the homogeneous system and we arrange, as usual, the two basis vectors of the space of the homogeneous solutions in the fundamental matrix

$$\tilde{\Xi}_{56} = \left(\begin{array}{c|c} \frac{(3r^4-1)}{r^4(r^4-1)} & \frac{r(6r^8-6r^4-1)}{r^3\sqrt{r^4-1}} - \frac{3r^4-1}{r^4(r^4-1)}\mathcal{F}(r) \\ \hline \frac{1}{r\sqrt{r^4-1}} & 1 - \frac{3r^4}{2} - \frac{1}{r\sqrt{r^4-1}}\mathcal{F}(r) \end{array}\right). \tag{7.61}$$

We define the two–component vector $\mathbf{g}^{\xi}_{56} = (g_5, g_6)$ of the non–homogeneous terms

$$g_5(r) = -2h^0 e^{-2u^0}\,\tilde{\xi}_7 - \frac{32}{3}(g_2^0 + g_3^0)\,\tilde{\xi}_7' = -\frac{32}{3} e^{u^0+2v^0} u_1 \tilde{\xi}_7' - 2h^0 e^{-2u^0}\tilde{\xi}_7\,,$$

$$g_6(r) = 2B_1 f_2\,\frac{r^4}{(r^4-1)^2}\,.$$

The solution will be

$$\tilde{\xi}_{56}(r) = \tilde{\Xi}_{56}(r)\mathbf{X}_{56} + \tilde{\Xi}(r)\int^r \tilde{\Xi}_{56}(y)^{-1}\mathbf{g}^{\xi}_{56}(y)\,dy$$

where $\mathbf{X}_{56} = (X_5, X_6)$ are the integration constants. We obtain

[5]Here we do a redefinition of the integration constant which appear by direct integration of equation (7.57) in order to reabsorb an imaginary constant which appears after manipulations with the elliptic function $\mathcal{F}$. We always consider real solutions.

$$
\begin{aligned}
\tilde{\xi}_5 =& \mathcal{F}(r)^2\left(\frac{B_1(1-3r^4)}{7r^4(r^4-1)}\right)\\
&+\mathcal{F}(r)\left(\frac{B_1K(-1)(3r^4-1)}{8r^4(r^4-1)}-\frac{(3r^4-1)(3X_6-2X_7)}{3r^4(r^4-1)}-\frac{3B_1(5r^8-5r^4-2)}{28r^3\sqrt{r^4-1}}\right)\\
&+\frac{B_1(15r^8-21r^4+10)}{28r^2(r^4-1)}-\frac{3B_1K(-1)}{8r^3\sqrt{r^4-1}}+\frac{(3r^4-1)}{r^4(r^4-1)}X_5+6r\sqrt{r^4-1}\,X_6-\frac{3X_6-2X_7}{3r^3\sqrt{r^4-1}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\xi}_6 =& \mathcal{F}(r)^2\left(-\frac{B_1}{7r\sqrt{r^4-1}}\right)\\
&+\mathcal{F}(r)\left(\frac{B_1(15r^8+3r^4-4)}{112(r^4-1)}+\frac{B_1K(-1)}{8r\sqrt{r^4-1}}-\frac{3X_6-2X_7}{3r\sqrt{r^4-1}}\right)\\
&-\frac{3B_1r(5r^4+4)}{112\sqrt{r^4-1}}-\frac{B_1K(-1)}{8(r^4-1)}+\frac{X_5}{r\sqrt{r^4-1}}+\left(1-\frac{3r^4}{2}\right)X_6-\frac{2}{3}X_7
\end{aligned}
\tag{7.62}
$$

7.4.4 $\tilde{\xi}_4$

$\tilde{\xi}_4$ has no homogeneous part but its non–homogeneous part depends on $\tilde{\xi}_5$ and $\tilde{\xi}_6$, the equation is the following:

$$
\tilde{\xi}_4' = \frac{3}{4}h^0 f_1\,\tilde{\xi}_5-\frac{3}{4}(f_2+f_3)\,\tilde{\xi}_6-\frac{3}{4}\,f_3\tilde{\xi}_7-\frac{B_1}{32}h^0e^{u_0}(2u_3-3)u_1\,. \tag{7.63}
$$

Having solved for $\tilde{\xi}_5$ and $\tilde{\xi}_6$ we can find a fully integrated solution which is:

$$
\begin{aligned}
\tilde{\xi}_4 =& \mathcal{F}(r)^3\left(\frac{3B_1(3r^4-1)}{448r^5(r^4-1)^{3/2}}\right)+\\
&+\mathcal{F}(r)^2\left(\frac{B_1(111r^{12}-222r^8+99r^4-16)}{3584r^4(r^4-1)^2}+\frac{(3r^4-1)}{3584r^5(r^4-1)^{3/2}}(168X_6-112X_7-45B_1K(-1))\right)\\
&+\mathcal{F}(r)\Bigg(-\frac{B_1(15r^8-12r^4+10)}{896r^3(r^4-1)^{3/2}}-\frac{B_1K(-1)(201r^{12}-402r^8+45r^4+44)}{7168r^4(r^4-1)^2}+\\
&+\frac{3r^4-1}{512r^5(r^4-1)^{3/2}}\left(-24X_5+K(-1)\left(3B_1K(-1)-24X_6+16X_7\right)\right)+\frac{9r^8-9r^4+4}{128r^4(r^4-1)}(3X_6-2X_7)\Bigg)\\
&-\frac{B_1(51r^4-32)}{3584r^2(r^4-1)}+\frac{B_1K(-1)(201r^8-231r^4+134)}{7168r^3(r^4-1)^{3/2}}-\frac{B_1K(-1)^2(9r^4-5)}{512r^4(r^4-1)^2}+\frac{3K(-1)(3r^4-1)}{65r^5(r^4-1)^{3/2}}X_5\\
&+\frac{3r^4-2}{128r^3\sqrt{r^4-1}}(3X_6-2X_7)-\frac{3X_5+K(-1)(3X_6-2X_7)}{64r^4(r^4-1)}+X_4\,.
\end{aligned}
\tag{7.64}
$$

7.4.5 $\tilde{\xi}_1$ and $\tilde{\xi}_2$

The functions $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are coupled and the system of equations is:

$$\tilde{\xi}_1' = h^0 e^{-u^0}\,\tilde{\xi}_1 + h^0 e^{u^0-2v^0}\,\tilde{\xi}_2 - 2(f_2 - f_3)\tilde{\xi}_6 + 2f_3\tilde{\xi}_7 - \frac{B_1}{8} r(2u_3 - 3)u_1 \tag{7.65}$$

$$\tilde{\xi}_2' = h^0 e^{-u^0}\,\tilde{\xi}_1 + 3h^0 e^{u^0-2v^0}\,\tilde{\xi}_2 - 2(3f_2 - f_3)\,\tilde{\xi}_6 + 2f_3\,\tilde{\xi}_7 + \frac{B_1}{8} r(2u_3 - 3)u_1 \tag{7.66}$$

The fundamental matrix $\tilde{\Xi}_{12}$ is

$$\tilde{\Xi}_{12} = \left(\begin{array}{c|c} r^4-1 & \frac{\sqrt{r^4-1}}{r}\left(1 - r\sqrt{r^4-1}(\mathcal{E}(r) - \mathcal{F}(r)\right) \\ \hline 2r^4 & -2r^4\left(\mathcal{E}(r) - \mathcal{F}(r)\right) \end{array}\right). \tag{7.67}$$

where $\mathcal{E}(r)$ denotes the incomplete elliptic integral of the second kind $\mathcal{E}(r) = E(\arcsin(1/r)|-1)$ and $E(k) = E(\pi/2|k)$. The solutions are quite cumbersome nevertheless we present them here for completeness:

$$\begin{aligned}
\tilde{\xi}_1 = {} & \mathcal{F}(r)^3\left(-B_1\frac{r^4+1}{112r^5(r^4-1)^{3/2}}\right) \\
& + \mathcal{F}(r)^2\left(B_1\frac{189r^{12} - 258r^8 + r^4 + 48}{1792r^4(r^4-1)} + (45B_1K(-1) - 168X_6 + 112X_7)\frac{r^4+1}{2688r^5(r^4-1)^{3/2}}\right) \\
& + \mathcal{F}(r)\Bigg(- B_1\frac{69r^{12} - 114r^8 + 61r^4 - 24}{896r^3(r^4-1)^{3/2}} - B_1K(-1)\frac{315r^{12} - 390r^8 - 53r^4 + 120}{3584r^4(r^4-1)} \\
& \qquad + X_2(r^4-1) - X_6\frac{63r^{12} - 78r^8 + 31r^4 - 8}{64r^4(r^4-1)} - X_7\frac{9r^{12} - 18r^8 - 7r^4 + 8}{96r^4(r^4-1)} \\
& \qquad + (24X_5 + K(-1)(24X_6 - 16X_7 - 3B_1K(-1)))\frac{r^4+1}{384r^5(r^4-1)^{3/2}}\Bigg) \\
& - B_1\frac{51r^8 - 75r^4 + 16}{1792r^2(r^4-1)} + B_1K(-1)\frac{315r^{12} - 516r^8 + 229r^4 - 60}{3584r^3(r^4-1)^{3/2}} + X_1(r^4-1) \\
& - B_1K(-1)^2\frac{63r^{12} - 126r^8 + 63r^4 - 4}{512r^4(r^4-1)} + X_2\frac{\sqrt{r^4-1}}{r} - X_2(r^4-1)\mathcal{E}(r) + X_5\frac{2r^4-1}{16r^4(r^4-1)} \\
& - X_5K(-1)\frac{r^4+1}{16r^5(r^4-1)^{3/2}} - X_6\frac{33r^8 - 35r^4 + 4}{64r^3\sqrt{r^4-1}} + X_6K(-1)\frac{63r^{12} - 78r^8 + 23r^4 - 4}{64r^4(r^4-1)} \\
& + X_7\frac{9r^8 - 11r^4 + 4}{96r^3\sqrt{r^4-1}} + X_7K(-1)\frac{9r^{12} - 18r^8 + r^4 + 4}{96r^4(r^4-1)}
\end{aligned} \tag{7.68}$$

$$
\begin{aligned}
\tilde{\xi}_2 =\ & \mathcal{F}(r)^3 \left(B_1 \frac{r^4-3}{112 r^5 (r^4-1)^{3/2}} \right) \\
& + \mathcal{F}(r)^2 \left(B_1 \frac{189 r^{16} - 438 r^{12} + 241 r^8 + 52 r^4 - 16}{896 r^4 (r^4-1)^2} - \frac{(45 B_1 K(-1) - 168 X_6 + 112 X_7)(r^4-3)}{2688 r^5 (r^4-1)^{3/2}} \right) \\
& + \mathcal{F}(r) \Bigg(- B_1 \frac{69 r^{12} - 132 r^8 + 25 r^4 + 20}{448 r^3 (r^4-1)^{3/2}} - B_1 K(-1) \frac{315 r^{16} - 750 r^{12} + 427 r^8 + 76 r^4 + 44}{1792 r^4 (r^4-1)^2} \\
& \qquad + X_2\, 2 r^4 - X_6 \frac{63 r^{12} - 87 r^8 + 40 r^4 - 12}{32 r^4 (r^4-1)} - X_7 \frac{9 r^{12} - 9 r^8 - 16 r^4 + 12}{48 r^4 (r^4-1)} \\
& \qquad + (K(-1)(3 B_1 K(-1) - 24 X_6 + 16 X_7) - 24 X_5) \frac{r^4-3}{384 r^5 (r^4-1)^{3/2}} \Bigg) \\
& - B_1 \frac{51 r^8 - 30 r^4 - 32}{896 r^2 (r^4-1)} + B_1 K(-1) \frac{315 r^{12} - 561 r^8 + 40 r^4 + 134}{1792 r^3 (r^4-1)^{3/2}} + X_1\, 2 r^4 - X_2\, 2 r^4 \mathcal{E}(r) \\
& - B_1 K(-1)^2 \frac{63 r^{16} - 126 r^{12} + 63 r^8 + 2 r^4 - 10}{256 r^4 (r^4-1)^2} + X_5 \frac{4 r^4 - 3}{16 r^4 (r^4-1)} + X_5 K(-1) \frac{r^4-3}{16 r^5 (r^4-1)^{3/2}} \\
& - X_6 \frac{33 r^8 - 38 r^4 + 6}{32 r^3 \sqrt{r^4-1}} + X_6 K(-1) \frac{63 r^{12} - 87 r^8 + 32 r^4 - 6}{32 r^4 (r^4-1)} \\
& + X_7 \frac{9 r^8 - 14 r^4 + 6}{48 r^3 \sqrt{r^4-1}} + X_7 K(-1) \frac{9 r^{12} - 9 r^8 - 8 r^4 + 6}{48 r^4 (r^4-1)} \qquad (7.69)
\end{aligned}
$$

7.5 ϕ functions. Equations and comments

As previously done when handling the ξ variables, we prefer to change the original ϕ into the new set $\tilde{\phi}$, defined as[6]:

$$
\tilde{\phi}^a = (\phi_1, \phi_1 - 2\,\phi_2, 8\,\phi_1 + 6\,\phi_3 - 3\,\phi_4, 8\,\phi_1 + 16\,\phi_2 + 30\,\phi_3 + \phi_4, \phi_5, \phi_6 + \phi_7, \phi_6 - \phi_7)\ .
$$

In this basis the equations are

[6]The inverse transformation is:

$$
\phi^a = \left(\tilde{\phi}_1, \frac{1}{2}\left(\tilde{\phi}_1 - \tilde{\phi}_2\right), -\frac{7}{12}\tilde{\phi}_1 + \frac{1}{4}\tilde{\phi}_2 + \frac{1}{96}\tilde{\phi}_3 + \frac{1}{32}\tilde{\phi}_4, \frac{3}{2}\tilde{\phi}_1 + \frac{1}{2}\tilde{\phi}_2 - \frac{5}{16}\tilde{\phi}_3 + \frac{1}{16}\tilde{\phi}_4, \tilde{\phi}_5, \frac{1}{2}\left(\tilde{\phi}_6 + \tilde{\phi}_7\right), \frac{1}{2}\left(\tilde{\phi}_6 - \tilde{\phi}_7\right) \right) .
$$

$$
\begin{aligned}
\tilde{\phi}_1' &= \frac{1}{20}\, h^0\, e^{-2u^0-4v^0}\, \left[\tilde{\xi}_1 + 2\,\tilde{\xi}_2 - 20\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 20\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \\
\tilde{\phi}_2' &= \frac{1}{20}\, h^0\, e^{-2u^0-4v^0}\, \left[4\,\tilde{\xi}_1 + 3\,\tilde{\xi}_2 - 20\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 60\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \\
\tilde{\phi}_3' &= \frac{1}{10}\, h^0\, e^{-2u^0-4v^0}\, \left[4\,\tilde{\xi}_1 + 8\,\tilde{\xi}_2 + \tilde{\xi}_3 - 32\,\tilde{\xi}_4 - 80\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 80\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \\
\tilde{\phi}_5' &= \frac{g_s^{1/2}}{4\,m^2}\, h^0\, e^{3z^0/2-3\Phi^0/4}\left[\ell^6\, e^{15z^0/2+\Phi^0/4}\,\tilde{\xi}_5 + g_s^{1/4}\, m^2\, \left(4\,\tilde{\phi}_6 - \left(g_2^0 + g_3^0\right)\, \left[8\,\tilde{\phi}_1 - \tilde{\phi}_3\right]\right)\right]\,, \\
\tilde{\phi}_6' &= \frac{1}{2\,g_s\,m^2}\, h^0\, e^{-2u^0-4v^0-3z^0/2+3\Phi^0/4}\left[\ell^6\, e^{15z^0/2+\Phi^0/4}\, \left(2\,e^{4u^0}\,\tilde{\xi}_6 + e^{4v^0}\,\tilde{\xi}_7\right)\right. \\
&\quad + 2\,g_s^{1/4}\, m^2\, e^{4u^0}\, \left[4\,\tilde{\phi}_5 + \left(c_2 + g_1^0\right)\, \left(8\,\tilde{\phi}_1 + 8\,\tilde{\phi}_2 - \tilde{\phi}_3\right)\right] \\
&\quad \left. + g_s^{1/4}\, m^2\, e^{4v^0}\, \left(4\,\tilde{\phi}_5 - \left(c_3 + g_1^0\right)\,\tilde{\phi}_3\right)\right]\,, \\
\tilde{\phi}_7' &= \frac{1}{2\,g_s\,m^2}\, h^0\, e^{-2u^0-4v^0-3z^0/2+3\Phi^0/4}\left[\ell^6\, e^{15z^0/2+\Phi^0/4}\, \left(2\,e^{4u^0}\,\tilde{\xi}_6 - e^{4v^0}\,\tilde{\xi}_7\right)\right. \\
&\quad + 2\,g_s^{1/4}\, m^2\, e^{4u^0}\, \left[4\,\tilde{\phi}_5 + \left(c_2 + g_1^0\right)\, \left(8\,\tilde{\phi}_1 + 8\,\tilde{\phi}_2 - \tilde{\phi}_3\right)\right] \\
&\quad \left. - g_s^{1/4}\, m^2\, e^{4v^0}\, \left(4\,\tilde{\phi}_5 - \left(c_3 + g_1^0\right)\,\tilde{\phi}_3\right)\right]\,, \\
\tilde{\phi}_4' &= -\frac{1}{10\,\ell^6}\, h^0\, e^{-2u^0-4v^0-15z^0/2-\Phi^0/4}\left[\ell^6\, e^{15z^0/2+\Phi^0/4}\, \left(8\,\tilde{\xi}_1 - 4\,\tilde{\xi}_2 - 5\,\tilde{\xi}_3\right)\right. \\
&\quad + 80\,\ell^6\, e^{u^0+4v^0+15z^0/2+\Phi^0/4}\,\tilde{\phi}_1 - 80\,\ell^6\, e^{3u^0+2v^0+15z^0/2+\Phi^0/4}\,\tilde{\phi}_2 \\
&\quad + 40\,g_s^{1/4}\, m^2 \left(4\, \left(g_2^0 + g_3^0\right)\,\tilde{\phi}_5 + 2\, \left(2\,g_1^0 + c_2 + c_3\right)\,\tilde{\phi}_6 + 2\,\left(c_2 - c_3\right)\,\tilde{\phi}_7\right. \\
&\quad \left.\left. - \left(g_2^0\, \left(c_2 + g_1^0\right) + g_3^0\, \left(c_3 + g_1^0\right)\right)\,\tilde{\phi}_4\right)\right]\,.
\end{aligned}
\tag{7.70}
$$

The ϕ system does not admit a fully integrated solution we are thus forced to proceed by series expansion which will be the subject of later Sections. Here we present the equations, we comment on the number of nested integrals and show that we can find solutions up to three nested integrals. We do not write them explicitly because the expressions are very complicated and not particularly enlightening, they can be computed using the expressions for the $\tilde{\xi}$ functions given in Section 7.4 and the solutions for the homogeneous part we will present in this Section. Again the presentation follows the order in which the equations have to be solved.

7.5.1 $\tilde{\phi}_1$ and $\tilde{\phi}_2$

The functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are coupled and the system is

$$
\begin{aligned}
\tilde{\phi}_1' &= -h^0 e^{-u^0}\tilde{\phi}_1 - h^0 e^{u^0-2v^0}\tilde{\phi}_2 + \frac{1}{20} h^0 e^{-2u^0-4v^0}(\tilde{\xi}_1 + 2\tilde{\xi}_2)\,, \\
\tilde{\phi}_2' &= -h^0 e^{-u^0}\tilde{\phi}_1 - 3h^0 e^{u^0-2v^0}\tilde{\phi}_2 + \frac{1}{20} h^0 e^{-2u^0-4v^0}(4\tilde{\xi}_1 + 3\tilde{\xi}_2)\,.
\end{aligned}
$$

The fundamental matrix is

$$\tilde{\Upsilon}_{12} = \left(\begin{array}{c|c} \frac{r^4+1}{r^3\sqrt{r^4-1}} & \frac{1}{r^4} + \frac{r^4+1}{r^3\sqrt{r^4-1}} \left(\mathcal{E}(r) - \mathcal{F}(r)\right) \\ \hline \frac{3-r^4}{r^3\sqrt{r^4-1}} & \frac{3}{r^4} + \frac{3-r^4}{r^3\sqrt{r^4-1}} \left(\mathcal{E}(r) - \mathcal{F}(r)\right) \end{array} \right) . \tag{7.71}$$

A formal solution is thus

$$\begin{pmatrix} \tilde{\phi}_1(r) \\ \tilde{\phi}_2(r) \end{pmatrix} = \tilde{\Upsilon}_{12}(r)\mathrm{Y}_{12} + \tilde{\Upsilon}_{12}(r) \int^y \tilde{\Upsilon}_{12}^{-1}(y)\mathbf{g}_{12}^{\phi}(y)dy \, . \tag{7.72}$$

Some of the integrals can be explicitly done but there are some terms for which we are unable to find a primitive. We thus have a solution up to one implicit integral.

7.5.2 $\tilde{\phi}_3$

We can use the following relation coming from the equation for $\tilde{\phi}_1$:

$$-h^0 e^{-u^0} \tilde{\phi}_1 - h^0 e^{u^0 - 2v^0} \tilde{\phi}_2 = \tilde{\phi}_1' - \frac{h^0}{20} e^{-2u^0 - 4v^0} \, , \tag{7.73}$$

to simplify the equation for $\tilde{\phi}_3$, which will take the following form:

$$\tilde{\phi}_3' = 8\tilde{\phi}_1' + \frac{h^0}{10} e^{-2u^0 - 4v^0} \left(\tilde{\xi}_3 - 32\tilde{\xi}_4 \right) . \tag{7.74}$$

It has the following solution:

$$\tilde{\phi}_3(r) = 8\tilde{\phi}_1(r) + \frac{8}{3} \int^r \frac{\tilde{\xi}_3}{(y^4-1)^{3/2}} \, dy - \frac{256}{5} \int^r \frac{\tilde{\xi}_4}{(y^4-1)^{3/2}} \, dy + Y_3 \, , \tag{7.75}$$

which is again implicitly defined in terms of a single integral.

7.5.3 $\tilde{\phi}_5$ and $\tilde{\phi}_6$

$\tilde{\phi}_5$ and $\tilde{\phi}_6$ are coupled and the system is

$$\begin{aligned} \tilde{\phi}_5' &= h^0 \tilde{\phi}_6 + \frac{\ell^6}{4m^2} h^0 H_0 \, \tilde{\xi}_5 - \frac{h^0}{4} f_1 \left(8\tilde{\phi}_1 - \tilde{\phi}_3 \right) \\ \tilde{\phi}_6' &= 2h^0 e^{2u^0} \left(2e^{-4v^0} + e^{-4u^0} \right) \tilde{\phi}_5 + \frac{\ell^6}{m^2} h^0 H_0 e^{2u^0 - 4v^0} \tilde{\xi}_6 + \frac{\ell^6}{2m^2} e^{-2u^0} h^0 H_0 \, \tilde{\xi}_7 \\ &\quad + \frac{f_2}{4} \left(8\tilde{\phi}_1 + 8\tilde{\phi}_2 - \tilde{\phi}_3 \right) - \frac{f_3}{4} \tilde{\phi}_3 \end{aligned}$$

The fundamental matrix is

$$\tilde{\Upsilon}_{56} = \left(\begin{array}{c|c} \frac{1}{r\sqrt{r^4-1}} & \frac{1}{21}\left(-2 + 3r^4\right) + \frac{2}{21r\sqrt{r^4-1}} \mathcal{F}(r) \\ \hline \frac{1-3r^4}{r^4(r^4-1)} & \frac{2(6r^8-6r^4-1)}{21r^3\sqrt{r^4-1}} + \frac{2(1-3r^4)}{21r^4(r^4-1)} \mathcal{F}(r) \end{array} \right) .$$

A formal solution will look the same as in (7.72). Recall that in $\mathbf{g}_{56}^{\phi}$ there are pieces defined in terms of one implicit integral coming from $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$, thus the expression we get are defined in terms of two nested integrals.

7.5.4 $\tilde{\phi}_7$

The equation for $\tilde{\phi}_7$ can be put in the form

$$\tilde{\phi}_7' = \tilde{\phi}_6' - \frac{\ell^6}{m^2} h^0 H_0 e^{-2u^0}\, \tilde{\xi}_7 + \frac{1}{2} f_3 \tilde{\phi}_3 - 4 h^0 e^{-2u^0} \tilde{\phi}_5\,.$$

A solution is given by

$$\tilde{\phi}_7 = \tilde{\phi}_6 - \frac{\ell^6}{m^2} \int h^0 H_0 e^{-2u^0}\, \tilde{\xi}_7 + \frac{1}{2} \int f_3\, \tilde{\phi}_3 - 4 \int h^0 e^{-2u^0} \tilde{\phi}_5\,.$$

Among the summands which appear under integral sign, the first contains no further integral while the second integrand is itself defined implicitly, thus it gives two nested integrals in our counting. The last summand is defined by three nested integrals (one explicit here and two coming from $\tilde{\phi}_5$). A simple integration by parts can reduce the number by one giving an expression for ϕ_7 which contains at most two nested integrals. We obtain

$$\begin{aligned}\tilde{\phi}_7(r) = \tilde{\phi}_6(r) &- \frac{\ell^6}{m^2} h^0 H_0 e^{-2u^0}\, \tilde{\xi}_7(r) + \frac{1}{2} f_3 \tilde{\phi}_3(r) \\ &+ 4 \int^r \left(- \frac{2y}{\sqrt{y^4-1}} - 2\mathcal{F}(y) \right) \tilde{\phi}_5'(y) dy + 8 \left(\frac{r}{\sqrt{r^4-1}} + \mathcal{F}(r) \right) \tilde{\phi}_5(r)\,.\end{aligned}$$

7.5.5 $\tilde{\phi}_4$

We can use the $\tilde{\phi}_1$, $\tilde{\phi}_2$ system to simplify the equation for ϕ_4 which one obtains from (7.13); it can be recast in the following form

$$\begin{aligned}\tilde{\phi}_4' = &- H_0^{-1} H_0'\, \tilde{\phi}_4 + 16 \tilde{\phi}_1' - 8 \tilde{\phi}_2' + \frac{1}{2} h^0 e^{-2u^0 - 4v^0} \tilde{\xi}_3 - \frac{16 m^2}{\ell^6} h^0 H_0^{-1} e^{-2u^0 - 4v^0} f_1 \tilde{\phi}_5 \\ &- \frac{4m^2}{\ell^6} e^{-4u^0} H_0^{-1} f_2 \tilde{\phi}_6 + \frac{3}{4} \frac{m^2}{\ell^6} e^{-2u^0 - 4v^0} h^0 H_0^{-1} \left(\tilde{\phi}_6 - \tilde{\phi}_7 \right)\,. \end{aligned} \tag{7.76}$$

The homogeneous equation admits the solution $\tilde{\phi}_{4,h} = H_0^{-1}$ and clearly a general solution is given by

$$\phi_4(r) = H_0^{-1}(r) Y_4 + H_0^{-1}(r) \int^r H_0(y) g_4^{\tilde{\phi}}(y) dy\,. \tag{7.77}$$

7.6 The force on a probe D2–brane

We compute the force on a D2–brane probing the backreaction of the CGLP background. At order zero in perturbation, the contribution from DBI cancels the contribution from WZ and there is no net force as expected for a probe D2–brane in this supersymmetric background. At a first glance, the expression for the force in the perturbed solution is rather complicated but we will show that, using the first order equations of motion, most of the terms cancel and the final expression is quite simple. The action of a D–brane we wrote in Section 3.6 was in string frame, we expressed it here in Einstein frame

$$S_{Dp} = -T_p \int d\xi^{p+1} e^{-\Phi/4} g_s^{-3/4} \sqrt{|\det(\iota^*[g] + \mathcal{F})|} + T_p \int \iota^*[C] \wedge e^{\mathcal{F}} \tag{7.78}$$

We define the force as follows

$$F = F^{DBI} + F^{WZ} \equiv -\frac{dV^{DBI}}{dr} - \frac{dV^{WZ}}{dr}\,. \tag{7.79}$$

We choose a static gauge for a brane aligned along $M_{1,2}$ and we do not turn on the gauge field on the brane. Thus, given that the B–field in (7.28) pulls–back to zero, there is no $\mathcal{F}$ and the DBI Lagrangian reduces to

$$\mathcal{L}_{DBI} = -V^{DBI} = -T_p e^{-\Phi/4} g_s^{-3/4} \sqrt{-g_{00} g_{11} g_{22}} = -T_p e^{-\Phi/4 - 15z/2} g_s^{-3/4}\,. \tag{7.80}$$

The only non–zero RR potential is C_3, the part which will not pullback to zero is given by

$$C_3 = \frac{1}{g_s} \mathcal{K}(r) dx^0 \wedge dx^1 \wedge dx^2 \qquad\qquad \frac{d\mathcal{K}(r)}{dr} = -K(r)\,, \tag{7.81}$$

where $K(r)$ is give in equation (7.38). The Wess–Zumino term thus reduces to

$$\mathcal{L}_{WZ} = -V^{WZ} = T_P \frac{1}{3!} \varepsilon^{i_1 i_2 i_3} (C_3)_{i_1 i_2 i_3} = -T_p \frac{1}{g_s} \mathcal{K}(r)\,. \tag{7.82}$$

We can now compute the force on a probe D2–brane (from now on we put $T_p = 1$). At zeroth order we have

$$\begin{aligned} F_0^{DBI} &= g_s^{-1/2} H_0' e^{-\Phi_0/2 - 15 z_0} = -\frac{4m^2}{\ell^6} g_s^{-1/2} e^{-\Phi_0/2 - 15 z_0 - 2u^0 - 4v^0} h^0 \left[c_2 g_2^0 + c_3 g_3^0 + g_1^0 (g_2^0 + g_3^0) \right] \\ F_0^{WZ} &= \frac{1}{g_s} K(r) = \frac{4m^2}{\ell^6} g_s^{-1/2} e^{-\Phi_0/2 - 15 z_0 - 2u^0 - 4v^0} h^0 \left[c_2 g_2^0 + c_3 g_3^0 + g_1^0 (g_2^0 + g_3^0) \right] \end{aligned}$$

and, as we have anticipated, the two contributions cancel.

At first order we obtain

$$\begin{aligned} F_1^{DBI} &= -F_0^{DBI} \left(\frac{1}{4} \phi_4 - \frac{15}{2} \phi_3 \right) + g_s^{-3/4} \left(\frac{1}{4} \phi_4' + \frac{15}{2} \phi_3' \right) e^{-\frac{\Phi_0}{4} - \frac{15}{2} z^0} \\ F_1^{WZ} &= -F_0^{WZ} \left(\frac{1}{2} \phi_4 + \frac{15}{2} \phi_3 - 2\phi_1 - 4\phi_2 \right) + g_s^{-3/4} \left(\frac{1}{4} \phi_4' + \frac{15}{2} \phi_3' \right) e^{-\frac{\Phi_0}{4} - \frac{15}{2} z^0} \\ &\quad + \frac{4m^2}{\ell^6} g_s^{-1/2} h^0 e^{-\frac{\Phi_0}{2} - 15 z^0 - 2u^0 - 4v^0} \left[c_2 \phi_6 + c_3 \phi_7 + \phi_5 (g_2^0 + g_3^0) + g_1^0 (\phi_6 + \phi_7) \right]\,. \end{aligned}$$

From these expressions and the fact that $F_0^{DBI} = -F_0^{WZ}$, using the equations of motion for ϕ_3 and ϕ_4 which can be easily read of the ones for $\tilde{\phi}_i$, one can notice that most of the terms at first order cancel so that the force on a probe D2–brane reduces to

$$\begin{aligned} F(r) &= F_1^{DBI} + F_1^{WZ} \\ &= \frac{1}{8 g_s^{3/4}} h^0 e^{-2u^0 - 4v^0 - \frac{15}{2} z^0 - \frac{1}{4}\Phi^0} \tilde{\xi}_3 \\ &= \frac{2}{g_s} \frac{X_3\, e^{-8z^0(1)}}{(r^4 - 1)^{3/2}} \end{aligned} \tag{7.83}$$

As an aside, the derivative of the Green's function for the CGLP background (7.85) matches the behavior of the force (7.83) (see [12] for comments on this point). Indeed, allowing only for a dependence on the radial variable, the solution to

$$\Box G = 0 \tag{7.84}$$

evaluated on the CGLP background is

$$G(r) = c_1 + c_2 \left(\frac{r}{\sqrt{r^4 - 1}} - F(\arcsin(1/r)| - 1) \right) . \tag{7.85}$$

7.7 IR expansion

We present here the IR expansion of the ϕ fields which are easily obtainable from the inverse transformation of Footnote 6. We write explicitly only the divergent and constant terms since terms which are regular in the IR (we recall here that it corresponds to the limit $r \to 1$ in our conventions) do not provide any constraint on the space of solutions. We also impose the zero energy condition (7.14) which gives $X_2^{IR} = 0$ as a constraint.

$$\begin{aligned}\phi_1 = \frac{1}{\sqrt{r-1}} \Bigg[& Y_1^{IR} + \Big(E(-1) - K(-1)\Big) Y_2^{IR} + \frac{\log(r-1)}{4480} \Bigg(- 3B_1 \Big(34 + 65K(-1)^2 \Big) \\ & + 1792X_1 + 336X_5 - 112K(-1)\Big(3X_6 - 2X_7\Big)\Bigg)\Bigg] + \mathcal{O}\Big((r-1)^{1/2}\Big)\end{aligned} \tag{7.86}$$

$$\begin{aligned}\phi_2 = \frac{1}{13440\sqrt{r-1}} \Bigg[& - 3B_1\Big(41 + 100K(-1)^2\Big) + 2688X_1 + 924X_5 \\ & - 308K(-1)\Big(3X_6 - 2X_7\Big)\Bigg] - Y_2^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big)\end{aligned} \tag{7.87}$$

$$\begin{aligned}\phi_3 = & \frac{1}{15482880\,(K(-1)^2 - 4)\sqrt{r-1}} \Bigg[480 \log(r-1) \left(K(-1)^2 - 4\right) \left(3B_1 \left(K(-1)^2 + 17\right)\right. \\ & \left. - 56(K(-1)(2X_7 - 3X_6) + 3X_5)\right) - 42K(-1)^2(21067B_1 - 49152X_4 + 17384X_5) \\ & + 87369B_1K(-1)^4 - 374856B_1 - 32K(-1)\Big(189168X_6 - 120117X_7 \\ & + 32(5210Y_2^{IR} - 33(7Y_3^{IR} + 80Y_6^{IR}))\Big) + 40K(-1)^3(36624X_6 - 22535X_7) \\ & + 1344\Big(4160X_1 - 19200X_4 + 311X_5 + 160(7Y_1^{IR} + 7Y_2^{IR}E(-1) + 132Y_5^{IR} - 144Y_7^{IR})\Big)\Bigg] \\ & + \frac{1}{256}\left(3B_1K(-1) - 8X_7\right)\log(r-1) \\ & - \frac{2Y_4^{IR}}{3\left(K(-1)^2 - 4\right)} + \frac{1}{96}\left(48Y_2^{IR} + Y_3^{IR}\right) + \mathcal{O}\Big((r-1)^{1/2}\Big)\end{aligned} \tag{7.88}$$

$$
\begin{aligned}
\phi_4 = {} & \frac{1}{7741440\,(K(-1)^2-4)\,\sqrt{r-1}}\Bigg[480\log(r-1)\,\Big(K(-1)^2-4\Big)\,\Big(3B_1\,\Big(K(-1)^2+17\Big) \\
& - 56(K(-1)(2X_7-3X_6)+3X_5)\Big) + 6K(-1)^2(9203B_1 - 56(92160X_4+6781X_5)) \\
& - 30135B_1K(-1)^4 - 2254920B_1 + 32\Big(K(-1)(488208X_6 - 331467X_7 \\
& + 32(-5210Y_2^{IR} + 231Y_3^{IR} + 2640Y_6^{IR})) + 42(4160X_1 + 79104X_4 + 4919X_5 \\
& + 160(7Y_1^{IR} + 7Y_2^{IR}E(-1) + 132Y_5^{IR} - 144Y_7^{IR}))\Big) + 8K(-1)^3(338909X_7 - 494256X_6)\Bigg] \\
& + \frac{1}{128}\,(3B_1K(-1)-8X_7)\log(r-1) - \frac{4Y_4^{IR}}{3\,(K(-1)^2-4)} + Y_2^{IR} - \frac{5Y_3^{IR}}{16} \\
& + \mathcal{O}\Big((r-1)^{1/2}\Big)
\end{aligned} \tag{7.89}
$$

$$
\begin{aligned}
\phi_5 = {} & \frac{1}{5160960}\frac{1}{\sqrt{r-1}}\Bigg[60\log(r-1)\,\Big(K(-1)^2-4\Big)\,\Big(3B_1\,\Big(K(-1)^2+17\Big) \\
& - 56\big(K(-1)(2X_7-3X_6)+3X_5\big)\Big) + 6K(-1)^2(1795B_1 - 3976X_5) + 2235B_1K(-1)^4 \\
& - 42024B_1 - 32K(-1)(6384X_6 - 3711X_7 + 4160Y_2^{IR} - 672Y_3^{IR} - 7680Y_6^{IR}) \\
& + 152K(-1)^3(336X_6 - 179X_7) + 1344(64X_1 - 768X_4 - 23X_5 - 160(Y_1^{IR} + Y_2^{IR}E(-1) \\
& - 12Y_5^{IR}))\Bigg] - \frac{3(K(-1)^2-4)}{2048}(3B_1K(-1)-8X_7) + \mathcal{O}\Big((r-1)^{1/2}\Big)
\end{aligned} \tag{7.90}
$$

$$
\begin{aligned}
\phi_6 = {} & \frac{1}{20643840}\Bigg[6K(-1)^2(36599B_1 + 30856X_5) - 5115B_1K(-1)^4 + 140376B_1 \\
& - 1344(1262X_1 - 2304X_4 + 185X_5 + 160(5Y_1^{IR} + 12Y_5^{IR} - 48Y_7^{IR} + 5Y_2^{IR}E(-1))) \\
& + 32K(-1)(28560X_6 - 18495X_7 + 44480Y_2^{IR} - 672Y_3^{IR} - 7680Y_6^{IR}) \\
& + 8K(-1)^3(16841X_7 - 26544X_6)\Bigg] + \mathcal{O}\Big((r-1)\Big)
\end{aligned} \tag{7.91}
$$

$$
\begin{aligned}
\phi_7 = \frac{1}{5160960(r-1)}\Big[&6K(-1)^2(5656X_5 - 295B_1) - 8K(-1)^3(7644X_6 - 4241X_7) \\
&- 2415B_1K(-1)^4 + 32K(-1)(7644X_6 - 4551X_7 + 32(130Y_2^{IR} - 21Y_3^{IR} - 240Y_6^{IR})) \\
&+ 8904B_1 - 1344(64X_1 - 768X_4 + 7X_5 - 160(Y_1^{IR} - 12Y_5^{IR} + Y_2^{IR}E(-1)))\Big] \\
&- \frac{\log(r-1)}{(r-1)}\,\frac{K(-1)^2-4}{86016}\Big[3B_1(17 + K(-1)^2) - 56(3X_5 - K(-1)(3X_6 - 2X_7))\Big] \\
&+ \frac{\log(r-1)}{860160}\Big[B_1(3468 + 4485K(-1)^2 - 15K(-1)^4) - 56(768X_1 + 204X_5 \\
&- 68K(-1)(3X_6 - 2X_7) - 15K(-1)^2X_5 + 5K(-1)^3(3X_6 - 2X_7))\Big] \\
&+ \frac{1}{20643840}\Big[32K(-1)(28650X_6 - 18495X_7 + 32(1390Y_2^{IR} - 21Y_3^{IR} - 240Y_6^{IR})) \\
&+ 6K(-1)^2(36599B_1 + 30856X_5) - 8K(-1)^3(26544X_6 - 16841X_7) - 5115K(-1)^4B_1 \\
&+ 140376B_1 - 1344(1216X_1 - 2304X_4 + 181X_5 \\
&+ 160(5Y_1^{IR} + 12Y_5^{IR} + 48Y_7 + 5Y_2^{IR}E(-1)))\Big] + \mathcal{O}\Big((r-1)^{1/2}\Big) \qquad (7.92)
\end{aligned}
$$

7.8 Analysis of the space of solutions

We want to study some configuration whose backreaction is described within the space of solutions we have found in the previous Section. In particular we are interested in the modes which arise in the backreaction of anti–D2-branes placed at the tip of the cone ($r = 1$) and which are smeared on the finite size S^4. We need to impose the correct infrared boundary conditions. As for the smearing this tells us that we need to impose boundary conditions such that the warp factor and the dilaton acquire a singularity of the type $1/\sqrt{r-1}$ because it has to be a solution of a wave equation in a three-dimensional transverse space[7].

7.8.1 Boundary conditions for BPS D2–branes

In this section, as a matter of exposing our method before we focus on the candidate backreaction by anti–D2 branes, we first derive the boundary conditions which correspond to the modes sourced by a stack of branes placed at the tip of the cone.

Let us then consider a set of N ordinary extremal D2 branes smeared on the S^4 at the

[7] This at a first glance is in contrast with the usual behavior $1/(r-1)$. This is due to to our choice of coordinate system in which the radial component of the transverse metric is h^2dr^2. In fact for $r \to 1$ we could choose a new radial variable $\tau \equiv \sqrt{r-1}$ such that $d\tau \sim dr/\sqrt{r-1}$ and have the usual behavior $d\tau^2$.

bottom of the throat. For the CGLP background, we can explicitly evaluate the Maxwell charge

$$\mathcal{Q}^{Max}_{CGLP}(r) = \frac{1}{(2\pi\sqrt{\alpha'})^5}\int_{M_6} e^{\Phi/2} * F_4 = \frac{4m^2 g_s^{-1/2}}{\ell(2\pi\sqrt{\alpha'})^5}\mathrm{vol}(M_6)[g_1(g_2+g_3)+c_2 g_2 + c_3 g_3]\,. \tag{7.93}$$

This quantity exhibits the following zeroth–order IR behavior:

$$\mathcal{Q}^{IR}_{CGLP} = 0\,, \tag{7.94}$$

as can be seen from

$$g_1^0(g_2^0+g_3^0)+c_2 g_2^0 + c_3 g_3^0 \simeq \frac{7}{128}(r-1)^{3/2} - \frac{77}{512}(r-1)^{5/2} + \mathcal{O}\left((r-1)^{7/2}\right)\,, \tag{7.95}$$

using equation (7.32).

Within the Ansatz we have been considering, a BPS solution describing the addition of N ordinary BPS D2–branes smeared on the S^4 in the IR can be found by shifting g_2 and g_3 such that the combination $g_2 + g_3$ — which is multiplied by g_1 in (7.93) — does not change:

$$g_2^0 \to g_2^0 + \frac{32N}{3}\,, \qquad g_3^0 \to g_3^0 - \frac{32N}{3}\,. \tag{7.96}$$

This way, the charge is shifted as

$$\mathcal{Q}^{Max}_{CGLP} \to \mathcal{Q}^{Max}_{CGLP} + \Delta\mathcal{Q}^{Max}_{D2}\,, \tag{7.97}$$

with

$$\Delta\mathcal{Q}^{Max}_{D2} = \frac{4Nm^2}{(2\pi\sqrt{\alpha'})^5}\frac{g_s^{-1/2}}{\ell}\mathrm{vol}(M_6)\,. \tag{7.98}$$

Note that the flux through S^4,

$$q_{S^4} = \frac{1}{(2\pi\sqrt{\alpha'})^3}\int_{S^4} F_4 = \frac{4m g_s^{-1}}{(2\pi\sqrt{\alpha'})^3}(g_1+c_2)\mathrm{vol}(S^4)\,, \tag{7.99}$$

stays unchanged under the shifts (7.96), while the warp factor shifts as

$$H_0(r) \to -\frac{4m^2}{\ell^6}\int^r h\, e^{-2u^0-4v^0}\left[g_1^0(g_2^0+g_3^0)+c_2 g_2^0 + c_3 g_3^0 + N\right] dy \tag{7.100}$$

and now is endowed with a singularity of the kind

$$H(r) \sim \frac{\Delta\mathcal{Q}_{D2}}{\sqrt{r-1}}\,. \tag{7.101}$$

This is the expected behavior of the harmonic function for Dp branes smeared on an S^r within an otherwise ten–dimensional flat space, which indeed behaves as $\frac{1}{\tau^{7-p-r}}$, where $p=2$ and $r=4$ for the CGLP background.

Let us now see in more detail how this BPS solution can be reproduced by the first–order perturbation apparatus. First of all, we set to zero all the modes related to supersymmetry–breaking, namely we impose that all the constants X_a and $B_1 \sim X_3$ (7.51), which enter upon integrating of $\tilde{\xi}$ equations, should vanish.

Furthermore, the zeroth–order combinations e^{2u^0} and e^{2v^0} reach constant or zero value in the IR ; since we expect that the geometry of the transverse space is not affected by the addition of BPS D2–branes we impose the perturbations associated to u and v to vanish as well. This fixes

$$Y_1^{IR} = Y_2^{IR} = 0. \tag{7.102}$$

In addition, non–singularity of ϕ_5 and ϕ_7 (we recall that they enter the fluxes of our Ansatz) is ensured by

$$Y_5^{IR} = -\frac{1}{840} K(-1)(7Y_3^{IR} + 80Y_6^{IR}). \tag{7.103}$$

The mode ϕ_6 is regular, and in view of the first–order contribution to (7.93)

$$\left(g_1^0 + c_2\right)\phi_6 + \left(g_1^0 + c_3\right)\phi_7 + \left(g_2^0 + g_3^0\right)\phi_5 \simeq -\frac{\sqrt{r-1}}{8}\phi_5 + \left(\frac{3}{32} - \frac{r-1}{16}\right)\phi_6 - \frac{r-1}{16}\phi_7 \,, \tag{7.104}$$

one should impose that $\phi_6(r \to 1)$ be proportional to the number N of BPS D2–branes spread over S^4 at the tip.

To recap, the above choices of integration constants (7.102)–(7.103) yield the expected behavior for BPS D2–branes added in the supersymmetric CGLP background:

$$\phi_1 = 0\,, \qquad \phi_2 = 0\,, \qquad \phi_5 = \mathcal{O}\left(r-1\right)\,, \tag{7.105}$$

$$\begin{aligned}
\phi_3 &= -\frac{2Y_7^{IR}}{4-K(-1)^2}\frac{1}{\sqrt{r-1}} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,, & \phi_4 &= -\frac{4Y_7^{IR}}{4-K(-1)^2}\frac{1}{\sqrt{r-1}} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,,\\
\phi_6 &= \frac{1}{2}Y_7^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,, & \phi_7 &= -\frac{1}{2}Y_7^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,.
\end{aligned}$$

We recall that $\phi_{1,2}$ denote perturbations of the stretching functions, $\phi_{3,4}$ label perturbations of the warp factor and dilaton, whilst $\phi_{5,6,7}$ are the modes corresponding to the linearized perturbations of the NSNS and RR fluxes of this IIA background.

The integration constant Y_7^{IR} is the only remaining one and is related to the number N of added BPS D2–branes: indeed, the equations for ϕ_6 and ϕ_7 reproduce the shift (7.96). The warp factor, along with the dilaton, acquires the expected singularity and

$$H = e^{8z_0}\left(1 + 8\,\phi_3\right)\,,\ \ e^{\Phi} = e^{\Phi_0}\left(1+\phi_4\right) = e^{2z_0}\left(1 + 2\,\phi_3\right), \tag{7.106}$$

in accordance with $e^{\Phi} \sim H^{1/4}$.

7.8.2 Assessing the anti-D2 brane solution

The final step and main aim of our analysis is to determine how, within the space of generic linearized deformations of the IIA CGLP background, one can account for the backreaction due to the addition of anti–D2 branes smeared on the S^4 at the tip of the warped throat.

As the prime physical requirement we should impose that the force felt by a D2–brane probing the backreaction due to this stack of anti–D2 branes be non-vanishing. So, we are forbidden from turning off the corresponding mode which appears in the expression (7.83) of the force, and enters the various expressions for the modes ϕ_a by means of the shorthand combination

$$B_1 = \frac{m^2}{\ell^6} X_3 e^{-8z_0(1)} \,. \tag{7.107}$$

As our next set of IR boundary conditions, let us recall that the modes ϕ_3 and ϕ_4 associated to the perturbation of the warp factor and the dilaton must exhibit no worse than a $1/\sqrt{r-1} \sim 1/\tau$ behavior (cf. Footnote 7). Such a behavior is in accordance with the Coulomb–like divergence associated to anti–D2 branes smeared over the S^4 at the tip of the warped throat.

Inspecting the IR expansions of the deformation modes ϕ_a, every piece that is more singular than the aforementioned $1/\sqrt{r-1}$ behavior will be culled by tuning appropriate combinations of the X's and the Y's integration constants parametrizing the space of generic linearized perturbations of the CGLP background.

Another, equivalent but slightly less liberal, criterion that we are about to consider focuses on allowing or discarding various pieces from the ϕ_a's IR expansions depending on their contribution to the energy. More precisely, we consider the kinetic energy (7.35) and the potential energy (7.39) obtained by reducing our IIA supergravity Ansatz to a one–dimensional sigma model.

For instance, the energy associated to the first–order perturbation of the dilaton and warp factor is obtained by expanding to second–order the corresponding terms from (7.35):

$$\begin{aligned} & \frac{e^{2\,(u^0+\phi_1)+4\,(v^0+\phi_2)}}{h} \Big[-30\,\left(z^{0\,\prime} + \phi_3'\right)^2 - \frac{1}{2}\left(\Phi^{0\,\prime} + \phi_4'\right)^2 \Big] \\ & \rightsquigarrow \\ & \frac{e^{2\,u^0+4\,v^0}}{h} \Big[-30\,\phi_3'^2 - \frac{1}{2}\,\phi_4'^2 - 2\,(\phi_1 + 2\,\phi_2)\,\left(\Phi^{0\,\prime}\,\phi_4' + 60\,z^{0\,\prime}\,\phi_3'\right) \Big] \end{aligned} \tag{7.108}$$

The energy associated to the deformation of the warp factor and dilaton exhibits the following singular behavior

$$(r-1)^{3/2}\left(\frac{d\phi_{3,4}}{dr}\right)^2 \sim \frac{1}{(r-1)^{3/2}}\ , \tag{7.109}$$

where as a matter of course we neglect less diverging terms. This behavior sets the threshold for what we consider an allowable singularity in the energy.

Note that, as it turns out, for all practical purposes we can neglect contributions of the type $\phi_a\phi_b$ and $\phi_a'\,\phi_b$ for $a \neq b$: they only contribute to sub–leading divergences. In addition, there is no contribution to the energy that is first–order in the SUSY–breaking parameters, since we are expanding around a saddle point.

Another remark is in order. We have considered linearized deformation for the fields entering our supergravity Ansatz, namely we have expanded as

$$\phi_a = \phi_a^0 + \phi_a^1(X,Y)\,, \tag{7.110}$$

with X_i and Y_i being implicitly the small supersymmetry–breaking expansion parameters. On the other hand, we are considering quadratic contributions of the ϕ_a^1's to the energy.

The reason why we do not stop at first–order contributions to the energy from those deformation modes is that we have expanded around a saddle point. Had we gone as far as computing 2nd order expansions of the deformation modes, namely

$$\phi_a = \phi_a^0 + \phi_a^1(X,Y) + \phi_a^2(X,Y,Z,W)\,, \tag{7.111}$$

which is an achievable if strenuous task, it might well happen that the singularities we are about to expose might cancel against truly second order contributions to the energy. By this we mean contributions of the type $\phi_a^2\phi_b^0$, in addition to those of the form $\left(\phi_a^1\right)^2\,\phi_b^0$ that we presently consider.

Everything is now in place to show that the candidate IIA supergravity dual to metastable supersymmetry–breaking that would be obtained out of backreacting $\overline{D_2}$'s spread over the S^4 in the far IR of the CGLP background comes with an irretrievable IR singularity. Indeed, we are going to show that it is not possible to simultaneously satisfy the two previously mentioned physical requirements.

In point of fact, there is a singularity associated to the NSNS and RR fluxes that is worse than the ones we allow, namely those that are physical and should be kept based on their identification with the effect of adding anti–D2 branes to uplift the AdS minimum of the potential. There is only one way of getting rid of that "unphysical" singularity: it entails setting to zero the single mode entering the force felt by a brane probing the non–supersymmetric backreaction by $\overline{D_2}$'s. So, our two sensible IR boundary conditions are incompatible.

Ensuring that there is a force exerted on a probe D2–brane by the anti–D2's at the tip results in a $\frac{1}{(r-1)^3} \sim \frac{1}{\tau^6}$ singular contribution to the energy, stemming from the NSNS or the RR field strength. Such a singularity is worse than the ones it is sensible to a priori allow, namely $\frac{1}{(r-1)^{3/2}}$ singularities or milder ones, associated to the smeared $\overline{D_2}$'s.

Let us see how this comes about with full details. First of all, note that the potentially most divergent deformation modes is ϕ_7: its IR series expansion (7.92) displays $\frac{1}{r-1}$ and $\frac{\log(r-1)}{r-1}$ pieces. That mode, ϕ_7, contributes only to the deformation of the NSNS 3–form field strength

$$\ell\,\delta H_3 = m\,\left[(\phi_6+\phi_7)\,U_3 + \phi_6'\,dr\wedge U_2 + \phi_7'\,dr\wedge J_2\right]\,. \tag{7.112}$$

In view of (7.35) and (7.39), the leading contribution to the energy from the deformation of the NSNS 3–form is

$$-\frac{m^2}{2\,\ell^6}\,\frac{e^{2\,u^0+4\,v^0-8\,z^0}}{h}\,\left[\phi_6'^2\,e^{-4\,u^0} + 2\,\phi_7'^2\,e^{-4\,v^0}\right] - 2\,\frac{m^2}{\ell^6}\,h\,e^{-8\,z^0}\,\left[\phi_6+\phi_7\right]^2\,. \tag{7.113}$$

There is another potential contribution from (7.39) which involves ϕ_6 and ϕ_7. It is easily seen that it is sub–leading. Now, what is the IR singular behavior of (7.113) ? We focus on the most singular piece of $\phi_7 \sim \frac{1}{r-1}$ and its derivative. It entails the following singular behavior

$$-\frac{m^2}{\ell^6}\,e^{-8\,z^0(r)}\,\left[\frac{e^{2\,u^0(r)}}{h(r)}\,\left(\frac{d}{dr}\,\frac{1}{(r-1)}\right)^2 + 2\,h(r)\,\left(\frac{1}{(r-1)}\right)^2\right] \sim \frac{1}{(r-1)^{5/2}}\,. \tag{7.114}$$

According to our physical criterion pertaining to the energy, we should then discard the most IR–divergent piece of ϕ_7, see (7.92). This is achieved by imposing

$$X_5 = \frac{1}{168}\left[3\left(17+K(-1)^2\right)B_1 + 56\,K(-1)\,(3\,X_6 - 2\,X_7)\right], \tag{7.115}$$

$$\begin{aligned} X_1 = \frac{1}{86016}\Big[&6048\,B_1 + 1032192\,X_4 + 215040\,Y_1^{IR} - 2580480\,Y_5^{IR} + 215040\,E(-1)\,Y_2^{IR} \\ &+ 235200\,K(-1)\,X_6 - 139360\,K(-1)\,X_7 + 133120\,K(-1)\,Y_2^{IR} \\ &- 21504\,K(-1)\,Y_3^{IR} - 245760\,K(-1)\,Y_6^{IR} + 8364\,K(-1)^2\,B_1 \\ &- 27216\,K(-1)^3\,X_6 + 11304\,K(-1)^3\,X_7 - 1809\,K(-1)^4\,B_1\Big], \end{aligned} \tag{7.116}$$

where (7.115) has been applied to obtain (7.116) out of the combination of X's and Y's from the $\frac{1}{(r-1)}$ part of ϕ_7's IR expansion.

We now turn our attention to getting rid of the singularities stemming from the RR flux and ϕ_5. First of all, note that the condition (7.115) washes out, at no extra cost, the leading $\frac{\log(r-1)}{\sqrt{r-1}}$ part of ϕ_5's IR asymptotics.

Still, one should enforce that the $\frac{1}{\sqrt{r-1}}$ part of ϕ_5's IR expansion be wiped out by appropriately tuning some of the X's and Y's. Indeed, if kept unchecked, that divergent piece would yield a singularity in the energy arising from the RR flux:

$$\begin{aligned} &-2\frac{m^2}{\ell^6}\,\frac{e^{-8\,z^0-9\,\phi_3+\phi_4/2}}{h}\,\left({g_1^0}' + \phi_5'\right)^2 \\ &-4\,\frac{m^2}{\ell^6}\,e^{-8\,z^0-9\,\phi_3+\phi_4/2+2\,u^0+2\,\phi_1}\,h\,\left[2\left(g_1^0+c_2+\phi_5\right)^2 e^{-4\,v^0-4\,\phi_2} + \left(g_1^0+c_3+\phi_5\right)^2 e^{-4\,u^0-4\,\phi_1}\right] \\ &\rightsquigarrow \frac{1}{(r-1)^{5/2}}\,, \end{aligned} \tag{7.117}$$

which is beyond the energy threshold (7.109) and should be culled. To get rid of that singular piece from ϕ_5, one must exact

$$\begin{aligned} &-32\,K(-1)\left(6384\,X_6 - 3711\,X_7 + 4160\,Y_2^{IR} - 672\,Y_3^{IR} - 7680\,Y_6^{IR}\right) \\ &6\,K(-1)^2\,(1795\,B_1 - 3976\,X_5) + 152\,K(-1)^3\,(336\,X_6 - 179\,X_7) \\ &+ 2235\,K(-1)^4\,B_1 - 42024\,B_1 + 1344\,\big[64\,X_1 - 768\,X_4 - 23\,X_5 \\ &- 160\,(Y_1^{IR} + E(-1)\,Y_2^{IR} - 12\,Y_5^{IR})\big] = 0\,. \end{aligned} \tag{7.118}$$

We have finally reached the punchline of our analysis: taking into account the conditions (7.115)–(7.116) that did arise from ensuring that no "unphysical" singularity pops out of the NSNS flux, it turns out that (7.118) yields

$$11340\,\left(4 - K(-1)^2\right)\,B_1 = 0\,, \tag{7.119}$$

in blatant opposition to the physical requirement that a D2–brane probing the non–supersymmetric deformation of the CGLP background experiences a non–vanishing force!

We have therefore come to the conclusion that a careful analysis of the backreaction of anti–D2 branes on the CGLP background inevitably results in an IR singularity. By focusing on two particular flux elements for which the energy contribution can be easily calculated, we have shown that it is not possible to avoid a singular behavior provided we want to keep the B_1 mode entering the expression for the force (7.83) to be non–vanishing.

One has to face that at least one of the perturbed NSNS or RR fluxes contributes to a divergent energy density and to a divergent action as well (given that the factor $\sqrt{g_{10}} \simeq \sqrt{r-1}$ appearing in the ten–dimensional action (7.1) is not enough to make the action finite in the IR), much as is the case in [11]. The key difference from [11] lies in the fact that in our case the singular behavior is not at all sub–leading.

The above type IIA analysis completes the program of investigating the would–be backreacted supergravity duals to metastable supersymmetry–breaking vacua, which was originally started in a type IIB setting [16], and next considered in [11] in an 11–dimensional context. It would be of much interest to consider other backgrounds and/or, as explained at the beginning of this Section, to go to higher–order in the perturbations around those BPS solutions. It might be that an absence of the nasty singularities we have kept on encountering so far could be used in order to discriminate among solutions of the landscape string theory vacua.

Chapter 8

Conclusions

The interplay between supersymmetry and geometry has been a guiding principle in string theory since its early days. As always, the presence of symmetries in a physical problem simplifies it and allows to find solutions to the equations describing such a system. The major advantage, among others, of supersymmetry in the context of superstring compactifications is that it allows to trade the second order differential equations of motion by some first order ones, whose content is the invariance of the solution under supersymmetry variations, and that, under some mild assumptions, are almost all[1] one needs to care about in order to have a solution. By itself this is already an important and deep feature of the theory, but it would be certainly less powerful if it was lacking the elegant relation with geometry.

In Chapter 2 we have introduced the main features of type II theories, their action and equations of motion together with a compactification ansatz and the relevant supersymmetry transformations. We also have briefly discussed the simplest case of fluxless Minkowski compactifications where we have seen the intimate relation between geometry and supersymmetry at work. We have reviewed how to reformulate the conditions imposed by supersymmetry as a geometric condition on the internal manifold; supersymmetry selects a specific class of manifolds, namely Calabi–Yau's [32]. It is well known that Calabi–Yau compactifications are not phenomenologically viable models because of the moduli problem. To look for a wider class of solutions is a natural step and we thus turn on some of the other fields of the theory (e.g. fluxes) and seek for solutions; the obvious drawback is that difficulty increases. For type II strings, which have been the object of interests (in particular type IIA) in this thesis, there is a geometric interpretation for a very general class of $\mathcal{N} = 1$ vacua[2], based on the tools provided by Generalized Complex Geometry, which has been initiated in [85, 86].

In Chapter 3 we have reviewed the mathematical aspects of Generalized Complex Geometry [106, 95] and their use in type II string compactifications. Also in presence of fluxes it is possible to fully characterize the class of manifolds selected by supersymmetry. The main point is the reformulation of the supersymmetry conditions (2.16) as equations (3.57) - (3.59), involving a pair of differential forms on the internal manifold, the pure spinors Φ_1 and Φ_2, which define the geometry and give the RR flux content. The class of manifolds selected by supersymmetry is now constituted by the so called generalized Calabi–Yau manifolds. D–branes and O–planes are

[1]As discussed in the text one should solve, in addition, for the Bianchi identities.

[2]It includes supersymmetric compactifications to Minkowski and AdS spaces with or without sources and with the most general flux content compatible with the ansatz discussed in Chapter 2. It does not take into account NS5–branes in a natural way.

important dynamical objects in string theory, both for phenomenological reasons and internal consistency. Generalized Complex Geometry provides a suitable extension of the calibration forms [126, 130, 149] which allows for a comprehensive description as reviewed in Section 3.6.

As we have seen it is hard to overestimate the importance of supersymmetry, nevertheless there is no evidence of it in the known particle spectrum and thus we live in a state where such a symmetry is broken. Phenomenology is of course a strong motivation to look for non–supersymmetric configurations but in this thesis we have served a less ambitious purpose; we have investigated the structure of non–supersymmetric vacua in type IIA string theory with the aim of understanding some features through the study of two examples without any pretension of constructing phenomenological reliable models. Our perspective has been, for the most of the time, focused on the ten dimensional physics, avoiding the much more challenging problem of a complete Kaluza–Klein reduction to understand the four dimensional physics of our models and we have limited ourselves to some considerations in Section 5.3. Our approach stems from the belief that the geometrical approach can be extended to a description of $\mathcal{N} = 0$ vacua and that Generalized Complex Geometry can still be a suitable language and organizing principle, maybe with suitable adjustments.

In Chapter 4 we have introduced solvmanifolds, a class of manifolds which admit a Generalized Calabi–Yau structure and which has been extensively used in flux compactification. We have reviewed their geometrical properties with a special attention to their compactness. We have reformulated some criterion to determine if they are compact or not, which was already present in the mathematical literature, via a more familiar (to physicists) language based on twist transformation [5]. Their fibration structure is then interpreted as successive steps of twist transformations from a torus and the compactness criterion related to its monodromy properties. We have concluded the Chapter by applying the twist transformation to a known solution to construct a new supersymmetric one on a solvmanifold labelled as $\mathfrak{g}_{5.17}$. This has been the starting point for the analysis of the following Chapter.

In Chapter 5 we have developed the details of our first example of non–supersymmetric configuration. Compactifications where the external space is a de Sitter space have received a growing attention in the last years, mostly because of recent cosmological observations which indicate that we live in a universe with a small positive cosmological constant. However, the understanding of such configurations is less developed because they are intrinsically non–supersymmetric. Starting from a Minkowski supersymmetric vacuum we have deformed it in order to obtain a de Sitter solution. Staying in the framework of Generalized Complex Geometry, we have seen how the supersymmetry breaking arises in the pure spinor equations and we have been forced to address the question of the behavior of the sources in such a background. Being non–supersymmetric, there was no a priori reason for the sources to arrange the same way as they do in supersymmetric backgrounds, namely wrapping calibrated supersymmetric cycles. We have thus taken a different approach and described the behavior of the sources via a general polyform X_- whose undetermined coefficients have been fixed by solving the equations of motion. We are aware of the limited validity due to the example based approach of our analysis but we can infer some general properties which we believed are shared by non–supersymmetric backgrounds (in general, not restricting to de Sitter compactifications).

In Chapter 6 we have collected a series of observations about the geometry and structure of non–supersymmetric backgrounds and reorganized them at a more general level than the single examples. The speculative nature of the Chapter makes it the right place where to discuss possible lines of development. A quite general parametrization of the supersymmetry breaking and its reformulation in bispinor language has been already developed in [137], but we believe that without any constrain this is of little help. Constrains should of course come from equations of motion and a partial analysis is done in [137], nevertheless a full understanding is not yet available, in particular we lack a bispinorial reformulation which would be the natural way to incorporate the issue in the Generalized Complex Geometry framework. We believe that to identify a suitable set of variables for an extension of the first order formalism also to a (sub)set of non–supersymmetric configurations is an important step to have a better control on their geometry. Another approach is based on T–duality, which also has a nice reformulation in terms of Generalized Complex Geometry. Being a duality of type II theories it sends solutions to solutions and it preserves supersymmetry under certain conditions [88]. We think that T–duality can be used to generate non–supersymmetric solutions from supersymmetric ones or to connect two non–supersymmetric ones; we have seen how the pure spinor equations get modified by supersymmetry breaking terms and argued about a set of bispinorial variables suitable for such cases. It would be important to find an example were to test some of our observations. As we have already said it was clear from the de Sitter example that the usual description of the sources has to be modified when supersymmetry is broken and that the question of stability of the sources lacks a full understanding for a general breaking. We have tried to suggest some possible development is Section 6.3 but a complete analysis is beyond the scope of this thesis.

Our second example of supersymmetry breaking solution has been presented in Chapter 7. We have been motivated by the interest in non–supersymmetric supergravity backgrounds as duals of non–supersymmetric metastable states in supersymmetric gauge theories via AdS/CFT duality. There the approach has been different, we have used the perturbative technique developed by Borokhov and Gubser in [24] and we devoted less attention to the geometrical properties of the solution. We have done a partial study of the space of first order deformations of a supersymmetric background found in [47]. Our motivations were mainly derived by the recent interest in these kind of deformations and techniques which have been applied to the renowned type IIB Klebanov–Strassler background [16, 13, 14, 62] and to an M–theory background [48, 11]. Some common features, like the fact that the force on a probe brane is related to a single perturbation mode and the presence of unwanted IR singularities, were pointed out in the previous analysis. To our knowledge there was no application to a type IIA configuration of such a technique and we have deemed important to fill the gap in order to have a wider range of examples. Our analysis pointed out the same features as the previous ones with a stronger indication that the perturbative approach could be problematic in some situations. As discussed in Section 7.8.2 the singularities we have found are more severe than in the other cases and its nature and admissibility is still source of debate and object of work. As they could be a drawback of first order perturbation theory it would be of certain interest to look at higher order corrections, unfortunately the system of equations one has to solve becomes rapidly complicated and since the only computationally available tool relies on a brute force approach we decided to not develop it here. The UV regime would deserve a separate discussion but due to the fact that we have been able to obtain a solution only in an $(r-1)$ power expansion a numeric analysis as in [13] is required to connect the two.Another possibility is that the IR singularities are intrinsic of perturbation theory and the addition anti–D branes cannot be considered as a perturbation of the supersymmetric background, a scenario which deserve a more careful investigation.

Appendix A

Conventions and useful formulae

A.1 Forms

We define a p–form A_p as:

$$A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \ldots \mathrm{d}x^{\mu_p} \,. \tag{A.1}$$

We define the contraction of a p–form $A = A_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_p}$ and a vector $v = v^\mu \partial_\mu$ by the following formula:

$$\iota_v A = v^\mu A_{\mu_1 \ldots \mu_p} \delta_\mu{}^{[\mu_1} \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_p]} \,. \tag{A.2}$$

For the coefficients of the wedge product of a forms A_p of degree p and a form B_q of degree q we adopt the following convention

$$\frac{1}{(p+q)!} (A_p \wedge B_q)_{\mu_1 \ldots \mu_{p+q}} = \frac{1}{p!q!} \, A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_q]} \,, \tag{A.3}$$

where the complete anti–symmetrization of the indexes of a tensor is taken with total weight one

$$A_{[\mu_1 \ldots \mu_p]} = \frac{1}{p!} \left(A_{\mu_1 \mu_2 \ldots \mu_p} - A_{\mu_2 \mu_1 \ldots \mu_p} + \ldots + (-)^{p(p-1)/2} A_{\mu_p \mu_{p-1} \ldots \mu_2 \mu_1} \right) \,. \tag{A.4}$$

Our choice for the Hodge dual operator in d dimensions is

$$*(\mathrm{d}x^{\mu_1} \ldots \mathrm{d}x^{\mu_p}) = \frac{\sqrt{|g|}}{(d-p)!} \, (-)^{(d-p)p} \varepsilon^{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_d} \, g_{\mu_{p+1} \nu_{p+1}} \ldots g_{\mu_d \nu_d} \mathrm{d}x^{\nu_{p+1}} \ldots \mathrm{d}x^{\nu_d} \,, \tag{A.5}$$

and on a p–form we have

$$** A_p = (-)^{(d-p)p+t} A_p \tag{A.6}$$

where $t = +1$ for Euclidean signature and $t = -1$ for Lorentzian signature. We conclude this section with an useful formula involving the Levi–Civita symbol:

$$\varepsilon_{\mu_1 \ldots \mu_q \mu_{q+1} \ldots \mu_d} \varepsilon^{\mu_1 \ldots \mu_q \nu_{q+1} \ldots \nu_d} = (-)^t q!(d-q)! \, \delta^{[\nu_{q+1} \ldots \nu_d]}_{[\mu_{q+1} \ldots \mu_d]} . \tag{A.7}$$

A.2 Gamma matrices and spinors

We report here some well known gamma matrix algebra formulae which can be useful to derive many expressions we have presented in this thesis. A more complete list can be found in [33], [155] or in standard supergravity textbooks.

$$
\begin{aligned}
&\{\gamma^m, \gamma^n\} = 2g^{mn} && [\gamma^m, \gamma^n] = 2\gamma^{mn} \\
&\{\gamma^{mn}, \gamma^p\} = 2\gamma^{mnp} && [\gamma^{mn}, \gamma^p] = -4\delta^{p[m}\gamma^{n]} \\
&\{\gamma^{mnp}, \gamma^q\} = 6\delta^{q[m}\gamma^{np]} && [\gamma^{mnp}, \gamma^q] = 2\gamma^{mnpq}
\end{aligned}
$$

From these expressions we can easily obtain:

$$\gamma^n\gamma^{pq} = 2g^{n[p}\gamma^{q]} + 2\gamma^{npq} \tag{A.8}$$

which we used in Section 2.3 to derive the Ricci flatness condition for a Calabi–Yau manifold.

According to the metric ansazt (2.11) the ten dimensional gamma matrices Γ_M decompose as follows:

$$\Gamma_\mu = e^A\gamma_\mu \otimes \mathbb{1} \qquad\qquad \Gamma_m = \gamma_5 \otimes \gamma_m \tag{A.9}$$

We define the chiral gamma matrices as:

$$\gamma_5 = i\gamma^{0123} \qquad\qquad \gamma = -i\gamma^{456789} \qquad\qquad \Gamma = \Gamma^{0\ldots9} \tag{A.10}$$

In four dimensions we can choose a basis such that the gamma matrices are real and hermitian, except γ^0 which is anti–hermitian. In six dimensions we can choose the gamma matrices γ^m to be purely imaginary and antisymmetric. In this way it is clear that they are hermitian.

Given this choice for the gamma matrices than the ten dimensional Cliff$(1,9)$ Majorana spinors are real. Moreover we can impose a Weyl condition which is compatible with the Majorana one given the fact that the chiral projectors

$$P_\Gamma^\pm = \frac{1}{2}(1 \pm \Gamma) \tag{A.11}$$

are real.

Under the $4+6$ splitting the space of ten dimensional spinors decompose into the tensor product $\Sigma_4 \otimes \Sigma_6$, thus a generic ten dimensional spinor ϵ is given by

$$\epsilon = \sum_{iJ} \alpha_{iJ}\, \zeta^I \otimes \eta^j \,, \tag{A.12}$$

where $\{\zeta_I\}_{I=1\ldots4}$ and $\{\eta_j\}_{j=1\ldots8}$ are a basis for the four and six dimensional spinors respectively. Majorana condition in ten dimensions $\epsilon = \epsilon^*$ is assured imposing:

$$(\zeta_+)^* = \zeta_- \qquad\qquad (\eta_+)^* = \eta_- \,. \tag{A.13}$$

A.3 SU(3)–structure

We collect here our conventions for SU(3) structures. In this section we assume the six dimensional globally defined nowhere vanishing spinors $\eta_\pm$ have been normalized as $\eta_\pm^\dagger\eta_\pm = |\eta_\pm| = 1$. From $\eta_\pm$ we can define an almost symplectic two–form J_{mn} and a $(3,0)$ decomposable form Ω_{mnp} as follows:

$$J_{mn} = \mp i\eta_\pm^\dagger\gamma_{mn}\eta_\pm \qquad\qquad \Omega_{mnp} = -i\eta_-^\dagger\gamma_{mnp}\eta_+. \tag{A.14}$$

They satisfy the compatibility condition $J \wedge \Omega = 0$ which means that J is a $(1,1)$ form with respect to the almost complex structure $I^p{}_q = g^{pm}J_{mq}$. Given a basis of eigenstates $\{\eta_\pm, \gamma^m\eta_\mp\}$, we can decompose the chiral projectors as

$$P_\gamma^\pm = \frac{1\pm\gamma}{2} = \eta_\pm\eta_\pm^\dagger + \frac{1}{2}\gamma^m\eta_\mp\eta_\mp^\dagger\gamma_m \tag{A.15}$$

and derive the following relations

$$\gamma_m\eta_+ = -iJ_{mn}\gamma^n\eta_+\,, \tag{A.16}$$

$$\gamma_{mn}\eta_+ = iJ_{mn}\eta_+ + \frac{i}{2}\Omega_{mnp}\gamma^p\eta_-\,, \tag{A.17}$$

$$\gamma_{mnp}\eta_+ = i\Omega_{mnp}\eta_- + 3iJ_{[mn}\gamma_{p]}\eta_+\,. \tag{A.18}$$

With these normalization we also have

$$*1 = \mathrm{vol}_6 = i\Omega\wedge\bar\Omega = \frac{4}{3}J^3\,. \tag{A.19}$$

The symplectic and complex structure defined by J and Ω are not necessarily integrable, the failure is measured by their non closure which is expressed in terms of the SU(3) decomposition of torsion classes [92, 43, 116]:

$$\mathrm{d}J = -\frac{3}{2}\,\mathrm{Im}(\bar W_1\Omega) + W_4\wedge J + W_3 \tag{A.20}$$

$$\mathrm{d}\Omega = W_1J^3 + W_2\wedge J + \bar W_5\wedge\Omega \tag{A.21}$$

Here we have that W_1 is a complex scalar, W_2 a complex primitive $(1,1)$ form, W_3 a real primitive $(1,2)+(2,1)$ form, W_4 a real one–form and W_5 a complex $(1,0)$ form.

A.4 Mukai pairing and Clifford map

We recall the definition of the Mukai pairing given in Section 3.3. Given two polyforms ϕ, $\sigma \in \Lambda^\bullet T^*M$ we define the Mukai pairing as

$$\langle\phi,\sigma\rangle = (\phi\wedge\lambda(\sigma))_d\,, \tag{A.22}$$

where $(\phi)_d$ means the projection of the polyform ϕ on the top–form component and the operator λ acts on a p–form by a complete reversal of its indexes:

$$\lambda(\sigma_p) = (-)^{\lfloor\frac{p}{2}\rfloor}\sigma_p = (-)^{\frac{1}{2}p(p-1)}\sigma_p\,. \tag{A.23}$$

The Mukai pairing is symmetric for $d \equiv 0$ or $1 \pmod 4$ and skew–symmetric otherwise. In particular for $d = 6$ it is antisymmetric. We list here some properties:

$$\lambda(e^B \sigma) = e^{-B} \lambda(\sigma) \,, \tag{A.24}$$

$$\langle e^{-B}\phi, e^{-B}\sigma\rangle = \langle \phi, \sigma\rangle \,, \tag{A.25}$$

$$\langle \phi, *\sigma\rangle = \langle \sigma, *\phi\rangle \,, \tag{A.26}$$

$$\langle \phi_\pm, \lambda(\sigma_\pm)\rangle = \pm\langle \phi_\pm, \lambda(\phi_\pm)\rangle \,, \tag{A.27}$$

$$\langle \phi, \mathbb{X}\cdot\sigma\rangle = (-)^{d+1}\langle \mathbb{X}\cdot\phi, \sigma\rangle \,, \tag{A.28}$$

$$\int_M \langle \mathrm{d}_H \phi, \sigma\rangle = (-)^d \int_M \langle \phi, \mathrm{d}_H \sigma\rangle \,. \tag{A.29}$$

We recall the definition of the Clifford map given in Section 3.5:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k} \mathrm{d}x^{i_1}\wedge\ldots\wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k}\gamma^{i_1\ldots i_k}_{\alpha\beta} \,. \tag{A.30}$$

For example the slashed fields that appear in the supersymmetric variations in (2.10) are:

$$\not{H}_M = \frac{1}{2} H_{MNP}\Gamma^{NP} \qquad \not{H} = \frac{1}{6} H_{MNP}\Gamma^{MNP} \qquad \not{F}_k = \frac{1}{k!} F_{M_1\ldots M_k}\Gamma^{M_1\ldots M_k} \,. \tag{A.31}$$

Following [87, 180] one can prove that the Clifford algebra Cliff(d,d) is isomorphic to two copies of the ordinary Clifford algebra Cliff(d). By gamma matrix algebra we have:

$$\begin{aligned} \gamma^m \gamma^{m_1\ldots m_k} &= \gamma^{m m_1\ldots m_k} + k g^{m[m_1}\gamma^{m_2\ldots m_k]} \,, \\ \gamma^{m_1\ldots m_k}\gamma^m &= (-)^k \left(\gamma^{m m_1\ldots m_k} - k g^{m[m_1}\gamma^{m_2\ldots m_k]}\right) \,. \end{aligned} \tag{A.32}$$

The gamma matrix action on bispinors corresponds to the following action on forms of degree k:

$$\gamma^m \not{\phi}_k = \ \not{(\mathrm{d}x^m \wedge + g^{mn}\iota_n)\phi_k} \qquad \not{\phi}_k \gamma^m = (-)^k \ \not{(\mathrm{d}x^m \wedge - g^{mn}\iota_n)\phi_k} \,. \tag{A.33}$$

An easy computation gives the action of Cliff(d,d) gamma matrices $\Gamma^\Sigma = \{\mathrm{d}x^m\wedge, \iota_m\}$ in terms of the two copies of Cliff(d) acting on the left and on the right:

$$\not{\mathrm{d}x^m \wedge \phi_\pm} = \frac{1}{2}[\gamma^m, \not{\phi}_\pm]_\pm \qquad\qquad \not{\iota_m \phi_\pm} = \frac{1}{2}[\gamma^m, \not{\phi}_\pm]_\mp \,. \tag{A.34}$$

The Mukai pairing (here we restrict to $d = 6$) under the Clifford map goes to:

$$\langle \phi_k, \sigma_{6-k}\rangle = \frac{i}{8}\mathrm{tr}\left(\gamma \not{\phi}_k^t \not{\sigma}_{6-k}\right) \mathrm{vol}_6 \,. \tag{A.35}$$

Other useful relations are:

$$\not{* \lambda(\phi)} = -i\gamma\not{\phi} \,, \qquad \not{*\phi} = -i\gamma\not{\phi}^{\,t} \,, \qquad \not{\phi}^{\,\dagger} = \ \not{\lambda(\bar{\phi})} \,. \tag{A.36}$$

We can express the normalization condition on the pure spinors $\Phi_\pm$ as:

$$\langle \Phi_-, \bar{\Phi}_-\rangle = \langle \Phi_+, \bar{\Phi}_+\rangle = -\frac{i}{8}\mathrm{tr}(\not{\Phi}_\pm \not{\Phi}^\dagger_\pm)\,\mathrm{vol} = -\frac{i}{8}||\eta^{(1)}_+||^2||\eta^{(2)}_\pm||^2\,\mathrm{vol} \,. \tag{A.37}$$

Some other useful property involving the operator λ are:

$$[\mathrm{d}, \lambda]\,\phi_{2k} = 0 \qquad\qquad \{d, \lambda\}\,\phi_{2k+1} = 0 \,, \tag{A.38}$$

$$[*_6, \lambda]\,\phi_{2k+1} = 0 \qquad\qquad \{*_6, \lambda\}\,\phi_{2k} = 0 \,. \tag{A.39}$$

A.5 The Generalized Hodge diamond

The space of complex Spin(6, 6) spinors decomposes into irreducible representations of the subgroup $\mathrm{SU}(3) \times \mathrm{SU}(3) \in \mathrm{Spin}(6,6)$ defined by the compatible pair $\Phi_\pm$. Via Clifford map it furnish a basis for the space of complex polyforms $\Lambda^\bullet T^*M \otimes \mathbb{C}$; it is given by [87]:

$$
\begin{array}{ccccccc}
 & & & \Phi_+ & & & \\
 & & \Phi_+\gamma^{i_2} & & \gamma^{\bar{\imath}_1}\Phi_+ & & \\
 & \Phi_-\gamma^{\bar{\imath}_2} & & \gamma^{\bar{\imath}_1}\Phi_+\gamma^{i_2} & & \gamma^{i_1}\bar{\Phi}_- & \\
\Phi_- & & \gamma^{\bar{\imath}_1}\Phi_-\gamma^{\bar{\jmath}_2} & & \gamma^{i_1}\bar{\Phi}_-\gamma^{j_2} & & \bar{\Phi}_- \\
 & \gamma^{\bar{\imath}_1}\Phi_- & & \gamma^{i_1}\bar{\Phi}_+\gamma^{\bar{\jmath}_2} & & \bar{\Phi}_-\gamma^{i_2} & \\
 & & \gamma^{i_1}\bar{\Phi}_+ & & \Phi_+\gamma^{\bar{\imath}_2} & & \\
 & & & \bar{\Phi}_+\,. & & &
\end{array}
\tag{A.40}
$$

Appendix B

T–dualising solvmanifolds

T–duality has been extensively used in flux compactifications in order to obtain solutions on nilmanifolds. Being iterations of torus bundles, these are obtainable from torus solutions with an appropriate B–field (the contraction of H with the isometry vectors should be a closed horizontal two–form that can be thought as a curvature of the dual torus bundle.). Correspondingly, the structure constants $f^a{}_{bc}$ have also a T–duality friendly form. For any upper index there is a well–defined isometry vector ∂_a with respect to which one can perform an (un–obstructed) T–duality.

In this section we would like to study some aspects of T–duality for solvmanifolds. In this case, the situation is more complicated. For instance, it can happen that the structure constants have the same index in the upper and lower position $f^a{}_{ac}$ and are not fully antisymmetric. Put differently, most of our knowledge about the global aspects of T–duality comes form the study of its action on (iterations of) principal $U(1)$ bundles. Since the Mostow bundles are not in general principal, the topology of the T–dual backgrounds is largely unexplored. We shall not attempt to do this here, but rather illustrate some of novel features by considering T–duality on the simplest cases of almost abelian manifolds.

Requiring that T–duality preserves supersymmetry imposes that the Lie derivatives with respect to any isometry vector v vanish, $\mathcal{L}_v\Psi_\pm = 0$ [88]. For the simple case of almost abelian solvmanifolds, it is not hard to check that all vectors $v_i = \partial_i$, where, in the basis chosen in this paper, $i = 1, ..., 4, 6$, satisfy this condition. However, these vectors are defined only locally[1], since they transform non–trivially under $t \sim t + t_0$. Hence, in general, the result of T–duality will be non–geometric. We shall see that there are subtleties even for the case when the supersymmetry–preserving isometries ∂_i are well defined.

We shall consider the action of T–duality on two solvmanifolds, $\mathfrak{g}_{5.17}^{0,0,\pm1} \times S^1$ (s 2.5) and $\mathfrak{g}_{5.7}^{1,-1,-1} \times S^1$. For $s\,2.5$, following [87], we write the algebra as $(25, -15, r45, -r35, 0, 0)$, $r^2 = 1$. The twist matrix $A(t)$ is made of periodic functions of $t = x^5$,

$$A = \begin{pmatrix} R_{r=1} & & \\ & R_r & \\ & & \mathbb{I}_2 \end{pmatrix}, \qquad R_r = \begin{pmatrix} \cos x^5 & -r \sin x^5 \\ r \sin x^5 & \cos x^5 \end{pmatrix}, \tag{B.1}$$

[1]As discussed, on the compact solvmanifolds there exists a set of globally defined one forms $\{e\} = \{A_M \mathrm{d}x\}$ and the dual basis $\{E\} = \{(A_M^{-1})^T \partial\}$ is made of globally defined vectors. However, the Lie derivative of the pure spinors with respect to these does not vanish.

and T–duality is un–obstructed. The various supersymmetric solutions found in [87, 4] are all related by two T–dualities

$$\begin{array}{ccccc} \multicolumn{2}{c}{\text{IIB}} & & \multicolumn{2}{c}{\text{IIA}} \\ \text{t:30} & \text{t:12} & & \text{t:30} & \text{t:12} \\ (13+24) \longleftrightarrow & (14+23) & \longleftrightarrow & (136+246) \longleftrightarrow & (146+236) \\ \text{T}_{12} & & \text{T}_6 & \text{T}_{12} & \\ (14+23) \longleftrightarrow & (13+24) & \longleftrightarrow & (146+236) \longleftrightarrow & (136+246) \end{array}$$

In the table we labelled each solution by the dominant O–plane charge. The sources are labelled by their longitudinal directions, e.g. (13 + 24) stands for a solution with two sources (one O5 and one D5) along directions $e^1 \wedge e^3$ and $e^2 \wedge e^4$. T–dualities (the subscripts indicate the directions in which they are performed) exchange the columns in the table; lines are exchanged by relabellings (symmetries of the algebra).

The T–dualities are type[2] changing, meaning a pair of type 0 and 3 (t:30) pure spinors is exchanged with a pair of type 1 and 2 (t:12) and vice versa.

It is natural to see what will it be the effect of a single T–duality. To be precise we take as starting point Model 3 of [87]. We shall concentrate on the NS sector and discuss the topology changes under T–duality. The NS flux is zero and the metric, in the $\mathrm{d}x^i$ basis is

$$\begin{aligned} \mathrm{d}s^2 &= \frac{t_1^2}{t_2}(\tau_2^1)^2 G(\mathrm{d}x^1 + \mathcal{A}\mathrm{d}x^2)^2 + \frac{t_1}{G}(\mathrm{d}x^2)^2 + t_1(\tau_2^1)^2 G(\mathrm{d}x^3 + r\mathcal{A}\mathrm{d}x^4)^2 \\ &\quad + \frac{t_2}{G}(\mathrm{d}x^4)^2 + t_3(\mathrm{d}x^5)^2 + t_3(\mathrm{d}x^6)^2 \end{aligned} \tag{B.3}$$

with

$$G = \cos^2(x^5) + \frac{t_2}{t_1(\tau_2^1)^2}\sin^2(x^5) \qquad \mathcal{A} = \frac{t_2 - t_1(\tau_2^1)^2}{2G\, t_1(\tau_2^1)^2}\, \sin(2x^5)\,. \tag{B.4}$$

A single T–duality along x^1 yields the manifold $T^3 \times \varepsilon_2$ ($\varepsilon_2 : (-23, 13, 0)$) with O7–D5 (or D7–O5) and an H–flux given by

$$H = -\mathrm{d}\mathcal{A} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\ . \tag{B.5}$$

Note that the H-flux (B.5) allows for topologically different choices of B–field. Being not completely solvable (see Footnote 1), s 2.5 can yield manifolds of different topology (different Betti numbers). Correspondingly, the results of T–duality should vary as well, and the application of the local Buscher rules might be ambiguous. The choice of B-field in (B.5), $B = -\mathcal{A}\mathrm{d}x^1 \wedge \mathrm{d}x^2$, corresponding to the application of the local rules to (B.3), is globally defined due to $\mathcal{A}(x^5 + l) = \mathcal{A}(x^5)$. There is a less trivial choice with $B = -x^1\, \partial_5 \mathcal{A}\, \mathrm{d}x^2 \wedge \mathrm{d}x^5$ which however does not arise from the application of local T–duality rules to (B.3) since the metric does not have off–diagonal elements between x^2 and x^5.

[2] A pure spinor Ψ can always be written as

$$\Psi = e^{B+ij}\Omega_k\,, \tag{B.2}$$

where Ω_k is a holomorphic k–form, B and j are real two–forms. The degree of Ω_k is the type of the pure spinor.

A further T–duality along x^2 gives back s 2.5 with O6–D6 sources, but the supersymmetry now is captured by a different pair of pure spinors.

For the manifold $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$, the twist matrix is

$$A(x^5) = \begin{pmatrix} R(x^5) & & \\ & R(-x^5) & \\ & & \mathbb{I}_2 \end{pmatrix} \qquad R(x^5) = \begin{pmatrix} \text{ch} & -\eta\text{sh} \\ -\frac{1}{\eta}\text{sh} & \text{ch} \end{pmatrix} , \tag{B.6}$$

where we set

$$\text{ch} = \cosh(\sqrt{q_1 q_2} x^5) \ , \ \text{sh} = \sinh(\sqrt{q_1 q_2} x^5) \ , \ \eta = \sqrt{\frac{q_1}{q_2}} \ . \tag{B.7}$$

Then it is straightforward to check that the isometry vectors $v_i = \partial_i$ are local. Any T–duality along these is thus obstructed, and hence the O6–D6 solution of [29, 87] does not have geometric T–duals. For this case we shall adopt the method applied to nilmanifolds in [88], and work out the action of T–duality on the generalized vielbeine.

The generalized vielbeine on $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$ can be obtained using twist transformation (see (4.52)) from the generalized vielbeine of the torus (on which we take for simplicity the identity metric)

$$\mathcal{E} = \left(\begin{array}{c|c} \mathbb{I}_6 & 0_6 \\ \hline 0_6 & \mathbb{I}_6 \end{array} \right) \left(\begin{array}{c|c} A & 0_6 \\ \hline 0_6 & A^{-T} \end{array} \right) . \tag{B.8}$$

To work out their T–duals, we act by

$$\mathcal{E}_T = O_T \times \mathcal{E} \times O_T \ , \tag{B.9}$$

where O_T is the O(d,d) matrix for T–duality. The O_T on the right is the regular action of T–duality, while the O_T on the left assures that the map has no kernel (see [88]). The T–duality is done in the x^1 direction, so the O_T is

$$O_T = \left(\begin{array}{ccc|ccc} T_1 & & & T_2 & & \\ & \mathbb{I}_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline T_2 & & & T_1 & & \\ & 0_2 & & & \mathbb{I}_2 & \\ & & 0_2 & & & \mathbb{I}_2 \end{array} \right) \ , \ T_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \ , \ T_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \ , \tag{B.10}$$

and then

$$\mathcal{E}_T = \left(\begin{array}{ccc|ccc} C_1 & & & B_1 & & \\ & R(-x^5) & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline B_2 & & & C_2 & & \\ & 0_2 & & & R(x^5)^T & \\ & & 0_2 & & & \mathbb{I}_2 \end{array} \right) , \tag{B.11}$$

with

$$C_1 = C_2 = \text{ch}\ \mathbb{I}_2 \ , \ B_1 = -\frac{1}{\eta}\text{sh}\ \epsilon \ , \ B_2 = \eta\text{sh}\ \epsilon \ , \ \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{B.12}$$

The generalized vielbeine $\mathcal{E}_T$ can be brought to the canonical lower diagonal form (4.50) by a local $O(d)\times O(d)$ transformation. When such a transformation cannot be made single–valued, we talk about non–geometric backgrounds (where the action of a non–trivial β cannot be gauged away). The result of the $O(d)\times O(d)$ transformation is

$$\mathcal{E}' = \left(\begin{array}{ccc|ccc} O_1 & & & O_2 & & \\ & \mathbb{I}_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline O_2 & & & O_1 & & \\ & 0_2 & & & \mathbb{I}_2 & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) \times \mathcal{E}_T = \left(\begin{array}{ccc|ccc} O_1C_1+O_2B_2 & & & O_1B_1+O_2C_2 & & \\ & R_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline O_2C_1+O_1B_2 & & & O_2B_1+O_1C_2 & & \\ & 0_2 & & & R_2^{-T} & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) , \tag{B.13}$$

where the non–trivial $O(d)\times O(d)$ components are

$$O_{1/2} = \frac{1}{2}(O_+ \pm O_-) \qquad O_\pm \in O(2) \ . \tag{B.14}$$

By solving $O_1B_1 + O_2C_2 = 0$, we can obtain O_2 and express $O_\pm$ in terms of O_1:

$$\begin{aligned} &O_\pm = O_1(\mathbb{I}_2 \pm u\ \epsilon) \ , \quad u = \frac{\mathrm{sh}}{\eta \mathrm{ch}} \ , \\ &O_\pm^T O_\pm = \mathbb{I}_2 \quad \Leftrightarrow \quad O_1^T O_1 = \frac{1}{1+u^2}\ \mathbb{I}_2 \ . \end{aligned} \tag{B.15}$$

A simple solution is given by

$$O_1 = \frac{1}{\sqrt{1+u^2}}\ \mathbb{I}_2 \quad \Rightarrow \quad O_2 = \frac{u}{\sqrt{1+u^2}}\ \epsilon \ . \tag{B.16}$$

Thus we can indeed bring $\mathcal{E}_T$ to a lower–diagonal form, but with an $O(d)\times O(d)$ transformation that is not globally defined. It is not hard to see that replacing the x^1 direction by others does not change much. Hence any T–dual to $\mathfrak{g}_{5.7}^{1,-1,-1} \times S^1$ is non–geometric.

A similar analysis for s 2.5 shows that one can easily solve the constraint $O_1B_1 + O_2C_2 = 0$ with O_1 and O_2 being globally defined (this is easy since the functions entering are all periodic).

Appendix C

Lie groups

Let consider a connected and simply–connected real Lie group G of identity element e [184, 183]. We denote by H, N and Γ subgroups of G and the associated Lie (sub)algebras of G, H, N by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$ respectively. Connected and simply–connected (sub)groups are in one–to–one correspondence with the corresponding (sub)algebras. Many properties of the (sub)algebras will have their counterpart in the (sub)groups and vice versa.

The ascending series $(G_k)_{k\in\mathbb{N}}$, the descending series $(G^k)_{k\in\mathbb{N}}$ and the derived series $(D^kG)_{k\in\mathbb{N}}$ of subgroups of G are defined as

$$G_0 = \{e\}\ ,\ G^0 = D^0G = G\ ,$$
$$G_k = \{g \in G | [g, G] \subset G_{k-1}\}\ ,\ G^k = [G, G^{k-1}]\ ,\ D^kG = [D^{k-1}G, D^{k-1}G]\ ,$$

where the commutator of two group elements g and h is $[g, h] = ghg^{-1}h^{-1}$. We define in the same way the ascending, descending and derived series of $\mathfrak{g}$ or its subalgebras, by using the Lie bracket instead of the commutator, and 0 instead of e.

G is *nilpotent* respectively *solvable* if there exist $k_0 \in \mathbb{N}$ such that $G^{k_0} = \{e\}$ respectively $D^{k0}G = \{e\}$. We define the same notions for the algebra $\mathfrak{g}$ replacing 0 with e. Lie (sub)algebras corresponding to nilpotent/solvable groups are nilpotent and solvable, respectively. The converse is also true. All nilpotent Lie algebras/groups are solvable (the converse is not true).

An ideal $\mathfrak{i}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ stable under the Lie bracket: $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. Obviously $\mathfrak{i}$ is also a subalgebra. The subalgebras given in the previously defined series are all ideals.

The nilradical $\mathfrak{n}$ of the algebra $\mathfrak{g}$ is the biggest nilpotent ideal of $\mathfrak{g}$. The nilradical is unique [6, 44] as will be the corresponding subgroup N of G, also named nilradical.

To ideals of $\mathfrak{g}$ will correspond normal subgroups of G. We recall that a subgroup N is said normal if $\forall g \in G$, $gNg^{-1} \subset N$, i.e. it is invariant under conjugation (inner automorphisms). This property is necessary in order to be able to define a group structure on the quotient G/N. Note that the nilradical N of a solvable Lie group G as well as the subgroups D^kG of the derived serie are normal subgroups.

C.1 The adjoint action

Let V be a vector space over a field $\mathbb{K}$ and let $\mathfrak{g}$ be a Lie algebra over the same field. A representation of $\mathfrak{g}$ is a map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that:

1. π is linear;
2. $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$.

There is a natural representation of a Lie algebra over itself called the adjoint representation:

$$\begin{aligned} ad \ &: \ \mathfrak{g} \to \mathrm{End}(|\mathfrak{g}|) \\ & \quad X \mapsto ad(X) = ad_X \ , \end{aligned}$$

where $|\mathfrak{g}|$ means the underlying vector space of the Lie algebra $\mathfrak{g}$, $\mathrm{End}(|\mathfrak{g}|)$ the space of all linear maps on it[1], and

$$\text{for } X \in \mathfrak{g} \ , \ ad_X \ : \ \mathfrak{g} \to \mathfrak{g}$$
$$\qquad Y \mapsto ad_X(Y) = [X,Y] \ .$$

We can obtain a matrix form of the adjoint representation from the structure constants in a certain basis of the Lie algebra. Let $\{E_a\}_{a=1,\ldots,d}$ be a basis of a Lie algebra $\mathfrak{g}$, and the structure constants in that basis given by

$$[E_b, E_c] = f^a{}_{bc} E_a \ . \tag{C.1}$$

Then the matrices (a is the row index and c is the column index)

$$(M_b)^a{}_c = f^a{}_{bc} \tag{C.2}$$

provide a representation of the Lie algebra $\mathfrak{g}$.

Let G be a Lie group and let V be a (real) vector space. A representation of G in V is a map $\pi : G \to \mathrm{Aut}(V)$ such that:

1. $\pi(e) = Id$;
2. $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, $\forall\, g_1, g_2 \in G$.

There is a natural representation of the group over its algebra called the adjoint representation:

$$\begin{aligned} Ad \ &: \ G \to \mathrm{Aut}(\mathfrak{g}) \\ & \quad g \mapsto Ad(g) = Ad_g \ , \end{aligned}$$

where $Ad_g = exp^{Aut(|\mathfrak{g}|)}(ad_{X_g})$ for $X_g \in \mathfrak{g}$, $exp^G(X_g) = g$. Actually one can show the following relations between the representations:

$$\begin{array}{ccc} G & \xrightarrow{Ad} & \mathrm{Aut}(\mathfrak{g}) \\ {\scriptstyle exp^G}\uparrow & & \uparrow{\scriptstyle exp^{Aut(|\mathfrak{g}|)}} \\ \mathfrak{g} & \xrightarrow{ad} & \mathrm{End}(|\mathfrak{g}|) \end{array}$$

The map ad then turns out to be the derivation[2] of Ad. At the level of the single elements, they act according to the following diagram:

[1] These maps do not necessarily respect the Lie bracket, or in other words, are not necessarily algebra morphisms. In particular, for $X \in \mathfrak{g}$, ad_X is a derivation and thus it cannot be an algebra morphism.

[2] It is the derivative with respect to the parameters of the group element g, taken at the identity.

$$\begin{array}{ccc} g & \xrightarrow{Ad} & Ad(g) = Ad_g \\ \uparrow & & \uparrow \\ X_g & \xrightarrow{ad} & ad(X_g) = ad_{X_g} \end{array}$$

One can show as well that the derivation of the inner automorphism I_g for $g \in G$ (the conjugation) is actually the adjoint action Ad_g:

$$d(I_g) = Ad_g \ . \tag{C.3}$$

Furthermore, for $\varphi : G \to G$ an automorphism, the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ {\scriptstyle exp^G}\uparrow & & \uparrow{\scriptstyle exp^G} \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{g} \end{array}$$

A Lie group is said to be exponential (the case for us) if the exponential map is a diffeomorphism. Denoting its inverse as log^G, then we deduce

$$I_g = exp^G \circ Ad_g \circ log^G \ . \tag{C.4}$$

C.2 Semidirect products

Most of the solvable groups we are interested in are semidirect products, we recall here some definitions.

Let us consider two groups H and N and a (smooth) action $\mu : H \times N \to N$ by (Lie) automorphisms. The semidirect product of H and N is the group noted $H \ltimes_\mu N$, whose underlying set is $H \times N$ and the product is defined as

$$(h_{i=1,2}, n_{i=1,2}) \in H \times N \ , (h_1, n_1) \cdot (h_2, n_2) = (h_1 \cdot h_2, n_1 \cdot \mu_{h_1}(n_2)) \ . \tag{C.5}$$

The semidirect product of Lie algebras can be defined in a similar way. Let $\mathfrak{d}(\mathfrak{h})$ be the derivation algebra of an algebra $\mathfrak{h}$ (for instance $ad \in \mathfrak{d}(\mathfrak{g})$). Let $\sigma : \mathfrak{g} \to \mathfrak{d}(\mathfrak{h})$, $X \mapsto \sigma_X$ be a representation of the Lie algebra $\mathfrak{g}$ in $|\mathfrak{h}|$. Then we can define the semidirect product $\mathfrak{g} \ltimes_\sigma \mathfrak{h}$ of the two Lie algebras with respect to σ in the following way:

- the vector space is $|\mathfrak{g}| \times |\mathfrak{h}|$
- the Lie bracket is $[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_{\mathfrak{g}}, [Y_1, Y_2]_{\mathfrak{h}} + \sigma_{X_1}(Y_2) - \sigma_{X_2}(Y_1))$.

This provides a Lie algebra structure to the vector space $|\mathfrak{g}| \times |\mathfrak{h}|$. Note that the fact σ is a derivation is important to verify the Jacobi identity for the new bracket. The map σ is related to the map μ in the following way: $\sigma = d_{e_H}\mu_1$ where $\mu_1 : H \to \mathrm{Aut}(\mathfrak{n})$ is given by $\mu_1(h) = d_{e_N}\mu(h, \ldots) = Ad_h^{H \ltimes_\mu N}$.

If we denote $\mathfrak{g}' = \mathfrak{g} \times \{0\}$ and $\mathfrak{h}' = \{0\} \times \mathfrak{h}$ then $\mathfrak{h}'$ is an ideal of the new algebra and $\mathfrak{g}'$ is a subalgebra of it. Furthermore

$$\mathfrak{g}' + \mathfrak{h}' = \mathfrak{g} \ltimes_\sigma \mathfrak{h} \ , \ \mathfrak{g}' \cap \mathfrak{h}' = 0 \ . \tag{C.6}$$

There is a unique decomposition of an element of $|\mathfrak{g}| \times |\mathfrak{h}|$ as a sum of an element of $|\mathfrak{g}|$ and one of $|\mathfrak{h}|$, thus we can think of it as the couple in $|\mathfrak{g}| \times |\mathfrak{h}|$ or as an element of a direct sum of vector spaces.

Let us consider a Lie group G and two subgroups H and N with N normal. If every element of G can be uniquely written as a product of an element in H and one in N, then one can show that $G \approx H \ltimes_\mu N$ with μ being the conjugation[3]. This point of view will be important for us. As discussed previously, the conjugation can be given in terms of the restriction of the adjoint action of H over $\mathfrak{n}$ as in (C.4), so we are able to determine μ in terms of $Ad_H(N)$. For exponential groups, as we consider here, the corresponding Lie algebra of $G = H \ltimes_\mu N$ is then clearly $\mathfrak{g} = \mathfrak{h} \ltimes_{ad_{\mathfrak{h}}(\mathfrak{n})} \mathfrak{n}$ (we just write ad in the following for simplicity).

Let us now consider a group G with a normal subgroup N of codimension 1. The Lie algebra $\mathfrak{g}$ has two components, $\mathbb{R}$ and $\mathfrak{n}$. We want to show that $\mathfrak{g}$ is isomorphic to $\mathbb{R} \ltimes_{ad} \mathfrak{n}$, and then, as discussed, we get that $G \approx \mathbb{R} \ltimes_\mu N$ with μ the conjugation. At level of the algebra, in terms of vector spaces, the isomorphism is obviously true. What needs to be verified is that the Lie brackets coincide. The Lie bracket of two elements of $\mathbb{R}$ or of $\mathfrak{n}$ clearly coincide with those of the corresponding two elements of $\mathbb{R} \ltimes_{ad} \mathfrak{n}$. Let us now take $X \in \mathbb{R},\ Y \in \mathfrak{n}$. We have for $\mathbb{R} \ltimes_{ad} \mathfrak{n}$:

$$[(X,0),(0,Y)] = (0, 0 + ad_X(Y) - ad_0(0)) = (0, [X,Y])\,, \tag{C.7}$$

which clearly coincides with the bracket $[X,Y]$ for $\mathfrak{g}$. We can conclude that $\mathfrak{g}$ is isomorphic to $\mathbb{R} \ltimes_{ad} \mathfrak{n}$ and thus the group is isomorphic to $\mathbb{R} \ltimes_\mu N$.

[3]In particular it is the case for a group $G = H \ltimes_\nu N$ with ν being not the conjugation.

Appendix D

Solvable algebras and solvmanifolds

D.1 Algebras admitting a lattice

We present here a list of indecomposable solvable, non–nilpotent unimodular algebras that admit a lattice (at least for certain values of the parameters p, q, r, for instance those chosen in table D.2.). For dimension up to four the algebras are almost nilpotent or almost abelian. For dimension 5 and 6, only almost abelian algebras have been considered. For the other six–dimensional indecomposable algebras, we do not know if a lattice exists.

Name	Algebra	
$\mathfrak{g}_{3.4}^{-1}$	$[X_1, X_3] = X_1, [X_2, X_3] = -X_2$	alm. ab.
$\mathfrak{g}_{3.5}^{0}$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	alm. ab.
$\mathfrak{g}_{4.5}^{p,-p-1}$	$[X_1, X_4] = X_1, [X_2, X_4] = pX_2, [X_3, X_4] = -(p+1)X_3,\ -\frac{1}{2} \leq p < 0$	alm. ab.
$\mathfrak{g}_{4.6}^{-2p,p}$	$[X_1, X_4] = -2pX_1, [X_2, X_4] = pX_2 - X_3, [X_3, X_4] = X_2 + pX_3,\ p > 0$	alm. ab.
$\mathfrak{g}_{4.8}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_4] = X_2, [X_3, X_4] = -X_3$	alm. nil.
$\mathfrak{g}_{4.9}^{0}$	$[X_2, X_3] = X_1, [X_2, X_4] = -X_3, [X_3, X_4] = X_2$	alm. nil.

Table D.1: Indecomposable non-nilpotent solvable unimodular algebras up to dimension 4, that admit a lattice

Name	Algebra
$\mathfrak{g}_{5.7}^{p,q,r}$	$[X_1,X_5]=X_1,[X_2,X_5]=pX_2,[X_3,X_5]=qX_3,[X_4,X_5]=rX_4,$ $-1\leq r\leq q\leq p\leq 1\ ,\ pqr\neq 0\ ,\ p+q+r+1=0$
$\mathfrak{g}_{5.8}^{-1}$	$[X_2,X_5]=X_1,[X_3,X_5]=X_3,[X_4,X_5]=-X_4$
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$[X_1,X_5]=X_1,[X_2,X_5]=-(1+2q)X_2,[X_3,X_5]=qX_3-rX_4,[X_4,X_5]=rX_3+qX_4,$ $-1\leq q\leq 0\ ,\ q\neq -\frac{1}{2}\ ,\ r\neq 0$
$\mathfrak{g}_{5.14}^{0}$	$[X_2,X_5]=X_1,[X_3,X_5]=-X_4,[X_4,X_5]=X_3$
$\mathfrak{g}_{5.15}^{-1}$	$[X_1,X_5]=X_1,[X_2,X_5]=X_1+X_2,[X_3,X_5]=-X_3,[X_4,X_5]=X_3-X_4$
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[X_1,X_5]=pX_1-X_2,[X_2,X_5]=X_1+pX_2,[X_3,X_5]=-pX_3-rX_4,[X_4,X_5]=rX_3-pX_4,$ $r\neq 0$
$\mathfrak{g}_{5.18}^{0}$	$[X_1,X_5]=-X_2,[X_2,X_5]=X_1,[X_3,X_5]=X_1-X_4,[X_4,X_5]=X_2+X_3$

Table D.2: Indecomposable solvable unimodular almost abelian algebras of dimension 5, that admit a lattice

Name	Algebra
$\mathfrak{g}_{6.3}^{0,-1}$	$[X_2,X_6]=X_1,[X_3,X_6]=X_2,[X_4,X_6]=X_4,[X_5,X_6]=-X_5$
$\mathfrak{g}_{6.10}^{0,0}$	$[X_2,X_6]=X_1,[X_3,X_6]=X_2,[X_4,X_6]=-X_5,[X_5,X_6]=X_4$

Table D.3: Indecomposable solvable unimodular almost abelian algebras of dimension 6, for which we know a lattice exists

D.2 Six–dimensional solvmanifolds in terms of globally defined one–forms

In the following table we present the solvmanifolds that we considered in this paper. They have the form $G/\Gamma = H_1/\Gamma_1 \times H_2/\Gamma_2$, i.e. they are products of (at most) two solvmanifolds. Each of these two solvmanifolds are constructed from the algebras given in the previous Tables (see Appendix D.1) and the three–dimensional nilpotent algebra $\mathfrak{g}_{3.1} : (-23, 0, 0)$. In particular, these are indecomposable solvable algebras for which the group admits a lattice. The difference with respect to Section D.1 is that the algebras are given here in terms of a basis of globally defined forms (see discussion in Section 4.2.3). They are related by isomorphisms to the algebras given in the Tables of D.1. The fact the forms are globally defined is important for studying the compatibility of orientifold planes with the manifold and for finding solutions. For $\mathfrak{g}_{4.5}^{p,-p-1} \oplus \mathbb{R}^2$ and $\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathbb{R}^2$, we were not able to find such a basis, even if a priori we expect it to exist.

The column Name indicates the label of the algebra and the corresponding solvmanifold. The column Algebra gives the corresponding six–dimensional algebra in terms of exterior derivative acting on the dual basis of globally defined one–forms (see Section **??**). The next two columns give the O5 and O6 planes that are compatible with the manifold. The column *Sp* indicates by a ✓ when the manifold is symplectic, according to [23, 30]. Notice that the same results can be obtained as conditions for the twisted pure spinors to solve the supersymmetry equations. In particular, for the even SU(3) pure spinor $\Phi_+ = \frac{1}{8}e^{-iJ}$ the condition

$$\mathrm{d}(O_{tw})\Phi_+ = 0 \tag{D.1}$$

is equivalent to the requirement that the manifold is symplectic.

There is an additional subtlety for not completely solvable manifolds, when one looks for solutions on them. This is due to the lack of isomorphism between the cohomology groups $H^*(\mathfrak{g})$ and $H^*_{dR}(G/\Gamma)$ for not completely solvable manifolds (see Footnote 1). In other words, the Betti numbers for the Lie algebra cohomology give only the lower bound for the corresponding numbers for de Rham cohomology. When looking for e.g. symplectic manifolds, we have considered only the forms in $H^2(\mathfrak{g})$, and hence might have missed some candidate two–forms in $H^2_{dR}(G/\Gamma)$.

Name	Algebra	O5	O6	Sp
$\mathfrak{g}_{3.4}^{-1} \oplus \mathbb{R}^3$	$(q_1 23, q_2 13, 0, 0, 0, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	123, 145, 146, 156, 245, 246, 256, 345, 346, 356	✓
$\mathfrak{g}_{3.5}^{0} \oplus \mathbb{R}^3$	$(-23, 13, 0, 0, 0, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	123, 145, 146, 156, 245, 246, 256, 345, 346, 356	✓
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}^{-1}$	$(-23, 0, 0, q_1 56, q_2 46, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.5}^{0}$	$(-23, 0, 0, -56, 46, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.5}^{0}$	$(q_1 23, q_2 13, 0, -56, 46, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.4}^{-1}$	$(q_1 23, q_2 13, 0, q_3 56, q_4 46, 0) \quad q_1, q_2, q_3, q_4 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_{3.5}^{0}$	$(-23, 13, 0, -56, 46, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{4.5}^{p,-p-1} \oplus \mathbb{R}^2$	?			-
$\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathbb{R}^2$	?			-
$\mathfrak{g}_{4.8}^{-1} \oplus \mathbb{R}^2$	$(-23, q_1 34, q_2 24, 0, 0, 0) \quad q_1, q_2 > 0$	14, 25, 26, 35, 36	145, 146, 256, 356	-
$\mathfrak{g}_{4.9}^{0} \oplus \mathbb{R}^2$	$(-23, -34, 24, 0, 0, 0)$	14, 25, 26, 35, 36	145, 146, 256, 356	-
$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$	$(q_1 25, q_2 15, q_2 45, q_1 35, 0, 0) \quad q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.8}^{-1} \oplus \mathbb{R}$	$(25, 0, q_1 45, q_2 35, 0, 0) \quad q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}$	$(q_1 25, q_2 15, -q_2 r 45, q_1 r 35, 0, 0) \; r \neq 0, \; q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$	$(-25, 0, -45, 35, 0, 0)$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.15}^{-1} \oplus \mathbb{R}$	$(q_1(25-35), q_2(15-45), q_2 45, q_1 35, 0, 0) \quad q_1, q_2 > 0$	14, 23, 56	146, 236	✓
$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$	$(q_1(p25+35), q_2(p15+45), q_2(p45-15), q_1(p35-25), 0, 0)$ $r^2 = 1, \; q_1, q_2 > 0$	14, 23, 56 $p = 0$: 12, 34	146, 236 $p = 0$: 126, 135, 245, 346	✓
$\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$	$(-25-35, 15-45, -45, 35, 0, 0)$	14, 23, 56	146, 236	✓
$\mathfrak{g}_{6.3}^{0,-1}$	$(-26, -36, 0, q_1 56, q_2 46, 0) \quad q_1, q_2 > 0$	24, 25	134, 135, 456	✓
$\mathfrak{g}_{6.10}^{0,0}$	$(-26, -36, 0, -56, 46, 0)$	24, 25	134, 135, 456	✓

Table D.4: Six–dimensional solvmanifolds considered in this thesis, in terms of globally defined one–forms

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