# UNIVERSITÉ PARIS DIDEROT (PARIS 7)
# et
# INSTITUT DE PHYSIQUE THÉORIQUE
# CEA/Saclay

ÉCOLE DOCTORALE: Physique de la Région Parisienne, ED107

## Thèse de doctorat

Spécialité: **Physique Théorique**


Présentée par

**Francesco ORSI**

Thèse dirigée par: M.me **Mariana GRAÑA**


Sujet:

## ÉTUDE DE VIDES DE LA THÉORIE DES CORDES AVEC FLUX

## VACUUM CONFIGURATIONS OF STRING THEORY IN THE PRESENCE OF FLUXES

Soutenue le 2 fevrier 2012
devant le jury composé de

| | |
|---|---|
| M. Jihad MOURAD | président du Jury, |
| M.me Mariana GRAÑA | directeur de thèse, |
| M. Ruben MINASIAN | examinateur, |
| M. Dimitrios TSIMPIS | rapporteur, |
| M. Daniel WALDRAM | rapporteur. |

## Acknowledgments

The time spent in writing a doctoral thesis provides a rather unique opportunity to look backwards and gain a definitive perspective on several years of work. I would therefore like to thank the many people that contributed in different ways to the realization of this work.

My first thanks go to my supervisor Mariana Graña, for having always encouraged and directed my work, as well for our fruitful collaboration. Her gentle personality and her helpful advices in dealing with research projects, together with her generosity in sharing with me her scientific experience, permitted to continuously improve my skills while allowing the progressive development of my autonomy. Without her help, patience, and guidance the work here presented would have been impossible.

I also have to thank Iosif Bena for his support and enthusiasm, which joint with a solid optimism really helped to transform even the hard work in an enjoyable experience. Also, he has always been available to discuss my progress and to give many useful scientific advices.

I am indebted to Nick Halmagyi, as I was really lucky to meet him as a graduate student at IPhT and collaborate with him my first research results. I also learned much from Nick's attitude: I particularly remember him emphasizing in several occasions the importance of being tenacious when approaching research problems. His sincere and active interest in sharing his result and making the project to grow were invaluable during our collaboration.

I would also like to thank Gregory Giecold for sharing many thoughts and starting new projects, which ended in a very productive collaboration, and for interesting, frequent and illuminating discussions. Aside from his outstanding ability to suggest and organize research work, I am grateful to him for giving many advices to my scientific activity, thus allowing me to benefit from his very deep experience and knowledge. Finally, I owe a big debt to him for invaluable proofreading of many parts of this manuscript.

I have to thank particularly Enrico Goi: the peculiar mix of friendship and professional collaboration which I have been establishing with him is a very important part of the human experience I have gone through during these last three years. He has always been exceedingly generous in sharing his time in plenty of stimulating scientific discussions, and his support has been always at hand. I believe he's a very talented scientist, which calm, self-control and talent in facing research tasks were of invaluable guidance and example for me. I wish to him all the best for his future career and personal life.

I thank Jihad Mourad, Daniel Waldram, Dimitrios Tsimpis and Ruben Minasian to be a part of my defense Jury. I especially thank the thesis referees for their careful report of my thesis manuscript.

I wish to spend also a few words on the colleagues who have shared with me the PhD experience, and make IPhT a really friendly place. In these years I have been given several occasions to get advantage from valuable discussions and advices from many scientists. Thanks to the string theory group, year after year bigger: Hagen Triendl, Bert Vercknocke, Sheer El-Showk, Stefano Giusto, Diego Marques, Thomas Van Riet. Thanks

also to the other PhD fellows in Saclay for their friendship: Emeline Cluzel, Jean-Marie Stephan, Helene Grandclaude, Sophie Mariadassou, Clement Gombeau, Clement Ruef, Roberto Bondesan, Piotr Tourkine, Stefano Massai, Andrea Puhm, Alexandre Lazarescu. Thanks also to the PhD fellows of the many labs I met across Paris and Europe: Flavien Kiefer, Lorenzo Battarra, Alexander Bohe, Alessandra Gnecchi, Giuseppe Torri, Giuseppe Dibitetto.
Many other members of the IPhT lab have been of incredible help during my PhD experience. To have an extension of my doctorate would have been not possible without their support and active help. I am really indebted to Olivier Golinelli for his guidance in dealing with French administration, as well for his kindness and discussions about the future. Thanks are also due to Henri Orland for support and many useful advices. I also thank Consorzio Ferrara Ricerche (CFR) for funding the final part of my PhD experience. Last but not least thanks to the administrative secretaries Catherine Cataldi, Laure Saboy, Sylvie Zaffanella and Anne Capdepon.
I must thank all my friends from Bologna who have continuously shown me their appreciation, and helping me not to lose my self-confidence: Carlo Alberto, Marco, Fabrizio, Emanuele, Stefano, Gianluca.
Thanks finally to my parents and my brother, for their invaluable support throughout my entire education and life.

**Abstract**

This thesis revolves around investigating some aspects of both supersymmetric and non-supersymmetric flux vacua of type II string theory. After providing the relevant definition of a vacuum in this setting, the framework of Generalized Complex Geometry is exposed: we review in particular the differential conditions for vacua in the presence of fluxes in this language, and we discuss their relation to integrability of the associated structures. We then survey a natural extension able to include the whole flux content into a geometrical picture, known as Exceptional Generalized Geometry. Fluxes are recovered in this context as a twisting of the Levi-Civita operator, from which a set of differential equations for the relevant algebraic structures is derived. These are carefully compared with the known supersymmetric constraints that a vacuum should satisfy in both cases of $\mathcal{N} = 1, 2$ supersymmetries. Motivated by the application of the AdS/CFT correspondence with a reduced amount of supercharges, we obtain an effective five-dimensional supersymmetric theory, and we demonstrate in particular how a specific ansatz largely used in the string theory literature can be naturally embedded in it. We then investigate the supergravity dual to a metastable supersymmetry-breaking state by considering the most general first-order deformations of a supersymmetric solution, in order to single out the backreaction of anti-D2 branes. We conclude that unavoidable infrared singularities arise in view of the presence of anti-D2 branes perturbing the underlying supersymmetric background.

**Résumé court**

Cette thèse se propose de présenter une contribution à l'étude des états fondamentaux tant supersymétriques que non-supersymétriques de la théorie des cordes. Nous introduisons la définition d'état fondamental en théorie des cordes, et nous passons ensuite en revue le formalisme de la Géométrie Complexe Généralisée. Nous discutons des conditions différentielles caractérisant les états fondamentaux en présence de flux dans ce contexte, et leur relation à l'intégrabilité des structures associées. Est ensuite introduite une extension de ces constructions connue sous le nom de Géométrie Généralisée Exceptionnelle. La totalité des flux est dérivé comme une torsion de l'opérateur de Levi-Civita, à partir de laquelle un ensemble d'équations différentielles pour les structures algébriques pertinentes est dérivé. Ces derniers sont ensuite comparées avec les restrictions qu'un état du vide supersymétrique doit satisfaire en présence de $\mathcal{N} = 1, 2$ supercharges. Ayant en vue l'étude de la correspondance AdS/CFT, on obtient une théorie effective en cinq dimensions d'espace-temps, et nous montrons comment un ansatz largement utilisé en théorie des cordes apparait comme faisant partie de notre théorie effective. Nous allons ensuite étudier la solution de supergravité dual à un état brisant la supersymétrie de manière métastable, par l'identification des déformations génériques au premier ordre, de ce qui nous isolons l'effet d'anti–D2 branes. Notre analyse révèle des singularités infrarouges inévitables en raison de la réaction des anti-branes.

# Résumé détaillé

La théorie des cordes est le principal candidat en lice pour l'unification de toutes les composantes fondamentales et leurs interactions, capable de surmonter le conflit de longue date entre la gravité et la mécanique quantique. Son hypothèse de base est que les particules élémentaires correspondent à différents modes de vibration d'un seul objet fondamental, une corde. L'existence d'une longueur caractéristique pour les cordes permet une régularisation des amplitudes de diffusion pour le graviton, qui en théorie quantique des champs ordinaire présentent un comportement divergent en raison de la nature ponctuelle de l'interaction.
D'autre part, toute théorie réaliste doit nécessairement être quantifiée. Dans le cas de la théorie des cordes, la préservation de certaines symétries au niveau classique en passant au niveau quantique n'est possible que par l'introduction de dimensions supplémentaires, ce qui est clairement en contradiction avec les données expérimentales. Ce fait établit une différence évidente entre le milieu où la théorie vit et l'expérience commune. Le point de contact entre notre réalité détectable et la théorie des cordes, bien définie et sous contrôle dans des dimensions supérieures, est réalisée par une *compactification* de la théorie.
L'étude des compactifications nécessite de faire une distinction entre un espace-temps externe, maximalement symétriques et non-compact, et un espace interne nécessairement infinitésimal et compact, la taille de celui-ci étant associée à une certaine échelle communément appelée échelle de compactification (ou de Kaluza-Klein (KK) ). Le lien entre la physique ordinaire en basse dimensions et la théorie formulée en dimensions supérieures est obtenu en procédant à une réduction de l'espace intérieur, en vue d'obtenir une théorie effective définie dans les dimensions externes non compacte seulement.
Une approche très fructueuse pour étudier les compactifications consiste à traduire les restrictions imposées par la supersymétrie en un langage géométriques. Les contraintes imposées par la supersymétrie rétrécit la nature de la variété compacte: elle doit être à holonomie réduite. Le premier exemple important de la relation intime entre géométrie de l'espace interne et supersymétries préservées a été réalisée dans le papier essentiel [40]. Pour une configuration où aucun des flux ne présente une valeur moyenne non-nulle, l'espace intérieur doit nécessairement être un espace de Calabi-Yau (CY), ou, autrement dit, une variété de six dimensions compactes avec holonomie $SU(3)$.
La compactification de théories de type II sur une variété de Calabi-Yau préserve génériquement $\mathcal{N} = 2$ supersymétries. Ces espaces répondent à une condition algébrique, qui est l'existence d'un spineur globalement défini et nulle disparaître dans l'espace interne, et une condition différentielle, consistant en ce que le spineur soit constant de manière covariante. Ces deux exigences sont indépendantes, tout comme la condition algébrique est nécessaire afin de obtenir une théorie supersymétrique efficace dans les quatre dimensions, tandis que la condition différentiel est requise afin d'avoir un état fondamental supersymétrique.
L'inclusion des flux à travers des boucles dans l'espace interne fournit différents avantages. Ceux-ci peuvent par exemple conduire à une rupture partielle de supersymétrie par rapport aux $\mathcal{N} = 2$ supersymétries des compactifications de Calabi-Yau jusqu'à $\mathcal{N} = 1$

par déformation de masse [195]. Pour cette raison, les milieux dans lesquels les valeurs moyennes à vide des flux sont non-nulles deviennent intéressants tant du point de vue théorique que phénoménologique. Par ailleurs, les compactifications dans des milieux à redshift géométrique en présence de flux se sont révélées être des cadres prometteurs pour réaliser une hiérarchie d'échelles [89, 182], ainsi que des ingrédients clés pour construire des configurations non-supersymétriques [139, 140].
Même si les flux introduisent nombre de bénéfices dans la compactification, ils ont des conséquences dramatiques et indésirables sur la géométrie de l'espace intérieur. Ils induisent en effet une réaction de retour, ayant pour effet que l'espace interne n'est plus une variété de Calabi-Yau. Ceci complique considérablement l'identification d'une théorie effective à basse énergie. Cependant, en comparaison avec le cas de compactification sur un CY, la condition algébrique que doit satisfaire un état du vide demeure inchangée, tandis que la condition différentielle se trouve être plus compliquée. Afin d'atteindre une compréhension plus profonde de la structure de la théorie des cordes, et sa relation avec le monde avec un nombre inférieur des dimensions, des efforts ont été entrepris pour étudier systématiquement, et classifier, les possibles compactifications avec fluxes. A cet effet, une analyse satisfaisante exige nécessairement de nouvelles techniques mathématiques.
Nous avons déjà commenté la manière dont la physique ordinaire est dérivée de la théorie des cordes en termes d'une théorie effective, ce qui correspond à un mécanisme de réduction particulière de la théorie des dimensions supérieures sur une variété interne. Lorsque l'espace interne est une variété de Calabi-Yau, et aucun des flux n'est considéré, l'action effective en basse dimension correspond à une supergravité non-jaugée. D'autre part, lors de l'examen des réductions sur des variétés plus compliquées que la réduction sur, par exemple, les tores, la théorie réduite corresponde à une supergravité jaugée: l'inclusion des flux, et l'éventuelle non-triviale courbure de l'espace, est traduit en termes de *jaugeages* du groupe de symétrie globale de la théorie originale non-jaugé.
Le jaugeage est un mécanisme qui favorise un sous-groupe d'un groupe de symétrie globale d'une théorie de supergravité à une symétrie locale, et peut être vu comme une déformation d'une supergravité non jaugés obtenue en remplaçant les dérivés ordinaires avec des dérivés covariantes minimalement couplés. Du point de vue de la théorie dans les dimensions supplémentaires, chacune des théories effectives jaugées et non jaugées correspondent à une réduction sur espaces intérieurs qui partagent certaines propriétés communes, car ils peuvent être décrits en utilisant le même structures *algébriques*, de sorte que les deux théories font le même contenu et la même supersymétrie off-shell, mais le conditions *différentielles* sont réellement différentes [80]. Il est en effet l'altération de ce dernier, avec la présence éventuelle de flux dans l'espace interne, qui conduisent à un jaugeage de la théorie. Jaugeages résultant de la non-triviales compactifications à quatre dimensions teories ont été étudiés dans [95, 96], tandis que, plus récemment, des configurations avec cinq dimensions ont atteint beaucoup d'attention [42, 80, 86, 87, 190].
Bien que de nombreux effets physiques intéressantes pourraient être capturés par l'étude effectuée purement en dimensions inférieures, la connaissance de la théorie sous-jacente en dimensions supérieures est requis pour une analyse théorique complète. Cependant, il n'est pas une simple étape de obtenir une théorie réduite qui peut également être elevée au

niveau des dimensions supérieures. En effet, en effectuant une réduction dimensionnelle générique, il n'est pas garanti d'avance que les solutions aux équations de mouvement dans les dimensions inférieures peuvent être promus à des solutions de la théorie pas réduite. Lorsque c'est le cas de la théorie effective est dit être une *troncature cohérente*, dû au fait que les champs qui ont été inclus dans la théorie bas-dimensionnelle ne sont pas source des autres champs de la théorie en dimensions supérieures. Par conséquent, les solutions éventuelles de la théorie en réduite dimensions sont automatiquement promus à des solutions à la théorie en dix dimensions. Les troncatures cohérentes fournit une approche efficace qui prend en compte qu'un nombre fini d'états dans la théorie effective et assure, en même temps, la levée de toutes leurs solutions à la théorie en dimensions supérieures.

En dépit d'être a priori un excellent principe guidant, pas toutes les théories peuvent être étudiés en se déplaçant pour les directions tracés par la symétrie. Motivé par l'application phénoménologique de la correspondance AdS/CFT pour des configurations avec des supersymétries réduit, nous avons déjà mentionné l'importance de se concentrer dans la recherche de solutions régulières, en considérant par exemple des configurations qui brisent la supersymétrie. Dans les situations où nous ne sommes plus en mesure de faire usage d'un principe de symétrie, la déformation des solutions connues est un compromis qui permet dans de nombreux cas dans l'exploration d'une physique nouvelle. Comme point de départ, on considère une configuration régulière, obtenue en plaçant des branes régulière dans un espace plat. La géométrie et les flux peuvent être modifiés d'une manière telle que leur combinaison donne encore une solution régulière. Une prescription générale suggère de modifier l'espace transversal à la brane avec un espace Ricci-plat, et dans le même temps de modifier le ansatz pour le flux [57]. Sous des hypothèses minimales, en imposant des contributions supplémentaires pour les flux proportionnelle à une certaine forme harmonique du nouvel espace trasverse, la configuration trouvés est compatible avec l'ajout de branes fractionnelle enveloppant des boucles dans la géométrie interne au milueu de départ. La régularité de la solution finale est alors carrément liées au comportement de la forme harmonique, car il n'est en effet pas garanti, bien sûr, que la déformation conduirait à une solution régulière. Beaucoup de solutions régulières ont été trouvés en utilisant cette stratégie [54, 57].

Une conjecture récente suggère qu'une pile de anti–D3 branes branes placé à la pointe de cône déformé de la solution de Klebanov-Strassler [144] créerait une vide méta-stable [141]. La supersymétrie est brisée en ajoutant une certaine quantité d'anti-branes qui sont attirés vers le fond de la gorge: les anti-branes seront anéantie ensuite avec les brane originaires dissouses dans le flux, puis finalement la vide va carier au configuration métastable dans la théorie du champ dual. Toutefois, une analyse au premier ordre en utilisant la méthode perturbative de Borokhov et Gubser [33] montre que, en imposant des conditions aux limites dans l'infrarouge (IR) compatible avec anti-D3 branes une singularité en apparence non-physique est clairement obtenu [17]. Une enquête analogique a été réalisée dans une configuration M-théorique [13], pour lequel un état métastable a été proposé de se poser, basée sur des calculs dans l'approximation de la sonde [143] similaire a celui discuté dans [141]. Le résultat obtenu confirme un comportement divergent de la solution du retour réagi dans la région infrarouge. Dans le cas Klebanov-Strassler la singularité a une action finie, tandis que dans l'analyse de M-théorie, il se révèle être plus sévère parce que l'action n'est pas finie dans l'IR. Un débat sur l'interprétation des singularités est en cours, la question étant de savoir si ces ont une signification physique ou non.

## Aperçu de la thèse

Dans le chapitre 2 nous collectons certaines conventions standard de la théorie de cordes de type II, que leurs contenu bosonique et fermionique, les actions en dix dimensions, les identités de Bianchi et l'équation du mouvement pour les deux types IIA et IIB. Nous écrivons explicitement les variations supersymétriques pour les champs fermioniques dans une formulation démocratique. En faisant un choix pour la compactification nous donnons notre définition du vide de la theorie et nous énonçons un résultat utile, qui simplifie les conditions que une configuration devrait satisfaire pour être une vide. Comme dernière étape, nous présentons la décomposition de la contrainte de supersymétrie pour un fractionnement particulier de l'espace-temps, dans le quel nous avons seulement quatre dimensions non-compactes. Ceux-ci seront le point de départ pour la reformulation géométriques présentées dans les chapitres suivants.
Dans le chapitre 3 nous passons en revue le cadre moderne de compactification dans la langue de Géométrie Complexe Généralisée, introduisant d'abord le formalisme mathématique nécessaire, pour ensuite passer dans son application à l'étude des milieux physiques en présence de flux. Nous allons commencer par introduire la notion de $G$-structure, ce qui permettra de presenter la classification standard des structures $SU(3)$ en utilisant les classes de torsion. Après avoir commenté deux exemples pertinents de compactifications avec et sans flux, nous introduisons le concept de structure presque complexe généralisée. Nous discutons de la manière dont une paire compatible de structures presque complexes généralisées définit une métrique et un champ $B$, et fournit ainsi une géométrisation du contenu de champ jauge NS-NS des théories de type II. La correspondance importante entre une structure presque complexe généralisée et un spineur pur est ensuite présenté. Conditions d'intégrabilité sont discutés dans les deux langues des structures complexes généralisés et de spineurs purs, et la partie conclusif du chapitre est consacrée à formuler des conditions différentiels pour le vides en présence de fluxes de contenu générique, à partir du cas de base où le flux est abstent, pour arriver a le cas le plus général, dont les contraintes différentielles sont les équations des spineurs purs.
Dans le chapitre 4, nous utilisons le langage de la Géométrie Généralisée Exceptionnelle pour proposer une généralisation des équations décrivant le compactification dans une espace externe $\text{Mink}_4$ pour les deux cases correspondant à $\mathcal{N} = 1$ [1] et $\mathcal{N} = 2$ [2] supersymétries préservée, respectivement. Premièrement, nous examinons comment la U-dualité émerge comme une symétrie à partir du point de vue de la supergravité, pour ensuite illustrer les aspects théoriques de base de groupe de $E_{7(7)}$. Nous discutons ensuite la construction des structures algébriques $(L, K_a)$ pertinente pour la discussion des conditions de vides dans ce formalisme. Après avoir introduit un opérateur différentiel approprié par le biais d'un dérivé torsadée, nous allons explicitement obtenir les équations qui décrivent un vide $\mathcal{N} = 1$. Dans ce qui suit, une comparaison détaillée avec la supersymétrie est réalisée. L'approche est également généralisé pour $\mathcal{N} = 2$ supersymétries, pour lequel un ensemble équivalent d'équations est élaboré, et une discussion sur leur relation avec la supersymétrie est faite. Nous concluons ce chapitre en commentant la relation entre les équations tordues et l'éventuelle intégrabilité des structures pertinentes

dans le contexte de la géométrie exceptionnelle.
Dans le chapitre 5 on obtient une troncature supersymétriques cohérente sur l'espace $T^{1,1}$, et nous montrent comment l'ansatz de Papadopoulos et Tseytlin, qui a été utilisé comme une théorie unidimensionnelle qui contient quelques solutions coniques de la theorie IIB, peut être naturellement dérivé comme une troncature supplémentaire de cette théorie [3]. Après avoir discuté quelques raisons phénoménologique pour se concentrer sur des configurations coniques en théorie des cordes, un mécanisme de résolution des singularités basé sur une déformation d'un configuration avec brane ordinaires à une configuration en ajoutant supplémentaires branes fractionnaire est présenté. Ensuite, on illustrent deux exemples explicites de solutions, le vide de Klebanov-Strassler et le vide Maldacena-Nuñez. Nous présentons aussi l'interpolation proposée par Papadopoulos et Tseytlin (PT). Les bases de supergravities jaugé sont alors présentés, et une attention particulière sera consacrée à la supergravité jaugé $\mathcal{N} = 4$ en cinq dimensions. Après avoir examiné les progrès récents sur la troncature sur les espaces de Sasaki-Einstein en cinq dimensions, nous spécialisons ces pour le cas de l'espace $T^{1,1}$. On déduit une théorie efficace cinq-dimensionnelle compatible avec une supergravité jaugé $\mathcal{N} = 4$ couplées à trois multiplets vectoriels. Nous allons enfin montrer comment la théorie peut être tronqué à l'ansatz de PT, et nous concluons en discutant d'autres troncatures supersymétriques.
Dans le chapitre 6 nous étudions l'espace de déformations linéarisé non supersymétriques autour d'une configuration supersymétrique. On se concentre en particulier sur la solution de supergravité IIA duale à un état métastable qui brise la supersymétrie compatible avec la présence d'anti-branes [4]. Nous présentons d'abord le fond qui sera examiné l'ordre zéro de la procédure perturbatif, donnent un aperçu de la dérivation de la superpotentiel et en présentant la méthode de perturbation générale développée par Borokhov et Gubser. L'ensemble complet des équations du premier ordre pour les scalaires $\phi^a$, ce qui paramétrer la milieu, ainsi que celui de leurs variables duales $\xi_a$, sont dérivés. Nous allons montrer comment, avec l'utilisation de ce dernier seulement, la force exercée sur une brane D2 utilisé comme sonde peut être calculée. Solutions analytiques sont explicitement trouvé pour le $\xi_a$ système, mais tout cela n'est pas possible dans le pour le $\phi^a$, pour lequelles nous proposons une solution perturbative dans la région IR. Nous allons ensuite donner une interprétation physique de la solution perturbée d'abord par l'étalonnage de notre résultat sur une pile de branes BPS D2, pour faire voire que dans le cas de l'anti-D2 branes un comportement singulier se pose inévitablement. Nous terminons le chapitre par l'examen du récent débat sur la nature de ces singularités.
Dans le chapitre 7 les résultats de la thèse sont examinés, et nous tirerons nos conclusions. On joigne les détails techniques dans les diverses Appendices.
Dans l' Appendice A on rassemble nos conventions sur les formes différentielles, la décomposition et la dualisation de Hodge.
Appendice B contient des aspects techniques du groupe $E_{7(7)}$, comme les représentations qui sont utilisés dans le texte principal, ainsi que les produits tensoriels concernés, tant dans les décompositions groupe $SL(2,\mathbb{R})\times O(6,6)$ et $SL(8,\mathbb{R})$. Les équations tordues pour des structures appartenant à la représentations fondamentale et adjointe sont présentés. Nous rapportons les variations supersymétriques et les preuves respectives de l'équivalence

entre les équations tordues proposées au chapitre 4 et la supersymétrie à la fois pour le $\mathcal{N} = 1$ et les $\mathcal{N} = 2$ cas.
Appendice C contient une révision standard de l'espace $T^{1,1}$, et le dictionnaire nécessaire entre la notation utilisée dans la literature (Maldacena-Nuñez, Klebanov -Strassler et Papadopoulos-Tseytlin) et la notation dans laquelle la troncature du chapitre 5 est effectuée.
Appendice D contient les détails de la troncature travaillé dans le chapitre 5, comme la réduction de la cinq forme, les équations de mouvement complète pour les trois formes et de la dilaton-axion de la théorie réduite, dont une partie sont pertinentes pour la subtroncature à la théorie efficace proposée par Papadopoulos-Tseytlin, ainsi que le dictionnaire de le scalaires entre notre théorie efficace et celle qui est utilisées pour leur ansatz.

# Contents

# Chapter 1

# Introduction

String theory is the main candidate theory for the unification of all the fundamental matter and interactions, able to overcome the long-standing conflict between gravity and quantum mechanics. Its basic assumption is that elementary particles correspond to different vibrational modes of a single fundamental object, a string. The existence of a characteristic length for strings permits an ultraviolet regularization of the graviton scattering amplitudes, which in quantum field theory have a divergent behavior due to the point-like nature of the interaction.

## 1.1 The compactification mechanism

Every realistic theory must necessarily be quantized. In the string theory case, preservation of some classical string symmetries at the quantum level is possible only by introducing extra dimensions, which is clearly in contradiction with the observational evidence. This fact establishes an evident discrepancy between the setting where the theory lives and common experience. The contact point between our detectable reality and string theory's higher dimensional nature is realized by *compactifying* the theory. To study a compactification amounts to investigating the theory on a geometric background which distinguishes between an external, maximally symmetric and non compact space-time, and an internal manifold necessarily small and compact, the size of it giving some scale usually referred to as compactification (or KK) scale . The connection between ordinary low dimensional physics and the higher dimensional theory is achieved by performing a reduction over the internal space, to recover an effective theory living in the external non-compact space only.

Unfortunately, the compactification mechanism introduces a high amount of ambiguity, as string theory allows in principle for many different choices of the internal manifold. As a consequence, the obtained effective theory strongly depends on the latter. In our present understanding string theory indeed allows for a vast number of metastable four-dimensional vacua, whose set of equally probable universes is often called the string Landscape. Out of it, one has to find vacua which have all the physics we currently observe (such as particle content, interactions and cosmological observables), a task which has not

been achieved yet.
It is a common request to ask a candidate four-dimensional configuration to preserve some supercharges in the compactification, on one hand for phenomenological reasons, and on the other hand because the study of supersymmetric backgrounds turns out to be simpler than non-supersymmetric ones. A very fruitful approach to compactifications is to translate the supersymmetry requirements in terms of geometric conditions. The constraints imposed by supersymmetry restrict the nature of the compact manifold: it must have reduced holonomy. The very first important example of the intimate relation between internal space geometry and preserved supersymmetries has been carried out in the seminal paper [40]. For a background configuration where none of the fluxes has an expectation value different from zero, the internal space has necessarily to be a Calabi-Yau manifold (CY), *i.e.* a six dimensional compact manifold with SU(3) holonomy. Compactification of type II theories on a Calabi-Yau manifold generically preserves $\mathcal{N} = 2$ supersymmetries. Such spaces satisfy an *algebraic* condition, namely the existence of a globally defined and nowhere vanishing internal spinor, and a *differential* one, that the spinor is covariantly constant. These two conditions come separately, as the algebraic condition is necessary in order to recover a supersymmetric effective *theory* in four dimensions, while the differential one is required in order to have supersymmetric *vacua*.
After compactification has been performed, one would require to obtain a realistic spectrum. Unfortunately, the reduced theory features massless neutral scalar fields whose VEV parameterizes the undetermined shape and size of the internal manifold. The stabilization of these apparently unconstrained quantities, generically named moduli, represents one of the most reknowed issues of string theory, as massless (or nearly massless) gravitationally coupled scalars would generate long range interactions that have been excluded by fifth force experiments. It then becomes of primarily importance to find a mechanism to generate a potential for them, in such a way that they acquire a mass and are not dynamical in the low energy action. Moreover, as supersymmetry is broken in our world, moduli stabilization should admit or even generate supersymmetry breaking.

## 1.2 Flux compactifications

Inclusion of fluxes through non-trivial cycles of the internal manifold provides a mechanism to make the moduli very massive. For this reason, backgrounds where non vanishing expectation values for the fluxes are turned on became rapidly interesting both from the theoretical and the phenomenological point of view.
Fluxes may also lead to a partial supersymmetry breaking of the original $\mathcal{N} = 2$ supersymmetry of Calabi-Yau compactifications down to $\mathcal{N} = 1$ by mass deformation [195]. Warped flux compactifications have turned out to be promising frameworks to allow for a hierarchy of scales [89, 182], as well as key ingredients to construct non-supersymmetric configurations [139, 140].
Even though fluxes introduce many benefits in the compactification, they have drastic consequences on the geometry of the internal space. They indeed induce a backreaction,

causing the internal space not to be a Calabi-Yau anymore. This considerably complicates the identification of a low-energy effective theory. However, it turns out that, in comparison with the CY case, the algebraic condition a vacuum have to satisfy stays intact, but the differential one becomes more intricate. In order to achieve a deeper understanding of the structure of string theory, and its relation with the lower dimensional world, efforts have been made to systematically study, and therefore classify, the possible compactifications with fluxes. To this purpose, a satisfactory analysis necessarily requires new mathematical techniques.

## 1.3 Realistic vacua

The discovery of D-branes allowed string theory to become a suitable setting for describing the properties of gauge theories, as these live on their world volume. The role of D-branes is therefore to provide non-Abelian gauge symmetries and chiral matter. On the other hand, they constitute the sources for R-R gauge fields. All D-brane supergravity solutions are examples of warped metrics due to the backreaction of the branes. Braneworld settings with flux compactification provide us natural toy models unifying aspects of string theory and cosmology. Particular attention was caught by the five-dimensional example suggested by Randall and Sundrum [182], in which gravity is localized on a four-dimensional brane, while the fifth dimension is infinitely extended. This is only possible because of an exponential warp factor in the metric.
D-branes also play a fundamental role in the Maldacena conjecture [164], which argues the exact equivalence between four dimensional $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory compactified on $AdS_5 \times S^5$.
Quite recently there has been great insight in the study of more realistic gauge theories which are less supersymmetric and non conformal. In this case an exact duality has not been established, but perturbative and non perturbative properties of the gauge theories living on the world-volume of D-branes were studied using their supergravity descriptions, and viceversa. Gravity duals of $\mathcal{N} = 1$ super Yang-Mills have particular phenomenological interest. At the present day only two examples of solutions of this type which are regular in the IR are known[1]: the one corresponding to a D5 brane wrapped on a nontrivial two-cycle of a Calabi-Yau space found by Maldacena and Nuñez (MN) [165] and the one corresponding to regular and fractional D3 branes wrapping the deformed conifold proposed by Klebanov and Strassler [144]. The second solution is also interesting for cosmological reasons, as it represents the starting point for the construction of dS solution proposed in [139]. Despite the two geometries look similar, these are actually different, the first case being a conformal Calabi-Yau manifold (the warped deformed conifold), and the second a member of the complex non-Kähler class. The aforementioned resemblance of the two configurations led to conceive an interpolating ansatz for the metric and a number of fluxes of type IIB theory, originally proposed by Papadopoulos and Tseytlin

[1] At least based on a $SU(3)$ structure. For instance there is indeed another known configuration, known as Polchinski-Strassler solution [179], which corresponds to an $SU(2)$ structure.

(PT) [178], capable of interpolating between the two. At the time of its proposition, no clear understanding of its consistency as a truncation was known and so it has been for almost ten years: it simply fits as a a generalized description which has endpoints in some known non-singular conifold solutions.

## 1.4 Effective theories and consistent truncations

In the compactification scheme discussed at the beginning of this Chapter we argued how the lower dimensional relevant physics is described in terms of an effective theory, which corresponds to a particular reduction mechanism of the higher dimensional theory on a given internal manifold. When the internal manifold is Calabi-Yau, and no fluxes are turned on, the lower-dimensional effective action corresponds to an un-gauged supergravity. On the other hand, when considering reductions on more complicated manifolds, such as twisted tori, the reduced theory is found to correspond to a gauged supergravity: the inclusion of fluxes, and the eventual non-trivial curvature of the manifold, is translated in terms of *gaugings* of the global symmetry group of the original un-gauged theory.
Gauging is a mechanism which promotes a subgroup of a global symmetry group of a supergravity theory to a local symmetry, and can be seen as a deformation of an un-gauged supergravity obtained by replacing ordinary derivatives with a minimally coupled covariant derivative. From an higher dimensional perspective, gauged and ungauged effective theories correspond to a reduction on specific internal manifolds which share some common properties, as these can be described using the same *algebraic* structures, so that the two theories feature the same field content and the same off-shell supersymmetry, but the *differential* conditions the two satisfy are actually different [80]. It is indeed the alteration of the latter, together with the eventual presence of background flux, which lead to a gauging of the theory. Gaugings arising from non-trivial compactifications to four dimensional teories have been investigated in [95,96], while more recently five dimensional cases have reached attention [42, 80, 86, 87, 190].
Although many interesting physical effects could be captured by studying purely lower-dimensional models, the knowledge of the underlying higher dimensional theory is required for a complete theoretical analysis. However, it is not an easy task to recover a reduced theory which can also be lifted to the higher dimensional level. Indeed by performing a generic dimensional reduction it is not a priori guaranteed that solutions to the lower dimensional equations of motion can be promoted to solutions of the non-reduced theory. When this is the case the effective theory is said to be a *consistent* truncation, as the fields that have been included in the lower-dimensional theory do not source other fields in the higher dimensional supergravity theory. Therefore, eventual solutions of the lower-dimensional theory are automatically promoted to solutions to the higher-dimensional theory. Consistent truncations provide an efficient approach which takes into account only a finite number of states in the effective theory and ensures at the same time the lift of all their solutions to the higher dimensional theory.

## 1.5 The role of symmetry

Most of our contemporary knowledge of physical theories has been improved by the systematic use of symmetry arguments. Recently, the perspective of translating necessary and sufficient conditions to have a vacuum configuration in a geometric language improved by translating at geometrical level some symmetries that the theory possess. T-duality has been one of the first symmetries of string theory to be discovered. A geometric realization of T-duality is possible when considering the sum of tangent and cotangent bundle of a $d$-dimensional manifold, as its structure group coincides with $O(d, d)$, which is indeed the T-duality group of type II theories compactified on $T^d$. Such a construction has been termed Generalized Geometry [105, 120–122], and has been largely developed expecially for the type II context [94, 98–100, 151, 169, 169, 196], although progresses in extending it to heterotic models are being made [9]. A generalized metric on the generalized tangent bundle encodes the metric and $B$-field of the manifold, which are exchanged by T-duality. Therefore, the theory geometricaly encodes the NS-NS sector of type II theories. Recent progress towards the geometrization of the R-R fields as well has proposed an extension of the generalized tangent bundle which includes higher exterior powers of the tangent and cotangent bundle [97, 128, 176]. In this case, the natural group acting on this extended bundle is not just $O(d, d)$, but the full U-duality group, which for $d = 6$ is $E_{7(7)}$.
Symmetry comes in help also in the construction of effective theories which are obtained by reduction on a particular manifold. A sufficient, but not necessary, condition for a truncation to be consistent may be seen as a natural consequence of symmetry [66]. Given a theory invariant under a symmetry group $G$, by considering some subgroup $K$ of $G$ it is possible to recover a consistent truncation by retaining the complete singlet content under the action of $K$[2]. Being this argument totally general, it may as well be applied to supergravity theories. In the modern language of $G$-structures, given an internal manifold with *reduced* structure group $G$, which as we will explain in Chapter 3 is equivalent to ask that globally defined and nowhere vanishing tensors can be defined on it, only $G$-singlets are used in constructing the corresponding ansatz on which the theory is truncated. The latter is obtained by expanding the internal fields in terms of the invariant tensors, and then by performing dimensional reduction. Clearly, the singlet modes could never source the truncated non-invariant modes. Furthermore, the reduction of a theory based on a $G$-invariant ansatz guarantees the possibility of dropping the dependence on the internal coordinates from the higher dimensional Lagrangian [65,66]. Many recent developments in effective theories make use of this argument for ensuring the consistency of the reduction mechanism [42, 80].

[2] Notice that all the $K$ singlet fields must in general be retained for this to be true.

## 1.6 Deforming solutions

Despite being an an excellent guiding principle, not all of the theories can be investigated by moving along paths traced by a symmetry principle. Motivated by the phenomenological application of the AdS/CFT correspondence in a setting with reduced supersymmetries, we already mentioned the importance to focus in searching for regular solutions. In situations where we are not anymore able to make use of a symmetry principle, deforming known solutions is a compromise which in many cases helps in exploring new physics. Consider we start from a regular brane background. Both the geometry and the fluxes may be modified in a way such that their combination gives once more a regular solution. A general prescription suggests to modify the transverse space of the brane solution with a Ricci-flat space, and at the same time to modify the ansatz for the fluxes [57]. Under mild assumptions, by imposing the extra contributions for the fluxes to be proportional to some harmonic form of the new trasverse space is compatible with the addition of fractional branes wrapping cycles in the internal geometry. The regularity of the solution is then straightforwardly related to the behavior of the harmonic form, as it is not indeed guaranteed, of course, that the deformation would lead to a regular solution. Many regular solutions have been found using this strategy [54, 57], which is reminiscent of the original construction of the Klebanov-Strassler solution.
Deformation could as well be explicitly used to recover some more realistic vacua like supersymmetry breaking configurations. A recent conjecture states that a stack of anti-D3 branes at the tip of the warped deformed conifold of the KS solution would create a meta-stable vacuum and ultimately decay by brane-flux annihilation to the BPS vacuum [141]. Supersymmetry is broken by adding a certain amount of anti-branes which are attracted to the bottom of the throat: the anti-branes annihilate then with the positive brane-charge dissolved in flux, then finally decay to the metastable vacuum in the dual field theory description. However, an analysis at first order using the perturbative method of Borokhov and Gubser [33] shows that by imposing IR boundary conditions compatible with anti-D3 branes a seemingly unphysical singularity arises [17]. An analogue investigation has been performed in an M-theoretical setup [13], for which a metastable state has been proposed to arise, based on calculations in the probe approximation [143] similar to the one discussed in [141]. The result obtained confirms a divergent behavior of the backreacted solution in the infrared region. In the Klebanov-Strassler case the singularity has finite action, while in the M-theory analysis it turns out to be more severe because as the action is not well behaved in the IR. A debate concerning the physical meaning of the singularities is being made, the intriguing question being whether these have physical meaning or not.

## 1.7 Outline of the thesis

The aim of this thesis is to expose advances in the analysis of vacua configurations with fluxes. We will illustrate progresses both in obtaining a formal description of the conditions a vacuum should satisfy, an explicit derivation of a five dimensional effective theory corresponding to a $\mathcal{N} = 4$ gauged supergravity and finally a perturbative analysis of the backreaction induced by anti-branes on a supersymmetric configuration.
In Chapter 2 we collect some standard conventions of type II string theories, as their bosonic and fermionic content, the action, the Bianchi identities and the equation of motion for both type IIA and type IIB. We write explicitly the supersymmetric variation for the fermionic fields in a democratic formulation for type II theories. By making a compactification ansatz we give our definition of vacuum and we enunciate a useful result which simplifies the conditions which a vacuum configuration should satisfy. As a last step, we present the decomposition of the supersymmetry constraint for a particular spacetime splitting. These will be the starting point for the geometrical reformulation presented in the later Chapters.
In Chapter 3 we review the modern framework of compactification in the language of Generalized Complex Geometry, first introducing the necessary mathematical formalism, for then moving into its application to the investigation of physical backgrounds in presence of fluxes. We will start by introducing the concept of $G$-structure manifolds, which will allow to outline the standard classification of $SU(3)$ structures by means of torsion classes. After commenting two relevant examples of compactifications without and with fluxes, we introduce the concept of generalized almost complex structure. We discuss how a compatible pair of generalized complex structures defines a metric and a $B$-field, and thus provides a geometrization of the NS-NS gauge field content of type II theories. The important correspondence between a generalized almost complex structure and a pure spinor is then presented. Integrability conditions are discussed in both generalized complex structure and pure spinor languages, and the concluding part of the the Chapter is devoted to formulate differential conditions for vacua in presence of a generic flux content, starting from the basic case where flux is abstent, to conclude with the most general case, for which the constraints are the pure spinor equations.
In Chapter 4 we use the language of Exceptional Generalized Geometry to propose a generalization of the vacua equations describing compactification to $\text{Mink}_4$ for both $\mathcal{N} = 1$ [1] and $\mathcal{N} = 2$ [2] preserved supersymmetries. We first review how U-duality emerges as a symmetry from the supergravity point of view, for then illustrate the basic group theoretical aspects of $E_{7(7)}$. We then discuss the construction of the algebraic structures $(L, K_a)$ relevant for the discussion of vacuum conditions in this formalism. After having introduced a suitable differential operator by means of the twisted derivative, we explicitly derive the equations for $\mathcal{N} = 1$ vacua. In what follows, a detailed comparison with supersymmetry is performed. The approach is also generalized to $\mathcal{N} = 2$ vacua conditions, for which an equivalent set of equations is worked out, and an analogue discussion about their relation with supersymmetry is made. We conclude the Chapter by commenting the relation between the twisted equations and the eventual integrability of the relevant

structures in the Exceptional Generalized Geometry setting.
In Chapter 5 we obtain a supersymmetric consistent truncation on the space $T^{1,1}$, and we illustrate how the PT ansatz, which was, as discussed above, nothing but an effective one-dimensional theory which contains some known solutions, can be naturally embedded in it [3]. After discussing some phenomenological reasons to focus on conical configurations, a singularity resolution mechanism based on a deformation of an ordinary brane background to a configuration featuring additional fractional branes is presented. We then illustrate two explicit examples of solutions, the Klebanov-Strassler and the Maldacena-Nuñez ones. In the following we outline the interpolation proposed by Papadopoulos and Tseytlin. The basics of gauged supergravities are presented, and special attention will be devoted to $\mathcal{N} = 4$ gauged supergravity in five dimensions. After reviewing recent advances on truncation on five dimensional Sasaki-Einstein manifolds, we specialize these to the $T^{1,1}$ case. We recover an effective five dimensional theory which can be matched to $\mathcal{N} = 4$ gauged supergravity in five dimensions coupled to three vector multiplets. We will show how the theory can be truncated to the PT ansatz, and we conclude by discussing other relevant supersymmetric truncations.
In Chapter 6 we investigate the space of linearized non-supersymmetric deformations around a supersymmetric IIA configuration, and focus on the supergravity solution dual to a metastable supersymmetry breaking state compatible with the presence of anti-branes [4]. We first introduce the background which will be considered the zeroth-order of the perturbation procedure, give an outline of the derivation of the superpotential and present the general perturbation method developed by Borokhov and Gubser. The complete set of first order equations for the scalars $\phi^a$ which parameterize the background, as well as the one for their dual variables $\xi_a$, are derived. We will show how, with the use of the latter only, the force on a probe D2 brane can be computed. Analytic solutions are explicitly found for the $\xi_a$ system, while this is not possible in the for the $\phi^a$ set, for which we propose a perturbative solution in the IR region. We then give a physical interpretation of the perturbed solution first by calibrating our result on a stack of BPS D2 branes, for arguing that in the case of anti-D2 branes an unavoidable singular behavior arises. We end the Chapter by reviewing the recent debate on the nature of these singularities.
In Chapter 7 the results of the thesis are reviewed, and we state our conclusions.
Appendix A collect our conventions on differential forms, spinor decomposition and Hodge dualization.
Appendix B contains technical aspects of the group $E_{7(7)}$, such the representations that are used in the main text as well as the relevant tensor products, both in the group decompositions $SL(2, \mathbb{R}) \times O(6, 6)$ and $SL(8, \mathbb{R})$. The twisted equations for structures belonging to the fundamental and the adjoint representations are presented. We report the supersymmetric variations and the respective proofs of the equivalence between the twisted equations proposed in Chapter 4 and supersymmetry both for the $\mathcal{N} = 1$ and the $\mathcal{N} = 2$ case.
Appendix C contains a standard review of the $T^{1,1}$ geometry, together with the necessary dictionary between the notation used in the a variety of papers (Maldacena-Nuñez, Klebanov-Strassler and Papadopoulos-Tseytlin) and the notation in which the truncation

of Chapter 5 is performed.
Appendix D features relevant details of the truncation worked out in Chapter 5, such the reduction of the five form, the complete equations of motion for the three forms and for the dilation-axion of the reduced theory, part of which are relevant for the subtruncation to Papadopoulos-Tseytlin, as well as the dictionary of the scalars between our effective theory and the ones used for the Papadopoulos-Tseytlin ansatz in [178].

The thesis is based on the following papers

[1] M. Graña, F. Orsi ,
"$\mathcal{N} = 1$ vacua in Exceptional Generalized Geometry,"
JHEP **1108**, (2011) 109
e-Print: ArXiv: hep-th/1105.4855

[2] M. Graña, F. Orsi ,
"$\mathcal{N} = 2$ vacua in Exceptional Generalized Geometry,"
in progress.

[3] I. Bena, G. Giecold, M. Graña, N. Halmagyi, F. Orsi ,
"Supersymmetric Consistent Truncations of IIB on $T^{1,1}$,"
JHEP **1104**, (2011) 021
e-Print: ArXiv: hep-th/1008.0983

[4] G. Giecold, E. Goi, F. Orsi ,
"Assessing a candidate IIA dual to metastable supersymmetry-breaking,"
e-Print: ArXiv: hep-th/1108.1789
Submitted to JHEP

and on the following proceeding:

[5] F. Orsi ,
"String vacua in exceptional generalized geometry,"
Nucl. Phys. Proc. Suppl. **216** (2011) 257-259.

# Chapter 2

# Overview of type II supergravity

The entire work exposed in this thesis deals with the study of type II string vacua. Type II theories are supersymmetric, their low-energy limit being the unique and maximally symmetric supergravity theories corresponding to $\mathcal{N} = 2$ supersymmetries in ten dimensions, which differ in the relative chirality of the supersymmetry parameters (opposite for IIA, same for IIB).

## 2.1 Theory content

We look separately at the bosonic and fermionic content of type II string theories.
The fermionic fields are: two gravitini $\psi_M^A$, $A = 1, 2$ of spin-$\frac{3}{2}$, of opposite chirality in IIA, and same chirality for type IIB. Furthermore there are two spin-$\frac{1}{2}$ dilatini $\lambda^A$, which have opposite chiralities with respect to the previously mentioned gravitini $\psi_M^A$.
The bosonic content further splits in the NS-NS sector and the R-R sector. In the NS-NS sector, universal for type II theories, we have

$$\begin{aligned} \text{Scalar field (dilaton)} &: \phi \\ \text{A metric (symmetric two-tensor)} &: g_{MN} \\ \text{An antisymmetric tensor} &: B_{MN} \end{aligned}$$

being $M, N, \cdots = 0, \ldots 9$. ten-dimensional indices, while in the R-R sector we have a collection of even/odd gauge field potentials, depending on the theory we are considering

$$\begin{aligned} \text{Type IIA:} \quad & C_1^{(10)}, C_3^{(10)}, \\ \text{Type IIB:} \quad & C_0^{(10)}, C_2^{(10)}, C_4^{(10)}. \end{aligned}$$

## 2.2 Ten-dimensional actions, equations of motion and Bianchi identities

We present here the bosonic sector of the ten-dimensional actions of type IIA and IIB supergravities in Einstein frame, following the conventions of [119], supplied by their equations of motion and Bianchi identities.
Starting from type IIA, the action reads

$$S_{\text{IIA}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{10}}R - \frac{1}{4\kappa^2}\int \big(d\Phi\wedge *_{10}d\Phi + g_s e^{-\Phi}H_3\wedge *_{10}H_3 + g_s^{1/2}e^{3\Phi/2}F_2\wedge *_{10}F_2 + g_s^{3/2}e^{\Phi/2}\tilde{F}_4\wedge *_{10}\tilde{F}_4 + g_s^2 B_2\wedge F_4\wedge F_4\big) \tag{2.1}$$

where

$$H_3 = dB_2\,, \tag{2.2}$$

and we used the following notation[1]

$$\tilde{F}_4 = F_4 - C_1\wedge H_3\,,\quad F_4 = dC_3\,,\quad F_2 = dC_1\,. \tag{2.3}$$

The equations of motion reads [183]

$$d*d\Phi = -\frac{g_s e^{\Phi}}{2}H_3\wedge *_{10}H_3 + \frac{3g_s^{1/2}e^{3\Phi/2}}{4}F_2\wedge *_{10}F_2 + \frac{g_s^{3/2}e^{\Phi/2}}{4}\tilde{F}_4\wedge *_{10}\tilde{F}_4\,, \tag{2.4}$$

$$d(e^{3\Phi/2} *_{10} F_2) = e^{\Phi/2}g_s H_3\wedge *_{10}\tilde{F}_4\,, \tag{2.5}$$

$$d(e^{\Phi/2} *_{10} \tilde{F}_4) = -g_s^{1/2}F_4\wedge H_3\,, \tag{2.6}$$

$$d(e^{-\Phi} *_{10} H_3 + g_s^{1/2}e^{\Phi/2}C_1\wedge *_{10}\tilde{F}_4) = \frac{g_s}{2}F_4\wedge F_4 \tag{2.7}$$

$$\begin{aligned} R_{MN} &= \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{g_s e^{-\Phi}}{4}(H_M{}^{PQ}H_{NPQ} - \frac{1}{12}G_{MN}H^{PQR}H_{PQR}) \\ &+ \frac{3g_s e^{3\Phi/2}}{2}\big(F_M{}^P F_{NP} - \frac{1}{16}G_{MN}F^{PQ}F_{PQ}\big) \\ &+ \frac{3g_s e^{\Phi/2}}{12}\big(\tilde{F}_M{}^{PQR}\tilde{F}_{NPQR} - \frac{3}{32}G_{MN}\tilde{F}^{PQRS}\tilde{F}_{PQRS}\big)e^{\Phi/2}\,. \end{aligned} \tag{2.8}$$

The Bianchi identities for the R-R fluxes read

$$d\tilde{F}_4 = -F_2\wedge H_3\,, \tag{2.9}$$

$$dF_2 = 0\,. \tag{2.10}$$

while for the NS-NS we have

$$dH_3 = 0\,. \tag{2.11}$$

[1] In this subsection, as we are presenting the ten-dimensional un-compactified theories, we dropped the superscript referring to the ten-dimensional nature of the gauge fields for notational convenience.

On the other hand, for type IIB we have [23]

$$S_{\rm IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_{10}}R - \frac{1}{4\kappa^2}\int \big(d\Phi\wedge *_{10}d\Phi + e^{2\Phi}dC_0\wedge *_{10}dC_0$$
$$+ g_s e^{-\Phi}H_3\wedge *_{10}H_3 + g_s e^{\Phi}\tilde{F}_3\wedge *_{10}\tilde{F}_3 + \frac{g_s^2}{2}\tilde{F}_5\wedge *_{10}\tilde{F}_5 + g_s^2 C_4\wedge H_3\wedge F_3\big) \tag{2.12}$$

where the following self-duality condition has to be imposed

$$*_{10}\,\tilde{F}_5 = \tilde{F}_5\,. \tag{2.13}$$

and we defined tilde quantities as follows

$$\tilde{F}_3 = F_3 - C_0 H_3\,,\quad F_3 = dC_2\,, \tag{2.14}$$
$$\tilde{F}_5 = F_5 - C_2\wedge H_3 \qquad F_5 = dC_4\,. \tag{2.15}$$

The equations of motion are [186]

$$d *_{10} d\Phi = e^{2\Phi}dC_0\wedge *_{10}dC_0 - \frac{g_s e^{-\Phi}}{2}H_3\wedge *_{10}H_3 + \frac{g_s e^{\Phi}}{2}\tilde{F}_3\wedge *_{10}\tilde{F}_3\,, \tag{2.16}$$
$$d(e^{2\Phi} *_{10} dC_0) = -g_s e^{\Phi}H_3\wedge *_{10}\tilde{F}_3\,, \tag{2.17}$$
$$d(e^{\Phi} *_{10} \tilde{F}_3) = g_s F_5\wedge H_3\,, \tag{2.18}$$
$$d(e^{-\Phi} *_{10} H_3 - C_0 e^{\Phi}\tilde{F}_3) = -g_s F_5\wedge F_3\,, \tag{2.19}$$
$$R_{MN} = \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{g_s e^{2\Phi}}{2}\partial_M C_0\partial_N C_0 + \frac{g_s^2}{96}\tilde{F}_{MNPQR}\tilde{F}_N{}^{PQRS}$$
$$+ \frac{g_s}{4}(e^{\Phi}H_{MPQ}H_N{}^{PQ} + e^{\Phi}\tilde{F}_{MPQ}\tilde{F}_N{}^{PQ})$$
$$- \frac{g_s}{48}G_{MN}\big(e^{-\Phi}H_{PQR}H^{PQR} + e^{\Phi}\tilde{F}_{MPQ}\tilde{F}^{PQE}\big)\,. \tag{2.20}$$

Finally, the Bianchi identities read

$$d\tilde{F}_3 = -dC_0\wedge H_3\,, \tag{2.21}$$
$$d\tilde{F}_5 = -F_3\wedge H_3\,. \tag{2.22}$$

together with (2.11). The fermionic counterparts are not displayed as not necessary for the analysis presented in the rest of the thesis.
We will overcome the lengh and the asymmetry of these equation between type IIA/IIB at the end of this Chapter, by introducing a concise reformulation which turns out to be more useful even when moving away from the supergravity limit by including background source terms such as D-branes or O-planes.

## 2.3 The compactification ansatz

As mentioned in the Introduction 1, connection with ordinary experience suggests string theories admit a compactification to four dimension. We will be mainly interested to the following compactification

$$M_{1,9} = M_{1,3} \times M_6 \,. \tag{2.23}$$

As a consequence we demand the ten-dimensional metric to be a product (non-necessarily direct) between a non-compact, maximally symmetric[2] external space-time $M_{1,3}$ and a compact internal space $M_6$. The most general metric which features a maximally symmetric subset is the warped metric [199]

$$ds^2_{10} = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n \,. \tag{2.24}$$

where $\mu, \nu = 0, \dots, 3$ and $m, n = 1, \dots, 6$.
The possible external maximally symmetric spaces are completely classified by their Ricci scalar[3]

$$\begin{aligned} R < 0: &\quad \text{AdS}_4 \\ R = 0: &\quad \text{Mink}_4 \\ R > 0: &\quad \text{dS}_4 \end{aligned} \tag{2.25}$$

We are then ready to enunciate the concept of vacuum. Intuitively, with this should correspond to a configuration for which no particles in the four-dimensional space-time are present. In the space-time splitting (2.23) we can summarize the definition of vacuum as follows [197]

**Definition 1.1** *A vacuum of type II supergravity is a solution of its equations of motion and Bianchi identities, such that $M_{1,9}$ is fibered over a spacetime $M_{1,3}$, and such that the whole solution (and not just the metric) enjoys maximal symmetry in four dimensions: that is, symmetry under $ISO(1,3)$ for $M_{1,3}$ = Mink$_4$, under $SO(2,3)$ for $M_{1,3}$ = AdS$_4$ and under $SO(1,4)$ for $M_{1,3}$ = dS$_4$.*

Although the choice (2.23) is the one which most naturally connects with the ordinary four-dimensional experience, an alternative repartition of external and internal dimensions admitting an external maximally symmetric space, and which enjoys the requirements of Definition 1.1, is an equivalently valid vacuum candidate[4].

---

[2]Only three possible Lie groups are generated by the maximal number Killing vectors, which in turn determine three possible choices for $M_{1,3}$: $SO(2,3)$, $ISO(1,3)$ (the Poincaré group in four dimensions) and $SO(1,4)$, which in turn identifies $M_{1,3}$ an AdS$_4$, Mink$_4$ or dS$_4$ respectively.

[3]This is a completely general statement, independent of the dimensions of the maximally symmetric space.

[4]In particular, in Chapter 5 we will discuss a trucation ansatz which include known vacua of with AdS$_5$ external space as solutions, while the perturbation analysis performed in Chapter 6 would take as zeroth-order background a configuration which features a Mink$_3$ space-time.

## 2.4 Supersymmetry and democratic formalism

As type II theories are supersymmetric, we are interested in the related transformation of the fields. The supersymmetric content of type II theories is encoded in two Majorana-Weyl supersymmetry parameters $\epsilon^A$, which have the same chiralities of the corresponding gravitinos. Requiring maximal symmetry for the external space implies the vacuum expectation value of the fermionic fields to vanish. The background should therefore be purely bosonic. For a supersymmetric vacuum with bosonic fields only a generic fermionic field $\chi$ has to satisfy $< Q_\epsilon \chi >=< \delta_\epsilon \chi >= 0$.
A useful compact reformulation of the ten-dimensional supersymmetry conditions in string frame is the so-called *democratic* formulation [24], which doubles the ten-dimensional R-R degrees of freedom by taking into account all even/odd forms:

$$\text{Type IIA: all odd forms gauge potentials } C_n^{(10)}\,, \quad n = 1, \ldots, 9$$
$$\text{Type IIB: all even forms gauge potentials } C_n^{(10)}\,, \quad n = 0, \ldots, 8$$

Although apparently confusing, this formalism allows to treat the two types II string theory in the same fashion. Indeed, denoting[5] $C^{(10)}, F^{(10)}$ as the formal sum of gauge the fields and the formal sum of their field strenghts respectively , we define[6]

$$\tilde{F}^{(10)} = F^{(10)} - H_3 \wedge C^{(10)} + e^B F_0^{(10)} = (d - H_3\wedge)C^{(10)} + e^B F_0^{(10)}. \tag{2.26}$$

The field-strenght are constrained by a Hodge duality as[7]

$$F_k^{(10)} = s(*_{10} F_{10-k}^{(10)})\,. \tag{2.27}$$

being $s$ the operator which revertes all the indices of a given form

$$s(A_p) = (-)^{\mathrm{Int}[p/2]} A_p\,. \tag{2.28}$$

The supersymmetric variations of the fermionic fields read then

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4} \not{H}_M \mathcal{P} \epsilon + \frac{e^\phi}{16} \sum_n \not{F}_n^{(10)} \Gamma_M \mathcal{P}_n \epsilon, \tag{2.29}$$

$$\delta\lambda = \left(\not{\partial}\phi + \frac{1}{2}\not{H}\mathcal{P}\right)\epsilon + \frac{e^\phi}{8} \sum_n (-1)^n (5-n) \not{F}_n^{(10)} \mathcal{P}_n \epsilon\,. \tag{2.30}$$

where the sum has to be taken over the even or odd $n$ if we are considering type IIA or type IIB theories respectively. In the equations (2.29)-(2.30) the spinors are arranged in column vectors

$$\psi_M = \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}. \tag{2.31}$$

[5]From now on, we will alway denote a form without subscripts as a formal sum in the way we define here, to be distingushed from the case where a subscript appear and we are indeed referring to a $n$-form.

[6]In the following we added by hand the Romans' mass parameter $F_0^{(10)}$.

[7]From this point and in the following, only tilded R-R $\tilde{F}$ quantities will be presented. We would from now on uniform the notation by showing no more tildes on top of these.

The matrices $\{\mathcal{P}, \mathcal{P}_n\}$ are defined differently for the two theories

$$\begin{aligned} \text{IIA} : \mathcal{P} &= \Gamma_{11}\,, \\ \mathcal{P}_n &= \Gamma_{11}^{(n/2)}\sigma^1\,. \end{aligned} \tag{2.32}$$

$$\begin{aligned} \text{IIB} : \mathcal{P} &= -\sigma^3\,, \\ \mathcal{P}_n &= \begin{cases} i\sigma^2 \text{ for } \dfrac{n+1}{2} \text{ even,} \\ \sigma^1 \text{ for } \dfrac{n+1}{2} \text{ odd.} \end{cases} \end{aligned} \tag{2.33}$$

A slash is to be understood in this context as a contraction with a ten-dimensional gamma matrix $\not{A} = A_{MNP\ldots}\Gamma^{MNP\ldots}$ in all the unwritten indices.
The assumption of ten-dimensional space-time splitting (2.23), together with the assumption of maximal symmetry, implies the fluxes to be non-trivial only on the internal manifold

$$F_k^{(10)} = F_k + \text{vol}_4 \wedge s(*_6 F_{6-k})\,, \tag{2.34}$$

This suggest us to refer in the following to $F$ as a formal sum of even/odd *internal* forms only

$$\begin{aligned} \text{IIA} : F &= F_0 + F_2 + F_4 + F_6\,. \\ \text{IIB} : F &= F_1 + F_3 + F_5\,. \end{aligned}$$

An immediate advantage of this formalism is that the Bianchi identities for the fluxes can be elegantly recast as

$$dH_3 = 0\,, \quad (d - H_3\wedge)F = \delta\,. \tag{2.35}$$

We allowed in the last equation for a generalization due to the presence of source-terms in the background. Whenever considering sources, both right-hand sides of these equations would in principle be modified allowing for charge terms. In particular, we keep the Bianchi identity for $H_3$ unmodified as long as we do not consider NS-5 branes to be present, while the second would in turn be modified whenever considering D-branes or O-planes source terms.
A suitable reformulation is also obtained for the equations of motion for the fluxes, which we can now write in a compact form for both IIA and IIB theories, and for which again we propose a form compatible to the case where source terms are present

$$d(e^{4A-2\phi} *_6 H) \pm e^{4A}\sum_k F_k \wedge *_6 F_{k+2} = \delta \tag{2.36}$$

$$(d + H\wedge)(e^{4A} *_6 F) = \delta \tag{2.37}$$

In principle, the status of a vacuum is totally unrelated to its supersymmetric properties. However, when restricting to supersymmetric configurations, the following useful result holds [82, 151, 160]

*For supersymmetric compactifications of type II theories to $M_{1,3}$ =Mink$_4$ or $M_{1,3}$ =AdS$_4$, a configuration which has vanishing supersymmetry constraints (2.29)-(2.30) supplied with the Bianchi identities for $H_3$ and $F$ (2.35) is a solution of all the other equations of motion (2.36)-(2.37) and Einstein equations (2.8) for type IIA, (2.20) for type IIB.*

As the reader may notice, we excluded from the dS$_4$ case in the previous statement. This is because it is incompatible with unbroken supersymmetry. A simple way to see it [197], when specializing to the case of $\mathcal{N} = 1$ supergravities is that the scalar potential of $\mathcal{N} = 1$ can be schematically pictured as

$$V = e^K(|DW|^2 - 3|W|^2) + D^2 \tag{2.38}$$

where $W$ is the superpotential, $K$ is the Kähler potential and $D$ are generic $D$-terms. Supersymmetry implies in this case $D = 0$ and $DW = 0$, allowing only negative or zero cosmological constant.
Before concluding the Chapter, we discuss briefly the explicit form of the constraints (2.29) and (2.30) under the splitting (2.23).
The ten-dimensional spinors decompose Spin(1, 9) $\to$ Spin(1, 3) $\times$ Spin(6), then in turn a spinor representation $\mathbf{16} \in$ Spin(1, 9) would split as $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}})$. As the supersymmetry parameters $\epsilon^A$ have different chiral splitting depending of the theory we are considering, we have the following decomposition[8]

$$\begin{aligned} \epsilon^1 &= \xi^1_+ \otimes \eta^1_- + \xi^1_- \otimes \eta^1_+ \,, \\ \epsilon^2 &= \xi^2_+ \otimes \eta^2_\pm + \xi^2_- \otimes \eta^2_\mp \,. \end{aligned} \tag{2.39}$$

where upper signs are for IIA and the lower ones for IIB.
Before concluding the Chapter we would give a specialization of the equations (2.29)-(2.30) in the case of preserved $\mathcal{N} = 1$ external supersymmetry (which corresponds to assume a proportionality between $\xi^1$ and $\xi^2$). These can indeed be completely translated in terms of differential conditions on $(\eta^1, \eta^2)$ [100]. Starting from the internal components, when taking $M = m\,, (m = 1, \ldots, 6)$ we have

$$\left(\nabla_m - \frac{1}{4}\not{H}_m\right)\eta^1_+ \mp \frac{e^\phi}{8}\not{F}\gamma_m\eta^2_\mp = 0\,, \tag{2.40}$$

$$\left(\nabla_m + \frac{1}{4}\not{H}_m\right)\eta^2_\mp - \frac{e^\phi}{8}\not{F}^\dagger\gamma_m\eta^1_+ = 0\,, \tag{2.41}$$

and the external ones for $M = \mu\,, (\mu = 1, \ldots, 4)$

$$\mu e^{-A}\eta^1_+ + \not{\partial}A\eta^1_+ - \frac{1}{4}e^\phi\not{F}\eta^2_\pm = 0\,, \tag{2.42}$$

$$\mu e^{-A}\eta^2_\mp + \not{\partial}A\eta^2_\mp + \frac{1}{4}e^\phi\not{F}^\dagger\eta^1_+ = 0\,, \tag{2.43}$$

[8] We are assuming withouth loss of generality that $M_6$ admits a Spin$_6$ structure.

while the dilatini equations (2.30) in the combination $\Gamma^M \delta\Psi_M^A - \delta\lambda^A$ , $A = 1, 2$ give in turn the following two equations

$$2\mu e^{-A}\eta_-^1 + \nabla\!\!\!/\,\eta_+^1 + \left(\partial\!\!\!/(2A-\phi) + \frac{1}{4}H\!\!\!/\right)\eta_+^1 = 0\,, \tag{2.44}$$

$$2\mu e^{-A}\eta_\pm^2 + \nabla\!\!\!/\,\eta_\mp^2 + \left(\partial\!\!\!/(2A-\phi) - \frac{1}{4}H\!\!\!/\right)\eta_\mp^2 = 0\,. \tag{2.45}$$

where only the NS-NS fields appear. In the following two Chapters we will mainly deal with the reformulation of these equations in a suitable geometrical form. We will demand the solutions to preserve $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetries in four dimensions, depending whether the external spinors $\xi^1, \xi^2$ are linearly dependent or not. We will see that the introduction of new mathematical tools will guide us into a smart rewriting of the conditions (2.40)-(2.45), which will unveil the intimate relation between fluxes and the internal geometry.

# Chapter 3

# Generalized Complex Geometry

The possible compactifications down to lower dimensions are considerably complicated, and strongly depend on the structure retained from the higher dimensional space. In the perspective of framing the compactification in a geometric language, demanding a configuration to be a vacuum amounts to a *topological* and a *differential* requirement to be satisfied. The efforts in identifying the nature of the internal space have been carried out gradually, starting from the case where the vacuum expectation values of all fields of the ten-dimensional theory except for the metric are set to zero: in this case the two conditions are found to meet in the Calabi-Yau condition. When including fluxes in the compactification, the corresponding differential conditions are considerably more complicated. A suitable tool for investigation of vacua in the presence of fluxes is the mathematical construction termed Generalized Complex Geometry (GCG for short) [105, 120–122]. This framework allows to give a geometrization of the NS-NS subsector of the gauge field content, and at the same time permits to formulate a concise statement of the geometric requirements for a vacuum in presence of background fluxes. In the literature many reviews discuss generalized geometry [94, 123, 147, 197]. The concept of $G$-structure is crucial, and we would start the Chapter by reviewing it. In the following we will discuss some concrete example of compactification which will be helpful in understanding how fluxes backreact on the geometry. We would then be ready for introducing Generalized Complex Geometry and discuss its application to type II string theory.

## 3.1 $G$-structures

Consider a background with metric splitting (2.23). The existence of a supercurrent in ten-dimension is in turn related to the existence of well defined spinors (in general more than one) $\eta^a$ on $M_6$.
The structure group of a manifold is defined as the group of transformations required to patch the orthonormal frame bundle. Notice that tensor bilinears $T_{i_1 \dots i_k}$ can be built out of the spinors $\eta^a$: the existence of $T_{i_1 \dots i_k}$ automatically implies that the structure group of the frame bundle is reduced, as in view of their existence is possible to construct a reduced frame bundle, where on the overlap between different patches only the rotations

that leave the bilinear invariant are allowed as transition functions. In the mathematical literature manifolds with a reduced structure group $G$ are called $G$-structure manifolds [79,81,85,136–138]. Given a generic six-manifold $M_6$ a possible definition of a $G$-structure with $G \subset \text{Spin}(6)$ can be given by the following Killing spinor equation

$$\nabla^T_m \eta^a = 0\,. \tag{3.1}$$

The possible groups $G$ are in this case the possible special holonomy groups, as requiring that a spinor is covariantly constant with respect to some connection $\nabla^T$ is equivalent to require that $\nabla^T$ has special holonomy. Geometrically, the necessary and sufficient conditions for having solutions of the particular supersymmetry constraints translate into the $G$-structure being of a certain type.
The reduced group structure $G$ does not necessarily coincide with the holonomy group of the Levi-Civita connection as the spinors are not necessarily covariantly constant with respect to it. The degree to which they fail to be covariantly constant is measured by a quantity known as the intrinsic torsion, which can be used to classify $G$-structures.
In the bilinear description of $G$-structures, the intrinsic torsion is a measure of the failure of the tensors to be covariantly constant with respect to the Levi-Civita connection of the metric defined by the structure. Then, it is a straighforward consequence the fact that all of the components of the intrinsic torsion are encoded in derivatives of the invariant tensors. We will discuss in subsection 3.2 how the deviation from the Levi-Civita case can help in classifiying the possible cases of structures which arise in type II string theories.
A $SU(n)$-structure in $d = 2n$ even real dimensions is completely specified by a real two-form $J$ of maximal rank and a complex $(n, 0)$-form $\Omega_n$ such that

$$J \wedge \Omega_n = 0\,, \qquad \Omega_n \wedge \bar{\Omega}_n = -i^{n(n+2)} \frac{2^n}{n!} \underbrace{J \wedge \ldots \wedge J}_{\text{n times}} \tag{3.2}$$

On a six dimensional manifold $M_6$, we deduce that we have an

***SU*(3) structure in $d = 6$:** A real two form $J$ and a complex $(3, 0)$-form $\Omega_3$ such that[1]

$$J \wedge \Omega_3 = 0, \qquad i\Omega_3 \wedge \bar{\Omega}_3 = \frac{4}{3} J^3\,. \tag{3.3}$$

Notice that, in terms of structures on $M_6$, $J$ alone defines an almost symplectic $\text{Sp}(6, \mathbb{R})$ structure (ASS), while $\Omega_3$ defines an almost complex $GL(3, \mathbb{C})$ structure[2] (ACS), the pair $(J, \Omega_3)$ intersecting in a $SU(3)$ structure. We also have

[1]The pair $(J, \Omega_3)$ satisfying (3.3) is mean to be nothing but a *formal* definition of the structure, specifying its defining objects and the algebraic relation these have to satify. Whenever we would have a real two-form $E$ and a complex $(3, 0)$-form $F$ satisfying (3.3) with respect to the interchange $E \leftrightarrow J$ and $F \leftrightarrow \Omega_3$, we would equally refer to $(E, F)$ as a six dimensional $SU(3)$ structure, and the same would be for other structures.

[2]The existence of a global, non-degenerate, holomorphic $(3, 0)$-form $\Omega_3$ defines an almost complex

$SU(2)$ **structure in** $d = 6$: A real 2-form $J$, a holomorphic $(2,0)$-form $\Omega_2$ and a complex form $\sigma$, satisfying (3.6) together with

$$J \wedge \Omega_2 = J \wedge \Omega_2 = 0 \,, \qquad \Omega_2 \wedge \bar{\Omega}_2 = 2J \wedge J \,, \tag{3.5}$$

$$\iota_\sigma J = 0 \,, \quad \iota_\sigma \Omega_2 = \iota_{\bar{\sigma}} \Omega_2 = 0 \,. \tag{3.6}$$

Two distinct $SU(3)$ structures intersect to obtain an $SU(2)$ structure, as

$$J^{(1,2)} = J \pm v \wedge w \,, \qquad \Omega_3^{(1,2)} = \Omega_2 \wedge (v \pm iw) \,. \tag{3.7}$$

and denoting the complex one-vector $k = v + iw$, it is straightforward to see that $(J, \Omega_2, k)$[3] satisfy (3.5)-(3.6) (see for instance [34]). The analogue of (3.5)-(3.6) in five dimensions reads

$SU(2)$ **structure in** $d = 5$: A real two-form $J$, a holomorphic $(2,0)$-form $\Omega_2$ and a complex vector (without zeros) $g$ satifying

$$\begin{aligned} &\iota_g J = \iota_g \Omega_2 = \iota_g \bar{\Omega}_2 = 0 \,, \\ &\Omega_2 \wedge \Omega_2 = \bar{\Omega}_2 \wedge \bar{\Omega}_2 = \Omega_2 \wedge J = 0 \,, \\ &\Omega_2 \wedge \bar{\Omega}_2 = 2J \wedge J \,. \end{aligned} \tag{3.8}$$

Of particular relevance for the truncation which we will deal with in Chapter 5 is the specific case where the real two-form $J$ and the complex one form $g$ could be further decomposed to $(J_1, J_2)$ and $(g_1, g_2)$ respectively, each of which satisfy (3.8) -(3.6) together with the same $\Omega_2$. The intersection of these two $SU(2)$ structures defines in turn an $U(1)$ structure

$U(1)$ **structure in** $d = 5$: Two real 2-forms $(J_1, J_2)$, a holomorphic $(2,0)$-form $\Omega_2$ and two complex one-forms $(g_1, g_2)$ which separately satisfy (3.8) with respect to $(J_i, \Omega_2, g_i)\,, i = 1, 2$.

## 3.2 $SU(3)$ torsion classes

We are interested in the most general differential relations the 2-form $J$ and the 3-form $\Omega$[4] can independenty satisfy as defining elements of an $SU(3)$-structure (3.3). Remarkably,

---

structure $I$ on $M_6$ [120]

$$I^m{}_n = 4 \frac{\mathrm{Re}\ \Omega_3 \wedge dy^n \wedge \iota_n \mathrm{Re}\ \Omega_3}{i\Omega_3 \wedge \bar{\Omega}_3} \,. \tag{3.4}$$

[3] See footnote 1.

[4] We drop from now on the subscript from $\Omega_3$ as all througout this Chapter we will be interested in the six dimensional case only.

these are completely specified by five torsion classes $\{W_i\}^{i=1,\dots,5}$ [48]. The proper torsion tensor lies in

$$T_{mn}{}^{p} \in \Lambda^1 \otimes \mathfrak{so}(6) \equiv \Lambda^1 \otimes (\mathfrak{su}(3) \oplus \mathfrak{su}(3)^{\perp})\,. \tag{3.9}$$

Because the action of the $SU(3)$ part would be trivial on $SU(3)$ invariant forms, we keep only the $\mathfrak{su}(3)^{\perp}$ part, recovering the representations which form the intrinsic torsion

$$T^{0}_{mn}{}^{p} \in \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} = (\mathbf{1} \oplus \bar{\mathbf{1}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}) \tag{3.10}$$

$$= (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus 2(\mathbf{3} \oplus \bar{\mathbf{3}}) \tag{3.11}$$

We define the following five torsion classes as elements of the various representations

$$W_1 \in (\mathbf{1} \oplus \mathbf{1})\,, \quad W_2 \in\ (\mathbf{8} \oplus \mathbf{8}), \quad W_3 \in (\mathbf{6} \oplus \bar{\mathbf{6}})\,, \quad W_4 \in (\mathbf{3} \oplus \bar{\mathbf{3}})\,, \quad W_5 \in (\mathbf{3} \oplus \bar{\mathbf{3}}) \tag{3.12}$$

Here $W_1$ is a complex scalar, $W_2$ is a complex primitive $(1,1)$ form, $W_3$ is a real primitive $(2,1)+(1,2)$ form[5] and $W_4$, $W_5$ are real vectors.
The differential relations on the $SU(3)$-invariant tensors is therefore parameterized in terms of the $W_i$ as follows

$$dJ = \frac{3}{2}\text{Im}\,(\bar{W}_1\Omega) + W_4 \wedge J + W_3\,, \tag{3.13}$$

$$d\Omega = W_1 \wedge J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega\,. \tag{3.14}$$

We briefly discuss in the following which kinds of internal manifolds correspond to specific choices of the torsion classes.
A manifold of $SU(3)$ structure is *complex* provided $W_1 = W_2 = 0$. Indeed, looking at which pieces contain $W_1$ and $W_2$ in $d\Omega$, it turns out these are $(2,2)$ forms. As in a complex manifold we expect the exterior derivative of a $(p,q)$-form to contain $(p+1,q)$ or $(p,q+1)$ parts, for a complex manifold only the third term in (3.14) is allowed. Requiring the manifold to be *symplectic* amounts to demand the fundamental 2-form $J$ to be closed. This can be accomplished by choosing the three elements ($W_1$, $W_3$, $W_4$) appearing in (3.13) to vanish separately.
When a manifold is complex and symplectic at the same time it is a *Kähler* manifold: in this case, the only data to specify is the value of its only non-vanishing element $W_5$. Whenever we specialize the Kähler condition to have a vanishing $W_5$ the relations completely match the ones in (3.25), describing a *Calabi-Yau* manifold. Finally, we would like to make a step backwards, looking back to a setting where we allow $(W_4, W_5)$ only to be non-vanishing. Under a conformal transformation of an $SU(3)$-structure

$$J \to e^{2F} J\,, \qquad \Omega \to e^{3F}\Omega \tag{3.15}$$

the metric scales as $g \longrightarrow e^{2F} g$, while $(W_1, W_2, W_3)$ are invariant [48,85]. The combination $3W_4 - 2W_5$ is also invariant under (3.15). Whenever we have vanishing $(W_1, W_2, W_3)$

[5] Primitivity for $W_2$ means $W_2 \wedge J \wedge J = 0$, while primitivity for $W_3$ is translated as $W_3 \wedge J = 0$.

together with exact $W_4$ (or $W_5$) satisfying $3W_4 - 2W_5 = 0$ the manifold is said to be a *conformal Calabi-Yau*. We will illustrate in Chapter 5 how the Klebanov-Strassler solution [144] is a concrete example belonging to this class. As supersymmetry imposes linear constraints among fluxes and torsion classes, whenever fluxes are turned on in the internal space these backreact: tables which summarize the possible vacua for type II string theories are displayed in [98]. On general grounds, we can state that type IIA solutions are twisted symplectic, while type IIB are twisted complex. This statement can be very simply derived from the pure spinor equations (3.103)-(3.105) we are going to present in the final part of the Chapter. Having studied which possible geometries an $SU(3)$ can describe, our aim is to illustrate the limitations of ordinary complex geometry in describing backgrounds with fluxes. To do so it will be helpful to concretely examine the two following explicit examples.

## 3.3 The Calabi-Yau case

As a start-up, let us investigate the compactification to a background for which the only field to have non vanishing expectation value is the metric. We search a compactification to four-dimensional maximally symmetric space $M_{1,3}$, in the specific case where the metric splitting (2.23) is just a direct product. This amounts to set to zero the warp factor $A(y)$ in (2.24). Demanding $M_{1,3}$ to be a maximally symmetric space reduces its Riemann tensor to be proportional to the external Ricci scalar

$$R_{\mu\nu\rho\lambda} = \frac{R}{12}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})\,. \tag{3.16}$$

where only three possible cases are allowed, corresponding to the maximally symmetric cases (2.25).

The variations of the bosonic terms in the supersymmetry transformations presented in the previous section considerably simplify. In particular, equation (2.29) would now look

$$\delta\psi_M = \nabla_M \epsilon\,. \tag{3.17}$$

Being interested in a solution preserving supersymmetry we demand the supersymmetry equations to vanish [40]. This corresponds to search solutions to the following Killing equation

$$\nabla_M \epsilon = 0 \tag{3.18}$$

We stress the fact that as the supersymmetric parameters should be globally defined on $M_{1,9}$ so must be the internal spinors $\eta^{1,2}$ for equation (2.39) to be a consistent splitting. Equation (3.18) can be used to deduce informations on both external and internal spaces. For the first, we are searching for covariantly constant spinors, parameterizing the preserved $\mathcal{N} = 2$ supersymmetries, which satisfy

$$\nabla_\mu \xi_+^{1,2} = 0 \tag{3.19}$$

This in turn yields the following integrability condition

$$[\nabla_\mu, \nabla_\nu]\xi^{1,2} = \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\xi^{1,2} = 0 \tag{3.20}$$

As we are restricting to maximally symmetric spaces, equation (3.20) together with equation (3.16) necessarily implies the external space to be Minkowski.
What is rather non-trivial is what happens for the internal components of (3.18), in view of the fact that that maximal symmetry is not assumed for $M_6$. Specializing this equations to the absence of fluxes, we deduce the following differential conditions

$$\nabla_m \eta_+^1 = \nabla_m \eta_+^2 = 0 \tag{3.21}$$

This equation restricts the internal manifold to have holonomy $G \subseteq SU(3)$ [104]. For the time being we restrict to situation $\eta^1 = \eta^2 \equiv \eta$, which exactly corresponds to an $SU(3)$ holonomy manifold. Equation (3.21) can be used to obtain an integrability condition as well

$$[\nabla_m, \nabla_n]\eta_+ = \frac{1}{4}R_{mnpq}\Gamma^{pq}\eta_+ = 0\,. \tag{3.22}$$

When dropping the assumption of maximal symmetry for $M_6$, the internal space has no longer vanishing curvature, but instead it must be Ricci-flat

$$R_{mn} = 0\,. \tag{3.23}$$

The existence of $\eta$ can be used to build explicitly the tensor bilinears defining the $SU(3)$ structure (3.3)

$$J_{mn} = \mp i\eta_\pm^\dagger \gamma_{mn}\eta_\pm\,, \qquad \Omega_{mnp} = -i\eta_-^\dagger \gamma_{mnp}\gamma_+\,. \tag{3.24}$$

Using (3.21), and the fact that the $\gamma$ matrices are covariantly constant with respect to the Levi-Civita connection, the following two differential relations hold

$$dJ = d\Omega = 0\,. \tag{3.25}$$

We found nothing but the Calabi-Yau condition (cfr. subsection 3.2). Notice also that the global existence of a holomorphic $(3,0)$-form $\Omega$ implies the first Chern class of $M_6$ to vanish:

$$c_1[M_6] = 0\,. \tag{3.26}$$

This last equation is a topological requisite. Indeed, there obviously exists manifolds which satisfies (3.26), but not necessarily (3.25). We will illustrate in the next example how these may be considered string vacua candidate. As discussed in the classification of subsection 3.2, whenever allowing fluxes with non vanishing expectation values in the internal manifold would imply this to depart from the Calabi-Yau class.

## 3.4 A first example of backreaction

The metric assumed in the previous paragraph, for which $A(y) = 0$, is able to describe a quite specific class of vacua. Nevertheless, we already pointed out that the most general metric describing a space compatible with the presence of a maximally symmetric subspace is the warped metric (2.24). In this specific example we will consider $H$ to be the only flux which has non-trivial values[6]. We will focus again on solutions featuring four-dimensional space-time but preserving this time $\mathcal{N} = 1$ supersymmetries, so that we can make use of the relation for internal gravitino (2.40) (as we are once more restricting to the case $\eta^1 = \eta^2 \equiv \eta$, so (2.40) and (2.41) are indeed the same equation). Out of the internal supersymmetric variations, by setting the gravitino equation (2.40) to zero, under the same assumptions used before, one recovers

$$\left(\nabla_m - \frac{1}{4}\not{H}_m\right)\eta_+ = 0 \tag{3.27}$$

Poincaré invariance of the external space-time requires some components of the $H$ flux to vanish

$$H_{\mu\nu\rho} = H_{\mu\nu p} = H_{\mu n p} = 0\,. \tag{3.28}$$

while supersymmetry implies the warp factor to be identified with the dilaton field $A(y) = \Phi(y)$ [194]. Equation (3.27) suggests an interpretation of the gravitino transformation (2.40) by using a torsion element defined by $H$

$$\nabla^T_m \eta_+ \equiv \left(\nabla_m - \frac{1}{4}\not{H}_m\right)\eta_+\,. \tag{3.29}$$

By defining spinor bilinears exactly as in the Calabi-Yau case (3.24), we investigate the differential relations these satisfy. By defining a rescaled complex three form, $\tilde{\Omega} = e^{-2\phi}\Omega$, and using the definitions (3.24) and (3.29) one finds

$$d\tilde{\Omega} = 0 \tag{3.30}$$

which tells that, as in the fluxless case, $M_6$ is a complex manifold. On the other hand $J$ turns out to be covariantly constant with respect to the torsionful connection $\nabla^T_m$

$$\nabla^T_m J_n{}^p = 0 \tag{3.31}$$

By using (3.29) one establishes the following relation between $H$ and $J$

$$H = i(\partial - \bar{\partial})J\,. \tag{3.32}$$

where $\partial, \bar{\partial}$ are the Dolbeault operators. Being $dJ \neq 0$ as long as $H$ is non vanishing, $M_6$ is necessarily non-Kähler. This type of solutions has been originally found by Strominger

[6] For notational convenience, from now on we will drop the subscript on $H_3$.

[194] in the context of heterotic theories. The type II analogue falls in class A of the table of [98] for both type II theories, corresponding to a complex, non-Kähler $M_6$. However, being the three-form $\Omega$ constructed as in (3.24) and globally defined, the first Chern class vanishes $c_1[M_6] = 0$ as it was for the Calabi-Yau case.
We conclude the analysis by claiming that, while the topological condition are unaltered with respect to (3.26), the differential conditions obeyed for the bilinears (3.24) indeed change [85, 167, 194]

$$\begin{aligned} d(e^{-2\phi}\Omega) &= 0\,, \\ e^{2\phi}d(e^{-2\phi}J) &= -*_6 H_3\,, \\ d(e^{-2\phi}J\wedge J) &= 0 \end{aligned} \tag{3.33}$$

The paper [194], in which this analysis was first presented, ended with the worrisome comment that the classification achieved in [40] by means of the condition (3.21) could not be done anymore in more general settings. This was basically due to the little knowledge of non-Kähler manifolds at the time the paper was written. In fact, it took almost twenty years to the general classification presented in subsection 3.2 to be realized using $G$-structures [79, 81, 85, 98].

## 3.5 Generalized complex geometry

We are now ready to explain how ordinary complex geometry can be enlarged in the perspective of giving a formal framework for studying vacua which deviate from the Calabi-Yau case.
Given a six-manifold $M_6$, let us consider the direct sum

$$E = TM_6 \oplus T^*M_6 \tag{3.34}$$

as an extension of its ordinary tangent bundle $TM_6$. Sections of $E$ are named generalized vectors, and correspond to formal sums of vectors $x \in TM_6$, and one-forms $\xi \in T^*M_6$. This extended bundle (which from now on will be named generalized tangent bundle) is endowed with a (symmetric) inner product

$$(x+\xi, y+\eta) = \frac{1}{2}(\iota_x\eta + \iota_y\xi), \qquad x, y \in TM_6, \quad \xi, \eta \in T^*M_6\,. \tag{3.35}$$

from which a natural metric $\mathcal{I}$ on $E$ may be written in the following matricial $(2\times 2)$ notation

$$\mathcal{I} = \begin{pmatrix} 0 & \mathbb{1}_6 \\ \mathbb{1}_6 & 0 \end{pmatrix} \tag{3.36}$$

$\mathcal{I}$ enters the pairing (3.35) between vectors and one-forms with indefinite signature $(6,6)$. As we just discussed in subsection 3.1, its global existence on $E$ automatically reduces

the structure group to $O(6,6)$. Considering such an extended bundle, and its symmetries, have a fundamental physical interest: the T-duality group of type II theories compactified on $T^6$ is indeed $O(6,6)$. It is then obvious that by introducing the generalized tangent bundle $E$ we adopt a formalism where the T-duality group emerges at the geometrical level, as it matches the reduced group structure of $E$.
A generic element $M \in O(6,6)$ can be represented in matrix form as:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.37}$$

We can decompose this in a number of subgroups, for which we will present the action of a generalized vector $X = x + \xi$.
First of all we have the $GL(6)$ action on the fibres of $TM_6$ and $T^*M_6$, which is embedded in $O(6,6)$ as follows

$$X \longrightarrow X' = \begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1} \end{pmatrix} X \tag{3.38}$$

Also, given a two-form $B$, the following transformation is a subgroup of (3.37)

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \qquad \text{such that} \qquad X = x + \xi \longrightarrow X' = x + (\xi - \iota_X B) \tag{3.39}$$

which is often referred to as $B$-transform. Similarly, one can use a bi-vector $\beta$ to construct the transformation

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \qquad \text{such that} \qquad X = x + \xi \longrightarrow X' = (x + \beta \cdot \xi) + \xi \tag{3.40}$$

Patch covering of $E$ can be done using elements of $GL(d)$ and $G_B$: these two indeed form a subgroup which is a semi-direct product $G_{geom} = GL(d) \rtimes G_B$. The structure group of the generalized tangent space $E$ actually reduces from $O(d,d)$ to $G_{geom}$. Eventual inclusion of $\beta$-shift elements such as (3.40) is related to non-geometric backgrounds [101].

## 3.6 Generalized almost complex structures

In complete analogy with usual complex geometry, we define a generalized almost complex structure (GACS for short) as a map from $E$ to itself[7]

$$\mathcal{J} : TM_6 \oplus T^*M_6 \longrightarrow TM_6 \oplus T^*M_6 \tag{3.41}$$

[7]We use here for the first time the calligraphic notation $\mathcal{J}$ for a generalized complex structure. We will use ordinary capitals $I, J, \ldots$ to label ordinary structures (complex, symplectic), and related objects, and calligraphic $\mathcal{J}, \mathcal{I}, \ldots$ for their generalized geometrical analogue.

such that the two following conditions are satisfied

$$\mathcal{J}^2 = -\mathbb{1}_{6+6}\,, \tag{3.42}$$

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I} \quad \text{(Hermiticity)} \tag{3.43}$$

In term of the $(2 \times 2)$ matricial picture already adopted above, $\mathcal{J}$ has the following expression

$$\mathcal{J} = \begin{pmatrix} I & P \\ L & -I^T \end{pmatrix} \tag{3.44}$$

where $(P, L)$ are antisymmetric matrices. The hermiticity condition imposes linear conditions on $(I, P, L)$, as for instance $I^2 + PL = -\mathbb{1}_6$. The existence of $\mathcal{J}$ reduces the structure group of $E$ from $O(6,6)$ to $U(3,3)$. By considering the complexification of the generalized tangent bundle

$$E^{\mathbb{C}} = E \otimes \mathbb{C}\,. \tag{3.45}$$

the GACS can be used to define a splitting of $E^{\mathbb{C}}$ into its $\pm i$-eigenbundles since one may define a projector by means of $\mathcal{J}$ as

$$\Pi = \frac{1}{2}(\mathbb{1}_{6+6} - i\mathcal{J})\,. \tag{3.46}$$

to define the i-eigenbundle $\mathcal{L}_{\mathcal{J}}$ as

$$\mathcal{L}_{\mathcal{J}} = \{A \in TM \oplus T^*M \,|\, \Pi A = A\}\,. \tag{3.47}$$

Notice that this space is null with respect to the metric $\mathcal{I}$, as given $A, B \in \mathcal{L}_{\mathcal{J}}$

$$(A, B) = A\mathcal{I}B = A\mathcal{J}^T\mathcal{I}\mathcal{J}B = (iA)\mathcal{I}(iB) = -A\mathcal{I}B = -(A, B) \tag{3.48}$$

This space has the maximal dimension that a null space can have in signature $(6,6)$, which is 6.
The general structure of (3.44), can be specialized to find back two common examples of ordinary geometry: an almost complex structure (ACS) $I$ can be embedded in this formalism as

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^T \end{pmatrix} \tag{3.49}$$

while an a non-degenerate two-form $J$ representing an almost symplectic structure (ASS) can be written as

$$\mathcal{J}_J = \begin{pmatrix} 0 & J \\ -J^{-1} & 0 \end{pmatrix} \tag{3.50}$$

## 3.7 Generalized metric

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be two generalized almost complex structures that commute and such that their product

$$G = -\mathcal{J}_1 \mathcal{J}_2 \,. \tag{3.51}$$

satisfies

$$G^2 = \mathbb{1}_{6+6} \,, \quad \mathcal{I} G = G^t \mathcal{I} \,. \tag{3.52}$$

When this is the case, two such structures are said to be *compatible*. In this case, the structure group it is further reduced to its maximal compact subgroup $U(3) \times U(3)$. Indeed, as these commute, one can diagonalize simultaneously, dividing $E^{\mathbb{C}}$ in four sub-bundles

$$\mathcal{L}_{++} = \mathcal{L}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_1} \,, \qquad \mathcal{L}_{+-} = \mathcal{L}_{\mathcal{J}_1} \cap \bar{\mathcal{L}}_{\mathcal{J}_1} \tag{3.53}$$

$$\mathcal{L}_{-+} = \bar{\mathcal{L}}_{\mathcal{J}_1} \cap \mathcal{L}_{\mathcal{J}_1} \qquad \mathcal{L}_{--} = \bar{\mathcal{L}}_{\mathcal{J}_1} \cap \bar{\mathcal{L}}_{\mathcal{J}_1} \tag{3.54}$$

One important results of [105] is that $G$ can be expressed without loss of generalities in the following form

$$G = -\mathcal{J}_1 \mathcal{J}_2 = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_d & 0 \\ B & \mathbb{1}_6 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_d & 0 \\ -B & \mathbb{1}_6 \end{pmatrix} \tag{3.55}$$

or in other words, two compatible GACS automatically provide a metric $g$ and $B$-field. It is interesting to notice that the metric $M = \mathcal{I} G$ was originally written in this form in the context of T-duality [163]. So far, by trading the tangent bundle for its generalization $E$, we see how generalized geometry easily encodes $(g, B)$ in a geometrical picture. These, together with the dilaton $\phi$, constitute the NS-NS sector of type II string theories. To understand in a better way how the dilaton enters the formalism of generalized geometry it is best to investigate an alternative description, which is in some sense dual to the GACS approach we have been discussing so far.

## 3.8 Pure spinors

The purpose of this subsection is to generalize the correspondence between a Weyl spinor and an almost complex structure from the perspective of generalized geometry. The important conclusions we will arrive at would link the previously introduced concept of GACS to quantities which are much more under control for making explicit calculations. We first recall that given a Spin(6) spinor $\eta_\pm$, an almost complex structure $I$ can be built as[8]

$$I_m{}^n = \mp i \eta_\pm^\dagger \gamma^m{}_n \eta_\pm \tag{3.56}$$

[8]Notice that for an $SU(3)$ structure, we can build $I$ out of $J$ defined in (3.24), as $I^m{}_n = g^{mp} J_{np}$ indeed gives (3.56).

In order to extend this to the generalized tangent bundle, the first natural request we ask is to be able to define spinors on $E$. We already commented that the existence of the natural metric (3.36) on $E$ reduces its structure group to $O(6,6)$. The associate covering group is Cliff$(6,6)$, whose algebra is defined by the relation

$$\mathbb{A}^2 = (\mathbb{A}, \mathbb{A}), \quad \forall \mathbb{A} \in TM_6 \oplus T^*M_6 \tag{3.57}$$

By considering the Clifford action of a section $X = x + \xi$ of $TM_6 \oplus T^*M_6$ on a spinor $\Phi_\pm \in S^\pm(E)$ (which is an algebra represention on $\Lambda^{\text{even/odd}} T^*M_6$)

$$X \cdot \Phi = x^m \iota_m \Phi + \xi_m \, dx^m \wedge \Phi \,, \tag{3.58}$$

and evaluating its squared action, one recovers (3.57)

$$(x+\xi) \cdot (x+\xi) \cdot \Phi = -(x+\xi, x+\xi) \cdot \Phi \,, \tag{3.59}$$

We just proved that the spinor bundle is isomorphic to the bundle of differential forms $\Lambda^\bullet T^*M_6$: whenever considering Cliff$(6,6)$ spinors we can treat them as differential forms. An inner product in the space of spinors is given by the Mukai paring [174]

$$\langle \Phi, \chi \rangle = (\Phi \wedge s(\chi))_6, \quad \text{for } \Phi, \chi \in \Lambda^{\text{even/odd}} T^*M_6. \tag{3.60}$$

where $s(\cdot)$ is the operation defined in (2.28).
However this isomorphism is non-canonical: under a $GL(6, \mathbb{R})$ action the Cliff$(6,6)$ decomposes as[9] [105, 128]

$$S^\pm(E) \simeq (\Lambda^6 T^*M_6)^{-1/2} \otimes \Lambda^{\text{even/odd}} T^*M_6 \tag{3.61}$$

This line bundle is naturally identified with the ten-dimensional dilaton, as this indeed defines the isomorphism between $S^\pm(E)$ and $\Lambda^{\text{even/odd}} T^*M_6$ [101]. This completes the illustration of how the entire NS-NS sector nicely fits in the generalized geometry setup. We define the *annihilator space* of a generic spinor $\Phi \in S(E)$ (we drop the chirality for convenience) as the kernel space with respect to the Clifford action

$$\mathcal{L}_\Phi = \{ X \in TM_6 \oplus T^*M_6 \,\big|\, X \cdot \Phi = 0 \} \tag{3.62}$$

In principle $\mathcal{L}_\phi$ is an isotropic subspace, but nothing guarantees it woud also be maximal, as it is not assured it has maximal dimension (which is six in the case at hand). Whenever its annihilator space is maximal we say that $\Phi$ is a *pure* spinor[10]. It is the good point to remark another maximally isotropic space has been already defined as the $+i$-eigenbundle $\mathcal{L}_\mathcal{J}$ (3.47) when discussing almost generalized complex structures. The matching of the two kernel spaces can be used to establish a link between two a-priori unrelated quantities

[9] It is worth commenting here that another possible spin structure exists [105], but we will use it in this context.

[10] Purity is a concept which exists of course even for ordinary Cliff$(d)$ algebras. In particular, every Cliff$(d)$ spinor is pure for $d \leq 6$.

in the following

**Proposition 3.1** *There is a correspondence between pure spinors and generalized almost complex structures on $TM_6 \oplus T^*M_6$ established by*

$$\mathcal{J} \leftrightarrow \Phi \quad \text{if} \quad \mathcal{L}_{\mathcal{J}} = \mathcal{L}_{\Phi} \,. \tag{3.63}$$

Notice that the correspondence is not properly one-to-one as the annihilator space is left unchanged after a rescaling of the pure spinor $\Phi$. The correspondence links an almost generalized complex structure $\mathcal{J}$ and a line bundle of pure spinors, in a complete analogy with the way the correspondence between a Weyl spinor and an almost complex structure works in ordinary complex geometry: we can even make it explicit as in (3.56), by writing, for a given chirality pure spinor $\Phi_\pm$

$$\mathcal{J}_\pm \;\leftrightarrow \Phi_\pm \,, \quad \mathcal{J}_{AB} = \langle \text{Re}(\Phi_\pm), \Gamma_{AB}\text{Re}(\Phi_\pm)\rangle \,. \tag{3.64}$$

where $A, B = 1, \ldots, 12$ are $O(6,6)$ indices, and $\Gamma_A \in \text{Cliff}(6,6)$.
Equation (3.61) gives evidence of the isomorphism between pure spinors and the bundle of differential forms. However, it would be more useful to identify pure spinors themselves as quantities living on $S^\pm(E)$. For this reason we now focus on finding an explicit expression for a pure spinors. Consider we start with a Cliff(6) spinor: we can re-state the maximality of its annihilator space (see footnote 10) as the existence of a set of 3 gamma matrices that annihilate it, which are by definition the holomorphic gamma matrices

$$\frac{1}{2}(1 \mp iI)^m{}_n \gamma^n \eta_\pm = 0. \tag{3.65}$$

The most natural way to shift to the Cliff(6,6) case would be to consider a tensoring of two different Cliff(6) spinors

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\dagger}_\pm \,. \tag{3.66}$$

we can make use of the argument of (3.65) twice times: purity of $\eta^1$ implies the existence of a set of 3 gamma matrices that annihilate it from the right, and purity of $\eta^2$ means there are 3 gamma matrices which do the same from the left. All in all, a total of $3+3$ annihilators of Cliff(6), is traded for 6 annihilator of Cliff(6,6). This represents a proof that the quantity $\Phi_\pm$ build as in (3.66) in indeed a Cliff(6,6) pure spinor. We will refer to spinors as in (3.66) as *naked*.
Naked pure spinors can be rotated by means of a generic $O(6,6)$ transformations (3.37). Of particular interest is the B-shift transformation (3.39): on spinors, it amounts to the exponential action $e^B \wedge \Phi$ on a naked spinor. The correspondence (3.64) still holds, and we can rewrite it as

$$\mathcal{J}_B = \mathcal{B}\mathcal{J}\mathcal{B}^{-1} \quad \leftrightarrow \quad \Phi^\pm_B = e^{-B}\Phi^\pm \,. \tag{3.67}$$

Spinors such $\Phi_B$ will be called *dressed* pure spinors[11]. In the pure spinor language, it is indeed a compatible pair $(\Phi^+_B, \Phi^-_B)$, corresponding to two GACS as (3.67) satisfying

[11]The the naked/dressed picture will be of particular relevance in Chapter 4.

(3.51)-(3.52), which define a positive definite metric (3.55) on the generalized tangent bundle, or in other words define a positive definite metric and a two form $(g, B)$ on the six-dimensional manifold. Given a pair of pure spinors $(\Phi_1, \Phi_2)$, their compatibility condition can be stated using the inner product (3.60) [95]

$$\langle \Phi_1, \Gamma^A \Phi_2 \rangle = 0\,, \quad A = 1, \ldots, 12. \tag{3.68}$$

This requirement simply translates the fact that the two pure spinors share half of their annihilators. It is straightforward to check that pure spinors constructed as (3.66) are automatically compatible, as by constructions share the required maximal number of gamma-matrices which annihilate them.
A useful translation between tensor product of spinors and differential forms can be obtained by using the isomorphism between the spinor bundle and the bundle of differential forms (often referred to as Clifford map):

$$A \equiv \sum_k \frac{1}{k!} A^{(k)}_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \longleftrightarrow \not{A} \equiv \sum_l \frac{1}{k!} A^{(k)}_{i_1,\ldots,i_k} \gamma^{i_1 \ldots i_k} \tag{3.69}$$

By using the following Fierz identity, namely an expansion of a generic tensor product of spinors expanded in terms of Cliff$(d, d)$ bilinears

$$\eta^1_\pm \otimes \eta^{2\dagger}_\pm = \frac{1}{8} \sum_k \frac{1}{k!} (\eta^{2\dagger}_\pm \gamma_{i_k \ldots i_1} \eta^1_\pm) \gamma^{i_1 \ldots i_k} \tag{3.70}$$

makes somehow explicit the correspondence between pure spinors and a formal sum of differential forms (3.61). We explore here how these concepts apply to the generalized structures in which we embedded ordinary complex (3.49) and symplectic (3.50) structures. The pure spinor which we associate to $\mathcal{J}_I$ is

$$\Phi_I = c_I \, \Omega\,, \tag{3.71}$$

for some nowhere-vanishing function $c_I$ and where $\Omega$ the complex decomposable three-form associated to $I$, while the pure spinor associated to $\mathcal{J}_J$ reads

$$\Phi_J = c_J \, e^{iJ}\,, \tag{3.72}$$

for some nowhere-vanishing function $c_J$ and where $J$ is the real two form which defines an almost symplectic structure. The pure spinors (3.71)-(3.72) correspond to a negative and a positive chiralities respectively. As can be seen by expanding (3.71)-(3.72) by using (3.70), we implicitly gave an expression for $\Phi_\pm$ in the particular case when $\eta^1 = \eta^2 \equiv \eta$ for the Calabi-Yau example (3.24) discussed in subsection 3.3.
One of the most important results of [105] is that these cases are the endpoints of the general expression pure spinor enjoys in arbitrary dimension

$$\Phi = \Omega_k \wedge e^{B+iJ} \tag{3.73}$$

We see from this expression how generalized geometry accounts for a democratic description of a complex-symplectic hybrid, interpolating between these two extremal points.

## 3.9 Algebraic condition

The previous introduction of generalized geometry has been proposed in general grounds for a $\text{Spin}_6$ manifold $M_6$, whithout discussing its setting in a physical theory. We expect the compactification mechanism on the space-time background (2.23) to give an effective four dimensional model for type II theories (or rather type II supergravity): in this subsection we discuss the corresponding requirements from the internal geometry point of view, which are best stated using the generalized geometrical language. In particular we will see how string theory naturally gives the necessary and sufficient elements to construct compatible pure spinors pairs.
Fermionic parameters in supergravity transform under the *reduced* maximal compact subgroup of the symmetry group. In the usual picture which looks at structures defined on $TM_6$, by recalling the decomposition of the spinor representation in ten dimensions under $\text{Spin}(1,3)\times\text{Spin}(6)$ proposed in (2.39), we obtain the pair $(\eta^1,\eta^2)$ which transform under $\text{Spin}(6)\times\text{Spin}(6)$. Each of the $\eta^i$ transform separately under the different $\text{Spin}(6)\simeq SU(4)$ groups which in turn separately reduce to two $SU(3)$ structures. As the decomposition of (2.39) feature two independent external spinors $(\xi^1,\xi^2)$ the four dimensional reduced theory would in general be an $\mathcal{N}=2$ theory. We can then summarize that

$$d=4,\ \mathcal{N}=2 \text{ effective theory} \quad\Leftrightarrow\quad M_6 \text{ admits a pair of } SU(3) \text{ structures}\,. \tag{3.74}$$

Notice that the two spinors $(\eta^1,\eta^2)$ span a two-dimensional subspace of the four-dimensional space of positive chirality $\text{Spin}(6)\simeq SU(4)$ spinors. Being this space invariant under $SU(2)\subset SU(4)$ rotations, under which both spinors are singlets, locally the presence of two $SU(3)$ structures defines an $SU(2)$ structure (3.5).
However, from the generalized geometry point of view, the pair of $SU(3)$ structures is most naturally described as a single generalized structure. The non-reduced structure group of $E$ is $O(6,6)$, and the existence of the pair $\Phi^\pm$, obtained from $(\eta^1,\eta^2)$ as in (3.66), reduces it to $SU(3)\times SU(3)$. From the point of view of generalized geometry we can state the following

$$d=4,\ \mathcal{N}=2 \text{ effective theory} \quad\Leftrightarrow\quad E \text{ admits an } SU(3)\times SU(3) \text{ structure}\,. \tag{3.75}$$

We actually expect that the reduced structure group is promoted to a local symmetry of the reformulated theory: if we had compactified on a torus $T^6$, we know that the low-energy theory has a local $O(6)\times O(6)$ symmetry, and a global $O(6,6)$ symmetry, reflecting the fact that the string theory has a T-duality symmetry. In principle, on $T^6$ any pair of constant spinors $(\eta^1,\eta^2)$ parameterizes a pair of preserved supersymmetries in four dimensions, and hence compactification gives an $\mathcal{N}=8$ effective theory. By isolating a single pair, this can be reformulated as an $\mathcal{N}=2$, where the local $O(6)\times O(6)$ symmetry should reduce to those symmetries that leave the pair invariant, a local $SU(3)\times SU(3)$. The following Figure 3.1 will serve as leading guide for the future purposes of reformulating the theory under a bigger duality group in Chapter 4.
We can summarize this as

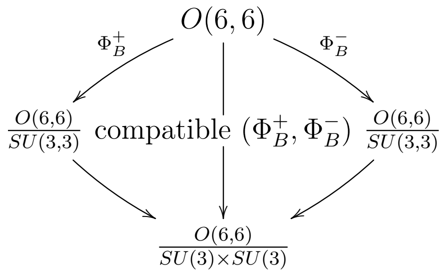


Figure 3.1: Algebraic structures in GG.

**Proposition 3.2** *To recover an $\mathcal{N} = 2$ effective theory in four dimensions, the generalized tangent bundle $E = TM_6 \oplus T^*M_6$ must have $SU(3) \times SU(3)$ reduced structure. Equivalently, the theory must admit a pair of compatible pure spinors $\Phi^{\pm}$ (3.66).*

## 3.10 The notion of integrability

Once we clarified how the existence of globally defined objects on the generalized tangent bundle yields a reduction of its structure group, we may demand differential conditions on them. A natural condition is integrability of the relevant algebraic structures. Once more, the analogy with ordinary complex geometry will be useful to discuss integrability for a GACS.

### 3.10.1 Integrability in the bracket language

Integrability conditions are commonly stated as an involution under a proper bracket. As a reminder, we refresh how it is defined for an ordinary almost complex structure. Once introduced the complexified tangent bundle $TM_6^{\mathbb{C}} = TM_6 \otimes \mathbb{C}$, when considering an ACS $I$ defined on it, we can consider its holomorphic subbundle $TM_6^{(1,0)}$ by restricting to the group of vector which satisfy $I^m{}_n v^n = iv^m$, or alternatively, if one defines the projector $P = \frac{1}{2}(\mathbb{1}_6 - iI)$

$$TM_6^{(1,0)} = \{v \in TM_6^{\mathbb{C}} \,|\, Pv = v\} \tag{3.76}$$

Integrability for ordinary complex geometry can then be formulated as follows: an ACS $I$ is integrable if and only if $TM^{(1,0)}$ is involutive under the Lie bracket

$$[TM_6^{(1,0)}, TM_6^{(1,0)}]_{\text{Lie}} \subset TM_6^{(1,0)} \tag{3.77}$$

or alternatively

$$\bar{P}[P(v), P(w)]_{\text{Lie}} = 0 \quad \forall\, (v, w) \in TM_6^{\mathbb{C}}\,. \tag{3.78}$$

We gave an alternative but equivalent condition in subsection 3.2, when arguing that the almost complex $GL(3,\mathbb{C})$ structure defined by $\Omega$ is indeed complex if and only if

$$d\Omega = \bar{W}_5 \wedge \Omega\,,. \tag{3.79}$$

and we discussed how this condition is satisfied for a Calabi-Yau manifold (3.25), and even for the non-Kähler background found in [194] (3.30).
In the perspective of generalize this, our aim is to give a suitable definition for the Lie bracket which can be easily generalized to the GCG case. The Lie bracket can be thought as a derived bracket [152], as we can define it, given two vectors $v, w \in TM_6$

$$[\{\iota_v, d\}, \iota_w] = \iota_{[v,w]_{\text{Lie}}} \tag{3.80}$$

where the brackets on the left hand side mean commutation/anti-commutation. All the variables in this equations should be interpreted as operators acting on differential forms. The most natural generalization on sections $A, B \in TM_6 \oplus T^*M_6$ is the Dorfman bracket [61]

$$[\{A\cdot, d\}, B\cdot] = A \circ B \tag{3.81}$$

which specializes to (having $A = v + \zeta$, $B = w + \eta$)

$$(v + \zeta) \circ (w + \eta) = [v, w]_{\text{Lie}} + \mathcal{L}_v\eta - \iota_w d\zeta\,. \tag{3.82}$$

We then define the Courant bracket [50, 51] as the anti-symmetrization of the Dorfman bracket (3.81)

$$\frac{1}{2}(A \circ B - B \circ A) = \frac{1}{2}\left([\{A\cdot, d\}, B\cdot] - [\{B\cdot, d\}, A\cdot]\right) \equiv [A, B]_{\text{Courant}}\,. \tag{3.83}$$

On explicit sections, we can compute

$$[v + \zeta, w + \eta]_{\text{Courant}} = [v, w]_{\text{Lie}} + \mathcal{L}_v\eta - \mathcal{L}_w\zeta - \frac{1}{2}d(\iota_v\eta - \iota_w\zeta)\,. \tag{3.84}$$

The most interesting feature of the Courant bracket it that it satisfies the Jacobi identity on isotropic subbundles $TM_6 \oplus T^*M_6$ (while there is in general no bracket which satifies it in general on $TM_6 \oplus T^*M_6$). It is pretty clear how from the expression of the Courant bracket one can recover the Lie bracket when specializing the first by taking $A = v\,, B = w$. The proper generalization of (3.78) is that a generalized almost complex structure $\mathcal{J}$ is integrable if its $i$-eigenbundle $\mathcal{L}_\mathcal{J}$ is closed under the Courant bracket

$$\bar{\Pi}[\Pi(v + \xi), \Pi(w + \eta)]_{\text{Courant}} = 0 \tag{3.85}$$

A common feature of Lie and Courant brackets is that they feature a differential in their definitions, the exterior differential $d$. The definition of a derived bracket allows for generalizations when considering an arbitrary differential [50, 152]. A possible extension is based on the fact that in type II theories without NS-fivebranes (in this case we have $dH = 0$), the differential $d_H \equiv d - H\wedge$ squares to zero, as it is for $d$ itself. Thus defining a twisted Dorfman bracket $A \circ_H B$, its antisymmetrization in $A, B$ in turn provides a definition of the twisted Courant bracket [188]

$$\frac{1}{2}(A \circ_H B - B \circ_H A) \equiv [A, B]_H \,. \tag{3.86}$$

which explicitly reads

$$[v + \zeta, w + \eta]_H = [v, w]_{\text{Lie}} + \mathcal{L}_v\eta - \mathcal{L}_w\zeta - \frac{1}{2}d(\iota_v\eta - \iota_w\zeta) + \iota_v\iota_w H \,. \tag{3.87}$$

Let just recall that the Lie bracket is invariant under diffeomorphisms: there are no further symmetries of $TM_6$ which preserve the Lie bracket. Whenever we look at the symmetries on $TM_6 \oplus T^*M_6$, the invariance under diffeomorphisms is preserved, but the richer structure of the bundle we are considering also allows for an additional symmetry, which is the $B$-tranform we already met (3.39). Indeed one can easily prove that

$$[e^B(v + \zeta), e^B(w + \eta)]_{\text{Courant}} = e^B[(v + \zeta), (w + \eta)]_{\text{Courant}} + \iota_v\iota_w dB \tag{3.88}$$

so $e^B$ is an automorphism of the Courant bracket if and only if B is closed [105]. For the twisted Courant bracket, a similar statement holds.

### 3.10.2 Integrability in pure spinor language

The principal reason why we rephrased every concept concerning GACS in the pure spinors language is that differential conditions one best stated in terms of these. We recall here some of the conclusions contained in [105].
Again following [100], we introduce the notion of integrability for a pure spinor. Let us consider an integrable $\mathcal{J}$, and two elements of $A, B \in \mathcal{L}_{\mathcal{J}}$. As the dual picture is defined by identifying the two annihilator spaces (3.47) and (3.62), this also means $[A, B]_{\text{Courant}} \in \mathcal{L}_\Phi$, which following the definition reads

$$0 = [A, B]_C\Phi = (AB - BA) \cdot d\Phi \,. \tag{3.89}$$

The first guess would be to consider $d\Phi = 0$, which in turn implies $[A, B]_C \in \mathcal{L}_\Phi = \mathcal{L}_{\mathcal{J}}$, and clearly corresponds to an integrable $\mathcal{J}$. We can however be less restrictive. Thinking of $A, B$ as gamma matrices, condition (3.89) is equivalent to impose that $d\Phi$ should be annihilated by two gamma matrices. Therefore that it must be at most at "level one" starting from the Clifford vacuum $\Phi$. We then recover

$$d\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J} \text{ integrable} \tag{3.90}$$

Consider we have a closed $\Phi$, which corresponds to a certain $\mathcal{J}$. Its twisting by a closed two-form $B$ is the dressed pure spinor $\Phi_B$ (3.67), which in turn correspond to the *twisting* of the original $\mathcal{J}$ by a $B$-transform (3.39). However, if $B$ is a B-field, that is not closed, $\Phi_B$ would not be closed under $d$ anymore. It would however be closed under a twisted differential

$$d_H = d - H\wedge\ . \tag{3.91}$$

We already commented in the previous subsection out how this differential has all the required properties to be used to define a derived bracket. It becomes now clear how the twisted and untwisted picture are equivalent: if a naked pure spinor is twisted closed, then the dressed pure spinor is closed under the ordinary exterior derivative, *i.e.*

$$0 = d_H\Phi = (d - dB\wedge)\Phi = e^B d(e^{-B}\Phi) = e^B d\Phi_B\ . \tag{3.92}$$

The differential (3.91) is the first case of twisted exterior derivative we meet, being the twisting due to the $B$-field as follows

$$d_H = e^B d e^{-B} \tag{3.93}$$

We would consider this as a first step towards the generalization to the twisting under all the field content (even including the R-R fluxes) which will be presented in next Chapter. As a conclusion, in the pure spinor language, twisted integrability reads

$$d_H\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J}\ \text{twisted integrable} \tag{3.94}$$

We can see the specialization of the pure spinor equation relative to $\mathcal{J}_I$ amounts to recover the already discussed equation (3.79). Moreover, when applying to the pure spinor related to an ASS $\mathcal{J}_J$ we simply find

$$dJ = 0\,. \tag{3.95}$$

Let us conclude by recalling a definition [105, 122] concerning a very specific case of integrable generalized complex structure in pure spinor language

**Definition 3.3** *A (twisted) generalized Calabi-Yau manifold à la Hitchin is a manifold that has a pure spinor closed under* $(d_H)\,d$*, whose norm does not vanish.*

## 3.11 Linking string vacua and integrability conditions

Having all the instruments needed to discuss necessary and sufficient conditions in order to have vacua in arbitrary backgrounds, we present how these can be described in a number of subcases for a different flux content. We would start by revisiting the fluxless case, for then discuss backgrounds of increasing complexity. We will consider the differential conditions imposed by requiring on-shell supersymmetry, or in other words, by demanding that the vacua are supersymmetric.
As we will show, these translate into integrability (or *non*-integrability) of some of the algebraic structures. First of all, let us make a preliminary choice for type II string theory with a pair of pure spinors of opposite chiralities, by putting an explicit dilaton factor in front of (3.66)

$$\Phi^+ = e^{-\phi}\eta^1_+ \otimes \eta^{2\dagger}_+ \,, \qquad \Phi^- = -ie^{-\phi}\eta^1_+ \otimes \eta^{2\dagger}_- \,. \tag{3.96}$$

in agreement with the decomposition (3.61).

**No fluxes** We stressed many times that in the absence of fluxes the integrability condition which is reinterpreted in terms of string vacua reduces to the familiar Calabi-Yau condition, which we presented for the very specific case $\eta^1 = \eta^2 \equiv \eta$ in (3.21), corrispondent to the case when the $SU(3)$ structure defined by $\eta$ is integrable, or in other words that the manifold has exactly $SU(3)$ holonomy [138]. The condition may also be stated that the complex and symplectic structures encoding the $SU(3)$ structure are both integrable.
However, this is a very specific case, as generic GACS reduces on the tangent bundle to a structure that is locally a product of lower dimensional complex and symplectic structures (3.73). Furthermore, as reviewed in 3.9, a pure spinor compatible pair ($\Phi^\pm$) defines an $SU(3)\times SU(3)$ structure. In this generic case, the requirement correspondent to a vacuum with no fluxes translates into the following differential relation on the pure spinors (3.96)

$$d\Phi^+ = 0 \;, \qquad d\Phi^- = 0 \;, \tag{3.97}$$

which means that both GACS are integrable (and both canonical bundles are trivial), or in other words that the $SU(3) \times SU(3)$ structure is integrable. Notice that in the general case these define a $\mathcal{N} = 2$ vacuum. Notice that manifolds satisfying (3.97) are obviously generalized Calabi-Yau[12].

**NS-NS fluxes only** The conditions arising varies depending on the number of supersymmetries we consider. Indeed, for vacua preserving four-dimensional $\mathcal{N} = 2$ supersymmetry in the presence of NS-NS fluxes, supersymmetry conditions in the presence of $H$-flux amount precisely to $H$-twisting the generalized Calabi-Yau metric condition (3.97), as they should satisfy [136]

$$d_H\Phi^\pm = 0 \,. \tag{3.98}$$

[12] In addition to be generalized Calabi-Yau manifolds à la Hitchin [122], as these satisfy Definition 3.3, these are also generalized Calabi-Yau à la Gualtieri [105] in that both pure spinors are closed.

*i.e.* they require $H$-twisted generalized Calabi-Yau metric structures.
We explicitly presented vacua with $\mathcal{N}=1$ supersymmetry in the presence of NS fluxes in subsection 3.4, discussing a case originally first analyzed in [194]. The equations (3.33) were reinterpreted in the language of $G$-structures in [85] as the following pure spinors conditions

$$\begin{aligned} d_H(e^{-\phi}\Phi^-) &= 0 \ , \\ d(e^{-\phi}\Phi^+) &= ie^{-2\phi} *_6 H \end{aligned} \tag{3.99}$$

Note that in the second equation $H$ does not enter as a twisting in the standard way, and therefore the even pure spinor is not twisted integrable. It would be interesting to get the right GCG description of $\mathcal{N}=1$ vacua with NS fluxes.

**NS-NS and R-R fluxes** The inclusion of the R-R degrees of freedom is presented in [98] as a generalization of the analysis used when only the NS-NS flux arise [72], following the prescription of decomposing the complete flux content in $SU(3)$ representations (which can be corrispondingly generalized to $SU(3)\times SU(3)$ [99]). Out of the possible compactifications with space-time splitting (2.23), we will focus in the following to compactifications on Mink$_4$ space preserving $\mathcal{N}=1$ supersymmetry in the presence of NS-NS and R-R fluxes. This compactification generally requires the spacetime to be a warped product (2.24) [99].
The preserved external spinor can be parameterized by means of the scalars $n_I=(a,\bar{b})$ as $\xi^1_-=a\,\xi_-$, $\xi^2_-=\bar{b}\,\xi_-$, so that the original ansatz (2.39)

$$\begin{aligned} \epsilon^1 &= a\,\xi_-\otimes\eta^1_+ + \text{c.c.}\,, \\ \epsilon^2 &= \bar{b}\,\xi_\pm\otimes\eta^2_- + \text{c.c.}\,. \end{aligned} \tag{3.100}$$

where we assume $|\eta^1|^2=|\eta^2|^2=1$ (while $|a|$ and $|b|$ are related to the warp factor, as we will see). For this reason, by shifting the originally normalized Cliff(6) internal spinors as $\eta^1\to a\eta^1$, $\eta^2\to b\eta^2$ we would recover the following pure spinors

$$\Phi'^+ = 2a\bar{b}\,\Phi^+ \ , \quad \Phi'^- = 2ab\,\Phi^- \ . \tag{3.101}$$

The vector $n_I$ distinguishes a $U(1)_R\subset SU(2)_R$ such that any triplet can be written in terms of a $U(1)$ complex doublet and a $U(1)$ singlet by means of the vectors[13]

$$\begin{aligned} (z^+,z^-,z^3) &= n_I(\sigma^a)^{IJ}n_J = (a^2,-\bar{b}^2,-2a\bar{b}) \ , \\ (r^+,r^-,r^3) &= n_I(\sigma^a)^I{}_J\bar{n}^J = (ab,\bar{a}\bar{b},|a|^2-|b|^2) \ . \end{aligned} \tag{3.102}$$

The conditions for flux vacua have been obtained in the language of GCG either using the ten-dimensional gravitino and dilatino variations [99], or by extremizing the superpotential of the four-dimensional $\mathcal{N}=1$ theory and setting the D-term to zero [41, 149]. For

[13] We introduce here a composite three-index $SU(2)$ notation we will largely use in the next Chapter: for instance for a vector $z$, $z^\pm$ denotes $z^1\pm iz^2$, while the third index is left as it is.

the case $|a| = |b|$, which arises when sources are present,

**Proposition 3.4 (Differential condition)** *The necessary and sufficient differential conditions for having an* $\mathcal{N} = 1$ *vacuum of type II theories compactified on a space-time* (2.23) *with* $M_{1,3}$ =*Mink*$_4$ *when both NS-NS and R-R fluxes have non-trivial expectation value is that the pair of pure spinors* (3.101) *satisfies the equations [99]*

$$d_H(e^{2A}\Phi_1) = 0 \tag{3.103}$$

$$d_H(e^{A}\text{Re}\Phi_2) = 0 \tag{3.104}$$

$$d_H(e^{3A}\text{Im}\Phi_2) = e^{4A} *_6 s(F^+) \tag{3.105}$$

*where*

$$\Phi_1 = \Phi'^{\pm}, \quad \Phi_2 = \Phi'^{\mp} \quad \text{for IIA/IIB (upper/lower sign respectively).} \tag{3.106}$$

*and have norms*

$$|a|^2 + |b|^2 = e^A \ . \tag{3.107}$$

Conditions (3.103)-(3.105) can be understood as coming from F and D-term equations. Equation (3.104) corresponds to imposing $\mathcal{D} = 0$, while (3.103) and (3.105) come respectively from variations of the superpotential with respect to $\Phi_2$ and $\Phi_1$.
The susy condition (3.103) says that the GACS $\mathcal{J}_1$ (corresponding to $\Phi_1$) is twisted integrable, and furthermore that the canonical bundle is trivial. The required manifold is a twisted generalized Calabi-Yau (see footnote 12). As commented in subsection 3.2, for type IIA, (3.103) specialized to (3.96) states the manifold should is twisted symplectic, while for type IIB it implies it should be rather twisted complex.
The other GACS appearing in (3.104)-(3.105) is "half integrable", *i.e.* its real part is closed, while the non-integrability of the imaginary part is due to the R-R fluxes. We note here that when considering the limit of (3.103)-(3.105) for R-R fluxes going to zero we recover (3.98) (for $F = 0$, (3.103)-(3.105) imply $A = 0$), *i.e.* $F \to 0$ is a singular limit of (3.103) where supersymmetry is enhanced to $\mathcal{N} = 2$.
In Chapter 2 we saw that, on top of supersymmetry conditions (3.103)-(3.105), the fluxes must satisfy the Bianchi identities (2.35) in the absence of sources ($\delta = 0$), while in the presence of D-branes or orientifold planes, the right hand sides get modified by the appropriate charge densities.
In the language of calibrations, the form $e^{3A-\phi}\text{Im}\,\Phi^2$ calibrates the cycle wrapped by a spacetime-filling brane or an orientifold [71,159,169,170]. By using the following property of the Mukai pairing (3.60)

$$\int \langle A, d_H B\rangle = \int \langle d_H A, B\rangle \tag{3.108}$$

integrating the following quantity and making use of (3.105) we obtain

$$\int \langle d_H F, e^{3A-\phi}\text{Im}\,\Phi^2\rangle = \langle F, d_H e^{3A-\phi}\text{Im}\,\Phi^2\rangle = \frac{1}{8}\int e^{4A}\langle F, \star_6 s(F)\rangle \,. \tag{3.109}$$

The term in the right hand side is always positive[14] [100, 165], which means that the scalar part of Bianchi identity for $F$ (2.35) has a positive contribution on a compact space. Negatively charged objects are then required to cancel this: in string theory, orientifold planes (O-planes) are such items. It was worth to make this comment as it constitutes an unavoidable consistency requirement easily deduced from the pure spinor equations, but we would not consider any explicit example of compactification which uses O-planes: indeed, the approach we follow here and in the next Chapter will be in term of a *local* description, which can be thought to be valid away from source loci.
The set of equations (3.103)-(3.105) has been generalized to $\mathcal{N} = 1$ compactification to $AdS_4$ [99]. Furthermore, extension to the heterotic case has been recently proposed [9]. The pure spinor formalism has been largely used to determine new solutions [161, 162], and to study deformations [148, 149, 169]. We also remark that this construction can also be extended to dimensions different from six [112, 159].
Once established the generalized Calabi-Yau nature of the internal manifold $M_6$ in supersymmetric solutions, the search of a suitable vacuum should in principle scan the various elements lying in this class. A set of candidate vacua which has been largely examined are nilmanifolds and solvmanifolds, as it has been proved that these are indeed generalized Calabi-Yau [46]. Related work has been done in studying both supersymmetric [100] and non-supersymmetric [10] vacua allowing for this internal structure. Recently, some effort have been devoted to reformulate the pure spinor equation for non-supersymmetric compactifications, thought the status of the approach is still at a very preliminary level [10, 158].
Despite the obvious advantages one gets in adopting this tool, the formalism shows various limitations. A straighforward question is whether one may enlarge the framework of generalized geometry in order to obtain a set of equations which read as *closed* differentials of certain algebraic structures. Naively, one may think that the explicit appearance of the R-R terms in the right hand side of (3.105) depends on the fact these are the only gauge fields which cannot be included in generalized geometry. From this point of view, generalized geometry fits in as an intermediate step: the complete program would be accomplished in a reformulation which fully encodes the flux content of type II theories in a generalized geometrical sense.

[14] We are using the following convention for Mukai pairing (3.60)

$$\langle A_k, B_{6-k} \rangle = \frac{1}{8}(-)^{k+1}(\not{A}_k *_6 \not{B}_{6-k})\text{vol}_6 \,. \tag{3.110}$$

According to our Hodge dualization conventions listed in Appendix A, we are using here the opposite convention with respect to the one adopted in [100]. Indeed, given a $p$-form $\omega$, its Hodge dual reads

$$*_6\, \omega = \frac{1}{(6-p)!}\epsilon^{i_{p+1}\dots i_d}{}_{i_1\dots i_p}\omega^{i_1\dots i_p} \,. \tag{3.111}$$

# Chapter 4

# Exceptional Generalized Geometry

We discussed in the previous Chapter the necessity of introducing a geometry defined on an extended tangent bundle in order to geometrize the NS-NS sector and how this allows to characterize more formal flux vacua.
The differential forms $\Phi_{1,2}$ determine the metric $g$ and the $B$-field on one hand, and on the other hand for supersymmetric vacua they obey the pure spinor equations (3.103)-(3.105). The last equation (3.105) describes how the R-R fluxes $F$ represents an obstruction to integrability of $\Phi_2$. We introduced the notion of integrability first in terms of derived brackets, and thed restated it using a (twisted) differential operator, which for ordinary generalized geometry is $d_H$.
As explained in subsection 3.5, we established that $E$ has a natural inner product preserved by the action of $O(6,6)$. To our present knowledge of string theory, T-duality invariance of type II theories belongs to the bigger set of much larger and hidden symmetries of type II supergravity. It was conjectured [129] that the duality group of the full string theory can be extended to the U-duality group $E_{7(7)}$ when compactified on a six-dimensional torus. Furthermore, the U-duality group acts on $g$ and $B$ together with the R-R gauge fields. Exceptional Generalized Geometry (EGG) [8, 97, 128, 176] is a candidate extension of the $O(6,6)$ (T-duality) covariant formalism of generalized geometry to an $E_{7(7)}$ (U-duality) covariant one, such that the R-R fields are incorporated into the geometry. The analysis presented in this Chapter is a rewriting of the supersymmetry differential equations in this enlarged formalism. To do so, we should first ensure to identify correctly the structures which are candidate to geometrically encode the supersymmetric background. Such an identification has been already studied from the $\mathcal{N}=2$ supergravity moduli space point of view [97]. In the same paper, a preliminary analysis of the differential conditions was proposed, in which, despite the full set of equations could not be completely linked to supersymmetry, a subset of it was able to reproduce the pure spinor equations. We will provide here a refinement of candidate $E_{7(7)}$-covariant equations which on one hand correspond to a rewriting of the pure spinor equations (3.103)-(3.105), which feature both NS-NS and R-R fluxes, and on the other hand can be easily compared to supersymmetry. We will start from a brief review of the electric-magnetic duality in field theories, for then moving to the rigorous formulation of Exceptional Generalized Geometry following

the lines of Chapter 3, first examining the algebraic conditions to admit an effective $\mathcal{N}=2$ theory in this language, and then discussing the differential conditions a vacuum configuration should satisfy. We will first of all investigate the $\mathcal{N}=1$ case, and then study the $\mathcal{N}=2$ one.

## 4.1 U-duality

Maxwell's equations in the vacuum are invariant under the exchange of the electric and magnetic vector fields $E \to B, B \to -E$. In covariant notation, this duality relation is expressed by the exchange of the gauge field-strenght and its Poincaré dual

$$\tilde{F} = *_4 F \tag{4.1}$$

Maxwell's vacuum can be written as

$$dF = 0\,, \tag{4.2}$$

$$d *_4 F = 0\,. \tag{4.3}$$

where the first equation is usually interpreted as a Bianchi identity for the curvature, and the second is its equation of motion. These can equivalently be rewritten in a symmetric way as

$$dF = 0\,, \tag{4.4}$$

$$d\tilde{F} = 0\,. \tag{4.5}$$

for which there is a *duality invariance* under the following transfomation

$$F \to *_4 F = \tilde{F}\,, \tag{4.6}$$

$$\tilde{F} \to *_4^2 F = -F\,. \tag{4.7}$$

Despite the fact that the equations of motion are invariant under this duality, this is not the case for the Lagrangian

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) = *_4(F \wedge \tilde{F})\,. \tag{4.8}$$

as it flips sign under the duality transformation.
A generalized form of duality transformation for abelian vector fields valid in presence of matter couplings can be straightforwardly proposed. Let us consider a Lagrangian containing $n_V$ abelian vectors through their field-strenghts $F^I$ and arbitrary coupling to other fields $\varphi^i$

$$\mathcal{L}(F^I, \varphi^i, \partial_\mu \varphi^i) \tag{4.9}$$

Being the vector field curvatures $F^I = dA^I$, the following Bianchi identities hold

$$dF^I = 0\,, \tag{4.10}$$

while, dualizing the fields

$$G_I = \frac{\partial \mathcal{L}}{\partial F^I}\,, \tag{4.11}$$

we describe the equations of motion for the $A^I$ as simple dual Bianchi identities

$$dG_I = 0\,. \tag{4.12}$$

We can summarize the set of equations of motion for the original $n_V$ vector fields and their duals as

$$dF^I = 0\,, \tag{4.13}$$
$$dG_I = 0\,. \tag{4.14}$$

In principle these equations are obviously invariant under any general linear transformation mixing the vector $(F^I, G_I)$

$$\begin{pmatrix} F^{I'} \\ G'_I \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^I \\ G_I \end{pmatrix} \tag{4.15}$$

where $\mathcal{S} \in GL(2n_V, \mathbb{R})$. Some consistency constraints on the matrix $\mathcal{S}$ have however to be imposed. First, we need to ensure that the definition of the transformed $G'_I$ is defined in terms of $F^{I'}$ as

$$G'_I = \frac{\partial \mathcal{L}}{\partial F^{I'}}\,. \tag{4.16}$$

Also, we need the duality transformation to preserve the equation of motion of the $\varphi^i$ fields. The restricted transformations define the *U-duality* group $\mathcal{U}$, $\mathcal{U} \subset \mathcal{S} \in GL(2n_V, \mathbb{R})$ as [77]

$$\mathcal{U} \subset \mathrm{Sp}(2n_V, \mathbb{R})\,. \tag{4.17}$$

Being this argument completely general, we can apply it to supergravity theories. The isometry group $G$ of the scalar manifold can be extended to act on vectors as a group of duality transformations if and only if $G \subset \mathrm{Sp}(2n_V, \mathbb{R})$. For $\mathcal{N} \geq 3$ this restricts the type of allowed scalar manifolds. The geometry is constrained in such a way that the scalar manifold is always a coset $G/H$. For $\mathcal{N} = 8$ there are $n_V = 28$ vectors, so we expect $G \subset \mathrm{Sp}(56, \mathbb{R})$. The effective action for type II string theory compactified on a six-torus is $\mathcal{N} = 8$ supergravity, which was indeed found to exhibit a $G = E_{7(7)}$ duality symmetry [52]. This is the duality symmetry which we would consider at the supergravity level, as it corresponds to the low energy limit of string theory in which the present analysis is performed.

## 4.2 Basic $E_{7(7)}$ theory

The group $E_{7(7)}$ can be defined as the subgroup of $\mathrm{Sp}(56,\mathbb{R})$ which preserves a particular symmetric quartic invariant $\mathcal{Q}$, in addition to the symplectic product $\mathcal{S}$ [52, 97, 176]. Decomposition under different subgroups can be used in order to study particular aspects of the theory. Out of the many, two of them will be relevant for the purposes of our analysis.

1. $SL(2,\mathbb{R}) \times O(6,6) \subset E_{7(7)}$ is the physical subgroup appearing as the factorization of S-duality[1] ant the T-duality group relevant in the framework of generalized geometry.

2. $SL(8,\mathbb{R}) \subset E_{7(7)}$ contains the product $SL(2,\mathbb{R}) \times GL(6,\mathbb{R})$, which can be in turn used to make contact with the decomposition under $SU(8)/\mathbb{Z}_2$. This is the reduced group under which the spinors transform in the U-duality perspective [128], and therefore the natural language to formulate supersymmetry via the Killing spinor equations.

In the first decomposition the fundamental representation decomposes as

$$\begin{aligned} \mathbf{56} &= (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{32}) \\ \lambda &= (\lambda^{iA}, \lambda^{+})\,, \end{aligned} \tag{4.18}$$

In order to explicit realize the program of extending the generalized tangent bundle to allow a natural action of the group $E_{7(7)}$, we use the decomposition under the $GL(6,\mathbb{R})$ subgroup (3.38), which describes the transformation of the fundamental **12** representation of $O(6,6)$. The only additional information we need is the transformation of an $SL(2,\mathbb{R})$ doublet $w^i$ under $GL(6,\mathbb{R})$, which we can embed as

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} (\det a)^{-1/2} & 0 \\ 0 & (\det a)^{1/2} \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \tag{4.19}$$

To decompose the **56** representation under the subgroup $O(6,6) \times SL(2,\mathbb{R})$, one should collect the transformation of $\lambda = (\lambda^{iA}, \lambda)$ to recover[2]

$$\begin{aligned} F = (\Lambda^6 T^*M)^{-1/2} \otimes \Big[ & TM \oplus T^*M \\ & \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \oplus \Lambda^{\text{even}} T^*M \Big] \end{aligned} \tag{4.20}$$

[1] The $SL(2,R)$ here is the "heteroitic S-duality", where the complex field that transforms by fractional linear transformations is $S = \tilde{B} + ie^{-2\phi}$, being $\tilde{B}$ a six form dual to $B_2$ (for which once more we will drop the subscript in the present Chapter), and the T-duality group which emerges in the framework of generalized geometry.

[2] In this Chapter we will discuss in detail type IIA theory, but most of the statements can be easily changed to type IIB.

It is worth to notice that the first line corresponds to the ordinary generalized tangent bundle $E$ (3.34).
Although it is useful to keep contact with the $O(6,6)$ picture in order to compare this formalism with the GCG one discussed in Chapter 3, the decomposition subgroup $SL(8,\mathbb{R})$ is particularly useful whenever a comparison with supersymmetry is needed. The decomposition of the relevant $E_{7(7)}$ representations in terms of $SL(8,\mathbb{R})$ are the following.
For the fundamental **56** we have

$$\begin{aligned} \nu &= (\nu^{ab}, \tilde{\nu}_{ab}) \\ \mathbf{56} &= \mathbf{28} + \mathbf{28}' \ . \end{aligned} \tag{4.21}$$

with $\nu^{ba} = -\nu^{ab}$.
Of particular importance would be, together with the fundamental, the adjoint representation **133**. Decomposing the adjoint **133** representation of $E_{7(7)}$ under $O(6,6)\times SL(2,\mathbb{R})$, we have

$$\begin{aligned} \mathbf{133} &= (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}') \\ \mu &= ({\mu^i}_j \ , \ {\mu^A}_B \ , \ \mu^{i-}) \end{aligned} \tag{4.22}$$

where $i = 1, 2$ is a doublet index of $SL(2,\mathbb{R})$, raised and lowered with $\epsilon_{ij}$, and the $O(6,6)$ fundamental indices $A, B = 1, ..., 12$ are raised and lowered with the metric $\mathcal{I}$ (3.36). For the purpose of constructing the algebraic structures, it is worth to decompose the adjoint representation under $SL(8,\mathbb{R})$ as

$$\begin{aligned} \mathbf{133} &= \mathbf{63} + \mathbf{70} \\ \mu &= ({\mu^a}_b, \mu_{abcd}) \end{aligned} \tag{4.23}$$

where ${\mu^a}_a = 0$ and $\mu_{abcd}$ is fully antisymmetric. Another representation, the **912** will be also relevant in the following, and we postpone its introduction in subsection 4.7.
Once we presented the fundamental and adjoint representations in the two main decompositions we illustrate how we can introduce objects belonging to these representations. We first of all focus on the gauge field content of type II theories and discuss how this can be entirely encoded in the Exceptional Generalized Geometry (EGG for short) setting. In order to do that, it is helpful to discuss in detail how it transforms under T-duality.

## 4.3 The gauge field embedding

Among the various symmetries string theory is known to exhibit, T-duality has been one of the first to be discovered. In simple terms, its action exchanges the size of the compactified space into its inverse in string unit.
Restricting ourselves to type II theories, we refresh how the symmetry acts on NS-NS and R-R fields. For the NS-NS sector, T-duality invariance can be easily inferred from the

rewriting of the corresponding actions (2.1) and (2.12) of Chapter 2, when considering a toroidal compactification on $T^6$ as [163]

$$\mathcal{L}_{\rm NS} = \frac{1}{8\kappa^2}{\rm Tr}\left(\partial_\mu M^{-1}\partial^\mu M\right) \tag{4.24}$$

being $M = \mathcal{I}G$, and $G$ the generalized metric (3.55). The story is more subtle for the R-R potentials. As a subgroup of the U-duality group, the T-duality group $O(6,6)$ is indeed the maximum subgroup which transforms NS-NS and R-R fields into themselves.
By decomposing $E_{7(7)}$ with respect to $O(6,6)$ it has been also shown that Majorana-Weyl representations of $O(6,6)$ should appear in the R-R sector [187]. However it is not straightforward to show that these directly transforms as Majorana-Weyl spinors [35,36]. To show this it is necessary to combine them with the NS-NS two-form $B$ to get new fields that have simple transformation properties under $O(6,6)$: the R-R action plus the Chern-Simons terms after toroidal compactification on $T^6$ is manifestly invariant under $O(6,6)$ *if and only if* the R-R fields transform as a Majorana-Weyl spinor [76].
The main reason to use the Exceptional Generalized Geometry formalism is to give a geometrization of the R-R gauge fields as well as the NS-NS ones. We saw in Chapter 3 that a particular $O(6,6)$ action (3.39), has a natural geometric interpretation in terms of shifts of the $B$-field. In EGG formulated for type IIA[3], shifts of the $B$-field as well as shifts of the sum of internal R-R fields $C^- = C_1 + C_3 + C_5$ , which transforms as a chiral $O(6,6)$ spinor, correspond to particular $E_{7(7)}$ *adjoint* actions. To form a set of gauge fields that is closed under U-duality, we also have to consider the shift of the six-form dual to $B$, which we will call $\tilde{B}$[4] (See also footnote 1) The $B$-transform action (3.39) naturally embeds in ${\mu^A}_B$, while the $C$-transformations belong to one of the two $\mathbf{32'}$ representations. Let us call $v^i$ the $SL(2,\mathbb{R})$ vector pointing in the direction of the $C$-field, which we can take without loss of generality to be

$$v^i = (1,0)\ . \tag{4.25}$$

The $GL(6,\mathbb{R})$ assignments of the different components shown in Appendix B.2, indicate that the shift symmetries are given by the following sum of generators

$$\left(\tilde{B}v^iv_j, \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, v^iC^-\right) \equiv \mathcal{A} \tag{4.26}$$

where $v_i = \epsilon_{ij}v^j$. Using (B.4) it is not hard to show that given this embedding we recover the following commutation relations

$$\left[B + \tilde{B} + C^-, B' + \tilde{B}' + C^{-\prime}\right] = 2\langle C^-, C^{-\prime}\rangle + B\wedge C^{-\prime} - B'\wedge C^-\ , \tag{4.27}$$

where the first term on the rhs is a six-form and therefore corresponds to a $\tilde{B}$ transformation, and the other two to an R-R shift.

[3] See footnote 2.

[4] Equivalently these are shifts of the dual axion $B_{\mu\nu}$.

We can as well identify the embedding of the gauge fields (4.26) in $SL(8,\mathbb{R})$ by using the $GL(6,\mathbb{R})$ properties of the different components of the adjoint representation given in (B.18). We get[5]

$$\mathcal{A} = \Big(e^{2\phi}\tilde{B}v^i v_j - v^i e^{\phi} C_m + e^{\phi}(*C_5)^m v_i \, , -\tfrac{1}{2}e^{\phi}C_{mnp}v_i - \tfrac{1}{2}B_{mn}\epsilon_{ij}\Big) \ , \tag{4.28}$$

or in other words

$$\begin{aligned} \mathcal{A}^1{}_2 = -e^{2\phi}\tilde{B} \ , \quad \mathcal{A}^1{}_m = -e^{\phi}C_m \ , \quad \mathcal{A}^m{}_2 = -e^{\phi}(*C_5)^m \\ \mathcal{A}_{mnp2} = \tfrac{1}{2}e^{\phi}C_{mnp} \ , \quad \mathcal{A}_{mn12} = -\tfrac{1}{2}B_{mn} \end{aligned} \tag{4.29}$$

where the factors and signs are chosen in order to match the supergravity conventions. Here and in the following, $*$ refers to a six-dimensional Hodge dual, while we use $\star$ for the eight-dimensional one.

## 4.4 $E_{7(7)}$ algebraic structures from spinor bilinears

The gauge fields which we previously embedded in the adjoint representation define an $SU(8)/\mathbb{Z}_2$ structure on $F$ [128]. As already commented in subsection 3.9, given a generic covariant theory with respect to a particular duality, its spinors transform under the maximal compact subgroup of the duality group. In the GCG case, this (non-reduced) subgroup is $O(6) \times O(6)$, which acts on the pair $(\eta^1, \eta^2)$. We also discussed how by tensoring two $O(6)$ spinors we can obtain the geometric quantities relevant for generalized geometry, *i.e.* the pure spinors pair $\Phi^{\pm}$, which transforms under the maximal compact subgroup of the reduced structure group $SU(3) \times SU(3)$. We want to generalize these considerations to an U-duality covariant formalism, for which the relevant, non-reduced group is $SU(8)$.
Let us focus again on type IIA theories for concretessness. The $\mathcal{N}=2$ supersymmetries of type II theories, assuming the spacetime splitting (2.23), can be generally parameterized as

$$\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} = \xi^1_- \otimes \theta^1 + \xi^2_- \otimes \theta^2 + \text{c.c.} \tag{4.30}$$

where $\theta^1, \theta^2$ are $SU(8)$ spinors. A generic pair $(\theta^1, \theta^2)$ defines to an $SU(6) \subset SU(8)$ structure. Notice that, provided we want to match (2.39), the two Spin(6) spinors must be diagonally embedded in the pair $(\theta^1, \theta^2)$ as

$$\theta^1 = \begin{pmatrix} \eta^1_+ \\ 0 \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} 0 \\ \eta^2_- \end{pmatrix} . \tag{4.31}$$

[5]To avoid introducing new notation, we are using the same as in (4.26), in particular $v_i \equiv \epsilon_{ij}v^j$, although indices in $SL(8,\mathbb{R})$ are raised and lowered with the metric $\hat{g}$ given in (B.16).

From now on, we will refer to (4.31) as the *restricted* ansatz. It is easy to see how in this case the $O(6,6)$ bi-spinors might be embedded in $SU(8)$ bi-spinors[6], and how these may arrange in the EGG formalism (see Appendix A.2 for conventions on transposed and conjugate spinors)

$$\theta^{1\alpha}\theta^{2\beta} = \begin{pmatrix} 0 & \eta^1_+ \otimes \eta^{2\dagger}_+ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Phi^+ \\ 0 & 0 \end{pmatrix} \in \mathbf{28} \subset \mathbf{56}\,, \tag{4.32}$$

$$\theta^{1\alpha}\bar{\theta}_{2\beta} = \begin{pmatrix} 0 & \eta^1_+ \otimes \eta^{2\dagger}_- \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Phi^- \\ 0 & 0 \end{pmatrix} \in \mathbf{63} \subset \mathbf{133}\,. \tag{4.33}$$

when we have written first the $SU(8)$ and then the $E_{7(7)}$ respresentation they belong to. We see that the two pure spinors necessarily belong to different representations. In the specific case of (4.31), relevant for GCG, these define an $SU(3)\times SU(3) \subset SU(4)\times SU(4)$ structure, diagonally embedded in $SU(6)$.
A generic $SU(6)$ structure is parameterized by four $O(6) \simeq SU(4)$ spinors $(\eta^1_+, \tilde{\eta}^1_-, \eta^2_-, \tilde{\eta}^2_+)$ as follows[7]

$$\theta^1 = \begin{pmatrix} \eta^1_+ \\ \tilde{\eta}^1_- \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} \tilde{\eta}^2_+ \\ \eta^2_- \end{pmatrix}\,. \tag{4.34}$$

We can take the $SU(4)$ spinors to be normalized to 1, and we choose the $SU(8)$ spinors to be orthonormal, namely

$$\bar{\theta}_I\, \theta^J = \delta_I{}^J\,. \tag{4.35}$$

where $I = 1,2$ is a fundamental $SU(2)_R$ index. The two $\theta^I$ spinors can be combined into the following $SU(2)_R$ singlet and triplet combinations, lying respectively in the fundamental **56** and in the adjoint **133** representations, and generalizing in turn (4.32)-(4.33)

$$L = e^{-\phi}\epsilon_{IJ}\theta^I\theta^J\,, \qquad K_a = \frac{1}{2}e^{-\phi}\sigma_{aI}{}^J\theta^I\bar{\theta}_J\,, \qquad K_0 = \frac{1}{2}e^{-\phi}\delta_I{}^J\theta^I\bar{\theta}_J\,, \tag{4.36}$$

Here we also introduced the quantity $K_0$, which will be of particular relevance for the $\mathcal{N} = 1$ case[8]. The triplet $K_a$ satisfies the $su(2)$ algebra with a scaling given by the dilaton, *i.e.*

$$[K_a, K_b] = 2ie^{-\phi}\epsilon_{abc}K_c \tag{4.37}$$

$L$ and $K_a$ are the $E_{7(7)}$ structures that play the role of the generalized almost complex structures $\Phi^+$ and $\Phi^-$. They belong respectively to the **28** and **63** representations of

[6]For notational convenience, we would not display the tensor product when tensoring $SU(8)$ spinors, but we will maintain it for $SU(3)$ spinor for consistency with the GCG picture of Chapter 3.

[7]Note that an $SU(6)$ structure can be built out of a single globally defined internal Spin(6) spinor $\eta$, taking $\eta^1 = \eta^2 = \eta$ and $\tilde{\eta}^1 = \tilde{\eta}^2 = 0$.

[8]Notice that even in this case we introduced a dilaton factor, as it was for the pure spinors (3.96) discussed in the GG case.

$SU(8)$, which are in turn part of the **56** and **133** representations of $E_{7(7)}$. Using the decompositions $\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$ and $\mathbf{133} = \mathbf{63} + \mathbf{35} + \overline{\mathbf{35}}$ shown in (A.12) and (A.13), they read

$$L = \left(e^{-\phi}\epsilon_{IJ}\theta^{I\alpha}\theta^{J\beta}, e^{-\phi}\epsilon_{IJ}\theta^{I*}_{\alpha}\theta^{J*}_{\beta}\right) \qquad K_a = \left(e^{-\phi}\tfrac{1}{2}\sigma_{aI}{}^{J}\theta^{I\alpha}\bar{\theta}_{J\beta}, 0, 0\right) \ . \tag{4.38}$$

In the following we give the explicit expressions of the bilinears (4.36) in the two main cases corresponding to the restricted ansatz (4.31) and the general ansatz (4.34).

### 4.4.1 Restricted ansatz

Using the ansatz (4.31), the definitions (4.36) give

$$L = \begin{pmatrix} 0 & \Phi^+ \\ -s(\bar{\Phi}^+) & 0 \end{pmatrix} \tag{4.39}$$

where the operation $s$ is introduced in (3.60), and $\Phi^+$ (as well as $\Phi^-$ which will appear below) is defined in (3.96) while for $K_\pm = K_1 \pm iK_2$ we get (see footnote 13 for index notation)

$$K_+ = \begin{pmatrix} 0 & \Phi^- \\ 0 & 0 \end{pmatrix} , \qquad K_- = \begin{pmatrix} 0 & 0 \\ -s(\bar{\Phi}^-) & 0 \end{pmatrix} , \tag{4.40}$$

and finally for $K_3$

$$K_3 = \begin{pmatrix} \Phi_1^+ & 0 \\ 0 & -\bar{\Phi}_2^+ \end{pmatrix}$$

where we used the following definition

$$\Phi_1^+ = e^{-\phi}\eta^1_+ \otimes \eta^{1\dagger}_+ \ , \qquad \Phi_2^+ = e^{-\phi}\eta^2_+ \otimes \eta^{2\dagger}_+ \ , \tag{4.41}$$

We see that $L$ contains the pure spinor $\Phi^+$, while $K_+$ is built from the pure spinor $\Phi^-$ (as expected from the chiralities of the spinor bundle respectively embedded in the fundamental and in the adjoint representations). $K_3$ contains on the contrary the even-form bilinears of the same $SU(4)$ spinor, or in other terms the symplectic structures defined by each spinor.

### 4.4.2 General ansatz

Using the parameterization (4.34) for a generic $SU(6)$ structure, we can make again contact with the pure spinor picture of GCG to get

$$L = \begin{pmatrix} \Lambda^- + s(\Lambda^-) & \Phi^+ - s(\tilde{\Phi}^+) \\ -s(\bar{\Phi}^+) + \bar{\tilde{\Phi}}^+ & \bar{\Lambda}'^- + s(\bar{\Lambda}'^-) \end{pmatrix} \tag{4.42}$$

where $\tilde{\Phi}^+$ (and $\tilde{\bar{\Phi}}^+$, which we will recover below) is defined in an analogous way as $\Phi^+(\Phi^-)$,

$$\tilde{\Phi}^+ = e^{-\phi}\tilde{\eta}^1_+ \otimes \tilde{\eta}^{2\dagger}_+, \qquad \tilde{\Phi}^- = e^{-\phi}\tilde{\eta}^1_+ \otimes \tilde{\eta}^{2\dagger}_- \tag{4.43}$$

where we used the operation $s$ defined in (2.28) , and we have defined

$$\Lambda^\pm = e^{-\phi}\eta^1_+ \otimes \tilde{\eta}^{2\dagger}_\pm \ , \quad \Lambda'^\pm = e^{-\phi}\tilde{\eta}^1_+ \otimes \eta^{2\dagger}_\pm \tag{4.44}$$

($\Lambda^+$ is defined for later use). For the particular $SU(6)$ structure corresponding to $\tilde{\eta}^1 = \tilde{\eta}^2 = 0$, the bispinor $L$ is given purely in terms of $\Phi^+$ (see (4.39)). In a generic $SU(6)$ structure, $L$ combines two even pure spinors $\Phi^+$ and $\tilde{\Phi}^+$ and is therefore a natural candidate to describe $\mathcal{N} = 2$ vacua. Furthermore, it contains 24 extra degrees of freedom (building up the $(\mathbf{2}, \mathbf{12})$ of $O(6,6) \times SL(2,\mathbb{R})$) encoded in $\Lambda^- + s(\Lambda^-)$ and $\Lambda'^- + s(\Lambda'^-)$ which contain the bilinear one and five- forms between $\eta^I$ and $\tilde{\eta}^J$. Using (4.34) we get for $K_\pm = K_1 \pm iK_2$

$$K_+ = \begin{pmatrix} \Lambda^+ & \Phi^- \\ \bar{\tilde{\Phi}}^- & \bar{\Lambda}'^+ \end{pmatrix} \ , \qquad K_- = \begin{pmatrix} s(\bar{\Lambda}^+) & -s(\tilde{\Phi}^-) \\ -s(\bar{\Phi}^-) & s(\bar{\Lambda}'^+) \end{pmatrix} \ , \tag{4.45}$$

where $\Lambda^+$ and $\Lambda'^+$ are defined in (4.44), while for $K_3$ we get

$$K_3 = \begin{pmatrix} \Phi^+_1 - \tilde{\Phi}^+_2 & \Lambda^-_1 + s(\Lambda^-_2) \\ -s(\bar{\Lambda}^-_1) - \bar{\Lambda}^-_2 & \bar{\tilde{\Phi}}^+_1 - \bar{\Phi}^+_2 \end{pmatrix} \tag{4.46}$$

where we have defined

$$\begin{aligned} \Phi^+_1 &= e^{-\phi}\eta^1_+ \otimes \eta^{1\dagger}_+ \ , & \Phi^+_2 &= e^{-\phi}\eta^2_+ \otimes \eta^{2\dagger}_+ \ , & \Lambda^-_1 &= e^{-\phi}\eta^1_+ \otimes \tilde{\eta}^{1\dagger}_- \ , \\ \tilde{\Phi}^+_1 &= e^{-\phi}\tilde{\eta}^1_+ \otimes \tilde{\eta}^{1\dagger}_+ \ , & \tilde{\Phi}^+_2 &= e^{-\phi}\tilde{\eta}^2_+ \otimes \tilde{\eta}^{2\dagger}_+ \ , & \Lambda^-_2 &= e^{-\phi}\eta^2_+ \otimes \tilde{\eta}^{2\dagger}_- \ . \end{aligned} \tag{4.47}$$

We see that $K_+$ contains the two pure spinors $\Phi^-$ and $\tilde{\Phi}^-$, which appear as independent degrees of freedom (unlike $\Phi^+$ and $\tilde{\Phi}^+$ in $L$), as well as two even pure spinors $\Lambda^+$ and $\Lambda'^+$ which contain the even-form bilinears between $\eta^I$ and $\tilde{\eta}^J$ (note though that the traceless condition removes one complex degree of freedom which is the 0 and 6-form in $\Lambda^+ + \Lambda'^+$). $K_3$ now contains, in addition to the even-form bilinears of the same spinor, built out of $\eta^I$ and $\tilde{\eta}^I$.

## 4.5 Algebraic condition

We commented in Chapter 3 how the pure spinors can be twisted by a B-shift transformation (3.67) . $B$-field twisted pure spinors (which we also named dressed pure spinors) could be geometrically interpreted in terms of an orbit starting from the un-twisted pure spinor, spanning a $\frac{O(6,6)}{SU(3,3)} \times \mathbb{R}^+$ space, where $SU(3,3)$ is the stabilizer of the pure spinor and the $\mathbb{R}^+$ factor corresponds to the norm [95]. Quotienting by the $\mathbb{C}^*$ action $\Phi_D \to c\Phi_D$, we get the space $\frac{O(6,6)}{U(3,3)}$ which is local Special Kähler. In a totally equivalent fashion, the (un-dressed) structures $L$ and $K_a$ can be twisted by the action of the gauge fields $B$, $\tilde{B}$ and $C^-$ in (4.26), (4.29), *i.e.* we define

$$L_D = e^C e^{\tilde{B}} e^{-B} L \ , \qquad K_{aD} = e^C e^{\tilde{B}} e^{-B} K_a \ . \tag{4.48}$$

In this case,the EGG structures $L_D$ and $K_{aD}$ span orbits in $E_{7(7)}$ which are respectively Special Kähler and Quaternionic-Kähler. As shown in [97], the structure $L_D$ is stabilized by $E_{6(2)}$, and the corresponding local Special Kähler space is $\frac{E_{7(7)}}{E_{6(2)}} \times U(1)$. The triplet $K_{aD}$ is stabilized by an $SO^*(12)$ subgroup of $E_{7(7)}$, and the corresponding orbit is the quaternionic space $\frac{E_{7(7)}}{SO^*(12)\times SU(2)}$, where the $SU(2)$ factor corresponds to rotations of the triplet. We can summarize the algebraic structures corresponding to the various spinor bilinears we discussed in the following table

| Theory | Algebraic Structure | Bispinor | Reduced structure group |
|---|---|---|---|
| GG | $\Phi_+$ | $\eta^1_+ \otimes \eta^{2\,\dagger}_+$ | $SU(3,3) \subset O(6,6)$ |
| GG | $\Phi_-$ | $\eta^1_+ \otimes \eta^{2\,\dagger}_-$ | $SU(3,3) \subset O(6,6)$ |
| EGG | $\lambda$ | $\epsilon_{IJ}\theta^{I\alpha} \otimes \bar{\theta}_{J\beta}$ | $E_{6(2)} \times U(1) \subset E_{7(7)}$ |
| EGG | $K_a$ | $\frac{1}{2}\sigma_{a\,I}{}^J \theta^{I\alpha} \otimes \bar{\theta}_{J\beta}$ | $SO^*(12) \times SU(2) \subset E_{7(7)}$ |

Table 4.1: Bispinor picture of generalized geometries.

In the approach we will introduce later on we are mainly interested in constructing untwisted quantities (4.36), as the derivative operator which we will adopt in subsection 4.7 would take into account the gauge field action in a similar fashion as it was in (3.93). To give a compatibility condition for $L$ and the triplet $K_a$ one should impose the respective group structures to share a common $SU(6)$ subgroup.

$$SO^*(12) \cap E_{6(2)} = SU(6) \, . \tag{4.49}$$

The corresponding quantities defined in (4.36) are thus compatible $SU(6)$ structures by construction. In the EGG language, the compatibility condition is translated in

$$L \cdot K_a \big|_{\mathbf{56}} = 0 \ , \tag{4.50}$$

where we have to apply the projection on the **56** on the product $\mathbf{56} \times \mathbf{133}$ (see Appendix B for the explicit tensor products in $O(6,6) \times SL(2,\mathbb{R})$ and $SL(8,\mathbb{R})$ decompositions). In the specific case where we use the parametrization (4.31), the decomposition under the $SL(2,\mathbb{R}) \times O(6,6)$ subgroup can be used to show that compatibility condition (4.50) has the following simple form

$$\begin{aligned} (K_+ \cdot L)^{iA} &= \langle \Phi^-, \Gamma^A \Phi^+ \rangle = 0 \\ (K_+ \cdot L) &= 0 \, . \end{aligned} \tag{4.51}$$

Notice that this condition is nothing but the compatibilty condition of a pure spinor pair in GCG language (3.68). To conclude, for the exceptional generalized geometrical case, we draw a scheme similar to the one displayed in Figure 3.1 which illustrates the reduction of the structure group of $F$ implied by the existence of the algebraic structures just introduced

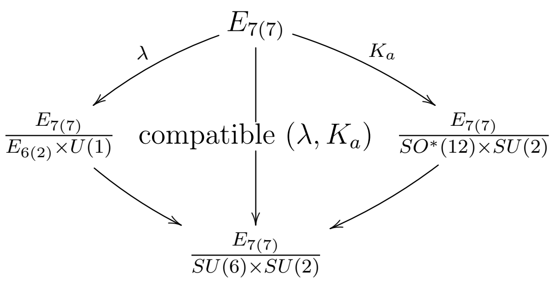


Figure 4.1: Algebraic structures in EGG.

We thus end by stating that

**Proposition 4.1 (Algebraic condition)** *To recover an $\mathcal{N}=2$ effective theory in four dimensions, the exceptional tangent bundle $F$ must have $SU(6)$ reduced structure. Equivalently the theory must admit the compatible structures $L$ and the triplet $K_a$ (4.36).*

It is worth to comment that we did not digress on a generalization of ordinary generalize geometry by defining exceptional generalized almost complex structures, but we rather decided to start from their dual description in terms of $SU(8)$ spinor bilinears. This choice was due on one hand because the bispinor picture is the most natural from a spacetime perspective, and on the other hand because this formulation is much more under control for explicit calculations. Furthermore, the analogue of the GACS in the $E_{7(7)}$ case has not been explored in much detail as it was for its generalized geometrical analogue, though progress in this direction is currently being made [97, 103].

## 4.6 Equations for vacua

We now focus on searching equations for vacua in the EGG formalism. Progress in connecting supersymmetric backgrounds and the language of exceptional generalized geometry has been proposed in [176] in an M-theoretical context. A preliminary formulation of the $\mathcal{N}=1$ vacua equations in the exceptional geometry framework (see section 4.2 of [97]) used the restricted ansatz (4.31) and the natural decomposition under the $SL(2,\mathbb{R})\times O(6,6)$ subgroup, in order to maintain contact with the generalized geometrical formulation, whose aim is to reproduce (in the spinor representation of each tensor product) the set of pure spinor equations (3.103)-(3.105). These can alternatively be reformulated (for type IIA) as [149]

$$d_H[e^{2A}\Phi'^{+}] = 0 \tag{4.52}$$

$$d_H[e^{A}\mathrm{Re}\Phi'^{-}] = 0 \tag{4.53}$$

$$[d_H(e^{2A}\mathrm{Im}\Phi'^{-}) - iF]_{(1,0)} = 0 \tag{4.54}$$

In this reformulation we trade $*_6$ for a projection on the $+i$ eigenspace of $J_{SK}$ (the complex structure on the **56**) in the last equation, which is the rewriting of (3.105). To recover these in the language of EGG it is necessary to consider projections on the adjoint and the fundamental representation of $E_{7(7)}$ respectively, and as will be explained in the following, even the Levi-Civita is embedded in the fundamental representation.
The spinor parts of the equations conjectured in [97] correspond to (4.52)-(4.54), but the vanishing of the vector components of the same equations could not be related to supersymmetry. The prescription used in this paper was to twist the structures first, as done in the algebraic constructions of the moduli space orbits, and then differentiate them using the Levi-Civita derivative. In the present Chapter we will rather follow the opposite approach, *i.e.* to twist the derivative (to recover a corresponding generalization of (3.91)), and let it act on the untwisted algebraic structures.
Before entering a detailed discussion, we outline how the corresponding differential equations are recovered, and how we overcome the issues encountered in their old version. To obtain the twisted equations, two separate tensor products are needed, the first in order to twist the derivative, and the second to act on the structure. In the original approach of [97], the twisting of the Levi-Civita differential by means of the gauge fields was generically kept on the corresponding tensor product $\mathbf{133} \times \mathbf{56}$. Doing so, gauge terms recovered from this procedure are not generally gauge-covariant, as not all of these are exterior differentials of the gauge fields. The fundamental improvement we propose here is to regularize this tensor product using an intermediate projection on the **912** representation. The flux content of the type II theory will then belong to this representation only. As we will see more in detail in Chapter 5, this is consistent with the nature of the $E_{7(7)}$ embedding tensor [60], which is indeed constrained by supersymmetry to belong to this representation.

In the following scheme we summarize the two projection procedures for the concrete example of the twisted equation for $L$

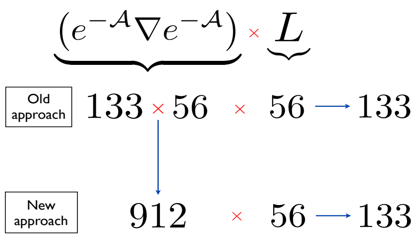


Figure 4.2: Comparison of the projection scheme used in [97] and the one we will adopt in the following. Red times correspond to tensor product representations, while blue arrows stand for projections of a tensor product to a particular representation.

## 4.7 The twisted derivative

Once the algebraic structures have been build, the next step would be to search for a generalization of the $H$-twisted differential used in generalized geometry $d_H$ (3.91). We define the candidate differential as a *twisted derivative* obtained by combining a differential operation with the gauge fields according to the tensor product rules of $E_{7(7)}$ representations. As a final result, as pictured in Figure 4.2, only gauge-invariant field strenghts arise by using this differential.
The putative conditions for supersymmetric vacua come from variations of the $E_{7(7)}$-covariant expression for the triplet of Killing prepotentials [97]

$$\mathcal{P}_a = \mathcal{S}(L_D, DK_{aD}) = \mathcal{S}(L, e^B e^{-\tilde{B}} e^{-C} D e^C e^{\tilde{B}} e^{-B} K_a) \ . \tag{4.55}$$

Here $\mathcal{S}$ is the symplectic invariant on the **56** whose decomposition in terms of $O(6,6) \times SL(2,\mathbb{R})$ and $SL(8,\mathbb{R})$ are given respectively in (B.1) and (B.8). In the second equality in (4.55) we have used the $E_{7(7)}$ invariance of the symplectic product to untwist the structures $L_D$ and $K_{aD}$ and express the Killing prepotentials in terms of naked structures, and a twisted derivative. We will now see how to properly define this twisted derivative, needed to get the equations for vacua.
For the gauge fields $\mathcal{A}$ and the derivative operator $D^{\mathcal{A}}$, $\mathcal{A} = 1, ..., 56$, one can define a connection $\phi^{\mathcal{AB}}{}_{\mathcal{C}} \in \mathbf{56} \times \mathbf{133}$ by the following twisting of the Levi-Civita one

$$(e^B e^{-\tilde{B}} e^{-C})^{\mathcal{B}}{}_{\mathcal{D}} D^{\mathcal{A}} (e^C e^{\tilde{B}} e^{-B})^{\mathcal{D}}{}_{\mathcal{C}} \equiv D^{\mathcal{A}} \delta^{\mathcal{B}}{}_{\mathcal{C}} + \phi^{\mathcal{AB}}{}_{\mathcal{C}} \ . \tag{4.56}$$

The connection $\phi$[9] contains derivatives of the gauge fields. The key point is that in the tensor product

$$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480} \tag{4.57}$$

only the terms in the **912** representation involve exterior derivatives of the gauge potentials [8], while the other representations contain non-gauge invariant terms (like divergences of potentials). We therefore define the twisted derivative as

$$\mathcal{D} = D + \mathcal{F} \ , \quad \text{where } \mathcal{F} = e^{B} e^{-\tilde{B}} e^{-C} D \, e^{C} e^{\tilde{B}} e^{-B} \big|_{\mathbf{912}} \ . \tag{4.58}$$

The fact that the fluxes lie purely in the **912** is consistent with the supersymmetry requirement that the embedding tensor of the resulting four-dimensional gauge supergravity be in the **912** [60]. We will return on this point in Chapter 5. The derivative $D$ is an element in the **56**, whose $O(6,6) \times SL(2,\mathbb{R})$ decomposition is

$$D = (D^{iA}, D^{+}) = (v^i \nabla^A, 0) \ , \qquad \text{where } \nabla^A = (0, \nabla_m) \ , \tag{4.59}$$

being $\nabla_m$ the Levi-Civita covariant derivative, while in $SL(8,\mathbb{R})$ we have

$$D = (D^{ab}, \tilde{D}_{ab}) = (0, v_i \nabla_m) \ . \tag{4.60}$$

(where we are using again $v_i = \epsilon_{ij} v^j = (0, -1)$), $DK_a$ in (4.55) is an element in the $\mathbf{56} \times \mathbf{133}$, which is projected to the **56** by the symplectic product. The **912** decomposes in the following $O(6,6) \times SL(2,\mathbb{R})$ representations

$$\begin{aligned} \mathcal{F} &= (\mathcal{F}^{iA}, \mathcal{F}^{i}{}_{j}{}^{+}, \mathcal{F}^{A-}, \mathcal{F}^{iABC}) \\ \mathbf{912} &= (\mathbf{2}, \mathbf{12}) + (\mathbf{3}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) + (\mathbf{2}, \mathbf{220}) \end{aligned}$$

where $\Gamma_A \mathcal{F}^{A-} = 0$ and $\mathcal{F}^{iABC}$ is fully antisymmetric in $ABC$. We show in the following how to obtain the connection from twisting the Levi-Civita covariant derivative (4.59) by the gauge fields $B$, $\tilde{B}$ and $C^-$ in the **133** representation. Using the Hadamard formula we get for any element $\mathcal{A}$ in the adjoint

$$e^{-\mathcal{A}} \nabla e^{\mathcal{A}} = \nabla + \nabla \mathcal{A} + \frac{1}{2} [\nabla \mathcal{A}, \mathcal{A}] + \frac{1}{6} [[\nabla \mathcal{A}, \mathcal{A}], \mathcal{A}] + \dots$$

We can decompose this expression using (4.27), and therefore get in the $O(6,6) \times SL(2,\mathbb{R})$ decomposition

$$\begin{aligned} (e^{B} e^{-\tilde{B}} e^{-C} \nabla e^{C} e^{\tilde{B}} e^{-B})^{i}{}_{j} &= \delta^{i}{}_{j} \nabla + v^i v_j \nabla \tilde{B} + v^i v_j \langle \nabla C^-, C^- \rangle \ , \\ (e^{B} e^{-\tilde{B}} e^{-C} \nabla e^{C} e^{\tilde{B}} e^{-B})^{B}{}_{C} &= \delta^{B}{}_{C} \nabla - \nabla B^{B}{}_{C} \ , \\ (e^{B} e^{-\tilde{B}} e^{-C} \nabla e^{C} e^{\tilde{B}} e^{-B})^{i-} &= v^i (e^{B} \nabla C^-) \ . \end{aligned} \tag{4.61}$$

[9] This object was originally introduced in [176], and first named generalized connection in [97].

and, as a very last step, we project to the **912** representation using the tensor product $\mathbf{56} \times \mathbf{133}\big|_{\mathbf{912}}$ for the subgroup $SL(2,\mathbb{R}) \times O(6,6)$ given in (B.5). We recover the simple result

$$\mathcal{F}^{1}{}_{2}{}^{+} = -e^{\phi} F^{+} \ , \qquad \mathcal{F}^{1}{}_{mnp} = -H_{mnp} \ , \tag{4.62}$$

where $F^{+} = e^{B} dC^{-}$, and all the other components are zero. Doing the same for the $SL(8,\mathbb{R})$ decomposition, we recover the generalized connection decomposition

$$\begin{gathered} \mathbf{912} = \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}' \\ \mathcal{F} = (\mathcal{F}^{ab}, \mathcal{F}^{abc}{}_{d}, \tilde{\mathcal{F}}_{ab}, \tilde{\mathcal{F}}_{abc}{}^{d}) \end{gathered} \tag{4.63}$$

where $\mathcal{F}^{ba} = \mathcal{F}^{ab}$ and $\mathcal{F}^{abc}{}_{c} = 0$ and similarly for the objects with a tilde. The NS-NS and R-R fluxes give the following non-zero components

$$\begin{aligned} \mathcal{F}^{mnp}{}_{2} &= -\frac{1}{2}(*H)^{mnp} \ , \qquad \mathcal{F}^{mn1}{}_{2} = -e^{\phi}\,\frac{1}{2}(*F_4)^{mn} \\ \tilde{\mathcal{F}}_{22} &= e^{\phi} *F_6 \ , \qquad \tilde{\mathcal{F}}_{mn2}{}^{1} = -e^{\phi}\,\frac{1}{2} F_{mn} \ . \end{aligned} \tag{4.64}$$

Notice that the mass parameter $F_{(0)}$ cannot be obtained this way, and should be added by hand. Using (B.19), we note that the component $\phi^{11}$ transforms as a scalar, and we therefore assign

$$\mathcal{F}^{11} = e^{\phi} F_0 \ . \tag{4.65}$$

## 4.8 The EGG equations

We present in this section the twisted equations in the EGG language. In applying the twisted derivative to the algebraic structures $L$ and $K$, the following tensor products appear

$$\begin{array}{llll} \mathcal{D}\cdot L = & D\cdot L \quad + \quad \mathcal{F}\cdot L \ , & \qquad \mathcal{D}\cdot K = & D\cdot K \quad + \quad \mathcal{F}\cdot K \\ & \mathbf{56}\times\mathbf{56} \ + \ \mathbf{912}\times\mathbf{56} & & \mathbf{56}\times\mathbf{133} \ + \ \mathbf{912}\times\mathbf{133} \end{array} \tag{4.66}$$

If we think of the vacua equations as coming from variations of the Killing prepotentials (4.55), out of these tensor products of representations, the equations should respectively lie in the **133** representation for $\mathcal{D}L$, and in the **56** in $\mathcal{D}K$. The explicit expression for the twisted equations is collected in Appendix B.3. In the following two particular examples will be investigated, and for each of them we will present the relevant set of equations specialized to the case we are considering. The comparison with supersymmetry will be presented for the two different cases of $E_{7(7)}$ structures built, namely the ones corresponding to the ansatz (4.31) and the ones recovered by using the more generic spinor ansatz (4.34). The discussion around the first is relevant for $\mathcal{N}=1$ vacua, and is presented in subsection 4.9, while the discussion of the case where we retain the structure necessary to $\mathcal{N}=2$ supersymmetry is presented in subsection 4.10.

## 4.9 EGG for $\mathcal{N}=1$ vacua

In this section we present an explicit form of the $E_{7(7)}$ algebraic structures constructed in [97] that play the role of the $O(6,6)$ pure spinors $\Phi^{\pm}$. Our intent is to give a specialization of the above ansatz in order to study the differential equations for $\mathcal{N}=1$ vacua, for which as did when we discussed the pure spinor equations in subsection 3.11 we will assume $\xi^1$ and $\xi^2$ to be linearly dependent. In that case, by using the restricted ansatz (4.31) we make contact between the $E_{7(7)}$ structures and the pure spinors of GCG. The equations we present are written in terms of $L$ and $K_a$ using the following parameterization for the spinors, which introduce a norm for each of the $SU(4)$ as

$$\theta^1 = \begin{pmatrix} a\eta^1_+ \\ 0 \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} 0 \\ \bar{b}\eta^2_- \end{pmatrix} \tag{4.67}$$

By following the same reasoning that leads from the superpotential to the equations for $\mathcal{N}=1$ vacua in the GCG case, a set of three equations were conjectured in [97] to be the EGG analogue of (3.103)-(3.105). To get the $SL(8)$ components of $L$ and $K_a$, we use (A.17). Using the decomposition of the gamma matrices given in (A.19), we get that the only non-zero components of $L$ and $K_a$ are

$$\begin{aligned} L: &\qquad L^{12}, L^{mn} \\ K_1, K_2: &\qquad {K_2}^{m1}, {K_2}^{m2}, {K_2}^{mnp1}, {K_2}^{mnp2} \\ K_0, K_3: &\qquad {K_3}^{mn}, {K_3}^{12}, {K_3}^{mnpq}, {K_3}^{mn12} \end{aligned} \tag{4.68}$$

where $L^{12}$ and $L^{mn}$ involve the zero and two-form pieces of $\Phi^+$, $K^{mi}_+, K^{mnpi}_+$ contain the one and three-form pieces of $\Phi^+$ (where the difference between the two $SL(2)$ components is a different $GL(6)$ weight), while $K_3$ contains the different components of $\Phi^+_1$ and $\Phi^+_2$. With this parameterization, the combinations that are relevant for $\mathcal{N}=1$ supersymmetry are

$$\begin{aligned} L' &\equiv e^{2A} L \ , \\ K'_1 &\equiv e^{A} r^a K_a = e^{A} K_1 \ , \\ K'_+ &\equiv e^{3A} z^a K_a = e^{3A}(K_3 + iK_2) \ . \end{aligned} \tag{4.69}$$

In the language of EGG, $\mathcal{N}=1$ supersymmetry requires for $L'$,

$$\mathcal{D}L'\big|_{\mathbf{133}} = 0 \ , \tag{4.70}$$

for $\mathcal{D}K'_1|_{\mathbf{56}}$[10]

$$\begin{aligned} (\mathcal{D}K'_1)^{mn} = 0, &\qquad \widetilde{(\mathcal{D}K'_1)}_{mn} = 0 \ , \\ (\mathcal{D}K'_1)^{12} = 0, &\qquad \widetilde{(\mathcal{D}K'_1)}_{12} = 0 \ , \\ (\mathcal{D}K'_1)^{m2} = 0, &\qquad \widetilde{(\mathcal{D}K'_1)}_{m1} = 0 \ , \end{aligned} \tag{4.71}$$

[10]We are using the notation in (4.21), where a tilde denotes the component in the $\mathbf{28}'$ representation.

and for $\mathcal{D}K'_+|_{\mathbf{56}}$

$$\begin{aligned}(\mathcal{D}K'_+)_{mn} - i\widetilde{(\mathcal{D}K'_+)}_{mn} &= 0 \ , \\ (\mathcal{D}K'_+)_{12} - i\widetilde{(\mathcal{D}K'_+)}_{12} &= 0 \ , \\ (\mathcal{D}K'_+)^{m2} &= 0 \ .\end{aligned} \tag{4.72}$$

The remaining components of $\mathcal{D}K$ (all with one internal index) are proportional to derivatives of the dilaton and warp factor as follows

$$(\mathcal{D}K'_1)^{m1} = 4e^{-2A}\partial_p A K'_+{}^{mp}, \quad \widetilde{(\mathcal{D}K'_1)}_{m2} = -4e^{-2A}\partial_p A\,(2K'_+{}^p{}_{m12} + i\delta^p_m K'_+{}^1{}_2), \tag{4.73}$$

$$(\mathcal{D}(e^{-\phi}K'_+))^{m1} = -4ie^{-\phi}g^{mp}\partial_p A K'_+{}^1{}_2 \ , \ (\mathcal{D}\widetilde{(e^{2A-\phi}K'_+)})_{m2} = -e^{2A-\phi}H_{mpq}K'_+{}^{12pq} \tag{4.74}$$

$$(\mathcal{D}\widetilde{(e^{-4A+\phi}K'_+)})_{m1} = 0 \ .$$

The equations for $L$, $K'_3$ and $K'_+$ in (4.70)-(4.72) are respectively the EGG version of (3.103), (3.104) and (3.105). The vectorial equations are a combination of (3.103)-(3.105) plus (3.107). Note that the symmetry group under which these equations are covariant is $GL(6,\mathbb{R}) \subset SL(8,\mathbb{R})$. First of all we specialize the equations for a generic $L$ and $K$ to the particular structure (4.68) compatible with $\mathcal{N}=1$ supersymmetry.
Using (4.68) in (B.20)-(B.27), we get that the only nontrivial components of Eq. (4.70) are[11]

$$(\mathcal{D}L')^1{}_2 = -e^{\phi}[iF_0 + (*F_6)]L'^{12} + \frac{e^{\phi}}{2}[F_{mn} + i(*F_4)_{mn}]L'^{mn} \ , \tag{4.75}$$

$$(\mathcal{D}L')^1{}_m = -\nabla_m L'^{12} \tag{4.76}$$

$$(\mathcal{D}L')^m{}_2 = -\nabla_p L'^{mp} + \frac{i}{2}(*H)^{mnp}L'_{np} \tag{4.77}$$

$$(\mathcal{D}L')_{mnp2} = \frac{3i}{2}\nabla_{[m}L'_{np]} + \frac{1}{2}H_{mnp}L'^{12} \ , \tag{4.78}$$

where we used (A.16), while for $K'_1$ we get

$$(\mathcal{D}K'_1)^{mn} = -2\nabla_p K'_1{}^{mnp2} + (*H)^{mnp}K'_1{}^2{}_p \tag{4.79}$$

$$\widetilde{(\mathcal{D}K'_1)}_{mn} = -2\nabla_{[m}K'_1{}^2{}_{n]} \tag{4.80}$$

$$\widetilde{(\mathcal{D}K'_1)}_{12} = -\nabla_n K'_1{}^n{}_1 - \frac{1}{3}H_{npq}K'_1{}^{2npq} \tag{4.81}$$

$$(\mathcal{D}K'_1)^{m1} = e^{\phi}F_0 K'_1{}^m{}_1 - e^{\phi}(*F_4)^{mn}K'_1{}^2{}_n - e^{\phi}F_{np}K'_1{}^{2npm} \tag{4.82}$$

$$\widetilde{(\mathcal{D}K'_1)}_{m2} = -e^{\phi}{*F_6}K'_1{}^2{}_m - e^{\phi}F_{mn}K'_1{}^n{}_1 + e^{\phi}(*F_4)^{np}K'_{1\,1npm} \tag{4.83}$$

[11] The interwining formula between $\lambda$, by means of which equations (B.20)-(B.27) are formulated, and $L$ is given in equation (A.16).

and, finally, for $K'_+$

$$(\mathcal{D}K'_+)^{mn} = -2\nabla_p K'_+{}^{mnp2} + (*H)^{mnp} K'_+{}^2{}_p + e^\phi (*F_4)^{mn} K'_+{}^2{}_1 \tag{4.84}$$

$$(\widetilde{\mathcal{D}K'_+})_{mn} = -2\nabla_{[m} K'_+{}^2{}_{n]} + e^\phi F_{mn} K'_+{}^2{}_1 \tag{4.85}$$

$$(\mathcal{D}K'_+)^{m1} = 2\nabla_p K'_+{}^{mp12} + e^\phi F_0 K'_+{}^m{}_1 - e^\phi (*F_4)^{mn} K'_+{}^2{}_n - e^\phi F_{np} K'_+{}^{2npm} \tag{4.86}$$

$$(\widetilde{\mathcal{D}K'_+})_{m1} = -\nabla_m K'_+{}^2{}_1 \tag{4.87}$$

$$\begin{aligned}(\widetilde{\mathcal{D}K'_+})_{m2} = & -\nabla_p K'_+{}^p{}_m - H_{mpq} K'_+{}^{pq12} - e^\phi * F_6 K'_+{}^2{}_m - e^\phi F_{mp} K'_+{}^p{}_1 \\ & + e^\phi (*F_4)^{pq} K'_{+1pqm}\end{aligned} \tag{4.88}$$

$$(\mathcal{D}K'_+)^{12} = -e^\phi F_0 K'_+{}^2{}_1 \tag{4.89}$$

$$(\widetilde{\mathcal{D}K'_+})_{12} = -\nabla_n K'_+{}^n{}_1 - \frac{1}{3} H_{npq} K'_+{}^{2npq} - e^\phi * F_6 K'_+{}^2{}_1 \tag{4.90}$$

where we should keep in mind that the components of $K_+$ with an odd (even) number of internal indices are proportional to $K_2$ ($K_3$) (see (4.68)).
We will outline here the proof that the twisted differential equations (4.70)-(4.72) are equivalent to $\mathcal{N} = 1$ supersymmetry.
We want to show first that supersymmetry requires (4.70), in particular the components appearing in (4.75) and (4.76). The proof for the rest of the components is reported in Appendix B.4.2. It is not hard to show that exactly the same combination of R-R fluxes appearing on the right hand side of (4.75) is obtained by multiplying Eq. (B.54), coming from the external gravitino variation, by $\Gamma^2$, and tracing over the spinor indices, namely

$$0 = \sqrt{2}\,\mathrm{Tr}\left(i\Gamma^2 \Delta_e \pi'\right) = -e^\phi [iF_0 + (*F_6)] L'^{12} + \frac{e^\phi}{2} \left[F_{mn} + i(*F_4)_{mn}\right] L'^{mn} = (\mathcal{D}L')^1{}_2$$

where in the second equality the term proportional to the derivative of the warp factor goes away by symmetry, and we have used (A.16) to relate the $SU(8)$ and $SL(8)$ components of $L$. Supersymmetry requires therefore $(\mathcal{D}L')^1{}_2 = 0$.
Consider then the equations which involve a covariant derivative of $L^{ab}$,. In this case, we can use (B.50) coming from the internal gravitino variation, multiplied by $\Gamma^{ab}$ and we trace over the spinor indices (see Eq. (A.16)). For $ab = 12$, for example, this gives

$$0 = \tfrac{\sqrt{2}}{4}\mathrm{Tr}\left(\Gamma^{12}\Delta_m L'\right) = \nabla_m L'^{12} - \partial_m(2A - \phi) L'^{12} - \frac{i}{4} H_{mnp} L'^{np} + \frac{e^\phi}{8}[F_{pq} + i(*F_4)_{pq}]\pi'^{2pq}{}_m$$

where $\pi'$ is defined in (B.51) and (B.52). Now we use Eqs. (B.53) and (B.55) multiplied by $\Gamma_m$ and traced over the spinor indices to cancel the terms containing derivatives of the dilaton and warp factor. In doing this, the term involving $H$ and $F$ fluxes completely cancel, *i.e.*

$$\begin{aligned}0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left(\Gamma^{12}\Delta_m L' + i\Gamma_m(-2\Delta_e L' + \Delta_d L')\right) \\ &= \nabla_m L'^{12} \\ &= (\mathcal{D}L')^1{}_m \,.\end{aligned}$$

We report in Appendix B.4.2 how supersymmetry requires the remaining equations (4.77) and (4.78) to vanish.
The equations for $K$ work similarly. In particular we found that when considering $K_1'$, equations (4.79)-(4.83) can be shown to be equivalent to supersymmetry by themselves, while when looking to $K_+'$ the corresponding equations (4.84)-(4.90) vanishes provided a holomorphic projection is imposed (4.72). For example, to show that (4.79) should vanish, we use (B.60) coming from internal gravitino, in the following way

$$\begin{aligned} 0 =& -\frac{i}{4}\mathrm{Tr}\left[\Gamma^{mnp2}(e^A\Delta_p K_1)\right] \\ =& -2e^{A-\phi}\nabla_p(e^{\phi}K_1{}^{mnp2}) + \frac{1}{2}H^{mnp}K_1'{}^1{}_p + \frac{3}{2}(*H)^{mnp}K_1'{}^2{}_p \\ & -2e^{-2A+\phi}F_0 K_+'^{mn12} - e^{-2A+\phi}F^{[m|p}K_{+p}{}^{|n]} \ . \end{aligned} \tag{4.91}$$

We combine this with external gravitino equations (B.115), (B.117) and dilatino equations (B.116), (B.118) to get (see more details in Appendix B.4.3)

$$\begin{aligned} 0 =& -\frac{i}{4}\,\mathrm{Tr}\left[\Gamma^{mnp2}(e^A\Delta_p K_1) + \{\Gamma^{mn1}, \Delta_e K_1' - \Delta_d K_1'\}\right] \\ =& -2\nabla_p K_1'^{mnp2} + (*H)^{mnp}K_1'{}^2{}_p \\ =& (\mathcal{D}K_1')^{mn} \end{aligned} \tag{4.92}$$

where we have used the notation in (B.78). We give the details about the rest of the components of the twisted derivative of $K_1'$ and $K_+'$ in Appendix B.4.3.
Some comment are relevant to connect the equations found to their generalized complex geometric counterparts, Eqs. (3.103)-(3.105) and (3.107).
Eqs. (4.76)-(4.78)) reduce to (3.103). The right hand side of Eq. (4.75) is proportional to $\langle F, \Phi^+\rangle$, which can be seen to vanish by wedging (3.103) with $C^-$ (this means that actually (4.75) can be derived from (4.76)-(4.78)). The $mn$ and 12 components of the EGG equations for $K_1'$ and $K_+'$ combine to build up respectively (3.104) and (3.105).
One interesting feature of the equations recovered is the explicit deduction of (3.107) from the twisted equations, while we recall that in GCG we had to add it by hand to the pure spinor equations. We see (3.107) has now become a condition recovered the second line of (4.74), as it can be seen by using (4.87) and the fact that $K_+'{}^2{}_1 = K_3'{}^2{}_1 = -\frac{i}{4}e^{3A-\phi}(|\eta_1|^2+|\eta_2|^2)$ The other vectorial components of $\mathcal{D}K$ involve for example terms of the form $\langle F, \Gamma^A\Phi^-\rangle$, which making use of (4.52)-(4.54), can be shown to be proportional to derivatives of the warp factor.
Making use of the fact that (3.103)-(3.107) were shown in [100] to be equivalent to supersymmetry conditions, we conclude that the EGG equations (4.70)-(4.74) are completely equivalent to requiring $\mathcal{N}=1$ supersymmetry, *i.e.*, supersymmetry requires (4.70)-(4.74), and (4.70)-(4.74) implies supersymmetry.
As mentioned in subsection 4.4, $L$ defines an $E_{6(2)}$ structure in $E_{7(7)}$. We have shown here that $\mathcal{N}=1$ supersymmetry requires this structure to be twisted closed, upon projection to the **133**. It would be very nice to show that this is equivalent to the structure being

integrable[12]. For constant warp factor and dilaton, also $K'_1$ is twisted closed. Most of the components of $K'_+$ are also twisted closed after projection onto holomorphic indices in the **56**. The vectorial components of $\mathcal{D}K$ are proportional to derivatives of the warp factor and dilaton, except the second equation in (4.74), which does not seem to be expressible in terms of such derivatives.

## 4.10 EGG for $\mathcal{N}=2$ vacua

Once the correspondence with the pure spinor equations corresponding to (4.52)-(4.54) has been explicitly derived in the previous subsection, we could generalize the procedure in the more general case of the structures (4.45) and (4.46). While in the $\mathcal{N}=1$ case one can always reparameterize $\eta^1+\tilde{\eta^1}\to\eta^1\,,\eta^2+\tilde{\eta^2}\to\eta^2$ so that the spinors (4.31) correspond to the most general ansatz for describing vacua with this number of supersymmetries, this is not the case for $\mathcal{N}=2$. In the pure spinor language, having an $\mathcal{N}=2$ vacuum would correspond to have two pairs of pure spinors $(\Phi^+,\Phi^-)$ and $(\tilde{\Phi}^+,\tilde{\Phi}^-)$ independently satisfying (3.103)-(3.105) (or (4.52)-(4.54)). In principle the ansatz (4.34) corresponds to a completely generic $SU(6)$ structure parameterized by four $SU(4)$ spinors $(\eta^1_+,\tilde{\eta}^1_-,\eta^2_-,\tilde{\eta}^2_+)$, which, as commented in subsection 4.4, can be used to describe an $\mathcal{N}=2$ vacuum for spacetime splitting (2.23) with $M_{1,3}=$Mink$_4$ as it was for $\mathcal{N}=1$. We will adopt in this subsection the following twisting

$$L'\equiv e^{-\phi}L \tag{4.93}$$
$$K'\equiv e^{A-\phi}K \tag{4.94}$$

One would have guessed the $\mathcal{N}=2$ conditions to be

$$\mathcal{D}L'\big|_{\mathbf{133}}=0 \tag{4.95}$$
$$[\mathcal{D}K'_a\big|_{\mathbf{56}}]_{(1,0)}=0 \tag{4.96}$$

as one may expect from the structure of equations (4.52)-(4.54). Nevertheless, in the EGG language, $\mathcal{N}=2$ supersymmetry requires
For $\mathcal{D}L'\big|_{\mathbf{133}}$:

$$(\mathcal{D}L')^1{}_m=0\,,\quad (\mathcal{D}L')^m{}_2=0\,,\quad (\mathcal{D}L')_{mnp2}=0\,, \tag{4.97}$$
$$(\mathcal{D}(e^{\phi}L'))^2{}_2=0\quad (\mathcal{D}(e^{-\phi}L'))^1{}_2=0\quad (\mathcal{D}(e^{\phi}L'))_{mn12}=0\,, \tag{4.98}$$

together with

$$(\mathcal{D}(e^{\frac{3}{2}(\phi-A)}L'))^n{}_m=i\frac{e^{\phi}}{2}\,(*F_4)^n{}_m L'^{12} \tag{4.99}$$

[12]Unlike the case of generalized complex structures, even if there is an exceptional Courant bracket [176], there is no known correspondence between the differential conditions on the structure and closure of a subset (defined by the structure) of the exceptional generalized tangent bundle under the exceptional Courant bracket.

while for $\mathcal{D}K'\big|_{\mathbf{56}}$:

$$(\mathcal{D}K')_{mn} - i(\widetilde{\mathcal{D}K'}_{mn}) = 0 \tag{4.100}$$

$$(\mathcal{D}K')_{12} - i(\widetilde{\mathcal{D}K'}_{12}) = 0 \tag{4.101}$$

$$(\mathcal{D}(e^{-\phi}K')^{m1} = 0\,, \tag{4.102}$$

$$(\mathcal{D}(\widetilde{e^{\phi}K'})_{m1} = 0\,, \tag{4.103}$$

$$(\mathcal{D}(e^{-\widetilde{(2A+\phi)}}K')_{m2} = -e^{-(2A+\phi)}H_{mpq}K'^{12pq}\,. \tag{4.104}$$

We will show how supersymmetry requires these equations. The richer structure of (4.45)-(4.46) makes all of the components of $L$ and $K_a$ to be non vanishing, and decouple the $K_a$ in a way which does not distinguish between the three, unlike it was for the $\mathcal{N}=1$ case (see (4.68)): the EGG equations effectively read as the generic ones reported in Appendix B.3 (B.20)-(B.27) and (B.28)-(B.35).

$$(\mathcal{D}L')^1{}_1 = -\frac{1}{4}\nabla_p L'^{p2} \tag{4.105}$$

$$(\mathcal{D}L')^2{}_2 = \frac{3}{4}\nabla_m L'^{m2} \tag{4.106}$$

$$\begin{aligned}(\mathcal{D}L')^1{}_2 = &-\nabla_m L'^{1m} - e^{\phi}(*F_6)L'^{12} - ie^{\phi}F_0 L'_{12} + \frac{e^{\phi}}{2}F_{mn}L'^{mn}\\ &+ i\frac{e^{\phi}}{2}(*F_4)^{np}L'_{np}\end{aligned} \tag{4.107}$$

$$(\mathcal{D}L')^m{}_2 = -\nabla_p L'^{mp} + \frac{i}{2}(*H)^{mnp}L'_{np} - e^{\phi}(*F_6)L'^{m2} + ie^{\phi}(*F_4)^{mn}L'_{n1} \tag{4.108}$$

$$(\mathcal{D}L')^1{}_m = \nabla_m L'^{12} - ie^{\phi}F_0 L'_{1m} + e^{\phi}F_{mn}L'^{n2} \tag{4.109}$$

$$(\mathcal{D}L')^n{}_m = \nabla_m L'^{n2} - \frac{1}{4}g^n{}_m \nabla_p L'^{p2} \tag{4.110}$$

$$(\mathcal{D}L')_{mnp2} = \frac{3i}{2}\nabla_{[m}L'_{np]} + \frac{1}{2}H_{mnp}L'^{12} + \frac{3}{2}ie^{\phi}F_{[mn|}L'_{|p]1} - \frac{e^{\phi}}{2}F_{mnpq}L'^{2q} \tag{4.111}$$

$$(\mathcal{D}L')_{mn12} = i\nabla_{[m}L'_{n]1} + \frac{1}{2}H_{mnp}L'^{p2}\;. \tag{4.112}$$

On the other hand we have

$$(\mathcal{D}K')^{mn} = -2\nabla_p K'^{mnp2} + (*H)^{mnp} K'^2{}_p + e^\phi (*F_4)^{mn} K'^2{}_1 \tag{4.113}$$
$$\widetilde{(\mathcal{D}K')}_{mn} = -2\nabla_{[m} K'^2{}_{n]} + e^\phi F_{mn} K'^2{}_1 \tag{4.114}$$
$$(\mathcal{D}K')^{m1} = 2\nabla_p K'^{mp12} + e^\phi F_0 K'^m{}_1 - e^\phi (*F_4)^{mn} K'^2{}_n - e^\phi F_{np} K'^{2npm} \tag{4.115}$$
$$\widetilde{(\mathcal{D}K')}_{m1} = -\nabla_m K'^2{}_1 \tag{4.116}$$
$$(\mathcal{D}K')^{m2} = 0 \tag{4.117}$$
$$\begin{aligned}\widetilde{(\mathcal{D}K)}'_{m2} = &-\nabla_p K'^p{}_m - H_{mpq} K'^{pq12} - e^\phi (*F_6) K'^2{}_m - e^\phi F_{mp} K'^p{}_1 \\ &+ e^\phi (*F_4)^{pq} K'_{1pqm}\end{aligned} \tag{4.118}$$
$$(\mathcal{D}K')^{12} = -e^\phi F_0 K'^2{}_1 \tag{4.119}$$
$$\widetilde{(\mathcal{D}K')}_{12} = -\nabla_n K'^n{}_1 - \frac{1}{3} H_{npq} K'^{2npq} - e^\phi (*F_6) K'^2{}_1 \tag{4.120}$$

The proof of equivalence of the twisted equations (4.97)-(4.104) and $\mathcal{N}=2$ supersymmetry proceeds in a very similar way to the $\mathcal{N}=1$ case: we listed in Appendices B.5.2 and B.5.3 generic supersymmetry transformation in $SL(8,\mathbb{R})$ language, as well as the explicit match between supersymmetry and each of the equations (B.20)-(B.27) for $L'$ defined in (4.93) and (B.28)-(B.35) for $K'$ defined in (4.94) respectively.
For instance we can see how supersymmetry implies (4.109) to vanish in this setting: multiplying respectively (B.94) by $\Gamma^{12}$, as well as (B.95) and (B.96) by $i\Gamma_m$, and tracing over spinor indices, we recover

$$\begin{aligned}0 &= \frac{\sqrt{2}}{4} \mathrm{Tr}\left[\Gamma^{12}\Delta_m L' + i\Gamma_m l_d \Delta_d L'\right] \\ &= \nabla_m L'^{12} + \partial_m \phi L'^{12} - l_d \partial_m \phi L'^{12} \\ &+ \frac{i}{4} H_{mnp} L'^{np}(-1 + l_d) \\ &+ \frac{e^\phi}{4}\left[iF_0(1 - 5l_d) + (*F_6)(1 - l_d)\right] L'^1{}_m \\ &+ \frac{e^\phi}{4}\left[F_{mp}(-1 - 3l_d) + i(*F_4)_{mp}(1 - l_d)\right] L'^{2p} \\ &= \nabla_m L'^{12} - ie^\phi F_0 L'^1{}_m - e^\phi F_{mp} L'^{2p} \\ &= (\mathcal{D}L')^1{}_m \,.\end{aligned} \tag{4.121}$$

where in the second passage we have explicitly taken $l_d = +1$. The rest of the detailed calculations which complete the equivalence scheme are collected in Appendix B.5.2.
In a similar way to what happened for $K'_+$ equations in the $\mathcal{N}=1$ case,we can show how the holomorphic projection (4.101) for $K'$ is implied by supersymmetry: taking indeed (B.114) $\Delta_p$ multiplied by $\Gamma^{p1}$, thus traced over the common internal six dimensional index,

as well as (B.115) an (B.116) multiplied by $-i\Gamma^2$, and tracing the overall sum we get

$$\begin{aligned}0 = &-\frac{1}{4}\text{Tr}\left[-\Delta_p K'\Gamma^{p1} - i\Gamma^2(n_d\Delta_d + n_e\Delta_e)L'\right]\\ &-\nabla_p K'^{p1} + \partial_p(A-\phi)K'^{p1} - \partial_p(n_eA + n_d\phi)K'^{1p} - \frac{1}{2}(1-\frac{n_d}{3})H_{mnp}K'^{2mnp}\\ &+\frac{e^\phi}{4}\left(iF_0(5n_d+n_e) + (*F_6)(6+n_e-n_d)\right)\\ &-\frac{e^\phi}{4}\left(F_{mn}(-2+3n_d+n_e) + i(*F_4)_{mn}(n_e+n_d)\right)\end{aligned} \tag{4.122}$$

by choosing here $n_d = 1$, $n_e = -1$ we recover

$$\begin{aligned}0 &= -\nabla_p K'^{p1} - \frac{1}{3}H_{mnp}K'^{2mnp} + e^\phi(iF_0 + (*F_6))K'^{12}\\ &= i\left[e^\phi F_0 - i(-\nabla_p K'^{p1} - \frac{1}{3}H_{mnp}K'^{2mnp} + e^\phi(*F_6)K'^{12})\right]\end{aligned} \tag{4.123}$$

where we recognize in the last last but one row the bracket to feature a combination of equations (4.119) and (4.120), so

$$0 = (4.123) = i[(\mathcal{D}K')^{12} - i(\widetilde{\mathcal{D}K'})^{12}] \tag{4.124}$$

finally implying that (4.101) is indeed equivalent to supersymmetry. Again, we collected the explicit details which explicitly match supersymmetry to the twisted equations (4.100)-(4.104) in Appendix B.5.3.

From comparison with the $\mathcal{N}=1$ case, we would have expected that even in the $\mathcal{N}=2$ case the $L$ structure would be twisted integrable. However, this is not the case, on one hand since different powers of the dilaton are necessary to recover twisted closure (see for instance the three equations in (4.98)), and on the other hand the component $(\mathcal{D}(e^{-\frac{3}{2}(\phi-A)}L))^n{}_m$ features an explicit R-R four form flux. Nonetheless, except for this last component, we recover a set of twisted closed equations.

On the other hand, the equations for $K_a$ turn out to be democratic, as expected, in that the full $SU(2)_R$ is preserved, contrarily to what happens in the $\mathcal{N}=1$ case where this is broken down to $U(1)_R$. As a consequence of this, the corresponding equations formally read the same.

In particular, let us compare the structure of the equations for $K_a$ in the $\mathcal{N}=1$ case (cfr. equations (4.73)-(4.74)) and their $\mathcal{N}=2$ analogue (4.100)-(4.104). The somehow undesired derivatives of dilaton and warp factor appearing can be further twisted by the corresponding differentials in the $\mathcal{N}=2$ case, while this could not be possible in the $\mathcal{N}=1$ case due to the splitting of these in $K'_+$ and $K'_1$, which have different components in the NS-NS sector of the internal gravitino variations (see equations (B.60),(B.61) and (B.62)). Furthermore, when massaging the equation (4.104), (which is nothing but the equivalent of the last equation in (4.74)) by using at best supersymmetry constraints we recover an explicit $H$ term. Once more, much as it happens for $L$, the quantities that we find to be equal to zero contain different powers of the dilaton, so we broadly refer to the set of the equations to be (almost) twisted closed.

# Chapter 5

# A consistent supersymmetric truncation on $T^{1,1}$

Non-compact geometries are of particular relevance in string theory, as these naturally appear in the AdS/CFT correspondence [164], but also in Randall-Sundrum models [182] which allow for a stringy origin of physical hierarchies [89]. This has been first investigated in the maximally supersymmetric and conformal case, where a precise duality can be established between the $\mathcal{N}=4$ SYM theory in four dimensions with gauge group $SU(N)$ and the type IIB supergravity in $\mathrm{AdS}_5 \times S^5$ which is the near horizon geometry of $N$ D3-branes in flat space.
Being $\mathcal{N}=4$ SYM in four dimensions a rather special theory due to its high amount of supersymmetry, a lot of effort has been devoted to find possible extensions of the Maldacena duality for less supersymmetric gauge theories, or at least to use the low-energy brane dynamics to extract information on the properties of such more realistic theories. We would refer in the following to the candidate vacuum as $\mathrm{AdS}_5 \times M_5$. One way to reduce the amount of supersymmetry is to place a stack of branes at the apex of conifold singularities. Out of the possible solutions realized in this way, two in particular are of relevance for the gauge/gravity duality, which are the Klebanov- Strassler [144] and the Maldacena - Nuñez [165] solutions. These represent the only known confining gauge gravity duals based on an $SU(3)$ structure manifold, which exhibit further interesting phenomenological features such as chiral symmetry breaking. Soon after the two solutions were found, an universal ansatz for the supergravity fields (to which from now on we would refer as the PT ansatz for short) was proposed by Papadopoulos and Tseytlin to interpolate between these and other conifold solutions [178]. In this Chapter, we will obtain an effective theory by using an ansatz justified by a symmetry principle only, which as discussed in the Introduction 1 guarantees the corresponding reduced theory to be a consistent truncation [66]: we will indeed systematically include all singlets under this symmetry and construct an explicitly supersymmetric action in five dimensions. This has been an open problem for some time, and indeed recent progress in Kaluza-Klein reductions [42, 80, 86, 87, 155, 190] appears to be fundamental for our scope.
We start by reviewing the two explicit solutions of [144] and [165], and after that discussing

how the PT ansatz interpolates between these. We then enter the details of the truncation.

## 5.1 Conifold solutions

We name *conical singularity* on ad $d$-dimensional manifold $Y_d$ a point (conventionally taken as $r = 0$) near which the metric can locally be put in the form

$$g_{mn}dx^m dx^n = dr^2 + r^2 \tilde{g}_{ij} dx^i dx^j \,, \tag{5.1}$$

where $\tilde{g}_{ij}$ is a metric on an $(d-1)$-dimensional manifold $M_{d-1}$, and the point $r = 0$ is singular (unless $M_{d-1}$ is a round sphere). Whenever the metric $g_{mn}$ on $Y_d$ is Ricci-flat, then $M_{d-1}$ is an Einstein manifold of positive curvature [146], as can be seen by performing a conformal transformation on the metric on $Y_d$

$$g_{mn}dx^m dx^n = d\phi^2 + \tilde{g}_{ij}dx^i dx^j \,, \qquad \phi = \ln r \,. \tag{5.2}$$

Provided $g_{mn}$ is a Ricci flat metric, by applying the conformal transformation to the Ricci tensor we find that $\tilde{g}_{ij}$ is an Einstein metric

$$R_{ij} = (d-2)\tilde{g}_{ij} \,. \tag{5.3}$$

Applying these considerations to the six dimensional case, we deduce that $M_5$ must be an Einstein space which corresponds to a Ricci-flat cone $Y_6$[1]. It has been discussed in [106] why gravity theories on Einstein spaces are dual to conformal field theories in four dimensions.
We could have seen also this the other way around: five-dimensional Einstein manifolds $M_5$[2] are in one-to-one correspondence with Ricci-flat cones $Y_6$, whose metric has the conical form

$$ds^2_{Y_6} = g_{mn}dy^m dy^n = dr^2 + r^2 ds^2_{M_5} \,. \tag{5.4}$$

Given a Ricci flat space $Y_6$ with metric (5.4), the regular D3-brane configuration

$$ds^2_{10} = h^{-1/2}(y) d^4x + h^{1/2}(y) g_{mn}(y) dy^m dy^n \,, \tag{5.5}$$

$$F_{(5)} = (1 + *_{10}) d^4x \wedge h(y) \,, \qquad \Phi = \text{const} \,, \tag{5.6}$$

is a solution of type IIB supergravity [7]. Here $h(y)$ is a harmonic function on the transverse Ricci-flat six-dimensional space

$$\frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{mn}\partial_n h) = 0 \tag{5.7}$$

[1]So far, we have used the notation $M_6$ for a generical internal six-dimensional space. In the following we will make use of $Y_6$ when referring to a cone.

[2]If the geometry supports a self-dual five-form, then $M_5$ must be Einstein with positive cosmological constant.

One may ask whether it is possible to generalize the solution (5.5)-(5.6) allowing for the introduction of fractional branes. Consider to replace the tranverse space with a new Ricci flat geometry. Eventually, if the new manifold admits a suitable non-trivial harmonic three-form, we can let the R-R and the NS-NS three-forms to be proportional to it. With this assumption, the type IIB supergravity equations can be satisfied, and the new configuration is compatible with the presence of fractional branes [63, 90]. The resulting background may then be thought as *deformations* of the standard brane solutions (5.5)-(5.6), in which additional flux generally associated with fractional branes is turned on [57]. The corresponding modified ansatz reads[3]

$$\begin{aligned} ds^2_{10} &= H^{-1/2}(y)dx_\mu dx^\mu + H^{1/2}(y)g_{mn}(y)dy^m dy^n \,, \\ F_{(5)} &= d^4x \wedge dH^{-1} + *_6 dH \,, \qquad G_{(3)} = F_{(3)} + iH_{(3)} \,. \end{aligned} \tag{5.8}$$

where again $g_{mn}$ describes any six-dimensional Ricci-flat Kähler metric admitting a non-trivial complex harmonic self-dual 3-form

$$G_{(3)} = i *_6 G_{(3)} \,. \tag{5.9}$$

These geometries fall into the conformal Calabi-Yau class discussed in subsection 3.2. Remarkably, it turns out that all the equations of motion are satisfied provided that

$$\Box H = -\frac{1}{12}|G_{(3)}|^2 \,. \tag{5.10}$$

Particularly interesting would be the cases when the harmonic forms are normalizable, since in this case the additional flux can smooth out eventual singularities in the original background.
In flat space, there is always an harmonic 3-form for which $G_{(3)} \wedge \overline{G}_{(3)}$ is proportional to the volume fom: however, as it contributes a term $-m^2r^2$, this choice corresponds to a singular solution. Notice that in view of (5.10) the eventual normalizability of $G_{(3)}$ translates in turn in the presence of singularities in $H$. We would focus on finding a suitable transverse space able to regularize the solution (5.8). As we will review in the following section, the progressive steps [145] which led to the Klebanov-Strassler solution [144] may be pictured exactly as a suitable replacing of the transverse space. Although we illustrated a specialization to the D3-branes case, the argument above is totally general: the straighforward analogue for the type IIA case is the CGLP solution [53, 57], which can be built using an equivalent construction for which a regular D2-brane background is deformed. The main difference is that the IIA configuration would now be dual to a three dimensional gauge theory, as the transverse space of a D2-brane is seven-dimensional. We will consider this in detail in Chapter 6.
Moreover, there are several other examples which fit as suitable generalizations of the resolution principle sketched in the discussion around (5.8), which we can briefly summarize in the table below for different theories.

[3] In this subsection we will denote the warp factor as $H$, not to be confused with the NS-NS three-form $H_{(3)}$.

| Holonomy group | Transverse space | Theory | Solution |
|---|---|---|---|
| $SU(3)$ | deformed conifold | IIB | KS [144] |
| $G_2$ | cone over squashed $\mathbb{CP}^3$ | IIA | CGLP [53] |
| $SU(4)$ | Stenzel space [193] | M theory | CGLP [55] |
| $Spin(7)$ | cone over a (cone over squashed $\mathbb{CP}^3$) | M theory | $A_8$ [54] |

Table 5.1: Main regular solutions featuring a brane resolution mechanism in diverse theories.

For the rest of this Chapter we will focus on IIB solutions with non-compact internal six-dimensional space $Y_6$.
Despite of the fact there are many solutions which can be built this way, the Maldacena-Nuñez configuration is an outsider in this perspective. Historically, it has been constructed in a totally different setting which we will review briefly in the following. Furthermore, this solution cannot at all been thought as a deformation of an ordinary brane configuration since it is obtained by wrapping fractional branes only. Nonetheless, although it can be pictured as the near brane region for five-branes wrapped on the two-sphere of the resolved conifold, its internal cone is not a Calabi-Yau manifold[4].

### 5.1.1 The Klebanov - Strassler solution

The first explicit example of string theory configuration based on a cone on $T^{1,1}$ has been studied in [146]. The $\mathcal{N} = 1$ superconformal field theory on N regular D3-branes placed at the singularity of the conifold has gauge group $SU(N) \times SU(N)$ and global symmetry $SU(2) \times SU(2) \times U(1)$, which is a symmetry of the metric of the cone[5]. The idea of placing branes in conifold singularities has been further developed [111, 142] using fractional D3-branes, which can be thought as D5-branes wrapped the $S^2$ of $T^{1,1}$. From the field theory point of view, such generalization leads to a four dimensional field theory which in no longer conformal and has gauge group $SU(N+M)\times SU(N)$. The supergravity solution describing a collection of $N$ regular D3-branes and $M$ fractional D3-branes on the (singular) conifold was the first attempt for a candidate gravity dual [145]. This solution features the ordinary D3-brane metric (5.5), where the harmonic function (*i.e.* the warp factor) is [145, 177]

$$h(r) = 1 + \frac{Q(r)}{r^4} \, , \qquad Q(r) = c_1 g_s N + c_2 (g_s M)^2 \ln \frac{r}{r_0} \, . \tag{5.11}$$

which evidently features a naked singularity in the IR, corresponding to small values of $r$.

[4] Despite the similarities of the internal space of the Maldacena-Nuñez solution and the resolved conifold, a replacement of the latter as transverse geometry in the procedure outlined around the deformed ansatz (5.8) gives a singular solution [177].

[5] We refer the interested reader to the Appendix C.1 for details on $T^{1,1}$.

The first consistent realization of the singularity resolution technique outlined in the previous subsection has been realized by Klebanov and Strassler [144]: to recover a regular solution it is sufficient to replace the singular conifold by the deformed conifold, keeping the same D3-brane structure of the ten-dimensional metric, and generalizing the three-form ansatz for the $H_3$-flux appropriately. This solution has the same UV (large $r$) asymptotic as the one featuring the singular conifold one [145], but at the same time it is regular at small $r$. The internal topology is unchanged from the singular conifold one, $\mathbb{R} \times S^2 \times S^3$, but now at the apex of the cone the $S^2$ shrinks to zero size while the $S^3$ remains finite.
The internal geometry is the deformed conifold with a conformal factor (referred to as throat) due to the presence of fluxes [173].

$$ds_6^2 = e^{-2A} ds_6^2(\text{Calabi-Yau})\,. \tag{5.12}$$

We can interpret this solution in the language of generalized geometry using the classification of subsection 3.2: it satisfies $W_1 = W_2 = W_3 = 0$ and the conformal condition $3W_4 = 2W_5$. This configuration falls into the B class of the Table of pag.15 of [98], with constant dilaton. The internal metric can be written explicitly by means of a set of left- and right- invariant one-forms of the conifold ($g^i$) (we collected in the technical Appendix C on $T^{1,1}$, where we explicit write these in (C.13) by means of the one-form set (C.12)) as

$$ds_6^2 = \frac{\epsilon^{4/3}}{2} K(\tau)\Big[\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2\frac{\tau}{2}((g^3)^2 + (g^4)^2) + \sinh^2\frac{\tau}{2}((g^1)^2 + (g^2)^2)\Big] \tag{5.13}$$

being

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \tag{5.14}$$

The solution has both NS-NS and R-R three form fluxes, as well as a R-R five form flux[6]

$$B_{(2)} = \frac{g_s N \alpha'}{2}\Big[f(\tau)(g^1 \wedge g^2) + k(\tau)(g^3 \wedge g^4)\Big] \tag{5.15}$$

$$F_{(3)} = \frac{g_s N \alpha'}{2}\Big[g^5 \wedge g^3 \wedge g^4 + d(F(\tau))(g^1 \wedge g^3 + g^2 \wedge g^4)\Big]\,, \tag{5.16}$$

$$F_{(5)} = \mathcal{F}_5 + \star_5 \mathcal{F}_5\,, \qquad \mathcal{F}_5 = \frac{g_s N^2 (\alpha')^2}{4} \ell(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \tag{5.17}$$

[6] With respect to the notation introduced in Chapter 2, we a slightly different notation here, as $H_3^{\text{there}} = H_{(3)}^{\text{here}}$, ${C_0}^{\text{there}} = {C_{(0)}}^{\text{here}}$, $\tilde{F}_3^{\text{there}} = F_{(3)}^{\text{here}}$, $\tilde{F}_5^{\text{there}} = F_{(5)}^{\text{here}}$, $\Phi^{\text{there}} = \phi^{\text{here}}$.

where the functions entering the flux ansatz read

$$F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau} \qquad f(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1) \tag{5.18}$$

$$k(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1)\,, \quad \ell(\tau) = \frac{\tau\coth\tau - 1}{4\sinh^2\tau}(\sinh 2\tau - 2\tau) \tag{5.19}$$

$$h(\tau) = (g_s N\alpha')^2 2^{2/3}\epsilon^{-8/3}\int_\tau^\infty dx\frac{x\coth x - 1}{\sinh^2 x}(\sinh 2x - 2x)^{1/3} \tag{5.20}$$

The full solution enjoys a particular $SU(2)\times SU(2)\times\mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2$ is ofter referred to as the $\mathcal{I}$-symmetry, and its action amounts to exchange the coordinates of the two spheres combined with a sign reversal of the NS-NS and R-R two-form gauge fields

$$(g,\phi,B_{(2)},C_{(0)},C_{(2)},C_{(4)}) \to (g,\phi,-B_{(2)},C_{(0)},-C_{(2)},C_{(4)}) \tag{5.21}$$

$$\sigma : (\theta_1,\phi_1,\theta_2,\phi_2) \to (\theta_2,\phi_2,\theta_1,\phi_1). \tag{5.22}$$

### 5.1.2 The Maldacena - Nuñez solution

This configuration has been found by promoting at the string level a monopole solution originally found by Chamseddine and Volkov (CV) [47], and for this reason it is sometime referred to as MN-CV. The field content is a metric, a dilaton, three $SU(2)_R$ gauge fields $(A^1, A^2, A^3)$ and a two-form $B_{(2)}$. As we did for the Klebanov-Strassler solution, we simply write the internal metric and the field content

$$\begin{aligned} ds_6^2 &= dr^2 + e^{2g(r)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\sum_a(\epsilon^a - A^a)^2\,, \\ H_{(3)} &= -\frac{1}{4}(\epsilon^1 - A^1)\wedge(\epsilon^2 - A^2)\wedge(\epsilon^3 - A^3) + \frac{1}{4}\sum_a F^a\wedge(\epsilon^a - A^a)\,, \\ e^{2\phi} &= e^{2\phi_0}\frac{2e^g}{\sinh 2r}\,. \end{aligned} \tag{5.23}$$

where $F^a = dA^a\,, a = 1,2,3$, and

$$\begin{aligned} A^1 &= \sigma^1 a(\rho)d\theta\,, \quad A^2 = \sigma^2 a(\rho)\sin\theta d\varphi\,, \quad A^3 = \sigma^3\cos\theta d\varphi\,, \\ a &= \frac{2\rho}{\sinh 2\rho}\,, \\ e^{2g} &= \rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}\,, \\ e^{2\phi} &= e^{2\phi_0}\frac{2e^g}{\sinh 2\rho}\,. \end{aligned} \tag{5.24}$$

being $\sigma^i\,, i = 1,2,3$ the ordinary Pauli matrices. Notice that the metric is now completely regular in the IR regime. This solution can easily be uplifted to ten-dimensional tipe IIB supergravity [56]. The metric has topology $\mathbb{R}\times S^2\times S^3$, as the one in KS (5.13), but the solution has no longer the $\mathcal{I}$-symmetry of the latter, being the first only $SU(2) \times SU(2)$ invariant. Notice that the resolution of the singularity happens in a very similar fashion to the resolution of the conifold as at the apex the $S^2$ remains finite, while the $S^3$ shrinks to zero (see also footnote 4) . It often appears in the literature, and would be useful for our purposes as well, the S-dual version of (5.23), which reads [27]

$$
\begin{aligned}
ds^2 &= e^{\Phi}\left[dr^2 + e^{2g(r)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\sum_a(\epsilon^a - A^a)^2\right], \\
F_{(3)} &= -\frac{1}{4}(\epsilon^1 - A^1)\wedge(\epsilon^2 - A^2)\wedge(\epsilon^3 - A^3) + \frac{1}{4}\sum_a F^a \wedge (\epsilon^a - A^a), \\
e^{2\phi} &= e^{2\phi_0}\frac{\sinh 2r}{2e^g}.
\end{aligned}
\tag{5.25}
$$

This last configuration (5.25) describes a type C solution in the Table of pag.15 of [98], with $\beta = i\alpha$, the only non trivial flux being the R-R three form. The first two components of the intrinsic torsion are zero, $W_1 = W_2 = 0$, but $W_3$ in general is not. The manifold is then complex but no longer a conformal Calabi-Yau as $W_4$ is also zero while $W_5$ is not. $W_5$ is then related to the dilaton, the vector component of the R-R flux and the warp factor.

## 5.2 An interpolating ansatz

Several trials have been done in order to connect the two aforementioned solutions, the insights coming both from the geometrical and from the gauge theory point of view. The completion of the program of replacing the transverse space with some other Ricci-flat cone has been done by [177] studying the resolution of the conifold, which led however to a singular solution in the IR, although it has the same UV behavior than the one in the KS case (see footnote 4). Furthermore, it seems the corresponding solution is not supersymmetric, following the criterion[7] of [102]. In the unification perspective, the deformed and the resolved conifold background are simply two one-parameter generalization of the singular conifold (respectively parametrized by the deformation $\epsilon$ and the resolution $a$).
The main difficulty a unified description of the two solutions necessarily meets is the substantial difference of the (metric and fluxes) symmetries of the two: as commented before, while the KS solution has a $SU(2) \times SU(2) \times \mathcal{I}$ symmetry, the MN has no longer the $\mathcal{I}$-invariance (this can also be deduced naively by the equivalence in the near-brane approximation of the MN solution on the resolved conifold [177]).
A candidate interpolation able to join the similar geometrical features of the MN and KS solutions has been proposed by Papadopoulous and Tseytlin (PT) [178]. The ten-dimensional metric has the desired topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$, and the internal geometry is specified by two sets of one-forms parameterizing respectively the two conifold topological cycles (we write down these in (C.17) where the standard forms (C.13) are in turn

[7] Indeed, the corresponding harmonic $G_3$ form features $(2,1)$ as well as $(1,2)$ parts.

explicitly expressed in terms of(C.12), see Appendix C.2 in terms of the standard forms $g_i$ used in [144]): $\{e_1, e_2\}$ describe the sphere $S^2$, while $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are right-invariant forms[8] on $S^3$. A convenient parametrization of the $\mathbb{Z}_2$-asymmetry between the two solutions is the introduction of the twisted one-forms

$$\tilde{\epsilon_1} \equiv \epsilon_1 - a(u)e_1\,, \qquad \tilde{\epsilon_2} \equiv \epsilon_2 - a(u)e_2\,, \tag{5.27}$$

being $a$ a function of the radial coordinate $u$ only. The metric ansatz in Einstein frame:

$$\begin{aligned} ds^2_{10} &= e^{2p-x}(e^{2A}dx_m dx^m + du^2) + ds^2_5\,, \\ ds^2_5 &= e^{x+g}(e_1^2 + e_2^2) + e^{x-g}(\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2) + e^{-6p-x}\tilde{\epsilon}_3^2 \\ &= (e^{x+g} + a^2 e^{x-g})(e_1^2 + e_2^2) + e^{x-g}[(\epsilon_1^2 + \epsilon_2^2) - 2a(\epsilon_1 e_1 + \epsilon_2 e_2)] + e^{-6p-x}\tilde{\epsilon}_3^2 \end{aligned} \tag{5.28}$$

features the functions $(p, x, A, g, a)$ depending on the radial coordinate $u$ only. We stress again the fact that the function $a$ multiplies the "off-diagonal" term $\epsilon_1 e_1 + \epsilon_2 e_2$. As expected the $\mathbb{Z}_2$ symmetry between the two spheres is broken unless $e^{x+g} + a^2 e^{x-g} = e^{x-g}$, which leads to $e^{2g} = 1 - a^2$. In the singular and resolved conifold case, we have $a = 0$. For the field content, we keep all three-forms and five-forms of the type IIB supergravity. Starting from the NS-NS sector we have

$$B_{(2)} = h_1(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + \chi(-\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + h_2(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)\,, \tag{5.29}$$

$$\begin{aligned} H_{(3)} = dB_2 &= h_2 \tilde{\epsilon}_3 \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) + du \wedge [h_1'(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) \\ &+ \chi'(-\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + h_2'(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)]\,, \end{aligned} \tag{5.30}$$

while the R-R fluxes are parameterized as

$$\begin{aligned} F_{(3)} &= P\tilde{\epsilon}_3 \wedge [\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)] \\ &\quad + du \wedge [b'(\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2)]\,, \end{aligned} \tag{5.31}$$

$$F_{(5)} = \mathcal{F}_5 + \star_{10}\mathcal{F}_5 \qquad \mathcal{F}_5 = K e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3\,. \tag{5.32}$$

where functions $(h_1, h_2, \chi, b, K)$ are scalars parameterizing the fluxes. A nicer form for the fluxes is obtained by using the basis $\{e_a, \tilde{\epsilon}_i\}$, $(a = 1, 2\,, i = 1, 2, 3)$ which makes the

---

[8] Right-invariance for the $S^3$ one-forms reads

$$d\epsilon_i = -\frac{1}{2}\epsilon_{ijk}\epsilon_j \wedge \epsilon_k\,. \tag{5.26}$$

See also Appendix C.2 for alternative left-invariant one-forms sets used in the literature.

metric diagonal

$$H_{(3)} = h_2\tilde{\epsilon}_3 \wedge (\tilde{\epsilon}_1) \wedge e_1 + \tilde{\epsilon}_2 \wedge e_2) + du \wedge \Big[(h_1' - \chi')\tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2$$
$$+ [h_1'(1+a^2) + 2h_2'a + \chi'(1-a^2)]e_1 \wedge e_2 + (ah_1' + h_2' - a\chi')(\tilde{\epsilon}_1 \wedge e_2 - \tilde{\epsilon}_2 \wedge e_1)\Big] , \tag{5.33}$$

$$F_{(3)} = P\Big[\tilde{\epsilon}_3 \wedge [\tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 + (a^2 - 2ab + 1)e_1 \wedge e_2$$
$$+ (a-b)(\tilde{\epsilon}_1 \wedge e_2 - \tilde{\epsilon} \wedge e_1)] + du \wedge [b'(\tilde{\epsilon}_1 \wedge e_1 + \tilde{\epsilon}_2 \wedge e_2)]\Big] , \tag{5.34}$$

$$\mathcal{F}_5 = K e_1 \wedge e_2 \wedge \tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 \wedge \tilde{\epsilon}_3 \, . \tag{5.35}$$

When explicitly substituting the ansatz for the fluxes in the type IIB supergravity action (2.12), together with the explicit use of the ten-dimensional ansatz metric (5.28), one recovers a mono-dimensional action where the functional dependence of all fields is on the radial coordinate only. The structure of the effective one-dimensional action is

$$S_{\text{1-d}} = \int du\, e^{4A}(3A'^2 + L) = \int du\, e^{4A}\Big[3A'^2 - \frac{1}{2}G_{ab}\varphi'^a\varphi'^b - V(\varphi)\Big] , \tag{5.36}$$

We start by reducing the gravity part of the ten-dimensional action (2.12)

$$\frac{1}{4}\int d^9x\sqrt{g}R \to e^{4A}(eA'^2 + L_{gr}) , \tag{5.37}$$

where

$$L_{gr} = -\frac{1}{2}x'^2 - \frac{1}{4}g'^2 - 3p'^2 - \frac{1}{4}e^{-2g}a'^2 - V_{gr} , \tag{5.38}$$

$$V_{gr} = -\frac{1}{2}e^{2p-2x}[e^g + (1+a^2)e^{-g}] + \frac{1}{8}e^{-4p-4x}[e^{2g} + (a^2-1)^2e^{-2g} + 2a^2] + \frac{1}{4}a^2e^{-2g+8p} , \tag{5.39}$$

while the so-called matter part has the following Lagrangian

$$L_m = -\frac{1}{8}\Big[\Phi'^2 + e^{-\Phi-2x}\Big(2h_2'^2 + 4e^{-2g}(h_1' + ah_2')^2$$
$$- 4[e^{2g} + (1-a^2)^2e^{-2g} + 2a^2]^{-1}[e^{-2g}(1-a^2)(h_1' + ah_2') - ah_2']^2 + 2e^{8p}h_2^2\Big)$$
$$+ P^2e^{\Phi-2x}\Big(e^{8p}[e^{2g} + e^{-2g}(a^2 - 2ab + 1)^2 + 2(a-b)^2] + 2b'^2\Big)$$
$$+ e^{8p-4x}K^2\Big] \tag{5.40}$$

We notice here that $K$ is not a dynamical field, and can be therefore eliminated using its equation of motion, which gives

$$K = Q + 2P[h_1 + bh_2] , \quad Q = \text{const.} . \tag{5.41}$$

Also $\chi$ turns out to be non-dynamical, and the corresponding equation of motion reads

$$\begin{aligned}&e^{2g}(h_1'-\chi')+e^{-2g}(a^2-1)[(1+a^2)h_1'+2ah_2'+(1-a^2)\chi']\\&+2a(ah_1'+h_2'-a\chi')=0\end{aligned} \tag{5.42}$$

To conclude, we illustrate how the solutions presented in subsections 5.1.1 and 5.1.2 respectively can be described by a specialization of this ansatz[9].

1. For the KS case, by expressing the radial variable $u$ as $\tau$, $g$ and $a$ can be expressed as

$$e^{-g}=\frac{1}{\tanh\tau}\,,\qquad a=-\frac{1}{\cosh\tau}\,. \tag{5.43}$$

   obviously satisfying the constraint $e^{2g}=1-a^2$, one fully recovers (5.13). For KS, in addition to the already written relation between $a$ and $g$ (5.43). Then, one should pick $\Phi=\text{const.}$ and $\chi=0$. One can then use a parameterization

$$h_1=\frac{1}{2}(f+k)\,,\quad h_2=\frac{1}{2}(k-f)\,,\quad b=P^{-1}F-1\,,\quad a^2=1-e^{2g}=\tanh^2 y\,. \tag{5.44}$$

2. If we want to recover the MN metric (5.23), the identification scheme is the following

$$\begin{aligned}A&=\frac{2}{3}(g+\Phi)\,,\qquad x=g+\frac{\Phi}{2}\,,\qquad p=-\frac{1}{6}(g+\Phi)\,,\\ a&=\frac{2u}{\sinh 2u}\,,\qquad dr=e^{-\frac{2}{3}(g+\Phi)}du\,,\\ e^{-2\Phi}&=\frac{2e^g}{\sinh 2u}\,,\qquad e^{2g}=u\coth 2u-\frac{1}{4}(1+a^2)\,.\end{aligned} \tag{5.45}$$

   In addition, one has to set to zero most of the flux scalars by imposing $h_1=h_2=\chi=0$ and to keep $a=b$. Finally the following relations hold

$$\Phi=-6p-g\,,\qquad x=g+\frac{1}{2}\Phi=\frac{1}{2}g-3p\,. \tag{5.46}$$

So far we introduced the PT ansatz in the way it has originally been conceived. Despite it is not a priori clear if this theory corresponds to a consistent truncation, it was largely used as one of the few tools for studying the physics connecting conifold solutions. By imposing the supersymmetric restrictions on a generic $SU(3)$ structure, [38] found explicitly a family of solutions inside the PT ansatz, representing the gravity dual of the baryonic branch of the Klebanov-Strassler gauge theory. These preserves the $SU(2)\times SU(2)$ ansatz for both metric and fluxes, while the original $\mathbb{Z}_2$ is broken as long as one moves away from

[9] A complete dictionary between the one-forms adopted in PT [178], KS [144] and MN [165] is given in Appendix C.2.

Klebanov-Strassler. As originally done in [178], the fluxes are constructed in such a way they automatically do satisfy Bianchi identities, so that solving the supersymmetric constraints for the PT ansatz guarantees to have a valid solution, *i.e.* a configuration that solves the second order equations of motion. A more recent family of solutions which contains both the MN and the KS backgrounds was found in [167] by wrapping fivebranes on the $S^2$ of the deformed conifold. This solution can be seen as a sort of non-Kähler analog of the conifold, as its geometry fits into the class of torsional, non-Kähler manifolds described in [194].

An important improvement in the way we should picture the PT ansatz has been its explicit generalization to a five-dimensional theory subject to a Hamiltonian constraint, rather than an effective mono-dimensional theory. This very first generalization of PT was inspired by holographic renormalization, as the bulk dynamics is five-dimensional (for a four-dimensional gauge theory) [21]. The main result of this paper is the addition of boundary momentum to the PT ansatz, which in turn leads to let the same scalar fields used in [178] be dependent also on the coordinates of the four dimensions of the gauge theory. This led in turns to embed PT in a five-dimensional effective theory, which can be shown to a correspond to a consistent truncation of type IIB supergravity.

A missing point which is put in evidence in this work is whether the truncation of the five-dimensional theory found can be made manifestly supersymmetric, and whether one can rewrite the general form of the potential in a five-dimensional gauged $\mathcal{N}=2$ supergravity form . Indeed, the truncation performed in [21] does not exhibit manifest supersymmetry as cannot be directly embedded in a supersymmetric action describing the bosonic degrees of freedom of the theory. The following part of the Chapter is devoted to show that indeed there is a five-dimensional $\mathcal{N}=4$ gauged supergravity theory candidate to supersymmetrize the PT ansatz. We briefly review the basics of gauged supergravity, for then moving to the specific derivation of our effective theory.

## 5.3 Gauged supergravity

We introduce gauged supergravities as a deformation of ungauged theories obtained by simple torus or Calabi-Yau reduction, where fluxes and geometric parameters act as deformation parameters. We will follow the general lines presented in the nice lecture notes [184], in which the gauging procedure is pictured in the following scheme

Looking to this mechanism from the four-dimensional theory level, in the gauging procedure a subgroup of the global symmetry group of the ungauged theory is promoted to a local gauge symmetry by coupling it to the (formerly abelian) vector fields of the theory. As a result, the matter fields of the theory are charged under the new gauge symmetry. In the above picture it is displayed how the gauging procedure, moving the horizontal line from left to right, should be totally equivalent to considering a general compactification from higher dimensions on some $G$-structure manifold in the presence of fluxes[10] with

[10]The reduced structure group $G$ of the internal manifold have not to be confused with the global symmetry group that will be gauged from the lower dimensional point of view.

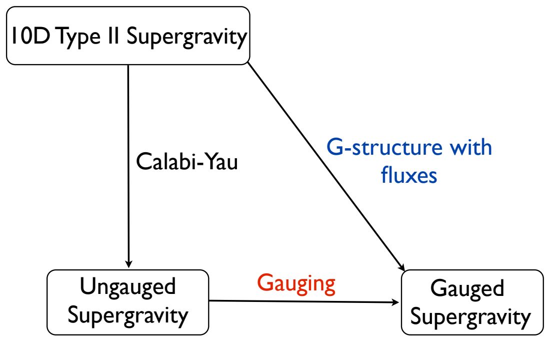


Figure 5.1: Gauged and Un-gauged supergravities

non-trivial structure.

### 5.3.1 Un-gauged supergravity

The bosonic field content of standard supergravity theories consists of the metric $g_{\mu\nu}$, a set of scalar fields $\phi^i$, a set of $n_v$ vector fields $A^M_\mu$ and eventual higher-rank antisymmetric $p$-forms $B^I_{\nu_1\ldots\nu_p}$. The corresponding dynamics is described in terms of a Lagrangian of the type

$$\mathcal{L}_{\text{bos}} = -\frac{1}{2}R - \frac{1}{2}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}\mathcal{M}_{MN}(\phi)F^M_{\mu\nu}F^{\mu\nu\,N} - \ldots\,, \tag{5.47}$$

being $e = \sqrt{|\det g_{\mu\nu}|}$, and the abelian field strengths $F^M_{\mu\nu} \equiv \partial_\mu A^M_\nu - \partial_\nu A^M_\mu$, for $M = 1,\ldots,n_v$. We will not consider the kinetic terms for the higher-rank $p$-forms and possible topological terms, which are neglected in the dotted part. Also, as we will consider the bosonic sector of the theory only.

The Lagrangian written is fixed by diffeomorphism and gauge covariance: still, the specification of the theory are the scalar and the vector kinetic matrices $G_{ij}(\phi)$ and $\mathcal{M}_{MN}(\phi)$. Scalar fields $\phi^i$ in half-maximal supergravity are described by a $G/K$ coset space sigma-model, being $G$ the global symmetry group of the theory and $K$ its maximal compact subgroup.

A convenient formulation of this sigma-model uses a $G$-valued matrix $\mathcal{V}$ to parameterize the scalar fields. We define the left-invariant current for $\mathcal{V}$ as

$$J_\mu = \mathcal{V}^{-1}\partial_\mu\mathcal{V} \in \mathfrak{g} = \text{Lie}\,G\,. \tag{5.48}$$

We decompose the coset space structure, $J_\mu$ as

$$J_\mu = Q_\mu + P_\mu\,,\quad Q_\mu \in \mathfrak{k}\,,\quad P_\mu \in \mathfrak{p} \tag{5.49}$$

where $\mathfrak{k} \equiv \mathrm{Lie}\, K$ and $\mathfrak{p}$ denotes its complement, *i.e.* $\mathfrak{g} = \mathfrak{k} \perp \mathfrak{p}$. The scalar Lagrangian is then given by

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\mathrm{Tr}(P_\mu P^\mu)\,. \tag{5.50}$$

which is invariant under global $G$ and local $K$ transformations acting as

$$\delta\mathcal{V} = \Lambda\mathcal{V} - \mathcal{V}k(x)\,, \qquad \Lambda \in \mathfrak{g} \qquad k(x) \in \mathfrak{k} \tag{5.51}$$

on the scalar matrix $\mathcal{V}$. Our interest is to rewrite the action (5.47) in terms of manifestly $K$-invariant objects, we can describe the scalar fields using the positive definite symmetric scalar matrix $\mathcal{M}$ defined by

$$M \equiv \mathcal{V}\Delta\mathcal{V}^T\,, \tag{5.52}$$

being $\Delta$ a constant $K$-invariant positive definite matrix[11]. The matrix $M$ is manifestly $K$-invariant and tranforms under G as

$$\delta M = \Lambda M + M\Lambda^T\,, \tag{5.53}$$

and the Lagrangian can be written in terms of $M$ as

$$\mathcal{L}_{\text{scalar}} = \frac{1}{8}\mathrm{Tr}(dM \wedge *dM^{-1})\,. \tag{5.54}$$

In un-gauged supergravity, the vectors (also referred to as gauge fields) transform in linear representations of the global symmetry group $G$:

$$\delta\mathcal{A}^M_\mu = -\Lambda^\alpha (t_\alpha)_N{}^M \mathcal{A}^N_\mu\,, \qquad M = 1, \ldots, n_v\,, \qquad \alpha = 1, \ldots, \dim G\,. \tag{5.55}$$

where $(t_\alpha)_N{}^M$ denote the generators of $\mathfrak{g}$ in a fundamental representation $\mathcal{R}_v$, with $\dim\mathcal{R}_v = n_v$. These are described by the action

$$\mathcal{L}_{\text{g,kin}} = -\frac{1}{4}M_{MN}\mathcal{H}^M_{\mu\nu}\mathcal{H}^{\mu\nu\,N}\,. \tag{5.56}$$

being $M_{MN}$ defined in (5.52), and $\mathcal{H} = d\mathcal{A}$ are the form field strenghts of the gauge fields.

### 5.3.2 The gauging procedure

We just illustrated how scalar fields and $p$-form fields respectively transform in a non-linear and in linear representations of the global symmetry group $G$. Gauging the theory amounts to choosing a subgroup $G_0 \subset G$ and promoting it to a local symmetry. The connection with what has been developed in the previous paragraph is best explained if considering the gauging as a deformation of the un-gauged theory. Consider a theory

[11] For our purposes, it will coincide with the identity matrix.

with an arbitrary number of vector fields $n_v$. These possess the standard abelian gauge symmetry $U(1)^{n_v}$:

$$\delta\mathcal{A}_\mu^M = \partial_\mu \Lambda^M \,, \tag{5.57}$$

with cooordinate-dependent parameters $\Lambda^M = \Lambda^M(x)$. Provided we pick a subgroup $G_0 \subset G$, the gauging can be made explicit by choosing a subset of generators $X_M$ within the global symmetry algebra $\mathfrak{g} = \mathrm{Lie}\, G$. To make the symmetry local, we replace ordinary covariant derivatives by covariant ones according to

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - g\mathcal{A}_\mu^M X_M \,, \tag{5.58}$$

where the set of generators can be written as

$$X_M \equiv \Theta_M{}^\alpha t_\alpha \in \mathfrak{g} \,. \tag{5.59}$$

The quantity $\Theta_M{}^\alpha$ of constant entries is known as embedding tensor, as it describes the explicit embedding of the gauge group $G_0$ into the global symmetry group $G$. The embedding tensor is a linear map from the vector space of vector gauge fields to the Lie algebra of invariances of the un-gauged theory. For practical reasons, one can think of this quantity as a constant $(n_v \times \dim G)$ matrix with its two indices $M$ and $\alpha$ in the fundamental and in the adjoint representation of $G$, respectively. When specifying $\Theta_M{}^\alpha$ we select a particular gauge group $G_0$, which breaks the global symmetry $G$. The theory should then be invariant under the following combined tranformations

$$\delta\mathcal{V} = g\Lambda^M X_M \mathcal{V} \,, \tag{5.60}$$

$$\delta\mathcal{A}_\mu^M = \partial_\mu\Lambda^M + g\mathcal{A}_\mu^N X_{NP}{}^M \Lambda^P = \mathcal{D}_\mu \Lambda^M \,. \tag{5.61}$$

with local parameter $\Lambda^M = \Lambda^M(x)$, and where

$$X_{MN}{}^P \equiv \Theta_M{}^\alpha (t_\alpha)_N{}^P \equiv X_{[MN]}{}^P + Z^P{}_{MN} \,. \tag{5.62}$$

are structure constants of the gauge algebra of the generators (5.59)

$$[X_M, X_N] = -X_{MN}{}^P X_P. \tag{5.63}$$

Independent of the number of supersymmetries or of the spacetime dimension the embedding tensor always has to satisfy the quadratic constraint (5.63). In addition, it has to satisfies a linear constraint whose form depends on the number of spacetime dimensions. We will postpone its introduction to the next subsection, when we will specialize our discussion to five dimensional gauged supergravity. To describe concretely an example, we consider the maximal $\mathcal{N} = 8$ $D = 4$ theory. Its global symmetry group $G = E_{7(7)}$ (cfr Chapter 4, subsection 4.1) has 133 generators, and the 28 vector fields transform in the fundamental **56** representation. Notice that due to its index structure, the embedding tensor $\Theta_M{}^\alpha$ lives in the tensor product of the fundamental and the adjoint representation

$$\Theta_M{}^\alpha : \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6840} \,. \tag{5.64}$$

It has been shown that compatibility of the deformation with supersymmetry formally restricts the embedding tensor to lie in the **912** representation [60] (see as well [184] for a brief review of the original argument). Therefore, in the natural comparison between the embedding tensor formalism and the generalized connection introduced in section 4.7 of Chapter 4, it should now be clear how the proposed regularization obtained by imposing an intermediate projection to the **912** indeed yields a pure flux content.
The gauged Lagrangian is dependent both on the number of the dimensions on which the theory is formulated as well as on the number of preserved supersymmetries. For this reason we introduce directly the specific case in which we will work out the truncation.

## 5.4 $\mathcal{N} = 4$ gauged supergravity in five dimension

We present here a general review of five dimensional $\mathcal{N} = 4$ gauged supergravity coupled to an arbitrary number of vector multiplets, mostly following [58,185]. The global symmetry group of ungauged $D = 5$, $\mathcal{N} = 4$ supergravity is $G = SO(1,1) \times SO(5, n_v)$ where $n_v \in \mathbb{N}$ now labels the number of *vector multiplets*[12].
The bosonic sector features the following multiplets

- graviton multiplet: $(g_{\mu\nu}, 6 \times A_\mu, \phi)$ - metric, 6 vectors and 1 real scalar
- vector multiplet: $(A_\mu, 5 \times \phi)$ - 1 vector and 5 real scalars

The scalar coset is [11]

$$\frac{SO(5, n_v)}{SO(5) \times SO(n_v)} \times SO(1,1) \tag{5.65}$$

The irreducible components of the embedding tensor are the quantities $f_{MNP}, \xi_{MN}, \xi_M$, where now $M, N = 1, \dots 5+n_v$ is a vector index of $SO(5, n_v)$. These quantities are tensors under the global symmetry group $SO(1,1) \times SO(5, n_v)$. Historically, the first gauging to be constructed was the one where the gauge group is a product of a semi-simple group and an Abelian factor [11].
The $SO(1,1)$ factor is described by a real scalar $\Sigma$ which is a singlet under $SO(5, n_v)$, and carries $SO(1,1)$ charge $-1/2$, while the remaining $5n_v$ scalars are described by a coset representative

$$\mathcal{V} \in \frac{SO(5, n_v)}{SO(5) \times SO(n_v)} \tag{5.66}$$

The $SO(5, n_v)$ generators are commonly given in a basis which we will call gauged supergravity basis ("gsg basis" for short) [184, 185] as

$$(t_{MN})_P{}^Q = \delta^Q_{[M}\eta_{N]P} \tag{5.67}$$

where $M, N \dots = 1, \dots, 5 + n_v$ and $\eta_{NP}$ is split sign signature $(5, n_v)$ diagonal identity matrix. We would ignore in the following higher $p$-form gauge fields: we will comment this

[12] Recall that in subsection 5.3 we used $n_v$ as label for the number of gauge fields.

when dealing with the covariant derivative of vector fields at the end of this subsection. The vector gauge fields form one vector $\mathcal{A}^M$, where $M \ldots = 1, \ldots, 5 + n_v$, and one scalar $\mathcal{A}^0$ under $SO(5, n_v)$. For notational convenience we use, as somehow standard in the supergravity literature, the calligraphic capital index $\mathcal{M} = (0, M)$, so that $\mathcal{A}^{\mathcal{M}} = (\mathcal{A}^0, \mathcal{A}^M)$.

The specific linear constraint of gauged supergravities in five dimensions is best written in terms of this compact notation using

$$d_{0MN} = d_{M0N} = d_{MN0} = \eta_{MN} \quad \text{all other component are vanishing,} \tag{5.68}$$

and

$$Z^{MN} = \frac{1}{2}\xi^{MN}\,, \quad Z^{0M} = -Z^{M0} = \frac{1}{2}\xi^{M}\,. \tag{5.69}$$

reading explicitly

$$X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}} = d_{\mathcal{M}\mathcal{N}\mathcal{Q}} Z^{\mathcal{P}\mathcal{Q}}\,. \tag{5.70}$$

while the corresponding generalization of (5.63) reads

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}. \tag{5.71}$$

where the gauge group generators can be explicitly related to the components $f_{MNP}, \xi_{MN}$ as [185]

$$X_{MN}{}^{P} = -f_{MN}{}^{P} - \frac{1}{2}\eta_{MN}\xi^{P} + \delta^{P}_{[M}\xi_{N]}\,, \quad X_{M0}{}^{0} = \xi_M\,, \quad X_{0M}{}^{N} = -\xi_M{}^{N}\,. \tag{5.72}$$

The general bosonic Lagrangian for the both un-gauged and gauged $\mathcal{N} = 4$ supergravity[13] is given by

$$\mathcal{L}_{\text{bos}} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{s,kin}} + \mathcal{L}_{\text{g,kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{top}}\,. \tag{5.73}$$

We first present the un-gauged theory. Separating the scalar from the tensor contributions we join (5.54) with a term for the $SO(1, 1)$ scalar $\Sigma$ to get[14]

$$\mathcal{L}^{\text{UG}}_{\text{s,kin}} = -3\Sigma^{-2} d\Sigma \wedge *d\Sigma + \frac{1}{8}\text{Tr}(dM^{-1} \wedge *dM)\,. \tag{5.74}$$

while for the gauge fields[15]

$$\mathcal{L}^{\text{UG}}_{\text{g,kin}} = -\tfrac{1}{2}\Sigma^2 M_{MN} \mathcal{H}^M_2 \wedge *\mathcal{H}^N_2 - \tfrac{1}{2}\Sigma^{-4}\mathcal{H}^0_2 \wedge *\mathcal{H}^0_2\,. \tag{5.75}$$

We introduce here the un-gauged topological term

$$\mathcal{L}^{\text{UG}}_{\text{top}} = \tfrac{1}{\sqrt{2}}\eta_{MN}\mathcal{A}^0 \wedge \mathcal{H}^M_2 \wedge \mathcal{H}^N_2\,. \tag{5.76}$$

[13] We will distinguish between the two theories with the $UG$ and $G$ superscripts respectively

[14] For the reader's convenience we will drop the $\mu\nu$ indices in the following.

[15] As we have introduced for the un-gauged theory, we will refer to $\mathcal{H}$ as the ordinary field strenghts of the gauge fields $\mathcal{A}$ obtained by exterior differentiation. When gauging is turned on, these will be replaced by covariant field strenghts which we will label as $\mathcal{F} = \mathcal{D}\mathcal{A}$.

which will be crucial, together with (5.75), for the identification of the vectors $\mathcal{A}^M$.
In agreement with the prescription (5.58), we expect the gauged theory would feature a covariantization of the derivatives appearing in the previous formulae.
It is indeed what we have for the gauged kinetic terms

$$\mathcal{L}^{\mathrm{G}}_{\mathrm{s,kin}} = -3\Sigma^{-2}(D\Sigma)^2 + \tfrac{1}{8}\mathrm{Tr}(DM^{-1} \wedge *DM)\,, \tag{5.77}$$

$$\mathcal{L}^{\mathrm{G}}_{\mathrm{g,kin}} = -\frac{1}{2}\Sigma^2 M_{MN}\mathcal{F}^M_{\mu\nu}\mathcal{F}^{N\,\mu\nu} - \frac{1}{2}\Sigma^{-4}\mathcal{F}^0_{\mu\nu}\mathcal{F}^{0\,\mu\nu}. \tag{5.78}$$

The potential Lagrangian is

$$\begin{aligned}\mathcal{L}^{\mathrm{G}}_{\mathrm{pot}} = \frac{1}{2}\Big[&f_{MNP}f_{QRS}\Sigma^{-2}(\tfrac{1}{12}M^{MQ}M^{NR}M^{PS} - \tfrac{1}{4}M^{MQ}\eta^{NR}\eta^{PS} + \tfrac{1}{6}\eta^{MQ}\eta^{NR}\eta^{PS})\\ &+ \tfrac{1}{4}\xi_{MN}\xi_{PQ}\Sigma^4(M^{MP}M^{NQ} - \eta^{MP}\eta^{NQ}) + \xi_M\xi_N\Sigma^{-2}M^{MN}\\ &+ \tfrac{1}{3}\sqrt{2}\xi_{MNP}\xi_{QR}\Sigma M^{MNPQR}\Big]\,.\end{aligned} \tag{5.79}$$

where we made use of the completely antisymmetric tensor

$$M_{MNPQR} = \epsilon_{abcde}\mathcal{V}_M^{\ a}\mathcal{V}_N^{\ b}\mathcal{V}_P^{\ c}\mathcal{V}_Q^{\ d}\mathcal{V}_R^{\ e}. \tag{5.80}$$

$a = 1, \ldots, 5$ are $SO(5)$ indices and $\mathcal{V}$ is the coset element (5.66). In the specific case we will analyze, the formula will be specialized to $\xi_M = 0$, describing the particular gaugings originally investigated in [58]. We will in particular make use of the following explicit covariant derivative of the scalars fields [42]:

$$\mathcal{D}M_{MN} = dM_{MN} + 2\mathcal{A}^P f_{P(M}{}^Q M_{N)Q} + 2\mathcal{A}^0\xi_{(M}{}^P M_{N)P}. \tag{5.81}$$

from which the component $(f_{MNP}, \xi_{MN})$ can be explicitly deduced. From the knowledge of the embedding tensor components one can read off the covariant field strength of the vector fields. In absence of two-form fields[16], the vector covariant derivative reads

$$\mathcal{D}\mathcal{A}^{\mathcal{M}} \equiv \mathcal{F}^{\mathcal{M}} = d\mathcal{A}^{\mathcal{M}} + \frac{1}{2}X_{\mathcal{N}\mathcal{P}}{}^{\mathcal{M}}\mathcal{A}^{\mathcal{N}} \wedge \mathcal{A}^{\mathcal{P}}\,. \tag{5.82}$$

Notice that when restricting to $\mathcal{M} = M$ this can be simplified by using (5.72), giving

$$\mathcal{F}^M = d\mathcal{A}^M + \frac{1}{2}f_{NP}{}^M\mathcal{A}^N \wedge \mathcal{A}^P + \frac{1}{2}\xi_P{}^M\mathcal{A}^0 \wedge \mathcal{A}^P \tag{5.83}$$

From this equation one can easily deduce the structure constants $X_{NP}{}^M$ of the gauging group.
The gauged topological action has as well a general form which we do not write down as we will not need it.

[16]The full formula for the covariant derivative would indeed include an extra contribution from $Z^{\mathcal{M}\mathcal{N}}B_{\mathcal{N}}$, where $B_{\mu\nu\mathcal{N}} = (B_{\mu\nu\,M}, B_{\mu\nu\,0})$ includes a vector with $SO(1,1)$ charge -1/2, while $B_{\mu\nu\,0}$ is a singlet carrying charge 1.

## 5.5 Truncations on Sasaki-Einstein manifolds

There has been a very intense recent work in performing universal reductions on manifolds belonging to specific geometry classes. We already pointed out the relevance of Einstein manifolds in the first part of the Chapter. We will specialize further this requirement, by reviewing some recent developments on truncation on five dimensional Sasaki-Einstein manifolds, for which the cone $M_6$ built over them is not only Ricci-flat, but a Calabi-Yau space [192]. Up to recent times, the only known explicit examples of Sasaki-Einstein five dimensional manifolds were the five-sphere $S^5$ and $T^{1,1}$. Recently, an infinite class, known as $Y^{p,q}$ has been constructed [83, 84].
The first universal Kaluza-Klein truncation on $SE_5$ spaces has been found by Buchel and Liu [37]. In this paper a detailed compactification of IIB on $T^{1,1}$ is worked out, and it was indeed established in that it corresponds to a gauged $\mathcal{N} = 2$ supergravity (with massless modes only). Restriction on the possible gauging arises whenever we want to consider only massless multiplets: as the isometry group of $T^{1,1}$ is $SU(2) \times SU(2) \times U(1)$, the massless gauge bosons transform under the same group. As the $U(1)$ gauge boson is the $\mathcal{N} = 2$ graviphoton coupling to $U(1)_R$, the massless sector may be described as $\mathcal{N} = 2$ gauged supergravity coupled to $SU(2) \times SU(2)$ vector multiplets. Nonetheless it was proved that it is inconsistent to retain these $SU(2) \times SU(2)$ vector multiplets in any truncation to the massless sector, being the only consistent truncation the one to pure $\mathcal{N} = 2$ gauged supergravity [125]. The authors of [37] indeed proceed to gauge the $U(1)$ out of the full isometry group. The obstruction to perform massless truncations retaining non-abelian gauge symmetries arises whenever only the supergravity multiplet is kept.
The way to overcome this limitation is the inclusion of massive modes. Taking these into account has made possible to construct string theory backgrounds with non-relativistic conformal symmetry [168], as well as emergent relativistic conformal symmetry in superfluids or superconducting states of strongly coupled gauge theories [78, 108]. Important steps towards a general truncation with massive modes have been performed in [42, 80, 87, 190] where reductions of type IIB supergravity on an arbitrary five-dimensional Sasaki-Einstein manifold $SE_5$ were considered.
$SE_5$ spaces have a five dimensional $SU(2)$ structure[17] specified by $(J, \Omega_2, g_5)$ and, in addition to satisfy the *algebraic* conditions (3.8), they also fulfill the following *differential* conditions

$$dJ = 0\,, \quad d\Omega_2 = 3ig_5 \wedge \Omega_2\,, \quad dg_5 = 2J\,. \tag{5.84}$$

A truncation to a five-dimensional $\mathcal{N} = 4$ gauged supergravity coupled to two vector multiplets has been found in [42, 80]. One of the crucial points of the reduction procedure was to expand the ten-dimensional fields in terms of the differential forms which define the structure (which are obviously singlets of the structure group they define), so that the resulting truncation is automatically consistent. In this case, the internal five-dimensional non-trivial geometry and the five-form flux induced a gauging on the effective

[17] In the general definition given subsection 3.1 we stated that the one-form $g$ should be complex: for a Sasaki-Einstein manifold it is however real, as it is the one-form dual to the Reeb vector [190, 192].

five-dimensional theory that correspond to the gauge group $G = \text{Heis}_3 \times U(1)_R$.
The forthcoming specialization to the $T^{1,1}$ case is a reduction of a particular case of Sasaki-Einstein space[18]: from the point of view that we will present in the next subsection, the previously mentioned work [42,87,190] is a fundamental preliminary step out of which we will isolate a concrete case. Investigating the $T^{1,1}$ case lead to a truncation which features an additional vector multiplet, but still fits in the framework of five-dimensional $\mathcal{N} = 4$ supergravity. Nonetheless, the gauge group turns out to be unaltered by the inclusion of the new vector multiplet, the gauging being due to the curvature of $T^{1,1}$ as well as the topological flux which we will include.

## 5.6 The $T^{1,1}$ truncation ansatz

Following the lines of the general picture of gauged supergravity 5.1, the Kaluza-Klein reduction on $T^{1,1}$ would be a deformation of the standard reduction of IIB on $T^5$ (where of course only massless modes are retained). However a more clear introduction of the ansatz we will adopt comes naturally from a symmetry principle. As a first step we propose the more general invariant metric on this space, then in the following, along the lines of [42,87] we will make an explicit choice for the three-forms and the five-form by expanding these on the structure group forms. In order to isolate the physical components for the fluxes, we will solve the corresponding Bianchi identities. We will consider as well the dilaton and axion as additional fields which have trivial Bianchi identities.

### 5.6.1 Derivation of the metric

We want to isolate a symmetry under which we can write down the most general metric. We restrict ourselves to a $\mathcal{K}$-invariant ansatz, where in standard coordinates on the conifold $\mathcal{K}$ is a particular $\mathbb{Z}_2$ symmetry acting on the conifold as (we collected standard conventions of $T^{1,1}$ in Appendix C.1)

$$\mathcal{K} : (\psi, \theta_2) \to (\psi + \pi, -\theta_2). \tag{5.85}$$

We can describe this action even using the usual complex coordinates (see equation (C.7)) on the cone over $T^{1,1}$

$$\mathcal{K} : (z_1, z_2, z_3, z_4) \to (z_2, -z_1, z_4, -z_3)\,. \tag{5.86}$$

[18] With respect to the works [42,87], for which a generic $SE_5$ is written as a fibration over a Kähler-Einstein four dimensional manifold $KE_4 \times S^1$, we are just specializing the analysis for the base $\mathbb{CP}^1 \times \mathbb{CP}^1$ (see also Appendix C.1).

We can alternatively parameterize $T^{1,1}$ by means of the left-invariant $SU(2)$ one-forms $(\sigma_i, \Sigma_j)$ defined in (C.10). $\mathcal{K}$-action on these read

$$\mathcal{K} \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = K \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$
$$\mathcal{K} \cdot \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} = K^{-1} \cdot \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} \tag{5.87}$$

being

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.88}$$

Let us define

$$E_1 = \frac{1}{\sqrt{6}}\big(\sigma_1 + i\sigma_2\big), \qquad E_2 = \frac{1}{\sqrt{6}}\big(\Sigma_1 + i\Sigma_2\big), \tag{5.89}$$
$$g_5 = \frac{1}{3}\big(\sigma_3 + \Sigma_3\big), \tag{5.90}$$

We now analyze how the $\mathcal{K}$-symmetry acts on these two-forms. The only one form which has real eigenvalues under $\mathcal{K}$ is $g_5$, which is $\mathcal{K}$-even. For the others, we have

$$\mathcal{K} \cdot E_1 \quad = iE_1\,, \tag{5.91}$$
$$\mathcal{K} \cdot E_2 \quad = -iE_2\,. \tag{5.92}$$

from which we can build the following $\mathcal{K}$-even two-forms

$$J_1 \quad = \tfrac{i}{2} E_1 \wedge \bar{E}_1\,, \tag{5.93}$$
$$J_2 \quad = \tfrac{i}{2} E_2 \wedge \bar{E}_2\,, \tag{5.94}$$
$$\Omega \quad = E_1 \wedge E_2\,, \tag{5.95}$$
$$\bar{\Omega} \quad = \bar{E}_1 \wedge \bar{E}_2\,. \tag{5.96}$$

Recalling the definition of $T^{1,1}$ as a coset space (see Appendix C.1)

$$T^{1,1} = \frac{\mathbb{CP}^1 \times \mathbb{CP}^1}{U(1)} \simeq \frac{SU(2) \times SU(2)}{U(1)}\,. \tag{5.97}$$

we recovered an explicit expression for the two independent Kähler forms of each $\mathbb{CP}^1$ $(J_1, J_2)$ and the global holomorphic two-form $\Omega$ of the Kähler-Einstein base $\mathbb{CP}^1 \times \mathbb{CP}^1$. The possible metric modes can also be arranged in terms of $\mathcal{K}$-symmetry: keeping only $\mathcal{K}$-even modes amounts to exclude six out of the ten possible metric modes, leaving just four relevant functions, which we list in the following table.

We are thus ready to write down our $\mathcal{K}$-invariant ansatz for the ten-dimensional metric. The only additional constraint we impose is that it matches the general form of the

| $\mathcal{K}$**-symmetry parity** | **Metric term** |
|---|---|
| even | $E_1\overline{E}_1, E_2\overline{E}_2, E_1E_2, \overline{E}_1\overline{E}_2$ |
| odd | $E_1\overline{E}_2, E_2\overline{E}_1, E_1E_1, \overline{E}_1\overline{E}_1, E_2E_2, \overline{E}_2\overline{E}_2$ |

Table 5.2: Parity of metric modes under $\mathcal{K}$ symmetry.

Papadopoulos-Tseytlin metric (5.28). First of all, we allow for the following frame twist beween $E_1$ and $E_2$, which is parameterized in terms of a complex scalar $v = v_1 + iv_2$

$$E'_1 = E_1 \qquad E'_2 = E_2 + v\overline{E}_1, \tag{5.98}$$

$$E_5 = g_5 + A_1\,. \tag{5.99}$$

where we introduced the real one-form $A_1$. Our candidate ten dimensional metric reads then

$$ds^2_{10} \;=\; e^{2u_3-2u_1}ds^2_5 + e^{2u_1+2u_2}E'_1\overline{E}'_1 + e^{2u_1-2u_2}E'_2\overline{E}'_2 + e^{-6u_3-2u_1}E_5E_5, \tag{5.100}$$

where $u_j\,, j = 1, 2, 3$ are three real scalar fields. The particular parameterization we have chosen for the scalar fields in (5.100) might seem convoluted but is motivated by getting canonical kinetic terms in the five-dimensional action. All the fields introduced lie in the reduced five-dimensional theory. By construction, this is the most general $SU(2)\times SU(2)$ invariant metric we can put on $T^{1,1}$ and consequently, by introducing a radial co-ordinate in five dimensions and allowing for the fields to have radially dependent profiles this ansatz contains the most general $SU(2)\times SU(2)$ invariant metric on the conifold (singular, resolved or deformed).
The symmetry principle dictated by $\mathcal{K}$-invariance led to isolate the invariant forms (5.90) (5.93)- (5.96). In the language of $G$-structures introduced in Chapter 3, these define a five dimensional $U(1)$ structure obtained as an intersection of the two different $SU(2)$, $(J_1, \Omega, \sigma_3)$ and $(J_2, \Omega, \Sigma_3)$ respectively satisfying (5.84). Clearly, by setting

$$J_\pm = J_1 \pm J_2\,, \tag{5.101}$$

we recover the $SE_5$ nature of $T^{1,1}$ in view of the fact $(J_+, \Omega, g_5)$ do satisfy (5.84) as well. The reduced structure group could help in isolating the relevant modes for the truncation, as these will simply be neutral fields with respect to the $U(1)$, which is embedded in $SO(5)$ as follows:

$$U(1) \subset SU(2)_D \subset SO(4) \subset SO(5) \tag{5.102}$$

and $SU(2)_D$ is diagonally embedded in $SO(4)$. Under this $U(1)$ the vielbeins (5.89)-(5.90) transform as

$$E_1 \to e^{i\alpha}E_1, \quad E_2 \to e^{-i\alpha}E_2, \quad E_5 \to E_5$$

so that obviously the fundamental forms (5.93)-(5.96) are $U(1)$-invariant.
It is worth to stress again how $\mathcal{K}$-symmetry helped in isolating both the set of invariant

forms of $T^{1,1}$ as well as the set of invariant metric modes. The first define a $U(1)$ structure, which can be used in order to perform a consistent truncation using the $G$-invariance argument outlined in the Introduction 1[19]. We are then ready to perform the Kaluza-Klein reduction by dimensionally reducing the ten-dimensional action, namely the kinetic terms for the dilaton-axion, three-forms, five-form and metric[20].

### 5.6.2 The Three-Forms

Motivated by the fact our symmetry principle turns out just to generalize the previously determined consistent truncations [42, 80], we proceed to expand the fluxes in terms of the $U(1)$-invariant structure forms. This simply amounts to take the already proposed expansions of [87] and generalize these for distinct $J_1$ and $J_2$. The three-forms read (refer to footnote 6 for the flux and metric convention used this Chapter, and their relation with respect to the general definitions of subsection 2.2 of Chapter 2)

$$\begin{aligned} H_{(3)} &= H_3 + H_2 \wedge (g_5 + A_1) + H_{11} \wedge J_1 + H_{12} \wedge J_2 + \\ &\quad + \Big( M_1 \wedge \Omega + M_0 \, \Omega \wedge (g_5 + A_1) + c.c \Big), \qquad (5.103) \\ F_{(3)} &= P(J_1 - J_2) \wedge (g_5 + A_1) + G_3 + G_2 \wedge (g_5 + A_1) + G_{11} \wedge J_1 + G_{12} \wedge J_2 + \\ &\quad + \Big( N_1 \wedge \Omega + N_0 \, \Omega \wedge (g_5 + A_1) + c.c \Big) \qquad (5.104) \end{aligned}$$

we have also included a topological term in $F_{(3)}$

$$P(J_1 - J_2) \wedge (g_5 + A_1) \qquad (5.105)$$

which is proportional to the volume form on the topologically nontrivial $S^3 \subset T^{1,1}$, which has a counterpart in the PT ansatz [178]. One can also include an independent topological term for the NS flux but by using the IIB $SL(2,\mathbb{Z})$ symmetry, this can always be rotated to a frame where the charge is just in $F_{(3)}$.

### 5.6.3 The Five-Form

Though the quantities $(J_+, \Omega, g_5)$ satisfy the relations (5.84), it turns out to be useful to introduce a twisted set of fundamental forms which behave nicely under the Hodge star operation (our conventions for Hodge dualizing are given in appendix A.3).

$$\begin{aligned} J'_1 &= \frac{i}{2} E'_1 \wedge \overline{E}'_1, \qquad J'_2 = \frac{i}{2} E'_2 \wedge \overline{E}'_2, \\ \Omega' &= E'_1 \wedge E'_2, \qquad \overline{\Omega}' = \overline{E}'_1 \wedge \overline{E}'_2. \qquad (5.106) \end{aligned}$$

[19] See for instance [32,66]. In the introduction of the latter paper the $G$-invariance argument is reviewed from recent point of view.

[20] All throghout this Chapter, we conventionally set $g_s, \kappa = 1$ with respect to the conventions used in the type IIB theory of Chapter 2.

We take the five-form to be manifestly self-dual (2.13)

$$\begin{aligned} F_{(5)} &= e^Z e^{8(u_3-u_1)}\text{vol}_5 + e^Z J_1' \wedge J_2' \wedge (g_5 + A_1) \\ &+K_1' \wedge J_1' \wedge J_2' - e^{-8u_1}(*_5 K_1') \wedge (g_5 + A_1) \\ &+K_{21}' \wedge J_1' \wedge (g_5 + A_1) + e^{-4u_2+4u_3}(*_5 K_{21}') \wedge J_2' \\ &+K_{22}' \wedge J_2' \wedge (g_5 + A_1) + e^{4u_2+4u_3}(*_5 K_{22}') \wedge J_1' \\ &+\big(L_2' \wedge \Omega' + c.c\big) \wedge (g_5 + A_1) + e^{4u_3}\big((*_5 L_2') \wedge \Omega' + c.c\big)\,, \end{aligned} \tag{5.107}$$

where we have defined the primed forms such as $K_1'$ in Appendix D.1.

## 5.7 Bianchi Identities

To establish the spectrum from our ansatz we must first solve the Bianchi identities, which we listed for type IIB theories in Chapter 2. From (2.11) we find

$$\begin{aligned} H_3 &= dB_2 + \frac{1}{2}(db - 2B_1) \wedge F_2, \\ H_2 &= dB_1, \\ H_{11} &= d(b + \tilde{b}) - 2B_1, \\ H_{12} &= d(b - \tilde{b}) - 2B_1, \\ 3iM_1 &= DM_0 \\ &= dM_0 - 3iA_1 M_0 \end{aligned} \tag{5.108}$$

where $F_2 = dA_1$. In the same way, from (2.21) we deduce

$$\begin{aligned} G_3 &= dC_2 - a\,dB_2 + \frac{1}{2}(dc - adb - 2C_1 + 2aB_1) \wedge F_2, \\ G_2 &= dC_1 - a\,dB_1, \\ G_{11} &= d(c + \tilde{c}) - 2C_1 - a\big(d(b + \tilde{b}) - 2B_1\big) - PA_1, \\ G_{12} &= d(c - \tilde{c}) - 2C_1 - a\big(d(b - \tilde{b}) - 2B_1\big) + PA_1, \\ 3iN_1 &= DN_0 + M_0 da \\ &= dN_0 - 3iA_1 N_0 + M_0 da \end{aligned} \tag{5.109}$$

where we labeled $a$ the R-R axion $C_{(0)}$. One can also write the three-form field strengths in terms of two-form potentials:

$$\begin{aligned} &H_{(3)} = dB_{(2)}, \\ \Rightarrow\ &B_{(2)} = B_2 + \frac{1}{2}bF_2 + B_1 \wedge (g_5 + A_1) + (b + \tilde{b})J_1 + (b - \tilde{b})J_2 \\ &\qquad + (\frac{1}{3i}M_0\Omega + c.c.), \end{aligned} \tag{5.110}$$

$$F_{(3)} = P(J_1 - J_2) \wedge (g_5 + A_1) + dC_{(2)} - a\, dB_{(2)},$$
$$\Rightarrow C_{(2)} = C_2 + \frac{1}{2} c F_2 + C_1 \wedge (g_5 + A_1) + (c + \tilde{c}) J_1 + (c - \tilde{c}) J_2 + (\frac{1}{3i} N_0 \Omega + c.c). \qquad (5.111)$$

We are then left with the five-form. Its Bianchi identity must be disentangled from the equation of motion (2.22), which after some calculation (all the relevant details are reported in Appendix D.1) we find the explicit solution

$$e^Z = Q - 2P\tilde{b} + \frac{4i}{3}(\overline{M}_0 N_0 - M_0 \overline{N}_0) \qquad (5.112)$$
$$K_1 = Dk + 2(bDc - \tilde{b}D\tilde{c}) + \frac{2i}{3}\Big(\overline{M}_0 N_1 - \overline{N}_0 M_1 - M_0 \overline{N}_1 + N_0 \overline{M}_1\Big) \qquad (5.113)$$
$$K_{21} = Dk_{11} + \frac{1}{2}\Big[Db \wedge Dc + Db \wedge D\tilde{c} + D\tilde{b} \wedge Dc\Big] \qquad (5.114)$$
$$K_{22} = Dk_{12} + \frac{1}{2}\Big[Db \wedge Dc - Db \wedge D\tilde{c} - D\tilde{b} \wedge Dc\Big] \qquad (5.115)$$

where we used the following notation

$$\begin{aligned} Dk &= dk + 4cB_1 - QA_1 - 2(k_{11} + k_{12}), \\ Dk_{11} = dk_{11} + PB_2, &\qquad Dk_{12} = dk_{12} - PB_2, \\ Db = db - 2B_1, &\qquad D\tilde{b} = d\tilde{b}, \\ Dc = dc - C_1, &\qquad D\tilde{c} = d\tilde{c} - PA_1 \end{aligned} \qquad (5.116)$$

and $Q$ is a constant corresponding to the D3 Page charge. As a side comment we point out that the explicit expressions for the fields which have been carried out in this section explicitly feature covariant derivatives. As we will explicitly show in the next subsection, these precisely correspond to the covariantization (5.58) of the derivatives of un-gauged supergravity, which will play fundamental role in the identification of the fields.
We should keep in mind that from dimensional reduction we recover also the dilaton $\phi$ and the axion scalar, which we label $a$. We summarize the field content coming both from from the dimensional reduction in following table

| | Axion | Dilaton | Metric | $H_3$ | $F_3$ | $F_5$ |
|---|---|---|---|---|---|---|
| Scalars | a | $\phi$ | $u_1, u_2, u_3, v, \overline{v}$ | $b, \tilde{b}, M_0, \overline{M}_0$ | $c, \tilde{c}, N_0, \overline{N}_0$ | $k$ |
| Vectors | - | - | $A_1$ | $B_1$ | $C_1$ | $k_{11}, k_{12}$ |
| Two-forms | - | - | - | $B_2$ | $C_2$ | $L_2, \overline{L}_2$ |

Table 5.3: Field content from dimensional reduction.

In total we thus recover 16 scalars, 5 vectors and 4 two-forms. Following the standard dualization procedure of $p$-forms into $(d-p-2)$ forms [184], in five-dimensional theories the two forms $(B_2, C_2)$ are dualized to one-forms $(\tilde{B}_1, \tilde{C}_1)$. Furthermore the physical data of the complex form $L_2$ is encoded in its gauge potential $D_1$, as we will illustrate when dealing with the identification of the gauge fields from in the un-gauged perspective. Putting together the information recovered from the dimensional reduction, and the related comments above, the symmetry principle together with the explicit solution of the Bianchi identities provides 16 scalars and 9 one-forms. As expected, these completely match the spectrum of singlets under the $U(1)$ structure group obtained, which we display for both scalars and one-forms in the following tables.

| Field | $SO(5) \to SO(4) \to SU(2)_D \to U(1)$ | $U(1)$ neutral field |
|---|---|---|
| $g_{mn}$ | $\mathbf{15} \to \mathbf{1}_0 + \mathbf{4} + \mathbf{1}_0 + \mathbf{9} \to \mathbf{1}_0 + 2\times\mathbf{2} + \mathbf{1}_0 + 3\times\mathbf{3}$ $\to \mathbf{1}_0 + 2(\mathbf{1}_1 + \mathbf{1}_{-1}) + \mathbf{1}_0 + 3(\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_{-1})$ | $v, \overline{v}, u_1, u_2, u_3$ |
| $B_{mn}$ | $\mathbf{10} \to \mathbf{4} + \mathbf{6} \to (\mathbf{2}+\mathbf{2}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + \mathbf{3})$ $\to 2(\mathbf{1}_1 + \mathbf{1}_{-1}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + (\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}))$ | $b, \tilde{b}, M_0, \overline{M}_0$ |
| $C_{mn}$ | $\mathbf{10} \to \mathbf{4} + \mathbf{6} \to (\mathbf{2}+\mathbf{2}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + \mathbf{3})$ $\to 2(\mathbf{1}_1 + \mathbf{1}_{-1}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + (\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}))$ | $c, \tilde{c}, N_0, \bar{N}_0$ |
| $C_{mnpq}$ | $\mathbf{5} \to \mathbf{1}_0 + \mathbf{4} \to \mathbf{1}_0 + (\mathbf{2}+\mathbf{2})$ $\to \mathbf{1}_0 + 2(\mathbf{1}_1 + \mathbf{1}_{-1})$ | $k$ |
| $\phi$ | $\mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0$ | $\phi$ |
| $a$ | $\mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0$ | $a$ |

Table 5.4: Decomposition of scalar fields under the structure group.

| Field | $SO(5) \to SO(4) \to SU(2)_D \to U(1)$ | $U(1)$ neutral field |
|---|---|---|
| $g_{\mu m}$ | $\mathbf{5} \to \mathbf{1}_0 + \mathbf{4} \to \mathbf{1}_0 + (\mathbf{2}+\mathbf{2}) \to \mathbf{1}_0 + 2(\mathbf{1}_1 + \mathbf{1}_{-1})$ | $A_1$ |
| $B_{\mu m}$ | $\mathbf{5} \to \mathbf{1}_0 + \mathbf{4} \to \mathbf{1}_0 + (\mathbf{2}+\mathbf{2}) \to \mathbf{1}_0 + 2(\mathbf{1}_1 + \mathbf{1}_{-1})$ | $B_1$ |
| $C_{\mu m}$ | $\mathbf{5} \to \mathbf{1}_0 + \mathbf{4} \to \mathbf{1}_0 + (\mathbf{2}+\mathbf{2}) \to \mathbf{1}_0 + 2(\mathbf{1}_1 + \mathbf{1}_{-1})$ | $C_1$ |
| $C_{\mu mnp}$ | $\mathbf{10} \to \mathbf{4} + \mathbf{6} \to (\mathbf{2}+\mathbf{2}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + 3)$ $\to 2(\mathbf{1}_1 + \mathbf{1}_{-1}) + (\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{1}_0 + (\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}))$ | $k_{11}, k_{12}, D_1, \overline{D}_1$ |
| $B_{\mu\nu}$ | $\mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0$ | $B_2$ |
| $C_{\mu\nu}$ | $\mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0 \to \mathbf{1}_0$ | $C_2$ |

Table 5.5: Decomposition of form fields under the structure group.

Concluding the ansatz section, we stress the fact that we have generalized the reduction on a generic $SE_5$ [42,87] allowing for 5 new scalars and one new vector[21]. We will discuss later in the manifest identification of the vectors and scalars from the supergravity point

[21]Using the language adopted in [87], we have indeed two more scalars from the three-forms fluxes $(\tilde{b}, \tilde{c})$, three more scalars from the metric $(u_2, v, \bar{v})$, and one more vector coming from the five-form flux (one between $k_{11}$ and $k_{12}$).

of view, but we can already notice how these just arrange in a new vector multiplet of $\mathcal{N} = 4$ supersymmetry in five dimensions.

## 5.8 The (dimensionally reduced) five-dimensional Lagrangian

Once the full ansatz has been written down, it is worth performing explicitly the reduction to a five dimensional theory. There are several subtleties in producing a five-dimensional Lagrangian whose equations of motion match those of the ten dimensional theory, largely due to the Chern-Simons terms in ten dimensions. We have checked that the full Lagrangian obtained by dimensional reduction of the type IIB action (2.12) reproduces correctly the ten dimensional equations of motion. The five-dimensional action which we obtain can be structured in the various contribution as in (5.73), which in turns correspond to the five-dimensional Einstein-Hilbert term, the scalar kinetic terms, the kinetic terms for the gauge fields and two-forms, the scalar potential and the Chern-Simons terms. In turns, we find the scalar kinetic terms to be

$$\begin{aligned}\mathcal{L}_{\text{s,kin}} \;=\; & -\frac{1}{2}e^{-4(u_1+u_2)-\phi}H'_{11}\wedge *_5 H'_{11} - \frac{1}{2}e^{-4(u_1-u_2)-\phi}H_{12}\wedge *_5 H_{12} - 4e^{-4u_1-\phi}M'_1\wedge *_5\overline{M}'_1 \\ & -\frac{1}{2}e^{-4(u_1+u_2)+\phi}G'_{11}\wedge *_5 G'_{11} - \frac{1}{2}e^{-4(u_1-u_2)+\phi}G_{12}\wedge *_5 G_{12} - 4e^{-4u_1+\phi}N'_1\wedge *_5\overline{N}'_1 \\ & -8du_1\wedge *_5 du_1 - 4du_2\wedge *_5 du_2 - 12du_3\wedge *_5 du_3 - e^{-4u_2}Dv\wedge *_5 D\overline{v} \\ & -\frac{1}{2}e^{-8u_1}K_1\wedge *_5 K_1 - \frac{1}{2}d\phi\wedge *_5 d\phi - \frac{1}{2}e^{2\phi}da\wedge *_5 da, \qquad (5.117)\end{aligned}$$

where we have twisted some of the one-forms

$$H'_{11} = H_{11} - |v|^2 H_{12} - 4\,\mathrm{Im}\,(vM_1), \qquad (5.118)$$

$$M'_1 = M_1 + \frac{i}{2}\overline{v}H_{12}, \qquad (5.119)$$

$$G'_{11} = G_{11} - |v|^2 G_{12} - 4\,\mathrm{Im}\,(vN_1), \qquad (5.120)$$

$$N'_1 = N_1 + \frac{i}{2}\overline{v}G_{12} \qquad (5.121)$$

and $\phi$ is the dilaton.

The kinetic terms for the gauge fields are

$$\begin{aligned}
\mathcal{L}_{\text{g,kin}} &= -\frac{1}{2}e^{-8u_3}F_2\wedge *_5F_2 - \frac{1}{2}e^{4u_1-4u_3-\phi}H_3\wedge *_5H_3 - \frac{1}{2}e^{4u_1+4u_3-\phi}H_2\wedge *_5H_2 \\
&\quad -\frac{1}{2}e^{4u_1-4u_3+\phi}G_3\wedge *_5G_3 - \frac{1}{2}e^{4u_1+4u_3+\phi}G_2\wedge *_5G_2 \\
&\quad -4e^{4u_3}\big(1+|v|^2e^{-4u_2}\big)L_2\wedge *_5\overline{L}_2 + 4e^{-4u_2+4u_3}\Big(v^2L_2\wedge *_5L_2 + c.c\Big) \\
&\quad -\frac{1}{2}e^{4u_2+4u_3}\big(1+|v|^2e^{-4u_2}\big)^2K_{22}\wedge *_5K_{22} - \frac{1}{2}e^{-4u_2+4u_3}K_{21}\wedge *_5K_{21} \\
&\quad +|v|^2e^{-4u_2+4u_3}K_{22}\wedge *_5K_{21} + 2e^{4u_3}\big(1+|v|^2e^{-4u_2}\big)\Big(ivK_{22}\wedge *_5L_2 + c.c\Big) \\
&\quad -2e^{-4u_2+4u_3}\Big(ivK_{21}\wedge *_5L_2 + c.c\Big) \qquad (5.122)
\end{aligned}$$

where the somewhat off-diagonal last four lines come from the five-form. The scalar potential has several contributions which we distinguish for clarity:

$$\begin{aligned}
\mathcal{L}_{\text{pot}} &= -\big(V_{gr} + V_{H_{(3)}} + V_{F_{(3)}} + V_{F_{(5)}}\big), \qquad (5.123)\\
V_{\text{gr}} &= -12e^{-4u_1-2u_2+2u_3}\big(1+|v|^2+e^{4u_2}\big) + 9|v|^2e^{-4u_2+8u_3} \\
&\quad +2e^{-8u_1-4u_3}\big(e^{4u_2}+e^{-4u_2}(1-|v|^2)^2+2|v|^2\big), \qquad (5.124)\\
V_{H_{(3)}} &= 4e^{-4u_1+8u_3-\phi}\Big(|M_0|^2 + 2e^{-4u_2}[\text{Im}\,(M_0v)]^2\Big), \qquad (5.125)\\
V_{F_{(3)}} &= \frac{1}{2}e^{-4u_1+8u_3+\phi}\Big(8|N_0'|^2 + e^{4u_2}P^2 + e^{-4u_2}\big(P(|v|^2-1)+4\,\text{Im}\,(N_0'v)\big)^2\Big), \qquad (5.126)\\
V_{F_{(5)}} &= \frac{1}{2}e^{2Z}e^{-8u_1+8u_3} \qquad (5.127)
\end{aligned}$$

where

$$N_0' = N_0 - \frac{i}{2}P\overline{v}. \qquad (5.128)$$

As expected this scalar potential is almost but not quite a sum of squares. The only term which spoils this property is $V_{gr}$. Finally the gravitational term is of course

$$\mathcal{L}_{gr} = R\,\text{vol}_5 \qquad (5.129)$$

where $R$ is the Ricci scalar in Einstein frame. The Chern-Simons terms are particularly long and unspectacular so we will not write them explicitly. Nonetheless, in the ungauged case which we deal with below, they are somewhat simpler and also extremely crucial so we will present them explicitly there.
In the following we explore the connection with gauged supergravity. To do so, the necessary steps are to identify the scalars $(\Sigma, M_{MN})$ and the gauge fields $(\mathcal{A}^0, \mathcal{A}^M)$ in the ungauged theory. Having obtained this, we would have all the necessary tools to discuss the gauging.

## 5.9 Identification of the fields via un-gauged $\mathcal{N}=4$ supegravity

A particularly insightful aspect of the works [42, 87] was the construction of manifest $\mathcal{N}=4$ supersymmetry (by which we mean 16 supercharges). In that case, the reason this was unexpected was that this particular gauging of $\mathcal{N}=4$ supergravity does not have a vacuum which preserves all the supercharges, the maximally supersymmetric vacuum is an $AdS_5$ which preserves only $\mathcal{N}=2$. Looking at the multiplet structure of $\mathcal{N}=4$ five-dimensional supergravity presented in subsection 5.4, we already notice our reduction features the bosonic field content of $\mathcal{N}=4$ gauged supergravity coupled to three vector multiplets. We are now going to show it explicitly. A particular basis for the coset (5.65) specialized for three vector multiplets

$$\frac{SO(5,3)}{SO(5)\times SO(3)}\times SO(1,1) \tag{5.130}$$

is given in [157] (eq. (3.31)): by explicitly computing the Lagrangian (5.74) using the matrix (5.52), one recovers

$$\begin{aligned}
-\operatorname{Tr}(dM\wedge *dM^{-1}) \;=\;& 2\big(d\phi_1^2+d\phi_2^2+d\phi_3^2\big)\\
&+4e^{-\phi_2+\phi_3}dx_1^2+4e^{-\phi_1+\phi_3}(dx_2-x_1dx_3)^2+4e^{-\phi_1+\phi_2}dx_3^2\\
&+4e^{\phi_1+\phi_2}(dx_4+x_7dx_8+x_{10}dx_{11})^2\\
&+4e^{\phi_1+\phi_3}\big(dx_5+x_7dx_9+x_{10}dx_{12}-x_1(dx_4+x_7dx_8+x_{10}dx_{11})\big)^2\\
&+4e^{\phi_1}dx_7^2+e^{\phi_2}\big(dx_8-x_3dx_7\big)^2\\
&+4e^{\phi_3}\big(dx_9-(x_2-x_1x_3)dx_7-x_1dx_8\big)^2\\
&+4e^{\phi_1}dx_{10}^2+e^{\phi_2}\big(dx_{11}-x_3dx_{10}\big)^2\\
&+4e^{\phi_3}\big(dx_{12}-(x_2-x_1x_3)dx_{10}-x_1dx_{11}\big)^2\\
&+4e^{\phi_2+\phi_3}\big[dx_6+x_2dx_4+x_2x_7dx_8+x_2x_{10}dx_{11}-x_3dx_5\\
&+(x_8-x_3x_7)dx_9+(x_{11}-x_3x_{10})dx_{12}\big]^2 .
\end{aligned} \tag{5.131}$$

It may be helpful to describe how the basis of [157] (which we will refer to as the "heterotic" basis) is related to a more common basis in the gauged supergravity literature [184, 185] (which we will refer to as the "gsg basis") which we wrote generally in (5.67), and we recall here for convenience in the explicit case of supergravity coupled to three vector multiplets

$$(t_{MN})_P{}^Q=\delta^Q_{[M}\eta_{N]P} \tag{5.132}$$

$M,N\ldots=1,\ldots,8$ and $\eta=\mathrm{diag}\{+++++---\}$. Of course only a subset of the $t_{MN}$ generate the coset $SO(5,3)/(SO(5)\times SO(3))$. The two basis (where the heterotic basis is completed to a full set of generators of $SO(5,3)$) are related by conjugation with $C$:

$$C=D_1+D_2+D_3+E_{44}+E_{55}, \tag{5.133}$$

where

$$D_i = \big(E_{i,i} - E_{i,i+5} + E_{i+5,i} + E_{i+5,i+5}\big)/\sqrt{2} \quad i = 1,\dots,3 \tag{5.134}$$

and where $E_{ij}$ is a matrix with 1 in the $i$-th row and $j$-th column and zero's elsewhere. This is most easily seen by relating $\eta$ and $\tilde{\eta}$ where

$$\begin{aligned} \tilde{\eta}_{16} = &\ \tilde{\eta}_{61} = -1 \\ \tilde{\eta}_{27} = &\ \tilde{\eta}_{72} = -1 \\ \tilde{\eta}_{38} = &\ \tilde{\eta}_{83} = -1 \\ \tilde{\eta}_{44} = &\ \tilde{\eta}_{55} = 1 \end{aligned} \tag{5.135}$$

and $\mathcal{V}^T \tilde{\eta} \mathcal{V} = \tilde{\eta}$. The only non-trivial step left is the explicit matching between the reduced action for scalar kinetic terms (5.117), previously obtained from dimensional reduction, and the un-gauged supergravity formula (5.131). We list then the complete identification of the scalars

$$\begin{aligned}
e^{2u_3} &= \Sigma \\
-4u_2 &= \phi_1 \\
-4u_1 - \phi &= \phi_2 \\
-4u_1 + \phi &= \phi_3 \\
\sqrt{2}\, v &= x_7 + i x_{10} \\
a &= x_1 \\
b - \tilde{b} &= x_3 \\
b + \tilde{b} &= -x_4 - \frac{1}{2} x_3 (x_7^2 + x_{10}^2) \\
c - \tilde{c} &= x_2 \\
c + \tilde{c} &= -x_5 - \frac{1}{2} x_2 (x_7^2 + x_{10}^2) \\
\frac{2\sqrt{2}}{3} M_0 &= -\big(x_8 - x_3 x_7\big) + i\big(x_{11} - x_3 x_{10}\big) \\
&= -(x_8 - i x_{11}) + x_3 (x_7 - i x_{10}) \\
\frac{2\sqrt{2}}{3} N_0 &= -\big(x_9 - (x_2 - x_1 x_3) x_7 - x_1 x_8\big) + i\big(x_{12} - (x_2 - x_1 x_3) x_{10} - x_1 x_{11}\big) \\
&= -(x_9 - i x_{12}) + (x_2 - x_1 x_3)(x_7 - i x_{10}) + x_1 (x_8 - i x_{11}) \\
k &= x_6 + x_2 x_4 + \frac{1}{2} x_2 x_3 (x_7^2 + x_{10}^2) \\
&\quad + \frac{1}{2}\Big(x_2\big(x_7 x_8 + x_{10} x_{11}\big) + x_9 (x_8 - x_3 x_7) + x_{12} (x_{11} - x_3 x_{10})\Big).
\end{aligned} \tag{5.136}$$

We now discuss how, from explicitly computing the gauge-kinetic (5.75) and Chern-Simons terms (5.74), we are able to identify the $SO(5,3)$ vectors and scalar content. The un-gauged theory is supported by a relaxation of the differential relations (5.84) to

$$dg_5 = 0, \quad dJ_{1,2} = 0, \quad d\Omega = 0. \tag{5.137}$$

together with setting the topological fluxes $(P,Q)$ to zero. With these conditions, the Lagrangians (5.75) and (5.76) can be used to recover explicitly the nine gauge fields. We must however first integrate out the pair of two-forms $(B_2, C_2)$. The central difference between the gauged reduction and the ungauged reduction is the Bianchi identities and their solution: instead of (5.108), (5.109) and (5.112)-(5.115) we have for $H_{(3)}$:

$$\begin{array}{ll} H_3 = dB_2 - B_1 \wedge F_2, & H_2 = dB_1, \\ H_{11} = d(b+\tilde{b}), & H_{12} = d(b-\tilde{b}), \\ 3iM_1 = dM_0, & \end{array} \tag{5.138}$$

for $F_{(3)}$

$$\begin{array}{ll} G_3 = dC_2 - adB_2 - (C_1 - aB_1)\wedge F_2, & G_2 = dC_1 - a\,dB_1, \\ G_{11} = d(c+\tilde{c}) - a\,d(b+\tilde{b}), & G_{12} = d(c-\tilde{c}) - a\,d(b-\tilde{b}), \\ 3iN_1 = dN_0 - adM_0, & \end{array} \tag{5.139}$$

and for $F_{(5)}$

$$\begin{array}{rl} K_1 = & dk + 2(bdc - \tilde{b}d\tilde{c}) + \frac{2i}{3}\Big(\overline{M}_0 N_1 - \overline{N}_0 M_1 - M_0\overline{N}_1 + N_0\overline{M}_1\Big), \\ K_{21} = & dk_{11} + (b+\tilde{b})dC_1 - (c+\tilde{c})dB_1, \\ K_{22} = & dk_{12} + (b-\tilde{b})dC_1 - (c-\tilde{c})dB_1, \\ L_2 = & dD_1 + \frac{1}{3i}\big(M_0 dC_1 - N_0 dB_1\big). \end{array} \tag{5.140}$$

We find that before integrating out $(B_2, C_2)$, the Chern-Simons terms are

$$\begin{aligned} \mathcal{L}_{top} \;=\; & -A_1 \wedge \Big[K_{22}\wedge K_{21} + K_1 \wedge(-C_1\wedge H_2 + B_1 \wedge G_2) + 4L_2\wedge \overline{L}_2 \\ & +K_{21}\wedge\big[d(b-\tilde{b})\wedge C_1 - d(c-\tilde{c})\wedge B_1\big] \\ & +K_{22}\wedge\big[d(b+\tilde{b})\wedge C_1 - d(c+\tilde{c})\wedge B_1\big] \\ & +\big((4i/3)L_2\wedge(\overline{M}_0 C_1 - \overline{N}_0 B_1) + c.c\big)\Big] \\ & -dC_2\wedge S_2 + dB_2 \wedge T_2 \end{aligned} \tag{5.141}$$

where

$$\begin{aligned} S_2 \;=\; & \big(k + \frac{4}{9}\mathrm{Re}(M_0\overline{N}_0)\big)dB_1 - (b^2 - \tilde{b}^2 + \frac{1}{9}|M_0|^2)dC_1 \\ & -(b-\tilde{b})dk_{11} - (b+\tilde{b})dk_{12} - \frac{8}{3}\mathrm{Im}\left(M_0 d\overline{D}_1\right) \end{aligned} \tag{5.142}$$

$$\begin{aligned} T_2 \;=\; & \big(k - \frac{4}{9}\mathrm{Re}(M_0\overline{N}_0)\big)dC_1 + (c^2 - \tilde{c}^2 + \frac{1}{9}|N_0|^2)dB_1 \\ & -(c-\tilde{c})dk_{11} - (c+\tilde{c})dk_{12} - \frac{8}{3}\mathrm{Im}\left(N_0 d\overline{D}_1\right). \end{aligned} \tag{5.143}$$

First we introduce Lagrange multiplers $(\widetilde{B}_1, \tilde{C}_1)$

$$\Delta\mathcal{L} = \tilde{C}_1 \wedge d\tilde{H}_3 + \tilde{B}_1 \wedge d\tilde{G}_3 \tag{5.144}$$

where

$$\tilde{H}_3 = dB_2, \quad \tilde{G}_3 = dC_2,$$

and then we integrate out $(\tilde{H}_3, \tilde{G}_3)$ and after some algebra we find

$$\begin{aligned}
\mathcal{L}_{g,kin} &= -\frac{1}{2}e^{-8u_3}F_2\wedge *_5F_2 - \frac{1}{2}e^{-4(u_1-u_3)-\phi}\tilde{H}_2\wedge *_5\tilde{H}_2 - \frac{1}{2}e^{-4(u_1-u_3)+\phi}\tilde{G}_2\wedge *_5\tilde{G}_2 \\
&\quad -\frac{1}{2}e^{4(u_1+u_3)-\phi}H_2\wedge *_5H_2 - \frac{1}{2}e^{4(u_1+u_3)+\phi}G_2\wedge *_5G_2 \\
&\quad -4e^{4u_3}\left(1+|v|^2e^{-4u_2}\right)L_2\wedge *_5\overline{L}_2 + 4e^{-4u_2+4u_3}\Big(v^2L_2\wedge *_5L_2 + c.c\Big) \\
&\quad -\frac{1}{2}e^{4u_2+4u_3}\left(1+|v|^2e^{-4u_2}\right)^2K_{22}\wedge *_5K_{22} - \frac{1}{2}e^{-4u_2+4u_3}K_{21}\wedge *_5K_{21} \\
&\quad +|v|^2e^{-4u_2+4u_3}K_{22}\wedge *_5K_{21} + 2e^{4u_3}\left(1+|v|^2e^{-4u_2}\right)\Big(ivK_{22}\wedge *_5L_2 + c.c\Big) \\
&\quad -2e^{-4u_2+4u_3}\Big(ivK_{21}\wedge *_5L_2 + c.c\Big), \qquad (5.145) \\
\mathcal{L}_{top} &= -A_1\wedge\Big[dk_{12}\wedge dk_{11} + 4dD_1\wedge d\overline{D}_1 - dB_1\wedge d\tilde{C}_1 - dC_1\wedge d\tilde{B}_1\Big], \qquad (5.146)
\end{aligned}$$

Identifying (5.146) with (5.76) we recognize the $SO(5,3)$ scalar

$$\mathcal{A}^0 = -A_1/\sqrt{2}\,, \qquad (5.147)$$

while by comparing (5.145) to (5.75) we get the $SO(5,3)$ vector $\mathcal{A}^M$ which components are the following one-form potentials:

$$\begin{array}{ll}
\mathcal{A}^1 = -k_{11}/\sqrt{2}, & \mathcal{A}^2 = \tilde{B}_1/\sqrt{2}, \\
\mathcal{A}^3 = \tilde{C}_1/\sqrt{2}, & \mathcal{A}^4 = 2\mathrm{Im}\,(D_1), \\
\mathcal{A}^5 = 2\mathrm{Re}\,(D_1), & \mathcal{A}^6 = k_{12}/\sqrt{2}, \\
\mathcal{A}^7 = C_1/\sqrt{2}, & \mathcal{A}^8 = B_1/\sqrt{2}.
\end{array} \qquad (5.148)$$

So far we have shown that the ungauged theory, corresponding to a particular consistent truncation on the five-torus $T^5$ has $\mathcal{N} = 4$ supersymmetry, and features three vector multiplets in addition to the gravity multiplet. Together with the scalar identification, we have in our hands the necessary tools for moving to the gauged picture. Of course this is not the theory of most interest to us but was a necessary step in developing the gauged theory, which corresponds to a consistent truncation on $T^{1,1}$. We are then ready to move along the horizontal arrow towards the right in the diagram draw in Figure 5.1.

## 5.10 The gauged theory

Having successfully demonstrated the manifest supersymmetry of the ungauged theory, the content of the gauged theory can be neatly summarized in the embedding tensor. We have found the heterotic basis (5.66) to be computationally efficient but the embedding tensor is most naturally expressed in the “gsg basis" (5.67) where it is completely antisymmetric (with all indices lowered):

$$f_{MNP} = f_{[MNP]}, \qquad \xi_{MN} = \xi_{[MN]}. \tag{5.149}$$

Note that our expressions (5.148) are in the heterotic basis. All the information about the gauging could be extracted from an explicit calculation of (5.81). It turns out that the only non-vanishing components in the basis (5.67) are

$$f_{123} = -f_{128} = f_{137} = f_{178} = 2, \tag{5.150}$$
$$\xi_{23} = -\xi_{28} = \xi_{37} = \xi_{78} = -Q/\sqrt{2}, \tag{5.151}$$
$$\xi_{45} = -3\sqrt{2} \tag{5.152}$$
$$\xi_{36} = \xi_{68} = \sqrt{2}P \tag{5.153}$$

and permutations. From this one can read off the covariant field strength of $\mathcal{N} = 4$ gauged supergravity (5.83), which for example specializes to

$$\begin{aligned} \mathcal{F}^1 + \mathcal{F}^6 &= d(\mathcal{A}^1 + \mathcal{A}^6) - 2\sqrt{2}\mathcal{A}^7 \wedge \mathcal{A}^8, \\ \mathcal{F}^7 &= d\mathcal{A}^7, \\ \mathcal{F}^8 &= d\mathcal{A}^8 \end{aligned} \tag{5.154}$$

here we have used the basis of gauged fields in the “heterotic basis" (5.148). From this we recognize the same Heisenberg algebra which was observed in [42, 87], and conclude that the gauge group is unaltered with respect to the one determined in those works. The only additional gaugings in our ansatz arise from the topological flux we have turned on (5.153). So the additional vector multiplet we have included has enhanced the complexity of the embedding tensor somewhat indirectly through the additional degrees of freedom required to allow for non-trivial topology and thus the flux $P$. As final check of our computations is to compute the scalar potential from the gauged supergravity formula. As we found that there is no relevant component $\xi_M$ different from zero, the scalar potential corresponds to a particular case of (5.79)

$$\begin{aligned} V \;=\; & \frac{1}{2} f_{MNP} f_{QRS} \Sigma^{-2}\Big(\frac{1}{12} M^{MQ} M^{NR} M^{PS} - \frac{1}{4} M^{MQ} \eta^{NR} \eta^{PS} + \frac{1}{6} \eta^{MQ} \eta^{NR} \eta^{PS}\Big) \\ & + \frac{1}{8} \xi_{MN} \xi_{PQ} \Sigma^4 \Big(M^{MP} M^{NQ} - \eta^{MP} \eta^{NQ}\Big) + \frac{1}{6}\sqrt{2} f_{MNP} \xi_{QR} \Sigma M^{MNPQR}. \end{aligned} \tag{5.155}$$

To work with the $SO(5)$ indices it is best to transform to the “gsg" basis for the coset. Note that the three separate terms in this expression are distinguished by the power of $\Sigma = e^{2u_3}$ and each such term is easily identified in (6.36). The check of agreement between the two expressions completes our prove that the supersymmetric truncation correspond to $\mathcal{N} = 4$ gauged supergravity coupled to three vector multiplets.

## 5.11 The PT ansatz truncation

The reason which led us to carry out the consistent truncation is to better understand the origin of the Kaluza-Klein reduction employed in [178]. Since we have the most general

$SU(2)\times SU(2)$ invariant reduction, it is guaranteed that the reduction of [178] lies within ours. Of course, as the former has no vector fields it clearly cannot be supersymmetric. In the scalar sector it is obtained from our truncation by setting the following fields to zero

$$(a, \mathrm{Im}\, M_0, \mathrm{Re}\, N_0, c, \tilde{c}, k, \mathrm{Im}\, v) \to 0 \tag{5.156}$$

leaving nine scalars.

We are now ready to explicitly show that the PT truncation is consistent. The most striking feature of the PT ansatz is that there is a distinct asymmetry between the R-R and NS-NS three-forms. One can understand this as a direct consequence of setting the axion to zero and satisfying the equation of motion for the axion (2.17). Following this logic explicitly in our new truncation, requires as a first step certain choices in setting the source for the axion to zero. Equation (2.17) reads

$$\begin{aligned} -d(*e^{2\phi}da) \;=\; & e^{\phi}\Big[e^{4(u_1-u_3)}H_3\wedge *G_3 + e^{-4(u_1-u_3)}H_2\wedge *G_2 \\ & +e^{-4(u_1+u_2)}H'_{11}\wedge *G'_{11} + e^{-4(u_1-u_2)}H_{12}\wedge *G_{12} \\ & +4e^{-4u_1}\big(M'_1\wedge *\overline{N}'_1 + c.c.\big) + \Big(4e^{-4u_1+8u_3}\big(M_0\overline{N}'_0 + c.c.\big) \\ & -4e^{-4u_1-4u_2+8u_3}\,\mathrm{Im}(M_0 v)\big(P(1-|v|^2) - 4\mathrm{Im}(N'_0 v)\big)\Big)\mathrm{vol}_5\Big], \end{aligned} \tag{5.157}$$

so that we are led to set

$$(\mathrm{Im}\ M_0, \mathrm{Re}\ N_0, \mathrm{Im}\, v, c, \tilde{c}) \to 0 \tag{5.158}$$

along with

$$(C_2, C_1) \to 0. \tag{5.159}$$

Further inspection of the equations of motion which arise from the ten-dimensional three form equations of motion (2.18) and (2.19) reveal that in addition we must have

$$(k, B_2, B_1, k_{11}, k_{12}, L_2, \overline{L}_2) \to 0. \tag{5.160}$$

From our formalism these steps are quite straightforward but one equation requires additional work. Since we have set $B_1 \to 0$ we must set the various source terms in (D.24) to zero and this would appear to give a differential constraint amongst several of the remaining scalars $(b, \tilde{b}, \mathrm{Re}\, v, \mathrm{Re}\, M_0)$:

$$\begin{aligned} 0 \;=\; & H_{11}(1-|v|^2)e^{-4u_2} + H_{12}\Big(e^{4u_2} - |v|^2(1-|v|^2)e^{-4u_2} + 2|v|^2\Big) \\ & +4\,\mathrm{Im}\,(vM_1)\Big(1-(1-|v|^2)e^{-4u_2}\Big). \end{aligned} \tag{5.161}$$

However if we take the exterior derivative of the Hodge star of (5.161) we in fact recover a linear combination of (D.25),(D.26),(D.27). By making use of the explicit dictionary between our truncation and the one originally adopted in PT reported in Appendix D.4, it turns out that (5.161) eactly matches (5.42). This is the only non-trivial step in showing that the ansatz employed in [178] is indeed a consistent truncation.

As very last step, let us see if there is some symmetry principle behind (5.156). We could expect it as we were able to recover the PT ansatz without particular effort as a further truncation of our original $\mathcal{N}=4$ truncated theory.
The minimal requirement for a candidate symmetry must differentiate between $B_{(2)}$ and $C_{(2)}$, so obviously it could not be the already discussed $\mathcal{I}$-symmetry. The best candidate is then world-sheet parity reversal under which

$$\Omega_p : (g, \phi, B_{(2)}, C_{(0)}, C_{(2)}, C_{(4)}) \to (g, \phi, -B_{(2)}, -C_{(0)}, C_{(2)}, -C_{(4)}). \tag{5.162}$$

A geometric symmetry is also needed and the best candidate appears to be reversal of all internal co-ordinates, which corresponds to exchanging the sphere coordinates:

$$\begin{aligned} \sigma : (\sigma_1, \Sigma_1, g_5) &\to -(\sigma_1, \Sigma_1, g_5), \\ (\sigma_2, \Sigma_2) &\to +(\sigma_2, \Sigma_2), \end{aligned} \tag{5.163}$$

which translates to

$$\begin{aligned} \sigma : (g_5, J_1, J_2, \text{Im } \Omega) &\to -(g_5, J_1, J_2, \text{Im } \Omega), \\ \text{Re } \Omega &\to \text{Re } \Omega. \end{aligned} \tag{5.164}$$

For what concerns the five-form, as $\text{vol}_5$ and $\text{vol}_{T^{1,1}}$ transform with opposite sign:

$$\Omega_p \cdot \sigma : (\text{vol}_5, \text{vol}_{T^{1,1}}) \to (-\text{vol}_5, \text{vol}_{T^{1,1}}) \tag{5.165}$$

so we are forced to introduce five-dimensional parity $P_5$ and use the composite symmetry

$$\mathcal{J} = \Omega_p \cdot \sigma \cdot P_5 \tag{5.166}$$

as candidate for a symmetry principle which restricts one to the PT ansatz. However $\mathcal{J}$ does not commute or anti-commute with the exterior derivative, to be more precise it anti-commutes with the external exterior derivative but commutes with the internal one. This means that even if terms in the potential have equal $\mathcal{J}$-charge, the field strength will not. We conclude we could not find any symmetry principle which restricts one to the PT ansatz, but nonetheless we have provided an explicitly supersymmetric embedding of it into a consistent truncation which *is* based on a symmetry principle.

## 5.12 The $\mathcal{I}$-truncation

One could even perform subtruncations of the gauged $\mathcal{N}=4$ supergravity theory just determined. When turning to zero the additional vector multiplet, we remain inside the $\mathcal{N}=4$ theory, finding back the truncations on a generic $SE_5$ [42, 87]. We can take once more a symmetry as guiding principle in performing a further truncation, by considering its additional restriction to the $\mathcal{I}$-invariant sector[22] (5.21). This is indeed the $\mathbb{Z}_2$ symmetry

[22] Notice that using the transformations (5.162)-(5.164) introduced above, $\mathcal{I}$ can be seen as the composite symmetry

$$\mathcal{I} = \Omega_p \cdot (-1)^{F_L} \cdot \sigma \, , .$$

which is broken away from the origin of the baryonic branch in the Klebanov-Strassler gauge theory [38, 107]. While the general $\mathcal{K}$-invariant ansatz has 16 scalars and 9 vectors, imposing the truncation to be also $\mathcal{I}$ -invariant reduces this to 2 vectors and 13 scalars. So, seven vectors must be eliminated: the two $\mathcal{I}$-invariant vectors which *survive* come from the metric, and in the five form as the term proportional to $J_+ \wedge (g_5 + A_1)$ only, which is now proportional to $(k_{11} + k_{12})$:

$$ds^2 \sim (g_5 + A_1)^2 \,, \tag{5.168}$$
$$F_5 \sim (k_{11} + k_{12}) \wedge J_+ \wedge (g_5 + A_1) \,. \tag{5.169}$$

Also, the three scalars have to be *killed.* Unfortunately, our metric parameterization is not the best choice to recover exactly the scalar which has to be projected out from the metric ansatz. In order to isolate the mode which is killed in the $\mathcal{I}$ truncation, we could reparametrize our metric functions $(u_1, u_2, u_3, v, \bar{v})$ in terms of the functions $(u, v, w, t, \theta)$ originally used in [43]. The explicit relations are

$$\begin{gathered} u_1 = u \,, \quad u_3 = -\frac{1}{3}(u+v) \,, \\ e^{2u_2} = \coth t \,, \\ v_1 = \cosh t \sin\theta \,, \quad v_2 = \cosh t \cos\theta \ . \end{gathered} \tag{5.170}$$

and the metric mode which is killed can be written in the language of our truncation as

$$e^{2w} = e^{2u_2}\sqrt{v_1^2 + v_2^2} \,. \tag{5.171}$$

We therefore recover the following scalars to be projected out

$$w \text{ in } ds^2 \sim e^{2w}(\sigma_1^2 + \sigma_2^2 - \Sigma^2 - \Sigma_2^2) \,, \tag{5.172}$$
$$b \text{ in } B_2 \sim bJ_+ \,, \tag{5.173}$$
$$c \text{ in } C_2 \sim cJ_+ \,. \tag{5.174}$$

The truncation to the $\mathcal{K} \times \mathcal{I}$- invariant sector is totally equivalent to truncate down the gravity multiplet from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ and require the equations of motion corresponding to the modes turned off to be satisfied. This can be indeed achieved by splitting the $\mathcal{N} = 4$ vector multiplet in a sum of a $\mathcal{N} = 2$ vector multiplet and a hypermultiplet, and by requiring to preserve the hyper while switching off the vector multiplet [43]. The converse, corresponding to keep the $\mathcal{N} = 2$ vector multiplet and to switching off the hyper multiplet, represents as well a consistent subtrucation, but we prefer would not discuss it here as it is not relevant for our purposes.
In five dimensional $\mathcal{N} = 2$ supergravity the bosonic sector features the following multiplets

---

where $(-1)^{F_L}$ is the world-sheet Witten index

$$(-1)^{F_L} : (g, \phi, B_{(2)}, C_{(0)}, C_{(2)}, C_{(4)}) \to (g, \phi, B_{(2)}, -C_{(0)}, -C_{(2)}, -C_{(4)}). \tag{5.167}$$

- graviton multiplet: $(g_{\mu\nu}, A_\mu)$ - metric and 1 vector
- hypermultiplet $(q_1, \overline{q}_1, q_2, \overline{q}_2)$ - 4 real scalars
- vector multiplet: $(V_\mu, v)$ - 1 vector and 1 real scalar

We thus recover the truncated theory to describe the bosonic content of an $\mathcal{N} = 2$ gauged supergravity in five dimensions coupled to one vector multiplet and three hypermultiplets

$$\begin{aligned} \text{Gravity multiplet + 1 Vector multiplet:} \quad & (g_{\mu\nu}, A_1, u+v, k_{11}+k_{12}) \\ \text{3 Hypermultiplets:} \quad & (M_0, \bar{M}_0, \tilde{b}, N_0, \bar{N}_0, \tilde{c}, \phi, C_0, u, t, \theta) \end{aligned} \tag{5.175}$$

Besides being an interesting subtruncation, the $(\mathcal{K} \times \mathcal{I})$-invariant sector of our theory obviously contains only Klebanov-Strassler out of the relevant solutions discussed in the beginning of this Chapter. Maldacena- Nuñez indeed features, using once more the language (5.170)-(5.171) of [43], both non-vanishing $w$ and $t$, meaning that in order to make the corresponding subsector of the Papadopoulos-Tseytlin ansatz supersymmetric, one should retain the full $\mathcal{N} = 4$ theory.
Since the PT ansatz can be embedded within our truncation, it is clear that the resolved, deformed and singular conifolds can all be found as its solutions: for these particular cases a superpotential was already proposed in [178]. While superpotentials for $\mathcal{N} = 2$ truncations have been constructed in [154], a more thorough analysis of the entire $\mathcal{N} = 4$ theory is required in order to determine the most general superpotential. It would also be desirable as it would also allow a direct connection with interpolating solutions such as those described in [38], and may help to further classify four and five-dimensional solutions of IIB supergravity.

# Chapter 6

# The backreaction of anti-D2 branes on the CGLP background

As supersymmetry is an unwanted feature from the phenomenological point of view, we want to investigate whether a supersymmetric vacuum can be perturbed in order to eventually break it. Generic supersymmetry-breaking configurations can be realized in string theory by placing anti-branes in warped throats backgrounds, such the ones we presented in Chapter 5. Being these gravity configurations dual to certain gauge theories, the brane/antibrane configuration should correspond to a supersymmetry-breaking state in the corresponding field theory.

Dynamical supersymmetry breaking (DSB) [130,189] is one of the main features of $\mathcal{N} = 1$ supersymmetric gauge field theories. DSB trades the classical supersymmetric vacuum for a global non-supersymmetric minimum of the potential. Whenever considering gauge theories on certain fractional branes at geometries without complex deformations [20,26,74], although one can still build non-perturbative superpotentials capable of removing the supersymmetric vacuum, the scalar potential of these theories leads to a runaway to infinity [74,131]. A realization of metastable SUSY–breaking starting from the KS background [144] has been proposed by Kachru, Pearson and Verlinde (KPV) in the probe approximation [141]. Supersymmetry is broken by adding a certain amount of anti-branes which are attracted to the bottom of the throat. In this paper a mechanism in which all of the anti-branes can annihilate (via polarization and the Myers effect [171]) with the positive brane-charge dissolved in flux is proposed, and it is argued how this process corresponds to the decay of the metastable vacuum in the dual field theory description [141]. This metastable configuration was involved to construct de Sitter vacua in the KKLT scenario [139], which starts from a solution where the moduli are stabilized at values which give a negative cosmological constant, and supersymmetry (SUSY) remains unbroken. In order to achieve a de Sitter minimum the authors introduce anti-D3 branes into the compactified volume: this uplifts the scalar potential to a positive value and breaks supersymmetry.

A recent program investigating the construction of metastable states beyond the probe approximation has been initiated in [13–17,67]. In general, the supergravity dual background to investigate is a locally stable, non–supersymmetric solution obtained by deformation (*i.e.* addition of anti-branes) of a supersymmetric one, following a generalization of the deformation procedure [57] outlined in Chapter 5 to recover (5.5)-(5.6). There are indeed backgrounds which share sufficient common features with the Klebanov–Strassler background to be candidate setups for arguing the presence of meta-stable vacua. In this perspective, a similar analysis to the already mentioned KPV for anti D3-branes on the conifold was proposed by Klebanov and Pufu in an M-theoretical setup [143]. Also in this case, a more careful analysis of the backreaction due to anti M2-branes branes led to a discovery of a singular behavior in the IR regime [13]. The final results are qualitatively the same despite the differences in the setups and the quite different calculations that led to them. Put together, these two examples illustrate the limits of the probe approximation: the conclusion is that there is an unavoidable singularity in the IR region of the backreacted solutions which have been considered so far.

We devote this Chapter to the study of the backreaction due to anti-D2 branes placed in a non-singular fractional+ordinary D2 branes supergravity solution found by Cvetič, Gibbons, Lü and Pope [53] (CGLP). We will illustrate how the candidate IIA supergravity dual to metastable SUSY–breaking[1] that we build is riddled with singularities arising from the linearized deformation of either the R-R or NS-NS field strengths. Of much concern, those are non-finite action singularities. A novelty of the case explored in this Chapter compared to [13, 17] is that those singularities are not sub-leading compared to the kind of singularities that are allowed as physically sensible ones, that is those stemming from the effect of anti–D2 branes smeared on the $S^4$ at the bottom of the tip. To proceed in perfect analogy with the IIB case, we may have wanted to start from a background describing the gravity dual of a four dimensional gauge theory. Configurations of intersecting branes in type IIA, in which a web of D-branes and NS5-branes encodes the dual of various four-dimensional gauge theories are known [70, 73, 75]. However, to establish if the brane configuration indeed describes a metastable state, it is necessary to investigate the relationship between the vacuum of the string theory and the vacuum of the gauge theory. The analysis of Bena, Gorbatov, Hellerman, Seiberg and Shih [18] shows that for the type IIA engineering, the supersymmetric vacuum and the brane construction differ in the running of the gauge theory coupling constant. The brane construction does not describe a metastable vacuum of a supersymmetric theory, but rather a non-supersymmetric vacuum of a non-supersymmetric theory. On the other hand, the same argument cannot be used to invalidate constructions of three-dimensional gauge theories, as in this case the vacuum of the supersymmetric and non-supersymmetric theory become the same at infinity. The gauge theory dual to the CGLP background that we will be considering is indeed a $2+1$ theory. Unfortunately, a missing point with respect to the KS case is that not much is known on the gauge theory dual to CGLP backgrounds: in spite of the fact the IR theory corresponding to wrapping of different fractional branes is apparently unique [53, 118], the UV behavior is not universal, and is in general less understood [156].

[1] See [92] for a generalization of the ISS model to lower dimension.

## 6.1 The CGLP solution

The supersymmetric background we will use as a zeroth order in perturbation theory is a very similar configuration to the KS background [144], for which the conical singularity is resolved along the general lines discussed in the previous Chapter. The CGLP background [53] describes regular deformed D2–branes together with fractional D2–branes. To outline its construction, consider we start with a standard D2–brane background with flat transverse space. As discussed in Chapter 5, in order to obtain a reasonable deformation which features no naked singularities one is led to replace the flat transverse space by a Ricci-flat manifold. If in addition the manifold is Kähler it admits a covariantly constant spinor and thus allows for a supersymmetric solution. We indeed require the geometry of the system to be a warped compactification of a $2+1$ dimensional Minkowski space and a complete, Ricci-flat, of $G_2$ holonomy and asymptotically conical seven dimensional space, which is an $\mathbb{R}^3$ bundle over a quaternionic Kähler Einstein base $M_4$.
The family of deformed D2–brane solutions of type IIA [57] is obtained from the following ansatz, which we may refer to as the IIA analogue of to the deformed solution (5.8) discussed in Chapter 5

$$\begin{aligned} ds^2_{10} &= H^{-5/8} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3/8} ds^2_7 \,, \\ g_s F_4 &= d^3x \wedge dH^{-1} + mG_4 \,, \quad \ell H_3 = mG_3 \,, \quad \Phi = \frac{1}{4}\log H \,. \end{aligned} \tag{6.1}$$

where $H_3$ is the NS-NS three form, $\Phi$ the dilaton and $H$ the warp factor. $G_3$ is an harmonic 3-form in the Ricci-flat 7-metric $ds^2_7$ and the ten dimensional trace of Einstein's equation relates it to $H$:

$$\Box H = -\frac{1}{6} m^2 |G_3|^2 \,. \tag{6.2}$$

As it was for the IIB case, there are two distinct contributions to the R-R four–form $F_4$. One is the usual form field sourced by the electric flux of regular D2–branes placed in the external $2+1$ dimensions, while the term proportional to $m$ is due to fractional D2–branes, which are D4–branes wrapping vanishing two–cycles in the seven–dimensional internal geometry. As discussed in [119], generic backgrounds of the form (6.1) can support fractional D2–branes if $M_6$ has a non-trivial 4–cycle around whose dual 2-cycle a D4–brane can wrap. Then if we let $\omega_4$ be the associated harmonic 4–form in $M_6$, we can set $G_4 = \omega_4$. The deformed D2–brane solution is completely specified in the class (6.1) by the choice of a Ricci-flat 7–manifold and an harmonic three form $G_3$[2].

[2]We will refer either to $G_3$ and to $G_4$, as we will illustrate in the following subsection how the flux equations of motion relate them.

## 6.2 $S^2$ bundles over $S^4$

In the following we will consider the explicit case of (6.1) for which the four-dimensional Kähler -Einstein base is $M_4 = S^4$. This choice corresponds to a seven-dimensional transverse space of co-homogeneity one and $\mathbb{CP}^3$ level surfaces. The harmonic 3-form can be chosen to be normalizable, so the solution is completely regular. For the case at hand, we can take $\omega_4 = J \wedge J$ being $J$ the Kähler form of $\mathbb{CP}^3$.
The complete Ricci–flat 7–metric on the bundle of self–dual 2–forms over $S^4$ [88], reads

$$ds_7^2 = h(r)^2 dr^2 + e^{2u(r)}(D\mu^m)^2 + e^{2v(r)} d\Omega_4^2 \,. \tag{6.3}$$

here $\mu^m$ are coordinates on $\mathbb{R}^3$ subject to $\mu^m \mu_m = 1$, $m = 1, 2, 3$ and the fibration is given by

$$D\mu^m = d\mu^m + \epsilon_{mnp} A^n \mu^p \,. \tag{6.4}$$

while $d\Omega_4^2$ is the metric on the unit 4-sphere. The quantities $A^m$ are self-dual $SU(2)$ instanton potentials on $S^4$ , whose field strenghts

$$J^m = dA^m + \frac{1}{2}\epsilon_{mnp} A^n \wedge A^p \,. \tag{6.5}$$

satisfy the algebra of the unit quaternions:

$$J^m_{\alpha\gamma} J^m_{\gamma\beta} = -\delta^{mn}\delta_{\alpha\beta} + \epsilon^{mn}{}_p J^p_{\alpha\beta} \,. \tag{6.6}$$

The following choice for the scalar functions $h(r)$, $u(r)$, $v(r)$

$$h(r)^2 = \left(1 - \frac{1}{r^4}\right)^{-1}, \qquad e^{2u(r)} = \frac{1}{4} r^2 \left(1 - \frac{1}{r^4}\right), \qquad e^{2v(r)} = \frac{1}{2} r^2 \,. \tag{6.7}$$

will assure that the metric (6.3) is Ricci-flat and has $G_2$ holonomy[3] . The radial coordinate runs from $r = 1$ (where the metric locally approaches $\mathbb{R}^3 \times S^4$) to the asymptotically flat region at $r = \infty$.

## 6.3 Zeroth–order background

To obtain a fractional D2–brane configuration it is sufficient to find a suitable harmonic form $G_3$, which is $L^2$-integrable at short distance and whose dual 4–form has a non–vanishing flux integral at infinity. According to [88], one can make the following ansatz

[3] The metric at large distance is asymptotic to the cone over the “squashed" Einstein metric on $\mathbb{CP}^3$, and not the Fubini–Study Einstein metric. Equation (6.3) is indeed an element of a family of conical metrics over $\mathbb{CP}^3$, which can be parametrized as

$$ds_6^2 = \lambda^2 (D\mu^m)^2 + d\Omega_4^2 \,. \tag{6.8}$$

where $\lambda^2 = 1$ corresponds to the Fubini-Study metric, and $\lambda^2 = 1/2$ to the squashed Einstein metric. This last case corresponds to a nearly Kähler metric [6].

for the harmonic three–form

$$G_3 = f_1 dr \wedge U_2 + f_2 dr \wedge J_2 + f_3 U_3 . \tag{6.9}$$

The forms in the previous expression are defined as

$$U_2 \equiv \frac{1}{2}\epsilon_{mnp}\,\mu^m D\mu^n \wedge D\mu^p, \qquad J_2 \equiv \mu^m J^m, \qquad U_3 = D\mu^m \wedge J^m \tag{6.10}$$

and satisfy the following differential relations

$$dU_2 = U_3, \qquad dJ_2 = U_3, \qquad dU_3 = 0\,. \tag{6.11}$$

The equations of motion give the following condition[4]

$$\begin{aligned} G_4 &= *_7 G_3 \\ &= f_1(r)h(r)\epsilon_{mnp}\mu^m dr \wedge D\mu^n \wedge J^p + \frac{f_3(r)}{h(r)}e^{2u(r)}U_2 \wedge J_2 + \frac{f_2(r)}{2h(r)}e^{4v(r)-2u(r)}J_2 \wedge J_2\,. \end{aligned} \tag{6.12}$$

By imposing harmonicity on $G_4$ (6.12), one obtains explicit expressions for $f_a$ , $a = 1, 2, 3$:

$$\begin{aligned} f_1(r) &= e^{u^0(r)+2v^0(r)}u_1(r)\,, \\ f_2(r) &= h(r)e^{2u^0(r)}u_2(r)\,, \\ f_3(r) &= h(r)e^{2v^0(r)}u_3(r)\,. \end{aligned} \tag{6.13}$$

where we defined

$$\begin{aligned} u_1(r) &= \frac{1}{4r^4(r^4-1)} - \frac{(3r^4-1)P(r)}{4r^5(r^4-1)^{3/2}}\,, \\ u_2(r) &= \frac{1}{r^4} + \frac{P(r)}{r^5(r^4-1)^{1/2}} \\ u_3(r) &= -\frac{1}{2(r^4-1)} + \frac{P(r)}{r(r^4-1)^{3/2}}\,. \end{aligned} \tag{6.14}$$

Here we introduced the quantity $P(r)$

$$P(r) = \int_1^r \frac{du}{\sqrt{u^4-1}} = K(-1) - F(\arcsin(1/r)|-1)\,. \tag{6.15}$$

where $F(\phi\,|\,m)$ denotes the incomplete elliptic integral of the first kind

$$F(\phi, m) = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-m^2 \sin^2\theta}}\,, \tag{6.16}$$

[4]The symbol $*_7$ denotes the seven–dimensional Hodge star operator, while a generic $*$ refers to the ten–dimensional one. From now on, superscripts within round brackets refer to the perturbation order, while subscripts label different functions; quantities which are not labelled by a (round bracketed) superscript will not enter the set of perturbed scalars which we will introduce in (6.38).

and $K(m) = F(\pi/2\,|\,m)$ is the complete elliptic integral of the first kind

$$K(m) = \int_0^{\phi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2\theta}} \tag{6.17}$$

Finally we define

$$g_1^0(r) = \int_1^r f_1(y)dy, \qquad g_2^0(r) = \int_1^r f_2(y)dy, \qquad g_3^0(r) = \int_1^r f_3(y)dy\,. \tag{6.18}$$

It turns out the $g_a^0$ can be expressed in terms of the $f_a$ only, as follows

$$g_1^0 = \frac{1}{4h^0} e^{-2u^0+4v^0} f_2 - c_2 = \frac{1}{2h^0} e^{2u^0} f_3 - c_3\,, \tag{6.19}$$
$$g_2^0 = -8e^{2u^0+2v^0} f_1\,, \tag{6.20}$$
$$g_3^0 = (1 + 8e^{2u^0+2v^0}) f_1\,. \tag{6.21}$$

As one can express $g_1$ either using $f_2$ or $f_3$, the constants introduced in (6.22) are related by

$$c_2 - c_3 = \frac{3}{32}\,. \tag{6.22}$$

The explicit parameterization of the CGLP background (6.1) which we will made use of reads [53, 118]:

$$ds_{10}^2 = e^{-5z^0(r)} \eta_{\mu\nu} dx^\mu dx^\nu + \ell^2 e^{3z^0(r)} ds_7^2\,, \tag{6.23}$$
$$g_s F_4 = K(r) d^3x \wedge dr + 2m(g_1^0(r) + c_2)\, J_2 \wedge J_2 + 2m(g_1^0(r) + c_3)\, U_2 \wedge J_2 + m g_1^{0\prime}(r) \epsilon_{abc} \mu^a dr \wedge D\mu^b \wedge J^c\,, \tag{6.24}$$
$$\ell B_2 = m\Big[g_2^0(r) U_2 + g_3^0(r) J_2\Big]\,, \quad F_2 = 0\,, \tag{6.25}$$
$$H_0(r) \equiv e^{8\,z^0} = \frac{m^2}{\ell^6} \int_r^\infty y^5\, [u_3(y) - u_2(y)]\; u_1(y)\, dy\,, \tag{6.26}$$
$$e^{\Phi^0(r)} = g_s H_0^{1/4}(r)\,. \tag{6.27}$$

Here $\eta_{\mu\nu}$ is the three-dimensional Minkowski flat metric with signature $(-,+,+)$, $ds_7^2$ is the fibration (6.3), the forms $U_2$, $J_2$, $D\mu^m$ are the ones defined in (6.10) and therefore satisfy (6.11). Recall that $H_0$ is the solution of (6.2). We notice that the unperturbed ten-dimensional metric in the IR reads

$$ds_{10}^2 = H_0(1)^{-5/8} ds^2_{\text{Mink}_3} + H_0(1)^{3/8} \left[\frac{1}{4(r-1)} dr^2 + (r-1)(D\mu^i)^2 + \frac{1}{2} d\Omega_4^2\right]\,. \tag{6.28}$$

and features a coordinate singularity at $r = 1$ which can be eliminated by redefining the radial coordinate as $\tau \equiv 2\sqrt{r-1}$.

It is easy to see how the addition of fractional D2–branes resolves the singularity of the solution featuring regular D2–branes only in the infrared [53, 118]. If we consider just ordinary D2–branes in this background, the radial-symmetric solution to $\Box H = 0$ is

$$H(r) = c_1 + c_2 \left( \frac{1}{r\sqrt{1-r^{-4}}} - F(\arcsin(1/r)| - 1) \right) . \tag{6.29}$$

which approaches a constant (plus $O(r^{-5})$) at large $r$, but diverges as $1/\sqrt{r-1}$ near $r = 1$. Whenever we consider fractional D2–branes, one has a corresponding non-vanishing right hand side in equation (6.2), whose solution is completely regular near the brane locus [118], as[5]

$$H_0(1) = \frac{m^2}{\ell^6} \int_r^\infty y^5 \, [u_3(y) - u_2(y)] \; u_1(y) \, dy = \frac{3m^2}{64\ell^6} \left(4 - K(-1)^2\right) . \tag{6.31}$$

Furthermore, $G_4$ is regular at $r = 1$

$$\lim_{r\to 1} G_4 = \frac{3}{8} \mathrm{vol}(S^4) \, . \tag{6.32}$$

## 6.4 The perturbative toolkit

In this section we outline the derivation of the superpotential for the CGLP background [118], and we will explain how its knowledge is the key ingredient in the perturbative method proposed by Borokhov and Gubser [33]. A fundamental assumption for the analysis of the next sections is that the symmetries of the starting background are powerful enough to impose that all the fields depend on a single radial coordinate. Let us consider the bosonic part of the type IIA supergravity action in Einstein frame (2.1). By inserting the corresponding expressions for the fields and metric (6.24)–(6.27) we find

$$\mathcal{S}_{\mathrm{IIA}} = \frac{\ell^5 \, \mathrm{Vol}\,(M_{1,2}) \; \mathrm{Vol}\,(M_6)}{2\,\kappa^2} \int dr \, \mathcal{L} \, , \tag{6.33}$$

where $M_{1,2}$, $M_6$ denote the $2+1$ dimensional Minkowski space and the level surfaces of the seven–dimensional $G_2$ holonomy manifold, respectively. As we are considering the unperturbed CGLP background, for the time being we will drop the superscript in the scalar functions. The kinetic term is

$$T = \frac{e^{2u+4v}}{h} \Big[ - 30 \, z'^2 + 2 \, u'^2 + 12 \, v'^2 + 16 \, u' \, v' - 2 \, g_s^{-1/2} \, \frac{m^2}{\ell^6} \, e^{-9z+\Phi/2-2u-4v} \, g_1'^2 \\ - \frac{g_s}{2} \, \frac{m^2}{\ell^6} \, e^{-6z-\Phi} \left( g_2'^2 \, e^{-4u} + 2 \, g_3'^2 \, e^{-4v} \right) - \frac{1}{2} \Phi'^2 \Big] . \tag{6.34}$$

[5]It is straightforward to check that the expression (6.26) for the warp factor is identical to the one provided by Herzog in [118]:

$$H_0(r) = \frac{m^2}{2\ell^6} \int_r^\infty \rho \, [2u_3(\rho) - 3] u_1(\rho) \, d\rho \, . \tag{6.30}$$

From this expression we see that both $K$ and $h$ are non–dynamical fields.
We will eliminate $K$ through its algebraic equation of motion, which gives the following explicit expression

$$K = \frac{4m^2}{\ell^6} g_s^{1/2} e^{2u+4v+15z-\Phi/2}\, h \Big[g_1(g_2+g_3) + c_2 g_2 + c_3 g_3\Big] \tag{6.35}$$

and evaluating the Lagrangian at the corresponding minimum for $K$, the potential reads

$$\begin{aligned} V =& -2\, h\, e^{-2u-4v} \left[e^{2u+8v} - e^{6u+4v} + 6\, e^{4u+6v}\right] + 2\, g_s\, h\, \frac{m^2}{\ell^6}\, e^{-6z-\Phi} \left[g_2+g_3\right]^2 \\ &+ 4\, g_s^{-1/2} \frac{m^2}{\ell^6}\, e^{-9z+\Phi/2+2u}\, h \left[2\, (g_1+c_2)^2\, e^{-4v} + (g_1+c_3)^2\, e^{-4u}\right] \\ &+ 8\, g_s^{1/2} \frac{m^4}{\ell^{12}}\, e^{-15z-\Phi/2-2u-4v}\, h\, \left[g_1\, (g_2+g_3) + g_2\, c_2 + g_3\, c_3\right]^2 . \end{aligned} \tag{6.36}$$

We parametrize the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\, G^{ab}\, (d\phi_a/dr)\, (d\phi_b/dr) - V\,, \tag{6.37}$$

where $G^{ab}$ is the field space metric read from (6.34), and where we denote the set of functions $\phi_a$, $a = 1, ..., 7$ in the following order

$$\phi_a = (u, v, z, \Phi, g_1, g_2, g_3)\ , \tag{6.38}$$

The following expression for the superpotential [118]

$$W = -8\, \left[e^{u+4v} + e^{3u+2v}\right] + 8\, \frac{m^2}{\ell^6}\, g_s^{1/4}\, e^{-\frac{15}{2}z-\frac{\Phi}{4}}\, \left[g_1\, (g_2+g_3) + g_2\, c_2 + g_3\, c_3\right] \tag{6.39}$$

accounts for all the terms in the potential (6.36), namely

$$V = \frac{1}{8}\, G_{ab}\, \frac{\partial W}{\partial \phi_a}\, \frac{\partial W}{\partial \phi_b}\,. \tag{6.40}$$

## 6.5 Borokhov–Gubser method

The method proposed by Borokhov and Gubser in [33] allows to find perturbative solutions to the equations of motion. The idea behind the technique is to trade the $n$ second–order equations for $n$ fields $\phi_a$ to $2n$ first–order equations for the fields $\phi_a$ and their "canonical conjugate variables" $\xi^a$.
We rewrite the Lagrangian by means of the superpotential (6.39) as follows

$$\mathcal{L} = -\frac{1}{2} \left(\frac{d\phi_a}{dr} - \frac{1}{2} G_{ac} \frac{\partial W}{\partial \phi_c}\right) \left(\frac{d\phi_b}{dr} - \frac{1}{2} G_{bd} \frac{\partial W}{\partial \phi_d}\right) - \frac{1}{2}\frac{dW}{dr}\,. \tag{6.41}$$

The equations of motion derived from $\mathcal{L}$ can be written as

$$-\frac{d}{dr}\left(\frac{\delta\mathcal{L}}{\delta\phi'^a}\right)+\frac{\delta\mathcal{L}}{\delta\phi^a}=\frac{1}{2}\left(\partial_a\partial_b W-\left(\partial_a G_{bc}\right)G^{cd}\partial_d W\right)\left(\phi'^b-\frac{1}{2}G^{be}\partial_e W\right)$$
$$-\frac{1}{2}\left(\partial_a G_{bc}\right)\left(\phi'^b-\frac{1}{2}G^{bd}\partial_d W\right)\left(\phi'^c-\frac{1}{2}G^{ce}\partial_e W\right)$$
$$+\frac{d}{dr}\left(G_{ab}\left(\phi'^b-\frac{1}{2}G^{bc}\partial_c W\right)\right)=0\,, \tag{6.42}$$

where a prime means derivative with respect to $r$.
The gradient flow equations deriver from the superpotential read

$$\frac{d\phi_a}{dr}=\frac{1}{2}G_{ab}\frac{\partial W}{\partial\phi_b}\,, \tag{6.43}$$

and the "zero-energy" condition coming from the $G_{rr}$ Einstein equation is:

$$-\frac{1}{2}G_{ab}\frac{d\phi_a}{dr}\frac{d\phi_b}{dr}+V(\phi)=0\,. \tag{6.44}$$

One can check that the zeroth-order CGLP background that we have previously summarized obeys the first-order BPS equations (6.43).
It is easy to show that solutions of (6.43) are also solutions to the equations of motion (6.42) derived from (6.41) if they satisfy the constraint (6.44) .
Following [33] one can use the superpotential to determine perturbations to a solution of (6.43) that satisfy the equations of motion but not necessarily (6.43) itself. Let us consider an expansion of the fields $\phi_a$ around their supersymmetric value $\phi_a^{(0)}$ (see footnote 4 for conventions used in this Chapter)

$$\phi_a=\phi_a^{(0)}+\phi_a^{(1)}(\alpha)+\mathcal{O}(\alpha^2) \tag{6.45}$$

for some set of parameters $\alpha$. Let us introduce the following notation

$$\xi^a=G^{ab}(\phi^{(0)})\left(\frac{d\phi_b^{(1)}}{dr}-N_b{}^d(\phi_0)\phi_d^{(1)}\right)\qquad\text{where}\qquad N_b{}^a(\phi^{(0)})=\frac{1}{2}\frac{\partial}{\partial\phi_a}\left(G_{bc}\frac{\partial W}{\partial\phi_c}\right) \tag{6.46}$$

If we now plug the expansion (6.45) in the equations of motion derived from the 1-dimensional Lagrangian, and we keep terms up to the linear order we obtain

$$\frac{d\xi^a}{dr}+\xi^b\,N_b{}^a(\phi^{(0)})=0\,, \tag{6.47}$$

$$\frac{d\phi_a^{(1)}}{dr}-N_a{}^b(\phi^{(0)})\phi_b^{(1)}=G_{ab}(\phi^{(0)})\xi^b\,, \tag{6.48}$$

while the constraint (6.44) can be written as

$$\xi^a\frac{d\phi_a^{(0)}}{dr}=0\,. \tag{6.49}$$

The functions $\xi^a$ are a measure of the deviation from the gradient flow equations (6.43). Notice that for a supersymmetric deformation all the $\xi^a$ vanish[6]. The obvious advantage

[6] The converse is not generally true.

of this method is that one can solve separately for the first–order subsystem (6.47) and then solve for (6.48) which are again first–order.

## 6.6 Explicit first–order equations

### 6.6.1 $\tilde{\xi}$ equations

We present the system (6.47) of first order equations in terms of the new variables

$$\tilde{\xi}_a = (\xi_1, \xi_1 - \xi_2, \xi_3 + 2\,\xi_4, \xi_4, \xi_5, \xi_6, -\xi_6 + \xi_7)\ . \tag{6.50}$$

for which the solution turns out to be much easier.

The equations are listed in the order one has to solve them.

$$\tilde{\xi}_3' = -4\frac{m^2 g_s^{1/4}}{l^6} h\, e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\Phi^0}{4}} \Big[c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0)\Big]\tilde{\xi}_3 \tag{6.51}$$

$$\tilde{\xi}_7' = -\frac{3m^2 g_s^{1/4}}{64 l^6} h\, e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\Phi^0}{4}}\tilde{\xi}_3 \tag{6.52}$$

$$\begin{aligned}\tilde{\xi}_5' = &-\frac{1}{2g_s^{3/4} l^6} h\, e^{-2u^0-4v^0-\frac{15z^0}{2}-\frac{\Phi^0}{4}} \Big[4l^6 e^{4v^0+6z^0+\Phi^0}(\tilde{\xi}_6+\tilde{\xi}_7) + 8l^6 e^{4u^0+6z^0+\Phi^0}\tilde{\xi}_6 \\ &- g_s m^2 (g_2^0+g_3^0)\tilde{\xi}_3\Big]\end{aligned} \tag{6.53}$$

$$\tilde{\xi}_6' = \frac{g_s^{1/4}}{2l^6} h\, e^{-2u^0-4v^0-\frac{3}{4}(10z^0+\Phi^0)}\Big[-2g_s^{1/2} l^6 e^{2u^0+4v^0+9z^0}\tilde{\xi}_5 + e^{\frac{\Phi^0}{2}} m^2 (c_2+g_1^0)\tilde{\xi}_3\Big] \tag{6.54}$$

$$\begin{aligned}\tilde{\xi}_4' = &\frac{h}{8g_s^{3/4}} e^{-\frac{3}{4}(10z^0+\Phi^0)}\Big[-24 e^{2u^0-4v^0+6z^0+\frac{3}{2}\Phi^0}(c_2+g_1^0)\tilde{\xi}_6 \\ &- 12 e^{-2u^0+6z^0+\frac{3}{2}\Phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) + 6e^{9z^0} g_s^{3/2}(g_2^0+g_3^0)\tilde{\xi}_5 \\ &- \frac{m^2 g_s}{l^6} e^{-2u^0-4v^0+\frac{\Phi^0}{2}}(c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\end{aligned} \tag{6.55}$$

$$\begin{aligned}\tilde{\xi}_1' = &\frac{1}{g_s^{3/4} l^6} h\, e^{-2u^0-4v^0-\frac{15}{2}z^0-\frac{\Phi^0}{4}}\Big[g_s^{3/4} l^6 e^{u^0+4v^0+\frac{15}{2}z^0+\frac{\Phi^0}{4}}\tilde{\xi}_1 + g_s^{3/4} l^6 e^{\frac{1}{4}(12u^0+8v^0+30z^0+\Phi^0)}\tilde{\xi}_2 \\ &- 8l^6 e^{4u^0+6z^0+\Phi^0}(c_2+g_1^0)\tilde{\xi}_6 + 4l^6 e^{4v^0+6z^0+\Phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) \\ &- g_s m^2 (c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\end{aligned} \tag{6.56}$$

$$\begin{aligned}\tilde{\xi}_2' = &\frac{1}{g_s^{3/4} l^6} h\, e^{-2u^0-4v^0-\frac{15}{2}z^0-\frac{\Phi^0}{4}}\Big[g_s^{3/4} l^6 e^{u^0+4v^0+\frac{15}{2}z^0+\frac{\Phi^0}{4}}\tilde{\xi}_1 + 3g_s^{3/4} l^6 e^{\frac{1}{4}(12u^0+8v^0+30z^0+\Phi^0)}\tilde{\xi}_2 \\ &- 24l^6 e^{4u^0+6z^0+\Phi^0}(c_2+g_1^0)\tilde{\xi}_6 + 4l^6 e^{4v^0+6z^0+\phi^0}(c_3+g_1^0)(\tilde{\xi}_6+\tilde{\xi}_7) \\ &+ g_s m^2 (c_2 g_2^0 + c_3 g_3^0 + g_1^0(g_2^0+g_3^0))\tilde{\xi}_3\Big]\end{aligned} \tag{6.57}$$

### 6.6.2 $\tilde{\phi}$ equations

As previously done for the $\tilde{\xi}$ equations, we shift the original $\phi$ into a linear combination $\tilde{\phi}$, defined as[7]

$$\tilde{\phi}_a = (\phi_1, \phi_1 - 2\,\phi_2, 8\,\phi_1 + 6\,\phi_3 - 3\,\phi_4, 8\,\phi_1 + 16\,\phi_2 + 30\,\phi_3 + \phi_4, \phi_5, \phi_6 + \phi_7, \phi_6 - \phi_7) \tag{6.59}$$

The set of equations (6.48) explicitly reads

$$\tilde{\phi}_1' = \frac{1}{20}\, h\, e^{-2u^0-4v^0}\, \left[\tilde{\xi}_1 + 2\,\tilde{\xi}_2 - 20\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 20\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \tag{6.60}$$

$$\tilde{\phi}_2' = \frac{1}{20}\, h\, e^{-2u^0-4v^0}\, \left[4\,\tilde{\xi}_1 + 3\,\tilde{\xi}_2 - 20\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 60\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \tag{6.61}$$

$$\tilde{\phi}_3' = \frac{1}{10}\, h\, e^{-2u^0-4v^0}\, \left[4\,\tilde{\xi}_1 + 8\,\tilde{\xi}_2 + \tilde{\xi}_3 - 32\,\tilde{\xi}_4 - 80\, e^{u^0+4v^0}\, \tilde{\phi}_1 - 80\, e^{3u^0+2v^0}\, \tilde{\phi}_2\right]\,, \tag{6.62}$$

$$\begin{aligned}\tilde{\phi}_5' = \frac{g_s^{1/2}}{4\,m^2}\, h\, e^{3z^0/2-3\Phi^0/4} \Big[&\ell^6\, e^{15z^0/2+\Phi^0/4}\, \tilde{\xi}_5 \\ &+ g_s^{1/4}\, m^2\, \left(4\,\tilde{\phi}_6 - \left(g_2^0 + g_3^0\right)\, \left[8\,\tilde{\phi}_1 - \tilde{\phi}_3\right]\right)\Big]\,,\end{aligned} \tag{6.63}$$

$$\begin{aligned}\tilde{\phi}_6' = \frac{1}{2\,g_s\,m^2}\, h\, e^{-2u^0-4v^0-3z^0/2+3\Phi^0/4} \Big[&\ell^6\, e^{15z^0/2+\Phi^0/4}\, \left(2\, e^{4u^0}\, \tilde{\xi}_6 + e^{4v^0}\, \tilde{\xi}_7\right) \\ &+ 2\, g_s^{1/4}\, m^2\, e^{4u^0}\, \left[4\,\tilde{\phi}_5 + \left(c_2 + g_1^0\right)\, \left(8\,\tilde{\phi}_1 + 8\,\tilde{\phi}_2 - \tilde{\phi}_3\right)\right] \\ &+ g_s^{1/4}\, m^2\, e^{4v^0}\, \left(4\,\tilde{\phi}_5 - \left(c_3 + g_1^0\right)\tilde{\phi}_3\right)\Big]\,,\end{aligned} \tag{6.64}$$

[7] The inverse transformation is

$$\begin{aligned}\phi^a = \Big(&\tilde{\phi}_1, \frac{1}{2}\left(\tilde{\phi}_1 - \tilde{\phi}_2\right), -\frac{7}{12}\,\tilde{\phi}_1 + \frac{1}{4}\,\tilde{\phi}_2 + \frac{1}{96}\,\tilde{\phi}_3 + \frac{1}{32}\,\tilde{\phi}_4, \frac{3}{2}\,\tilde{\phi}_1 + \frac{1}{2}\,\tilde{\phi}_2 - \frac{5}{16}\,\tilde{\phi}_3 + \frac{1}{16}\,\tilde{\phi}_4, \\ &\tilde{\phi}_5, \frac{1}{2}\left(\tilde{\phi}_6 + \tilde{\phi}_7\right), \frac{1}{2}\left(\tilde{\phi}_6 - \tilde{\phi}_7\right)\Big)\,.\end{aligned} \tag{6.58}$$

$$
\begin{aligned}
\tilde{\phi}_7' = & \frac{1}{2\, g_s\, m^2}\, h\, e^{-2u^0-4v^0-3z^0/2+3\Phi^0/4} \Big[ \ell^6\, e^{15z^0/2+\Phi^0/4} \left(2\, e^{4u^0}\, \tilde{\xi}_6 - e^{4v^0}\, \tilde{\xi}_7\right) \\
& + 2\, g_s^{1/4}\, m^2\, e^{4u^0} \left[4\, \tilde{\phi}_5 + \left(c_2 + g_1^0\right) \left(8\, \tilde{\phi}_1 + 8\, \tilde{\phi}_2 - \tilde{\phi}_3\right)\right] \\
& - g_s^{1/4}\, m^2\, e^{4v^0} \left(4\, \tilde{\phi}_5 - \left(c_3 + g_1^0\right)\, \tilde{\phi}_3\right) \Big] ,
\end{aligned}
\tag{6.65}
$$

$$
\begin{aligned}
\tilde{\phi}_4' = & -\frac{1}{10\, \ell^6}\, h\, e^{-2u^0-4v^0-15z^0/2-\Phi^0/4} \Big[ \ell^6\, e^{15z^0/2+\Phi^0/4} \left(8\, \tilde{\xi}_1 - 4\, \tilde{\xi}_2 - 5\, \tilde{\xi}_3\right) \\
& + 80\, \ell^6\, e^{u^0+4v^0+15z^0/2+\Phi^0/4}\, \tilde{\phi}_1 - 80\, \ell^6\, e^{3u^0+2v^0+15z^0/2+\Phi^0/4}\, \tilde{\phi}_2 \\
& + 40\, g_s^{1/4}\, m^2 \Big(4\, \left(g_2^0 + g_3^0\right)\, \tilde{\phi}_5 + 2\, \left(2\, g_1^0 + c_2 + c_3\right)\, \tilde{\phi}_6 + 2\, \left(c_2 - c_3\right)\, \tilde{\phi}_7 \\
& - \left(g_2^0\, \left(c_2 + g_1^0\right) + g_3^0\, \left(c_3 + g_1^0\right)\right)\, \tilde{\phi}_4\Big)\Big] .
\end{aligned}
\tag{6.66}
$$

## 6.7 Force on a probe D2

In this section we compute the force on a D2–brane probing the deformation of the CGLP background. At zeroth–order in perturbation theory, the Dirac–Born–Infeld (DBI) term cancels the contribution from Wess–Zumino (WZ), and as expected there is no net force for a probe D2–brane in this supersymmetric background. At first glance, the expression for the force in the perturbed solution is rather complicated but we will show that using the first–order equations of motion most of the terms cancel and the final expression is quite simple, being related to the one mode only as expected by analogy with the previous analysis [13, 14, 17].

Although D$p$-brane are not truly supergravity objects, they admit a description in terms of an effective action, which Einstein frame reads [62, 153]

$$
S_{Dp} = -T_p \int d\xi^{p+1} e^{-\Phi/4} g_s^{-3/4} \sqrt{|\det(\iota^*[g] + \mathcal{F})|} + T_p \int \iota^*[C] \wedge e^{\mathcal{F}} \tag{6.67}
$$

where $T_p = \frac{\pi}{\kappa_{10}^2}(4\pi^2\alpha')^{3-p}$ is the brane tension, $\Sigma$ denotes the world-volume of the brane, $\iota^*$ denote the pullback of the world-volume of bulk tensors and $C$ are the the R-R gauge potentials as well. Furthermore, it is conventional to denote $\mathcal{F} = \iota^*[B] + 2\pi\alpha' F$ the gauge invariant combination of the field strenght $F$ of the world volume gauge field and the pullback of $B_2$.

We then define the force as follows

$$
F = F^{DBI} + F^{WZ} \equiv -\frac{dV^{DBI}}{dr} - \frac{dV^{WZ}}{dr} . \tag{6.68}
$$

We choose a static gauge for a brane aligned along $M_{1,2}$ and we do not turn on the gauge field on the brane. The DBI Lagrangian reduces to

$$\mathcal{L}_{DBI} = -V^{DBI} = -T_p e^{-\Phi/4} g_s^{-3/4} \sqrt{-g_{00} g_{11} g_{22}} = -T_p e^{-\Phi/4 - 15z/2} g_s^{-3/4} . \tag{6.69}$$

The only non-zero R-R potential is $C_{MNP}$, and the part which gives non–vanishing contribution is given by

$$C_3 = \frac{1}{g_s} \mathcal{K}(r) dx^0 \wedge dx^1 \wedge dx^2 , \qquad \frac{d\mathcal{K}(r)}{dr} = -K(r) . \tag{6.70}$$

where $K(r)$ is given in equation (6.35). On the other hand, the Wess-Zumino contribution reduces to

$$\mathcal{L}_{WZ} = -V^{WZ} = T_p \frac{1}{3!} \varepsilon^{i_1 i_2 i_3} (C_3)_{i_1 i_2 i_3} = -T_p \frac{1}{g_s} \mathcal{K}(r) . \tag{6.71}$$

We can now compute the force on a probe D2–brane (from now on we put $T_p = 1$). At zeroth order we have

$$F^{(0)\,DBI} = g_s^{-1/2} H_0' e^{-\Phi^0/2 - 15z^0} = -\frac{4m^2}{\ell^6} g_s^{-1/2} e^{-\Phi^0/2 - 15z^0 - 2u^0 - 4v^0} h \left[ c_2 g_2^0 + c_3 g_3^0 + g_1^0 (g_2^0 + g_3^0) \right]$$

$$F^{(0)\,WZ} = \frac{1}{g_s} K(r) = \frac{4m^2}{\ell^6} g_s^{-1/2} e^{-\Phi^0/2 - 15z^0 - 2u^0 - 4v^0} h \left[ c_2 g_2^0 + c_3 g_3^0 + g_1^0 (g_2^0 + g_3^0) \right]$$

so, as anticipated at the beginning of the section, the two contributions cancel. At first order we obtain

$$\begin{aligned}
F^{(1)\,DBI} &= -F^{(0)\,DBI} \left( \frac{1}{4}\phi_4 - \frac{15}{2}\phi_3 \right) + g_s^{-3/4} \left( \frac{1}{4}\phi_4' + \frac{15}{2}\phi_3' \right) e^{-\frac{\Phi^0}{4} - \frac{15}{2} z^0} \\
F^{(1)\,WZ} &= -F^{(0)\,WZ} \left( \frac{1}{2}\phi_4 + \frac{15}{2}\phi_3 - 2\phi_1 - 4\phi_2 \right) + g_s^{-3/4} \left( \frac{1}{4}\phi_4' + \frac{15}{2}\phi_3' \right) e^{-\frac{\Phi^0}{4} - \frac{15}{2} z^0} \\
&\quad + \frac{4m^2}{\ell^6} g_s^{-1/2} h \, e^{-\frac{\Phi^0}{2} - 15z^0 - 2u^0 - 4v^0} \left[ c_2\phi_6 + c_3\phi_7 + \phi_5 (g_2^0 + g_3^0) + g_1^0 (\phi_6 + \phi_7) \right] .
\end{aligned}$$

From these expressions and the fact that $F^{(0)\,DBI} = -F^{(0)\,WZ}$, using the first–order equations (6.62), (6.66) for $\phi_3$ and $\phi_4$ one can notice that most of the terms at first order cancel so that the force on a probe D2-brane reduces to

$$\begin{aligned}
F(r) &= F^{(1)\,DBI} + F^{(1)\,WZ} \\
&= \frac{1}{8 g_s^{3/4}} h \, e^{-2u^0 - 4v^0 - \frac{15}{2} z^0 - \frac{1}{4}\Phi^0} \tilde{\xi}_3 \\
&= \frac{2}{g_s} \frac{X_3 \, e^{-8z^0(1)}}{(r^4 - 1)^{3/2}}
\end{aligned} \tag{6.72}$$

where we used equation (6.73) which will be derived in the next subsection. As a side comment, we notice that the derivative of (6.29) matches the behavior of the force (6.72) (see [14] for comments on this point). Among the 14 modes which parameterize the first order deformation space, only one of these enters in the expression of the force that a probe D2 brane feels in this background. Hence, since anti-D2 branes attract probe branes, if the perturbed solution may have any chance to describe back-reacted anti-D2 branes, a necessary requisite is that this mode must be non-vanishing.

## 6.8 The solution space

### 6.8.1 Solution of the $\tilde{\xi}_a$ system

We present here some remarks on solving for the set of equations (6.51)-(6.57). In the case at hand we were able to find a fully analytic expression[8] for all the $\tilde{\xi}_a$. We present the comments in the order in which the corresponding equations have to be solved.
The first equation we have to solve is the one for $\tilde{\xi}_3$, which after some manipulation can be expressed as

$$\tilde{\xi}_3' = \frac{H_0'}{H_0}\tilde{\xi}_3 \tag{6.73}$$

Its solution is:

$$\tilde{\xi}_3(r) = X_3 H_0(r) e^{-8z^0(1)} \, . \tag{6.74}$$

We define the constant $B_1$ which we find convenient to use in the the following as

$$B_1 = \frac{m^2}{\ell^6} X_3 e^{-8z^0(1)} \, . \tag{6.75}$$

The next step is to explicitly do the integration entering the definition of the function $H_0$, which we rewrite here as

$$H_0(r) = \frac{m^2}{\ell^6}\int_r^\infty y^5 \left[u_3(y) - u_1(y)\right] u_1(y) dy \tag{6.76}$$

The integrand has the following structure[9]:

$$\alpha_2 \mathcal{F}(r)^2 + \alpha_1 \mathcal{F}(r) + \alpha_0 \tag{6.80}$$

[8]It is important to have a solution expressed in terms of the least possible number of nested integrals. As happens in the previous analysis [13, 15, 17], it is not possible to find a fully integrated solution and one solves in series expansion, if the number of nested integrals is high it could be computationally heavy. In the counting of nested integrals we do not take into account the one which enters the definition of the elliptic functions.

[9]We adopt here, and for the remainder of the Chapter, the following calligraphic notation for the incomplete elliptic integral of the first kind $F$:

$$\mathcal{F}(r) \equiv F(\arcsin(1/r), -1)) \tag{6.77}$$

and, similarly, later on we will refer to

$$\mathcal{E}(r) \equiv E(\arcsin(1/r), -1)) \tag{6.78}$$

where $\alpha_i$ are some functions of $r$ which do not contain $\mathcal{F}$. We simply use integration by parts (in the following we drop the radial dependence for ease of notation):

$$\begin{aligned} \int \alpha_2 \mathcal{F}^2 + \alpha_1 \mathcal{F} + \alpha_0 &= A_2 \mathcal{F}^2 + \int (\alpha_1 - 2\mathcal{F}' A_2) + \int \alpha_0 \\ &= A_2 \mathcal{F}^2 + A_3 \mathcal{F} + \int (\alpha_0 - \mathcal{F}' A_3) \\ &= A_2 \mathcal{F}^2 + A_3 \mathcal{F} + A_4 \end{aligned} \tag{6.81}$$

where the notations denotes the following:

$$\begin{aligned} \mathcal{F}' &= \frac{d}{dy} F(\arcsin(1/y)| - 1) = -\frac{1}{\sqrt{y^4 - 1}}\,, \\ \alpha_3 &= \alpha_1 - 2\mathcal{F}' A_2\,, \\ \alpha_4 &= \alpha_0 - \mathcal{F}' A_3\,, \\ A_i &= \int \alpha_i\,. \end{aligned} \tag{6.82}$$

Once we have a primitive we have just to evaluate it at the two extrema of integration to get an analytic expression for $H_0$, and therefore for $\tilde{\xi}_3$.
The equations for $\tilde{\xi}_7$ is:

$$\tilde{\xi}_7' = -\frac{3}{64}\frac{m^2}{\ell^6} h\, e^{-2u^0 - 4v^0} H_0^{-1} \tilde{\xi}_3 = -\frac{3}{4} B_1 \frac{1}{(r^4 - 1)^{3/2}} \tag{6.83}$$

which can be directly integrated.
The functions $\tilde{\xi}_5$ and $\tilde{\xi}_6$ are coupled in a subsystem, which we can rewrite as

$$\tilde{\xi}_5' = -2h\,(2e^{2u^0 - 4v^0} + e^{-2u^0})\,\tilde{\xi}_6 - 2h\, e^{-2u^0}\,\tilde{\xi}_7 - \frac{32}{3} f_1\, \tilde{\xi}_7' \tag{6.84}$$

$$\tilde{\xi}_6' = -h\,\tilde{\xi}_5 - \frac{8}{3}\frac{1}{h} e^{-2u^0 + 4v^0} f_2\, \tilde{\xi}_7' \tag{6.85}$$

To get a solution, we first solve for the homogeneous system and arrange the two basis vectors of the space of the homogeneous solutions in the so–called fundamental matrix

$$\tilde{\Xi}_{56} = \left( \begin{array}{c|c} \frac{(3r^4-1)}{r^4(r^4-1)} & \frac{r(6r^8-6r^4-1)}{r^3\sqrt{r^4-1}} - \frac{3r^4-1}{r^4(r^4-1)}\mathcal{F}(r) \\ \hline \frac{1}{r\sqrt{r^4-1}} & 1 - \frac{3r^4}{2} - \frac{1}{r\sqrt{r^4-1}}\mathcal{F}(r) \end{array} \right). \tag{6.86}$$

The solution of the inhomogeneous system is then

$$\begin{pmatrix} \tilde{\xi}_5(r) \\ \tilde{\xi}_6(r) \end{pmatrix} = \tilde{\Xi}_{56}(r) X_{56} + \tilde{\Xi}(r) \int^r \tilde{\Xi}_{56}(y)^{-1} g^{\xi}_{56}(y)\, dy$$

---

as the incomplete elliptic integral of the second kind $E$, which is defined as

$$E(\phi, m) = \int_0^\phi \sqrt{1 - m^2 \sin^2\theta}\, d\theta. \tag{6.79}$$

where $X_{56} = (X_5, X_6)$ are integration constants, and $g^\xi_{56} = (g^\xi_5, g^\xi_6)$ collects non-homogeneous terms in (6.84)–(6.85).
The equation for $\tilde\xi_4$ has no homogeneous part but its non-homogeneous part depends on $\tilde\xi_5$ and $\tilde\xi_6$. We can rewrite it as follows:

$$\tilde\xi_4' = \frac{3}{4}h\, f_1\, \tilde\xi_5 - \frac{3}{4}(f_2+f_3)\,\tilde\xi_6 - \frac{3}{4}\,f_3\tilde\xi_7 - \frac{B_1}{32}h\, e^{u^0}(2u_3-3)u_1\,. \tag{6.87}$$

which can be solved.
Finally, the functions $\tilde\xi_1$ and $\tilde\xi_2$ are coupled in the following differential system:

$$\tilde\xi_1' = h\, e^{-u^0}\,\tilde\xi_1 + h\, e^{u^0-2v^0}\,\tilde\xi_2 - 2(f_2-f_3)\tilde\xi_6 + 2f_3\tilde\xi_7 - \frac{B_1}{8}r(2u_3-3)u_1\,, \tag{6.88}$$

$$\tilde\xi_2' = h\, e^{-u^0}\,\tilde\xi_1 + 3h\, e^{u^0-2v^0}\,\tilde\xi_2 - 2(3f_2-f_3)\,\tilde\xi_6 + 2f_3\,\tilde\xi_7 + \frac{B_1}{8}r(2u_3-3)u_1\,. \tag{6.89}$$

whose fundamental matrix $\tilde\Xi_{12}$ reads

$$\tilde\Xi_{12} = \left(\begin{array}{c|c} r^4-1 & \frac{\sqrt{r^4-1}}{r}\left(1-r\sqrt{r^4-1}(\mathcal{E}(r)-\mathcal{F}(r)\right) \\ 2r^4 & -2r^4\left(\mathcal{E}(r)-\mathcal{F}(r)\right)\end{array}\right)\,. \tag{6.90}$$

We conclude by listing in the following the analytic solutions for the $\tilde\xi$ system[10]

$$\begin{aligned}
\tilde\xi_1 &= \mathcal{F}(r)^3\left(-B_1\frac{r^4+1}{112r^5(r^4-1)^{3/2}}\right)\\
&+\mathcal{F}(r)^2\left(B_1\frac{189r^{12}-258r^8+r^4+48}{1792r^4(r^4-1)} + (45B_1K(-1)-168X_6+112X_7)\frac{r^4+1}{2688r^5(r^4-1)^{3/2}}\right)\\
&+\mathcal{F}(r)\Bigg(-B_1\frac{69r^{12}-114r^8+61r^4-24}{896r^3(r^4-1)^{3/2}} - B_1K(-1)\frac{315r^{12}-390r^8-53r^4+120}{3584r^4(r^4-1)}\\
&\qquad + X_2(r^4-1) - X_6\frac{63r^{12}-78r^8+31r^4-8}{64r^4(r^4-1)} - X_7\frac{9r^{12}-18r^8-7r^4+8}{96r^4(r^4-1)}\\
&\qquad + (24X_5+K(-1)(24X_6-16X_7-3B_1K(-1)))\frac{r^4+1}{384r^5(r^4-1)^{3/2}}\Bigg)\\
&- B_1\frac{51r^8-75r^4+16}{1792r^2(r^4-1)} + B_1K(-1)\frac{315r^{12}-516r^8+229r^4-60}{3584r^3(r^4-1)^{3/2}} + X_1(r^4-1)\\
&- B_1K(-1)^2\frac{63r^{12}-126r^8+63r^4-4}{512r^4(r^4-1)} + X_2\frac{\sqrt{r^4-1}}{r} - X_2(r^4-1)\mathcal{E}(r) + X_5\frac{2r^4-1}{16r^4(r^4-1)}\\
&- X_5K(-1)\frac{r^4+1}{16r^5(r^4-1)^{3/2}} - X_6\frac{33r^8-35r^4+4}{64r^3\sqrt{r^4-1}} + X_6K(-1)\frac{63r^{12}-78r^8+23r^4-4}{64r^4(r^4-1)}\\
&+ X_7\frac{9r^8-11r^4+4}{96r^3\sqrt{r^4-1}} + X_7K(-1)\frac{9r^{12}-18r^8+r^4+4}{96r^4(r^4-1)}
\end{aligned} \tag{6.91}$$

[10]In general, we did a redefinition of the integration constant which appear by direct integration in order to reabsorb an imaginary constant which appears after several manipulations. We always consider real solutions.

$$
\begin{aligned}
\tilde{\xi}_2 &= \mathcal{F}(r)^3 \left( B_1 \frac{r^4-3}{112 r^5 (r^4-1)^{3/2}} \right) \\
&\quad + \mathcal{F}(r)^2 \left( B_1 \frac{189 r^{16} - 438 r^{12} + 241 r^8 + 52 r^4 - 16}{896 r^4 (r^4-1)^2} - \frac{(45 B_1 K(-1) - 168 X_6 + 112 X_7)(r^4-3)}{2688 r^5 (r^4-1)^{3/2}} \right) \\
&\quad + \mathcal{F}(r) \Bigg( - B_1 \frac{69 r^{12} - 132 r^8 + 25 r^4 + 20}{448 r^3 (r^4-1)^{3/2}} - B_1 K(-1) \frac{315 r^{16} - 750 r^{12} + 427 r^8 + 76 r^4 + 44}{1792 r^4 (r^4-1)^2} \\
&\qquad\quad + X_2\, 2r^4 - X_6 \frac{63 r^{12} - 87 r^8 + 40 r^4 - 12}{32 r^4 (r^4-1)} - X_7 \frac{9 r^{12} - 9 r^8 - 16 r^4 + 12}{48 r^4 (r^4-1)} \\
&\qquad\quad + (K(-1)(3 B_1 K(-1) - 24 X_6 + 16 X_7) - 24 X_5) \frac{r^4-3}{384 r^5 (r^4-1)^{3/2}} \Bigg) \\
&\quad - B_1 \frac{51 r^8 - 30 r^4 - 32}{896 r^2 (r^4-1)} + B_1 K(-1) \frac{315 r^{12} - 561 r^8 + 40 r^4 + 134}{1792 r^3 (r^4-1)^{3/2}} + X_1\, 2r^4 - X_2\, 2r^4 \mathcal{E}(r) \\
&\quad - B_1 K(-1)^2 \frac{63 r^{16} - 126 r^{12} + 63 r^8 + 2 r^4 - 10}{256 r^4 (r^4-1)^2} + X_5 \frac{4 r^4 - 3}{16 r^4 (r^4-1)} + X_5 K(-1) \frac{r^4-3}{16 r^5 (r^4-1)^{3/2}} \\
&\quad - X_6 \frac{33 r^8 - 38 r^4 + 6}{32 r^3 \sqrt{r^4-1}} + X_6 K(-1) \frac{63 r^{12} - 87 r^8 + 32 r^4 - 6}{32 r^4 (r^4-1)} \\
&\quad + X_7 \frac{9 r^8 - 14 r^4 + 6}{48 r^3 \sqrt{r^4-1}} + X_7 K(-1) \frac{9 r^{12} - 9 r^8 - 8 r^4 + 6}{48 r^4 (r^4-1)}
\end{aligned} \tag{6.92}
$$

$$
\tilde{\xi}_3(r) = X_3 e^{-8 z_0(1)} H_0(r)\,, \tag{6.93}
$$

where

$$
\begin{aligned}
H_0(r) &= \frac{m^2}{2\ell^6}\mathcal{F}(r)^2\left(\frac{3}{32}-\frac{1}{8r^4(r^4-1)^2}\right) \\
&\quad -\frac{m^2}{2l^6}\mathcal{F}(r)\left(\frac{3r^8+3r^4-4}{16r^3(r^4-1)^{3/2}}+\frac{K(-1)}{16}\left(3-\frac{4}{r^4(r^4-1)^2}\right)\right) \\
&\quad +\frac{m^2}{2l^6}\left(\frac{3r^4-4}{32r^2(r^4-1)}+\frac{3r^8+3r^4-4}{16r^3(r^4-1)^{3/2}}K(-1)-\frac{K(-1)^2}{8r^4(r^4-1)^2}\right), \qquad (6.94)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\xi}_4 &= \mathcal{F}(r)^3\left(\frac{3B_1(3r^4-1)}{448r^5(r^4-1)^{3/2}}\right)+ \\
&+\mathcal{F}(r)^2\left(\frac{B_1(111r^{12}-222r^8+99r^4-16)}{3584r^4(r^4-1)^2}+\frac{(3r^4-1)}{3584r^5(r^4-1)^{3/2}}(168X_6-112X_7-45B_1K(-1))\right) \\
&+\mathcal{F}(r)\Bigg(-\frac{B_1(15r^8-12r^4+10)}{896r^3(r^4-1)^{3/2}}-\frac{B_1K(-1)(201r^{12}-402r^8+45r^4+44)}{7168r^4(r^4-1)^2}+ \\
&+\frac{3r^4-1}{512r^5(r^4-1)^{3/2}}\left(-24X_5+K(-1)\left(3B_1K(-1)-24X_6+16X_7\right)\right)+\frac{9r^8-9r^4+4}{128r^4(r^4-1)}(3X_6-2X_7)\Bigg) \\
&-\frac{B_1(51r^4-32)}{3584r^2(r^4-1)}+\frac{B_1K(-1)(201r^8-231r^4+134)}{7168r^3(r^4-1)^{3/2}}-\frac{B_1K(-1)^2(9r^4-5)}{512r^4(r^4-1)^2}+\frac{3K(-1)(3r^4-1)}{64r^5(r^4-1)^{3/2}}X_5 \\
&+\frac{3r^4-2}{128r^3\sqrt{r^4-1}}(3X_6-2X_7)-\frac{3X_5+K(-1)(3X_6-2X_7)}{64r^4(r^4-1)}+X_4\,, \qquad (6.95)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\xi}_5 &= \mathcal{F}(r)^2\left(\frac{B_1(1-3r^4)}{7r^4(r^4-1)}\right) \\
&\quad +\mathcal{F}(r)\left(\frac{B_1K(-1)(3r^4-1)}{8r^4(r^4-1)}-\frac{(3r^4-1)(3X_6-2X_7)}{3r^4(r^4-1)}-\frac{3B_1(5r^8-5r^4-2)}{28r^3\sqrt{r^4-1}}\right) \\
&\quad +\frac{B_1(15r^8-21r^4+10)}{28r^2(r^4-1)}-\frac{3B_1K(-1)}{8r^3\sqrt{r^4-1}}+\frac{(3r^4-1)}{r^4(r^4-1)}X_5+6r\sqrt{r^4-1}\,X_6-\frac{3X_6-2X_7}{3r^3\sqrt{r^4-1}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\xi}_6 &= \mathcal{F}(r)^2\left(-\frac{B_1}{7r\sqrt{r^4-1}}\right) \\
&\quad +\mathcal{F}(r)\left(\frac{B_1(15r^8+3r^4-4)}{112(r^4-1)}+\frac{B_1K(-1)}{8r\sqrt{r^4-1}}-\frac{3X_6-2X_7}{3r\sqrt{r^4-1}}\right) \\
&\quad -\frac{3B_1r(5r^4+4)}{112\sqrt{r^4-1}}-\frac{B_1K(-1)}{8(r^4-1)}+\frac{X_5}{r\sqrt{r^4-1}}+\left(1-\frac{3r^4}{2}\right)X_6-\frac{2}{3}X_7 \qquad (6.96)
\end{aligned}
$$

$$
\tilde{\xi}_7(r) = X_7+\frac{3}{8}B_1\left[\frac{r}{\sqrt{r^4-1}}-\mathcal{F}(r)\right] \qquad (6.97)
$$

### 6.8.2 Solving the $\phi_a$ equations

The $\tilde{\phi}$ system does not admit a fully analytic solution. We are thus forced to proceed by series expansion: here we present the equations, and show that we can find solutions up to three nested integrals. Once more, the presentation follows the order in which the equations have to be solved. We will report our final result back again the $\phi$ basis, as the related discussion will be done on these variables.

The functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are coupled and the system is

$$\tilde{\phi}_1' = -h\, e^{-u^0}\tilde{\phi}_1 - h\, e^{u^0-2v^0}\tilde{\phi}_2 + \frac{1}{20}h\, e^{-2u^0-4v^0}(\tilde{\xi}_1 + 2\tilde{\xi}_2)\,, \tag{6.98}$$

$$\tilde{\phi}_2' = -h\, e^{-u^0}\tilde{\phi}_1 - 3h\, e^{u^0-2v^0}\tilde{\phi}_2 + \frac{1}{20}h\, e^{-2u^0-4v^0}(4\tilde{\xi}_1 + 3\tilde{\xi}_2)\,. \tag{6.99}$$

The fundamental matrix is

$$\tilde{\Upsilon}_{12} = \left(\begin{array}{c|c} \frac{r^4+1}{r^3\sqrt{r^4-1}} & \frac{1}{r^4} + \frac{r^4+1}{r^3\sqrt{r^4-1}}\left(\mathcal{E}(r) - \mathcal{F}(r)\right) \\ \hline \frac{3-r^4}{r^3\sqrt{r^4-1}} & \frac{3}{r^4} + \frac{3-r^4}{r^3\sqrt{r^4-1}}\left(\mathcal{E}(r) - \mathcal{F}(r)\right) \end{array}\right)\,. \tag{6.100}$$

A formal solution is thus

$$\begin{pmatrix} \tilde{\phi}_1(r) \\ \tilde{\phi}_2(r) \end{pmatrix} = \tilde{\Upsilon}_{12}(r)Y_{12} + \tilde{\Upsilon}_{12}(r)\int^y \tilde{\Upsilon}_{12}^{-1}(y)g_{12}^{\phi}(y)dy\,. \tag{6.101}$$

where $Y_{12} = (Y_1, Y_2)$ are integration constants, and $g_{12}^{\phi} = (g_1^{\phi}, g_2^{\phi})$ encodes the non-homogeneous terms in the couple of equations (6.98)–(6.99) above. Some of the integrals can be explicitly done but sadly there are some terms for which we are unable to find a primitive. We thus have a solution up to an implicit integral.

We can use the following relation arising from the equation for $\tilde{\phi}_1$:

$$-\,h\, e^{-u^0}\tilde{\phi}_1 - h\, e^{u^0-2v^0}\tilde{\phi}_2 = \tilde{\phi}_1' - \frac{h}{20}e^{-2u^0-4v^0}\,, \tag{6.102}$$

to simplify the equation for $\tilde{\phi}_3$, which will take the following form:

$$\tilde{\phi}_3' = 8\tilde{\phi}_1' + \frac{h}{10}e^{-2u^0-4v^0}\left(\tilde{\xi}_3 - 32\,\tilde{\xi}_4\right)\,. \tag{6.103}$$

and has the following solution

$$\tilde{\phi}_3(r) = 8\tilde{\phi}_1(r) + \frac{8}{3}\int^r \frac{\tilde{\xi}_3}{(y^4-1)^{3/2}}\,dy - \frac{256}{5}\int^r \frac{\tilde{\xi}_4}{(y^4-1)^{3/2}}\,dy + Y_3\,, \tag{6.104}$$

which is again implicitly defined in terms of a single integral.

$\tilde{\phi}_5$ and $\tilde{\phi}_6$ are coupled and the system is

$$\begin{aligned}
\tilde{\phi}_5' &= h\,\tilde{\phi}_6 + \frac{\ell^6}{4m^2}h\,H_0\,\tilde{\xi}_5 - \frac{h}{4}f_1\left(8\tilde{\phi}_1 - \tilde{\phi}_3\right) \\
\tilde{\phi}_6' &= 2h\,e^{2u^0}\left(2e^{-4v^0} + e^{-4u^0}\right)\tilde{\phi}_5 + \frac{\ell^6}{m^2}h\,H_0 e^{2u^0-4v^0}\tilde{\xi}_6 + \frac{\ell^6}{2m^2}e^{-2u^0}h\,H_0\,\tilde{\xi}_7 \\
&\quad + \frac{f_2}{4}\left(8\tilde{\phi}_1 + 8\tilde{\phi}_2 - \tilde{\phi}_3\right) - \frac{f_3}{4}\tilde{\phi}_3
\end{aligned}$$

whose fundamental matrix reads

$$\tilde{\Upsilon}_{56} = \left( \begin{array}{c|c} \frac{1}{r\sqrt{r^4-1}} & \frac{1}{21}\left(-2+3r^4\right) + \frac{2}{21r\sqrt{r^4-1}}\mathcal{F}(r) \\ \hline \frac{1-3r^4}{r^4(r^4-1)} & \frac{2(6r^8-6r^4-1)}{21r^3\sqrt{r^4-1}} + \frac{2(1-3r^4)}{21r^4(r^4-1)}\mathcal{F}(r) \end{array} \right) .$$

A formal solution will have the same structure as (6.101). Recall that $g^{\phi}_{56}$ features quantities defined in terms of one implicit integral coming from $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_3$, thus the expressions we get are defined in terms of two nested integrals.
The equation for $\tilde{\phi}_7$ can be put in the form

$$\tilde{\phi}_7' = \tilde{\phi}_6' - \frac{\ell^6}{m^2} h\, H_0 e^{-2u^0}\, \tilde{\xi}_7 + \frac{1}{2} f_3 \tilde{\phi}_3 - 4h^0 e^{-2u^0}\, \tilde{\phi}_5 \,.$$

and the solution is given by

$$\tilde{\phi}_7 = \tilde{\phi}_6 - \frac{\ell^6}{m^2} \int h\, H_0 e^{-2u^0}\, \tilde{\xi}_7 + \frac{1}{2} \int f_3\, \tilde{\phi}_3 - 4 \int h\, e^{-2u^0}\, \tilde{\phi}_5 \,.$$

Among the summands which appear under integral sign the first contains no further integral while the second integrand is itself defined implicitly and so it gives two nested integrals in our counting. The last summand is defined by three nested integrals (one explicit here and two coming from $\tilde{\phi}_5$). A simple integration by parts can reduce the number by one giving an expression for $\tilde{\phi}_7$ which contains at most two nested integrals. We obtain

$$\begin{aligned} \tilde{\phi}_7(r) = {} & \tilde{\phi}_6(r) - \frac{\ell^6}{m^2} h\, H_0 e^{-2u^0}\, \tilde{\xi}_7(r) + \frac{1}{2} f_3 \tilde{\phi}_3(r) \\ & + 4 \int^r \left( -\frac{2y}{\sqrt{y^4-1}} - 2\mathcal{F}(y) \right) \tilde{\phi}_5'(y) dy + 8 \left( \frac{r}{\sqrt{r^4-1}} + \mathcal{F}(r) \right) \tilde{\phi}_5(r) \,. \end{aligned}$$

We can now use the $\tilde{\phi}_1$, $\tilde{\phi}_2$ system to simplify the equation for $\tilde{\phi}_4$ which one obtains from (6.48), which can be recast in the following form

$$\begin{aligned} \tilde{\phi}_4' = {} & -H_0^{-1} H_0'\, \tilde{\phi}_4 + 16 \tilde{\phi}_1' - 8 \tilde{\phi}_2' + \frac{1}{2} h\, e^{-2u^0-4v^0}\, \tilde{\xi}_3 - \frac{16m^2}{\ell^6} h\, H_0^{-1} e^{-2u^0-4v^0} f_1 \tilde{\phi}_5 \\ & - \frac{4m^2}{\ell^6} e^{-4u^0} H_0^{-1} f_2 \tilde{\phi}_6 + \frac{3}{4} \frac{m^2}{\ell^6} e^{-2u^0-4v^0} h\, H_0^{-1} \left( \tilde{\phi}_6 - \tilde{\phi}_7 \right) . \end{aligned} \tag{6.105}$$

The homogeneous solution to this equation reads $\tilde{\phi}_{4,hom} = H_0^{-1}$. Labelling by $g_4^{\phi}$ the non–homogeneous term in (6.105), a general solution is given by

$$\tilde{\phi}_4(r) = H_0^{-1}(r) Y_4 + H_0^{-1}(r) \int^r H_0(y) g_4^{\phi}(y) dy \,. \tag{6.106}$$

### 6.8.3 IR asymptotics of the $\phi_a$ equations

We present here the IR expansion of the $\phi_a$ fields which are easily obtainable from the inverse transformation (6.58). We write explicitly only the divergent and constant terms since terms which are regular in the IR (we recall here that it corresponds to the limit $r \to 1$ in our conventions) do not provide any constraint on the space of solutions. We also impose the zero energy condition (6.49) which gives $X_2^{IR} = 0$ as a constraint.

$$\begin{aligned}\phi_1 = &\frac{1}{\sqrt{r-1}}\Bigg[Y_1^{IR} + \Big(E(-1) - K(-1)\Big)Y_2^{IR} + \frac{\log(r-1)}{4480}\Bigg( - 3B_1\Big(34 + 65K(-1)^2\Big) \\ &+ 1792X_1 + 336X_5 - 112K(-1)\Big(3X_6 - 2X_7\Big)\Bigg)\Bigg] + \mathcal{O}\Big((r-1)^{1/2}\Big) \end{aligned} \tag{6.107}$$

$$\begin{aligned}\phi_2 = &\frac{1}{13440\sqrt{r-1}}\Bigg[ - 3B_1\Big(41 + 100K(-1)^2\Big) + 2688X_1 + 924X_5 \\ &- 308K(-1)\Big(3X_6 - 2X_7\Big)\Bigg] - Y_2^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big) \end{aligned} \tag{6.108}$$

$$\begin{aligned}\phi_3 = &\frac{1}{15482880\left(K(-1)^2-4\right)\sqrt{r-1}}\Bigg[480\log(r-1)\left(K(-1)^2-4\right)\Big(3B_1\left(K(-1)^2+17\right) \\ &- 56(K(-1)(2X_7 - 3X_6) + 3X_5)\Big) - 42K(-1)^2(21067B_1 - 49152X_4 + 17384X_5) \\ &+ 87369B_1K(-1)^4 - 374856B_1 - 32K(-1)\Big(189168X_6 - 120117X_7 \\ &+ 32(5210Y_2^{IR} - 33(7Y_3^{IR} + 80Y_6^{IR}))\Big) + 40K(-1)^3(36624X_6 - 22535X_7) \\ &+ 1344\Big(4160X_1 - 19200X_4 + 311X_5 + 160(7Y_1^{IR} + 7Y_2^{IR}E(-1) + 132Y_5^{IR} - 144Y_7^{IR})\Big)\Bigg] \\ &+ \frac{1}{256}\left(3B_1K(-1) - 8X_7\right)\log(r-1) \\ &- \frac{2Y_4^{IR}}{3\left(K(-1)^2-4\right)} + \frac{1}{96}\left(48Y_2^{IR} + Y_3^{IR}\right) + \mathcal{O}\Big((r-1)^{1/2}\Big) \end{aligned} \tag{6.109}$$

$$\phi_4 = \frac{1}{7741440\,(K(-1)^2-4)\sqrt{r-1}}\Bigg[480\log(r-1)\left(K(-1)^2-4\right)\Big(3B_1\left(K(-1)^2+17\right)$$
$$- 56(K(-1)(2X_7-3X_6)+3X_5)\Big) + 6K(-1)^2(9203B_1 - 56(92160X_4+6781X_5))$$
$$- 30135B_1K(-1)^4 - 2254920B_1 + 32\Big(K(-1)(488208X_6 - 331467X_7$$
$$+ 32(-5210Y_2^{IR} + 231Y_3^{IR} + 2640Y_6^{IR})) + 42\big(4160X_1 + 79104X_4 + 4919X_5$$
$$+ 160(7Y_1^{IR} + 7Y_2^{IR}E(-1) + 132Y_5^{IR} - 144Y_7^{IR}))\Big) + 8K(-1)^3(338909X_7 - 494256X_6)\Bigg]$$
$$+ \frac{1}{128}\,(3B_1K(-1) - 8X_7)\log(r-1) - \frac{4Y_4^{IR}}{3\,(K(-1)^2-4)} + Y_2^{IR} - \frac{5Y_3^{IR}}{16}$$
$$+ \mathcal{O}\Big((r-1)^{1/2}\Big) \tag{6.110}$$
$$\phi_5 = \frac{1}{5160960}\frac{1}{\sqrt{r-1}}\Bigg[60\log(r-1)\left(K(-1)^2-4\right)\Big(3B_1\left(K(-1)^2+17\right)$$
$$- 56\big(K(-1)(2X_7-3X_6)+3X_5\big)\Big) + 6K(-1)^2(1795B_1 - 3976X_5) + 2235B_1K(-1)^4$$
$$- 42024B_1 - 32K(-1)(6384X_6 - 3711X_7 + 4160Y_2^{IR} - 672Y_3^{IR} - 7680Y_6^{IR})$$
$$+ 152K(-1)^3(336X_6 - 179X_7) + 1344\big(64X_1 - 768X_4 - 23X_5 - 160(Y_1^{IR} + Y_2^{IR}E(-1)$$
$$- 12Y_5^{IR}))\Bigg] - \frac{3(K(-1)^2-4)}{2048}(3B_1K(-1) - 8X_7) + \mathcal{O}\Big((r-1)^{1/2}\Big) \tag{6.111}$$
$$\phi_6 = \frac{1}{20643840}\Bigg[6K(-1)^2(36599B_1 + 30856X_5) - 5115B_1K(-1)^4 + 140376B_1$$
$$- 1344\big(1262X_1 - 2304X_4 + 185X_5 + 160(5Y_1^{IR} + 12Y_5^{IR} - 48Y_7^{IR} + 5Y_2^{IR}E(-1))\big)$$
$$+ 32K(-1)(28560X_6 - 18495X_7 + 44480Y_2^{IR} - 672Y_3^{IR} - 7680Y_6^{IR})$$
$$+ 8K(-1)^3(16841X_7 - 26544X_6)\Bigg] + \mathcal{O}\Big((r-1)\Big) \tag{6.112}$$

$$\phi_7 = \frac{1}{5160960(r-1)}\Bigg[6K(-1)^2(5656X_5 - 295B_1) - 8K(-1)^3(7644X_6 - 4241X_7)$$
$$- 2415B_1K(-1)^4 + 32K(-1)\big(7644X_6 - 4551X_7 + 32(130Y_2^{IR} - 21Y_3^{IR} - 240Y_6^{IR})\big)$$
$$+ 8904B_1 - 1344\big(64X_1 - 768X_4 + 7X_5 - 160(Y_1^{IR} - 12Y_5^{IR} + Y_2^{IR}E(-1))\big)\Bigg]$$
$$- \frac{\log(r-1)}{(r-1)}\frac{K(-1)^2-4}{86016}\Bigg[3B_1(17 + K(-1)^2) - 56\big(3X_5 - K(-1)(3X_6 - 2X_7)\big)\Bigg]$$
$$+ \frac{\log(r-1)}{860160}\Bigg[B_1(3468 + 4485K(-1)^2 - 15K(-1)^4) - 56\big(768X_1 + 204X_5$$
$$- 68K(-1)(3X_6 - 2X_7) - 15K(-1)^2X_5 + 5K(-1)^3(3X_6 - 2X_7)\big)\Bigg]$$
$$+ \frac{1}{20643840}\Bigg[32K(-1)\big(28650X_6 - 18495X_7 + 32(1390Y_2^{IR} - 21Y_3^{IR} - 240Y_6^{IR})\big)$$
$$+ 6K(-1)^2(36599B_1 + 30856X_5) - 8K(-1)^3(26544X_6 - 16841X_7) - 5115K(-1)^4B_1$$
$$+ 140376B_1 - 1344\big(1216X_1 - 2304X_4 + 181X_5$$
$$+ 160(5Y_1^{IR} + 12Y_5^{IR} + 48Y_7 + 5Y_2^{IR}E(-1))\big)\Bigg] + \mathcal{O}\Big((r-1)^{1/2}\Big) \qquad (6.113)$$

## 6.9 Discussion

We are about to extract physical considerations out of the family of perturbations determined. We focus on the candidate backreaction by anti-D2 branes. We first derive the boundary conditions which correspond to the modes sourced by a stack of branes placed at the tip of the cone, for then determining how, within the space of generic linearized deformations of the IIA CGLP background, one can account for the backreaction due to the addition of anti-D2 branes smeared on the $S^4$ at the tip of the warped throat.

### 6.9.1 Boundary conditions for BPS D2 branes

The solution of the differential system (6.47)-(6.48) encodes the first order perturbation and, as we have seen, depends on 13 integration constants (once we take into account the zero energy condition (6.49)). In this section we derive the boundary conditions which correspond to the modes sourced by a stack of branes placed at the tip of the cone. Let us then consider a set of $N$ ordinary BPS D2–branes smeared on the $S^4$. For the CGLP background, we can explicitly evaluate the Maxwell charge

$$\mathcal{Q}^{Max}_{CGLP}(r) = \frac{1}{(2\pi\sqrt{\alpha'})^5}\int_{M_6} e^{\Phi/2} * F_4 = \frac{4m^2 g_s^{-1/2}}{\ell(2\pi\sqrt{\alpha'})^5}\text{vol}(M_6)[g_1(g_2+g_3)+c_2 g_2 + c_3 g_3] \,.$$

where $\text{vol}(M_6) = D\mu^1 \wedge D\mu^2 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4$. Furthermore, we can evaluate the flux through the $S^4$ as

$$q_{S^4} = \frac{1}{(2\pi\sqrt{\alpha'})^3}\int_{S^4} F_4 = \frac{4mg_s^{-1}}{(2\pi\sqrt{\alpha'})^3}(g_1+c_2)\text{vol}(S^4) \,. \tag{6.114}$$

where we used $2\text{vol}(S^4) = J_2 \wedge J_2$. These quantities have the following behavior in the IR:

$$\mathcal{Q}^{IR}_{CGLP} = 0, \qquad q^{IR}_{S^4} = \frac{3mg_s^{-1}}{8(2\pi\sqrt{\alpha'})^3}\text{vol}(S^4) \,. \tag{6.115}$$

as one can see from

$$\lim_{r\to 1}[g_1(g_2+g_3)+c_2 g_2 + c_3 g_3] = \frac{7}{128}(r-1)^{3/2} - \frac{77}{512}(r-1)^{5/2} + \mathcal{O}\left((r-1)^{7/2}\right) \tag{6.116}$$

where (6.22) was used. The addition of $N$ ordinary BPS D2 branes amounts to consider the following modifications

$$g_2^0 \to g_2^0 + \frac{32N}{3} \,, \qquad g_3^0 \to g_3^0 - \frac{32N}{3} \,. \tag{6.117}$$

In turn, the charge is shifted as

$$\mathcal{Q}^{Max}_{CGLP} \to \mathcal{Q}^{Max}_{CGLP} + \Delta\mathcal{Q}^{Max}_{D2} \,. \tag{6.118}$$

and indeed one finds that

$$\Delta\mathcal{Q}^{Max}_{D2} = \frac{4Nm^2}{(2\pi\sqrt{\alpha'})^5}\frac{g_s^{-1/2}}{\ell}\text{vol}(M_6) \,. \tag{6.119}$$

while the flux through $S^4$ is unaffected. The warp factor will shift as

$$H_0(r) \to -\frac{4m^2}{\ell^6}\int^r h\, e^{-2u^0-4v^0}\left[g_1^0(g_2^0+g_3^0)+c_2 g_2^0 + c_3 g_3^0 + N\right] dy \,, \tag{6.120}$$

and its deformation acquires now a singularity of the kind

$$H(r) \sim \frac{\Delta\mathcal{Q}_{D2}}{\sqrt{r-1}} \tag{6.121}$$

as expected. We can now see how this BPS solution can be reproduced by the first–order perturbation apparatus. First of all, we set all the modes related to supersymmetry breaking to zero, namely we set to zero all the constants $X_a$ and $B_1 \sim X_3$ which are related to integration constants of the $\tilde{\xi}$ modes. The factors $e^{3z_0+2u^0}$ and $e^{3z_0+2v^0}$ reach constant or zero value in the IR and since we expect that the geometry of the transverse space is not affected we impose that the perturbation associated to $u$ and $v$ are zero. This amounts to setting

$$Y_1^{IR} = Y_2^{IR} = 0. \tag{6.122}$$

Remarkably, one can regularize $\phi_5$ and $\phi_7$ at the same time by imposing

$$Y_5^{IR} = -\frac{1}{840} K(-1)(7Y_3^{IR} + 80Y_6^{IR}). \tag{6.123}$$

This above choice of integration constants (6.122)–(6.123) gives the following perturbations:

$$\phi_1 = 0\,, \qquad\qquad \phi_2 = 0\,, \qquad\qquad \phi_5 = \mathcal{O}\Big((r-1)\Big)\,, \tag{6.124}$$

$$\phi_3 = -\frac{2Y_7^{IR}}{4-K(-1)^2}\frac{1}{\sqrt{r-1}} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,, \quad \phi_4 = -\frac{4Y_7^{IR}}{4-K(-1)^2}\frac{1}{\sqrt{r-1}} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,,$$
$$\phi_6 = \frac{1}{2}Y_7^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,, \qquad \phi_7 = -\frac{1}{2}Y_7^{IR} + \mathcal{O}\Big((r-1)^{1/2}\Big)\,.$$

It is clear that $Y_7^{IR}$ is related to the number $N$ of added D2–branes: the equations for $\phi_6$ and $\phi_7$ reproduce the shift (6.117). The warp factor, as well as the dilaton, acquires the expected singularity as

$$H = e^{8z_0}\left(1+8\phi_3\right)\,, \qquad e^{\Phi} = e^{\Phi_0}\left(1+\phi_4\right) = e^{\Phi_0}(1+2\phi_3) \;\text{ as }\; e^{\Phi} \sim H^{1/4}\,.$$

## 6.10 Assessing the anti–D2 brane solution

Motivated by the physical requirement that force on a probe D2–brane exerted by a stack of anti–D2 branes is non-vanishing, we are not allowed to turn off the corresponding mode which appears in the expression (6.72) of the force, and enters the various equations by means of the shorthand combination

$$B_1 = \frac{m^2}{\ell^6} X_3 e^{-8z_0(1)}\,. \tag{6.125}$$

To physically investigate the presence of singularities, we start from the behavior of the background scalars which enter the flux ansatz, and the effect on the related energy density induced by their perturbation.
Inspecting the IR expansions of the deformation modes $\phi_a$, every piece that is more singular than the aforementioned $1/\sqrt{r-1}$ behavior will be culled by tuning appropriate combinations of the $X$'s and the $Y$'s integration constants parametrizing the space of

generic linearized perturbations of the CGLP background.
Another, equivalent but slightly less liberal, criterion that we are about to consider focuses on allowing or discarding various pieces from the $\phi_a$'s IR expansions depending on their contribution to the energy. More precisely, we consider the kinetic energy (6.34) and the potential energy (6.36) obtained by reducing the IIA supergravity Ansatz (6.24) to a one–dimensional sigma model. For instance, the energy associated to the first–order perturbation of the dilaton and warp factor is obtained by expanding to second–order the corresponding terms from (6.34):

$$\frac{e^{2\left(u^0+\phi_1\right)+4\left(v^0+\phi_2\right)}}{h}\Big[-30\,\left(z^{0\prime}+\phi_3'\right)^2-\frac{1}{2}\left(\Phi^{0\prime}+\phi_4'\right)^2\Big]$$
$$\rightsquigarrow$$
$$\frac{e^{2\,u^0+4\,v^0}}{h}\Big[-30\,\phi_3'^2-\frac{1}{2}\,\phi_4'^2-2\,\left(\phi_1+2\,\phi_2\right)\,\left(\Phi^{0\prime}\,\phi_4'+60\,z^{0\prime}\,\phi_3'\right)\Big] \tag{6.126}$$

We recall that the modes $\phi_3$ and $\phi_4$ associated to the perturbation of the warp factor and the dilaton must respectively exhibit a $1/\sqrt{r-1}$ behavior in order to reproduce the expected singularities due to the presence of anti–D2–branes. The energy associated to the deformation of the warp factor and dilaton exhibits the following singular behavior

$$(r-1)^{3/2}\,\left(\frac{d\phi_{3,4}}{dr}\right)^2\sim\frac{1}{(r-1)^{3/2}}\,,$$

where we are neglecting less diverging terms. This behavior sets the threshold for what we consider an allowable singularity in the energy, and very more divergent piece in the IR expansion will have to be avoided by appropriately tuning the $X$'s and the $Y$'s.
Note that, as it turns out, for all practical purposes we can neglect contributions of the type $\phi_a\phi_b$ and $\phi_a'\,\phi_b$ for $a\neq b$: they only contribute to sub–leading divergences. In addition, there is no contribution to the energy that is first–order in the SUSY–breaking parameters, since we are expanding around a saddle point.
We have considered linearized deformation for the fields entering the supergravity Ansatz (6.23)-(6.27), in that, recalling (6.45), we have expanded as

$$\phi_a=\phi_a^{(0)}+\phi_a^{(1)}(X,Y)\,, \tag{6.127}$$

with $X_i$ and $Y_i$ being implicitly the small supersymmetry–breaking expansion parameters. On the other hand, we are considering quadratic contributions of the $\phi_a^{(1)}$'s to the energy. The reason why we do not stop at first–order contributions to the energy from those deformation modes is that we have expanded around a saddle point. Had we gone as far as computing 2nd order expansions of the deformation modes, namely

$$\phi_a=\phi_a^{(0)}+\phi_a^{(1)}(X,Y)+\phi_a^{(2)}(X,Y,Z,W)\,, \tag{6.128}$$

which is an achievable if strenuous task, it might well happen that the singularities we are about to expose might cancel against truly second order contributions to the energy. By this we mean contributions of the type $\phi_a^{(2)}\phi_b^{(0)}$, in addition to those of the form $(\phi_a^{(1)})^2\,\phi_b^{(0)}$ that we presently consider.

Everything is now in place to show that the candidate IIA supergravity dual to metastable supersymmetry–breaking that would be obtained out of backreacting anti-D2's spread over the $S^4$ in the far IR of the CGLP background comes with an irretrievable IR singularity. Indeed, we are going to show that it is not possible to simultaneously satisfy the two previously mentioned physical requirements. In point of fact, there is a singularity associated to the NS-NS and R-R fluxes that is worse than the ones we allow, namely those that are physical and should be kept based on their identification with the effect of adding anti–D2 branes to uplift the AdS minimum of the potential. There is only one way of getting rid of that "unphysical" singularity: it entails setting to zero the single mode entering the force felt by a brane probing the non–supersymmetric backreaction by anti-D2's. So, our two sensible IR boundary conditions are incompatible. Ensuring that there is a force exerted on a probe D2–brane by the anti–D2's at the tip results in a $\frac{1}{(r-1)^3} \sim \frac{1}{\tau^6}$ singular contribution to the energy, stemming from the NS-NS or the R-R field strength. Such a singularity is worse than the ones it is sensible to a priori allow, namely $\frac{1}{(r-1)^{3/2}}$ singularities or milder ones, associated to the smeared anti-D2's.

Let us see how this comes about with full details. First of all, by looking at the IR series expansion (6.113) we see that $\phi_7$ is the potentially most divergent mode: it indeed displays $\frac{1}{r-1}$ and $\frac{\log(r-1)}{r-1}$ pieces. That mode contributes only to the deformation of the NS-NS 3–form field strength

$$\ell\,\delta H_3 = m\,\left[(\phi_6+\phi_7)\,U_3 + \phi_6'\,dr\wedge U_2 + \phi_7'\,dr\wedge J_2\right]\,. \tag{6.129}$$

In view of (6.34) and (6.36), the leading contribution to the energy from the deformation of the NS-NS 3–form is

$$-\,\frac{m^2}{2\,\ell^6}\,\frac{e^{2\,u^0+4\,v^0-8\,z^0}}{h}\,\left[\phi_6'^2\,e^{-4\,u^0}+2\,\phi_7'^2\,e^{-4\,v^0}\right]-2\,\frac{m^2}{\ell^6}\,h\,e^{-8\,z^0}\,\left[\phi_6+\phi_7\right]^2\,. \tag{6.130}$$

There is another potential contribution from (6.36) which involves $\phi_6$ and $\phi_7$. It is easily seen that it is sub–leading. Now, what is the IR singular behavior of (6.130) ? We focus on the most singular piece of $\phi_7 \sim \frac{1}{r-1}$ and its derivative. It entails the following singular behavior

$$\begin{aligned}&-\,\frac{m^2}{\ell^6}\,e^{-8\,z^0(r)}\,\left[\frac{e^{2\,u^0(r)}}{h(r)}\,\left(\frac{d}{dr}\,\frac{1}{(r-1)}\right)^2+2\,h(r)\,\left(\frac{1}{(r-1)}\right)^2\right]\\&\rightsquigarrow\frac{1}{(r-1)^{5/2}}\,.\end{aligned} \tag{6.131}$$

According to our physical criterion pertaining to the energy, we should then discard the most IR–divergent piece of $\phi_7$, see (6.113). We are therefore led to set

$$X_5 = \frac{1}{168}\,\left[3\,\left(17+K(-1)^2\right)\,B_1+56\,K(-1)\,\left(3\,X_6-2\,X_7\right)\right]\,, \tag{6.132}$$

$$X_1 = \frac{1}{86016}\Big[6048\,B_1 + 1032192\,X_4 + 215040\,Y_1^{IR} - 2580480\,Y_5^{IR} + 215040\,E(-1)\,Y_2^{IR}
+ 235200\,K(-1)\,X_6 - 139360\,K(-1)\,X_7 + 133120\,K(-1)\,Y_2^{IR}
- 21504\,K(-1)\,Y_3^{IR} - 245760\,K(-1)\,Y_6^{IR} + 8364\,K(-1)^2\,B_1
- 27216\,K(-1)^3\,X_6 + 11304\,K(-1)^3\,X_7 - 1809\,K(-1)^4\,B_1\Big]\,, \tag{6.133}$$

where (6.132) has been applied to obtain (6.133) out of the combination of $X$'s and $Y$'s from the $\frac{1}{(r-1)}$ part of $\phi_7$'s IR expansion.

Having set the relevant conditions to regularize the $\phi_7$ mode, we now focus to getting rid of the singularities associated to R-R flux. The relevant mode is in this case $\phi_5$. Before imposing further conditions, notice how equation (6.132) eliminates the leading $\frac{\log(r-1)}{\sqrt{r-1}}$ part of $\phi_5$'s IR asymptotics. Nevertheless, one should enforce that the $\frac{1}{\sqrt{r-1}}$ part of $\phi_5$'s IR expansion be wiped out by appropriately tuning some of the $X$'s and $Y$'s. Indeed, if kept unchecked, that divergent piece would yield a singularity in the energy arising from the R-R flux:

$$- 2\frac{m^2}{\ell^6}\,\frac{e^{-8\,z^0-9\,\phi_3+\phi_4/2}}{h}\,\left(g_1^{0\,\prime}+\phi_5'\right)^2
- 4\,\frac{m^2}{\ell^6}\,e^{-8\,z^0-9\,\phi_3+\phi_4/2+2\,u^0+2\,\phi_1}\,h\,\Big[2\,\left(g_1^0+c_2+\phi_5\right)^2\,e^{-4\,v^0-4\,\phi_2} + \left(g_1^0+c_3+\phi_5\right)^2\,e^{-4\,u^0-4\,\phi_1}\Big]
\rightsquigarrow \frac{1}{(r-1)^{5/2}}\,, \tag{6.134}$$

which is beyond the energy threshold (6.127) and should be culled. To get rid of that singular piece from $\phi_5$, one must impose

$$- 32\,K(-1)\,\left(6384\,X_6 - 3711\,X_7 + 4160\,Y_2^{IR} - 672\,Y_3^{IR} - 7680\,Y_6^{IR}\right)
6\,K(-1)^2\,(1795\,B_1 - 3976\,X_5) + 152\,K(-1)^3\,(336\,X_6 - 179\,X_7)
+ 2235\,K(-1)^4\,B_1 - 42024\,B_1 + 1344\,\big[64\,X_1 - 768\,X_4 - 23\,X_5
- 160\,(Y_1^{IR} + E(-1)\,Y_2^{IR} - 12\,Y_5^{IR})\,\big] = 0\,. \tag{6.135}$$

We have finally reached the punchline of our analysis. Taking into account the conditions (6.132)–(6.133) that did arise from ensuring that no "unphysical" singularity pops out of the NS-NS flux, it turns out that the regularization of the R-R mode (6.135) becomes

$$11340\,\left(4 - K(-1)^2\right)\,B_1 = 0\,, \tag{6.136}$$

in opposition to the physical requirement that a D2–brane probing the non–supersymmetric deformation of the CGLP background experiences a non–vanishing force.

We have finally come to the conclusion that a careful analysis of the backreaction of anti–D2 branes yields a singularity in the IR region, provided we want to keep the $B_1$ mode entering the expression for the force (6.72) to be non–vanishing. We end our analysis

by finding that at least one among these perturbed flux components has to contribute both to a divergent energy density, as just explained, but as well to a divergent action, as the factor $\sqrt{-g_{10}} \simeq \sqrt{r-1}$ appearing in the ten dimensional action (2.1) is not enough to make the action finite in the IR, similarly to what found for an analogue analysis performed in an M-theoretical setup [13]. Furthermore, with respect to this last case, the singular behavior found in the present analysis is not at all subleading.
Our type IIA analysis completes the investigation of backreacted solutions which can be used to infer the existence of meta–stable vacua in string theory, which was originally done in a type IIB setting [17], and in the M-theory investigation of [13].

## 6.11 The nature of singularities

It in unclear to the present day whether singularities like the ones we presented in the previous subsection are physical or not. Whereas for the backreaction of anti–D3's on the Klebanov–Strassler solution one could have expected, with hindsight, a singularity to arise in analogy with the IIA brane engineering of four–dimensional gauge theories, a similar argument does not hold for string theory constructions of 2+1–dimensional gauge theories. Indeed, the profile of the NS5–branes featured in those brane engineering constructions is generally not rigid but is instead sourced by the stack of Dp branes in–between. For four–dimensional field theories living on D4–branes between two NS5's, the profile determined upon solving a Laplace equation is logarithmically running. This corresponds to the log–running of the gauge coupling for asymptotically free theories. For three–dimensional field theories living on D3–branes between two NS5's, the profile decays as $1/r$ away from the location of the D3's on the NS5. Such a mode does not have the potential ability to enhance small IR fluctuations into log–running ones, an ability to which one might roughly ascribe the singularities encountered in the holographic approach to realizing metastable states in string theory, if those singularities are deemed as truly pathological. Proceeding in analogy with brane engineering constructions, for 2+1–dimensional IR perturbations should be expected not to affect the UV asymptotics of the background. This is not quite the case for the candidate supergravity dual to a 2+1–dimensional metastable state studied in this Chapter: the IR singularities we find are affecting the UV behavior, in the sense that they cannot be completely tamed without switching off at the same time the force felt by a probe D2–brane in the UV. Also, having their legs in the wrong directions, those IR divergences cannot be identified as the remnant signature of an NS5 instanton through which the metastable state is been argued to decay in the probe approximation [141,143], and it seems these cannot be identified either with those characterizing fractional branes on Ricci–flat transverse geometries before the resolution or deformation of those manifolds (solutions of the Klebanov–Tseytlin [145] type, whose singularities get resolved in the Klebanov–Strassler solution).
To give an interpretation of singularities such the one encountered in the previous subsection stimulated a recent debate. One of the first remarks we had to make is that in the analysis presented we had to smear branes on the $S^4$. The role of smearing in all of this

is quite puzzling, and it might well be that those singularities are an artifact of having to smear anti-branes in order to make the problem tractable. Some recent results [28, 29] suggest however that a localization procedure is bound to make things worse, rather than alleviating them. In view of the fact the perturbation method works for the linear approximation, we propose the following analogy for linking an eventual singular behavior and its physical causes. In QCD, there are free quarks in the linearized approximation. Their "backreaction" results in a Coulomb–like singularity. We know that this is an indication that quarks are not good approximations at all to finite–energy states from the spectrum of QCD, which instead consists of confined, colorless states. Now, the singularities we find involve in particular an IR–divergent NS-NS flux–density, so in the linear approximation we can, with a terminology abuse, dub them as Coulomb–like as well. We would like to suggest, following the above–mentioned situation in QCD, that those singularities perhaps hint that some of the scenarios that have been proposed in a probe approximation to uplift an AdS–vacuum to a metastable de Sitter one using brane sources do not engineer acceptable states of the "spectrum" of string theory. But how is the analogy to hold with the Coulomb–like singularities of QCD, given that anti-branes are not expected to source NS-NS flux at full non-linear level? Indeed, they certainly do not if we stick to the guideline and intuition drawn from the supergravity solutions describing such sources in flat space. Nevertheless, it has been proposed that they naturally do [67, 172] in perturbation theory around a complicated, warped geometry such as the Klebanov–Strassler solution or the CGLP solution we investigated in the present Chapter.

The authors of [67, 172] argue rather convincingly that the IR–divergencies that seem to affect, at linearized order in the SUSY–breaking parameter, the backreaction by antibranes of an underlying warped background should disappear at full non–linear order. Their claim goes as follows: the singularities in the flux densities are naturally sourced at linear–order in perturbation theory by the acceptable $1/\tau$ behavior[11] of the deformation of the warp factor due to, say, the bunch of anti–D3's smeared on the $S^3$ of the Klebanov–Strassler geometry. It is then advocated that the $1/\tau$ contribution of first–order perturbation theory, summed up with the $1/\tau^2$ contribution at second–order, and so on with all the other contributions, are nothing but terms in the expansion of the inverse warp factor of the backreacted background modified by the presence of antibranes. The whole sum of the individual contributions at each order is then claimed to be a perfectly regular quantity, $h^{-1}$, the inverse of the backreacted warp factor. As a result, it is argued that singularities in the fluxes are a footprint of perturbation theory that should wash out at full non–linear order. It is currently a daunting task to check if this possibility is indeed realized. However, we commented on this possibility in subsection 6.10 when we argued how the issue of those singularities in the smeared approximation could perhaps be settled by considering 2nd–order expansions for the deformation modes of a BPS background, a task which has not been attempted so far. But, to come back to the tentative analogy with QCD, it might seem after all, in view of the above chain of argument involving the full backreaction by antibranes (something we do not attempt;

[11] $\tau$ denotes the radial variable in the bulk.

we stick at linearized deformation), that this analogy does not hold when applied to the backreaction of antibranes at full non–linear order. Nonetheless, the reason we maintain that this analogy might possibly hold is that the aforementioned argument seeking to explain how the singularities in the fluxes should vanish at full non–linear order is not entirely water–tight. Indeed, a recent paper [30] very convincingly shows that in some instance a singularity in the $H_3$ flux density is still present at full non–linear order.

# Chapter 7

# Conclusions

Despite the fact that string theory is a highly fruitful construction, its formulation starting from first principles has not been consistently found yet. Several limits of the theory are known, which are related by means of what is often called a *web* of dualities, able to tie together the different aspects of the theory. These are evidence of the fundamental role of symmetries in string theory, as although the many descriptions of the theory could change under duality transformations, the whole structure is left invariant.
All throughout the exposition of the present thesis, we made large use of symmetries as a leading guide for investigating various aspects of type II theories. As a first concretization, we explored the possibility of giving a covariant reformulation of type II string theory by introducing a set of frameworks in which a physical symmetry is promoted to the geometrical level. The setting of Generalized Complex Geometry which we outlined in Chapter 3 is an interesting realization, for which T-duality is naturally encoded in the geometry as the structure group of the generalized tangent bundle. We reviewed how generalized geometry makes possible the geometrization of the NS-NS sector, and provides a useful reformulation of necessary and sufficient conditions for a configuration to be a vacuum of the theory. Generalized geometry plays a fundamental but at the same time intermediate role, as it fits as a preliminary step in order to find a covariant theory with respect to the most general duality of string theory, U-duality. The aim of Chapter 4 was indeed to reformulate the condition for string vacua in this extended formalism. The study of Exceptional Generalized Geometry is somehow a harder task with respect to its generalized geometrical companion, on one hand in view of its increased complexity imposed by the more complicate structure of the $E_{7(7)}$ group with respect to $O(6,6)$, and on the other hand due to the lack of the mathematical developments which were of fundamental help in establishing the conditions to have a vacuum in the ordinary generalized geometrical setting. We obtained the differential constraints on the algebraic structures $(L, K_a)$ required by $\mathcal{N}=1$ on-shell supersymmetry and presented the results for the generalization to the $\mathcal{N}=2$ vacua case. The first important task was to identify the appropriate twisted derivative that generalizes the operator $d - H\wedge$, commonly used in Generalized Geometry, in order to include the R-R fluxes. As explained in the main text, this amounts to correctly identify the generalized connection: such object is obtained as in standard

differential geometry by the operation $g^{-1}\nabla g$, where now $g$ are the $E_{7(7)}$-adjoint elements corresponding to the shift symmetries (the so-called $B$- and $C$-transforms), and the derivative operator $\nabla$ is embedded as well in the fundamental representation of $E_{7(7)}$ [97]. We furthermore imposed that this object, a priori transforming as a generic tensor product of adjoint and fundamental representations, should only belong to a particular irreducible representation in this tensor product, which in the case at hand is the **912**. As in Chapter 4 we explained how supersymmetry constrains the embedding tensor to belong to this particular representation for $\mathcal{N} = 8$ supergravity in four dimensions, this suggests that these two quantities are nothing but different descriptions of the same object. Once having identified the appropriate connection, we matched supersymmetry conditions with twisted closure of the structures for two main cases. The equations for $\mathcal{N} = 1$ supersymmetry require on one hand closure of $L$, as conjectured in [97], and on the other hand, the components of the twisted derivative of $K'_1$ with an even number of internal indices have to vanish, while those with an odd number are proportional to derivatives of the warp factor. We thus recover the relation between the norms of the spinors and the warp factor (which is imposed by hand in the GCG formulation) as one of the equations in the vector components. Furthermore, when considering $K'_+$, twisted closure occurs upon projecting onto the holomorphic sub-bundle defined by $L$.

We then analyzed the domain of $\mathcal{N} = 2$ compactifications. Preserving the full $SU(2)_R$ permits to reformulate in a democratic way the equations for the triplet $K_a$. Also, these structures are much closer to being closed structures with respect to the twisted differential than it was in the $\mathcal{N} = 1$ case. We recall in fact that, in this latter, the substantial asymmetry between $K'_+$ and $K'_1$ structures (cfr. equation (4.68)) led to a rather complicated decoupling of the twisted equations for these. Unexpectedly, issues arise in conjecturing analogue equations for vacua for the $L$ structure, where in the $\mathcal{N} = 1$ was easily found to correspond to a twisted integrable structure. If we ovecome this by further massaging supersymmetry requirements, we would be able to claim that $\mathcal{N} = 2$ vacua correspond to twisted closed structures $(L, K_a)$.

In Chapter 5 we constructed a five-dimensional gauged supergravity theory by explicit dimensional reduction on $T^{1,1}$, including the entire set of modes which are singlets under the global $SU(2) \times SU(2)$. While the five-dimensional $\mathcal{N} = 4$ theory we have constructed is uniquely specified by the number of vector multiplets and the embedding tensor, the consistency of the truncation guarantees the possibility of uplifting any solution of the theory to type IIB. This is something which is often quite difficult to obtain: for example a closely related system is the $SU(2) \times U(1)$ gauged supergravity constructed in [49] which contains in its solution space the Klebanov-Witten flow [146], but explicit uplift formula are not available and one is forced to work directly in ten dimensions [115]. In fact it would be interesting to see if the Heisenberg algebra found arises as a contraction of the $SU(2)$ gauging in [49].

As recalled in that Chapter, one central motivation to perform the $T^{1,1}$ truncation is to compute a superpotential in five dimensions for a reduction which includes PT. We also remarked that the connection between the Klebanov-Strassler and the Maldacena-Nuñez solution is quite non-trivial from the five dimensional point of view, in that it does not exist a subtruncation of the theory which contains a supersymmetrization of both solutions. Nonetheless finding a superpotential for this $\mathcal{N} = 4$ supersymmetric theory we have presented is a small step further than would be extremely useful and should have direct application to the physics of flux backgrounds. Such a quantity would provide an organizing principle for the spectrum of supersymmetric and non-supersymmetric modes [33], and can be used to investigate a perturbative analysis similar to the one outlined in Chapter 6. There are several direct generalizations of the method proposed which could provide new insight into the physics of lower dimensional gauged supergravities and string theory. There exists a family of Einstein manifolds related to $T^{1,1}$ called $T^{p,q}$ [173], all of which can be viewed as $U(1)$ fibrations over $S^2 \times S^2$ however these do not admit a covariantly constant Killing spinor and thus appear not to preserve any supersymmetry. It would be interesting to determine whether reduction on these manifolds results in a non-supersymmetric theory or a supersymmetric theory with no (canonical) supersymmetric vacuum. Another interesting direction is to consider reductions of IIA on $T^{1,1}$ and compare the resulting embedding tensor of the gauged supergravity to the one found here. As by $T$-duality the spectrum of the ungauged theory is identical to that found in Chapter 5, the only difference must lie in the embedding tensor. This will presumably shed some light on the work [113] where evidence was presented that KS cannot have a mirror which is even locally geometric.

In the last Chapter 6 we investigated how the perturbation of a supersymmetric configuration could describe some metastable state of the theory obtained by placing anti-branes on it. By applying the technique proposed by Borokhov and Gubser to the type IIA CGLP background featuring both regular and fractional branes [53], we conclude the unavoidable presence of singularities in the IR regime which cannot be excluded provided the mode related to the force is turned on. This cannot physically be set to zero when considering the corresponding backreacted solution: since anti-branes attract probe branes, this mode must be present in order to have a meaningful backreaction. The backreacted solution of Chapter 6 features non-subleading divergencies: this is the first case ever singularities of this kind are obtained. While in the Klebanov-Strassler case the singularity has finite action[1] [17], in the M-theory analysis of [13] it turns out to be more severe because also the action is not well behaved in the IR. The singularities found for the perturbation of the CGLP background has behavior properties similar to this last case. As discussed in the last part of that Chapter the interpretation of those singularities represents one of the most recent interesting debates inside the string theory community. The role of the smearing as a cause may have been ruled out in view of the recent works of [28, 29], which suggest that considering localized sources would not alleviate the singular behavior. There are two main opposite points of view. The first states that an analysis beyond the linear order would eliminate the singularities as these are an artifact of perturbative theory at the linear level, while the second claims that is that those Coulomb–like singularities (as directly sourced by anti-branes in the linear approximation) are indeed of physical significance and could be used to discriminate among solutions of the string theory landscape.

[1] The finite action does not automatically guarantee that the singularity is acceptable as the negative mass Schwarzschild counterexample shows [124].

We conclude by remarking that a recent analysis at full non–linear level [30] has been performed, and its conclusion gives evidence of how a divergent behavior is still present, at least when completely localized sources are considered. It is however beyond the scope of the present thesis to offer more credence to vindicate or dispel this possibility but it is very tempting to imagine that the IR singularities we keep on finding upon backreacting antibranes on some BPS background are similarly a hint that some of the constructions which have been proposed as duals to metastable SUSY–breaking might instead belong to some "swampland" [198] once the backreaction of the SUSY–breaking ingredients is duly taken into account.

# Appendix A

# Conventions

## A.1 Differential Forms

Let $M_d$ be an orientable manifold, *i.e.* it admits a globally defined and nowhere vanoshing top form $\mathrm{vol}_d$, called volume form. We fix the orientation requiring the coefficient of $\mathrm{vol}_d$ to be positive: given a local coframe $\{e^{\underline{i_p}}\}^{i=1,\dots,d}$ we take

$$e^0 \wedge e^1 \wedge \ldots \wedge e^{d-1} = +\mathrm{vol}_d\,. \tag{A.1}$$

The Levi-Civita tensor is defined in flat indices, for which we will use underlined indices, by

$$\epsilon_{\underline{m_1\dots m_d}} \equiv \epsilon_{[\underline{m_1\dots n_d}]}\,, \quad \epsilon_{\underline{1\dots d}} = +1\,. \tag{A.2}$$

where the complete anti-symmetrization of the indices is taken as

$$A_{\mu_1\dots\mu_p} = A_{[\mu_1\dots\mu_p]} = \frac{1}{p!}(A_{\mu_1\mu_2\dots\mu_p} - A_{\mu_2\mu_1\dots\mu_p} + \cdots + (-)^{p(p-1)/2}A_{\mu_p\mu_{p-1}\dots\mu_2\mu_1})\,. \tag{A.3}$$

while in curved indices (the ones mainly used in the main text) we have

$$\epsilon_{m_1\dots m_d} \equiv \epsilon_{n_1\dots n_d} e^{\underline{n_1}}{}_{m_1} \cdots e^{\underline{n_d}}{}_{m_d}\,, \quad \epsilon_{12\dots d} = \det e^{\underline{n}}{}_m = \sqrt{|\det g_{mn}|}\,. \tag{A.4}$$

We define a $p$-form $A_p$ as

$$A_p = \frac{1}{p!}A_{m_1\dots m_p}dx^{m^1} \wedge \ldots \wedge dx^{m^p}\,. \tag{A.5}$$

The contraction of a $p$-form $A_p$ and a vector $v = v^m\partial_m$ read

$$\iota_v A_p = v^m A_{m_1\dots m_p}\delta^{[m_1}_m dx^{m_2} \wedge \ldots \wedge dx^{m_p]}\,. \tag{A.6}$$

The wedge product of a form $A_p$ of degree $p$ and a form $B_q$ of degree $q$ is defined as

$$\frac{1}{(p+q)!}(A_p \wedge B_q)_{m_1\dots m_{p+q}} = \frac{1}{p!q!}A_{[m_1\dots m_p}B_{m_{p+1}\dots m_q]}\,. \tag{A.7}$$

The Hodge dual of a $p$-form $A_p$ reads

$$*_d\, A_p = \frac{1}{p!(d-p)!}\epsilon_{m_{p+1}\dots m_d}{}^{m_1\dots m_p} A_{m_1\dots m_p} dx^{m_{p+1}} \wedge \dots \wedge dx^{m_d} \,. \tag{A.8}$$

which satisfies

$$*_d *_d A_p = (-)^{p(d-p)+t} \tag{A.9}$$

where $t=0$ if $M_d$ is Riemaniann, and $t=1$ if $M_d$ is Lorentzian.

## A.2 $SU(8)$ and $SU(4)\times SU(2)$ conventions

The spinor $\theta^\alpha$ transforms in the fundamental representation of $SU(8)$. The standard interwining relations

$$\Gamma_M^\dagger = A\Gamma_M A^{-1}, \qquad \Gamma_M^T = C^{-1}\Gamma_M C, \qquad (\Gamma_M)^* = -D^{-1}\Gamma_M D \tag{A.10}$$

allow to define the conjugate spinors

$$\bar\theta = \theta^\dagger A \;, \quad \theta^t = C\theta^T \;, \quad \theta^c = D\theta^* \;. \tag{A.11}$$

Under $SU(8)$, the **56** decomposes according to

$$\begin{aligned} \nu &= (\nu^{\alpha\beta}, \bar\nu_{\alpha\beta}) \\ \mathbf{56} &= \mathbf{28} + \bar{\mathbf{28}} \end{aligned} \tag{A.12}$$

while for the adjoint **133** we have

$$\begin{aligned} \mu &= (\mu^\alpha{}_\beta, \mu^{\alpha\beta\gamma\delta}, \bar\mu_{\alpha\beta\gamma\delta}) \\ \mathbf{133} &= \mathbf{63} + \mathbf{35} + \bar{\mathbf{35}} \;. \end{aligned} \tag{A.13}$$

where $\mu^\alpha{}_\alpha = 0$ and $\bar\mu_{\alpha\beta\gamma\delta} = \star\mu_{\alpha\beta\gamma\delta}$, being $\star$ the eight-dimensional Hodge star. Note that these are very similar to the $SL(8,\mathbb{R})$ decompositions (4.21), (4.23). To go from one to the other, we use for the **56** [176]

$$\nu^{ab} = \frac{\sqrt{2}}{8}(\nu^{\alpha\beta} + \bar\nu^{\alpha\beta})\Gamma^{ab}{}_{\beta\alpha} \;, \tag{A.14}$$

$$\tilde\nu_{ab} = -\frac{\sqrt{2}}{8} i(\nu^{\alpha\beta} - \bar\nu^{\alpha\beta})\Gamma^{ab}{}_{\beta\alpha} \;. \tag{A.15}$$

In the main text we use a complex **28** object, defined from its real pieces $\lambda^{ab}, \tilde\lambda_{ab}$ in the obvious way

$$L^{ab} = \lambda^{ab} + i\tilde\lambda^{ab} = \frac{\sqrt{2}}{4} L^{\alpha\beta}\Gamma^{ab}{}_{\beta\alpha} \tag{A.16}$$

From the **63** adjoint representation of $SU(8)$ (*i.e.* taking $\mu_{\alpha\beta\gamma\delta} = 0$) one recovers the following $SL(8,\mathbb{R})$ components

$$\begin{aligned}\mu_{ab} &= -\frac{1}{4}\mu^{\alpha}{}_{\beta}\Gamma_{ab}{}^{\beta}{}_{\alpha} \\ \mu_{abcd} &= \frac{i}{8}\mu^{\alpha}{}_{\beta}\Gamma_{abcd}{}^{\beta}{}_{\alpha}\end{aligned} \tag{A.17}$$

where $\mu_{ba} = -\mu_{ab}$ and $\star\mu_{abcd} = -\mu_{abcd}$ (the symmetric and self-dual pieces are obtained from the **70** representation $\mu^{\alpha\beta\gamma\delta}$) and $\mu_{ab} = g_{ac}\mu^{c}{}_{b}$ (at this point there is a metric since $SL(8) \cap SU(8) = SO(8)$).
The duality relation for a Cliff$(8,0)$ gamma matrix read

$$\Gamma^{a_1\dots a_p} = -\frac{1}{(8-p)!}\epsilon^{a_1\dots a_p a_{p+1}\dots a_8}\Gamma_{a_{p+1}\dots a_8} \tag{A.18}$$

When breaking $SU(8) \to SU(4) \times SU(2)$, the spinor index decomposes in a pair of indices $\alpha = \hat{\alpha}I$, where $\hat{\alpha}$ is an $SU(4)$ spinor index. For the Cliff$(8,0)$ gamma matrices, we have used the following basis in terms of Cliff$(6,0)$ and Pauli sigma-matrices

$$\begin{aligned}\Gamma^{m\alpha}{}_{\beta} &= \gamma^m \otimes \sigma_3 \\ \Gamma^{1\alpha}{}_{\beta} &= \mathbb{1}_6 \otimes \sigma_1 \\ \Gamma^{2\alpha}{}_{\beta} &= \mathbb{1}_6 \otimes \sigma_2\ .\end{aligned} \tag{A.19}$$

The intertwiners $A, C, D$ also split into Cliff(6)$\otimes$Cliff(2) intertwiners. In particular, $C$ splits as

$$C = \hat{C} \otimes c \tag{A.20}$$

where $\hat{C}$ is the intertwiner

$$\gamma^{mT} = -\hat{C}^{-1}\gamma^m\hat{C}\ . \tag{A.21}$$

We get that

$$C_{\alpha\beta} = \hat{C} \otimes \sigma_1 \tag{A.22}$$

We will use a basis for the Cliff$(6,0)$ gamma matrices in which $\hat{A} = \hat{C} = \hat{D} = \mathbb{I}$, and therefore the $SU(4)$ conjugate spinors are just

$$\bar{\eta} = \eta^{\dagger}\ , \quad \eta^t = \eta^T\ , \quad \eta^c = \eta^* \tag{A.23}$$

and $\eta_- \equiv \eta_+^*$. In this basis, the $SU(8)$ spinors in (4.34) have conjugates

$$\theta^{1t} = (0, \eta_+^{1T}) \tag{A.24}$$

$$\bar{\theta}_1 = \theta^{1\dagger} = (\eta_-^{1\dagger}, 0)\ . \tag{A.25}$$

## A.3 Five dimensional Hodge dualization

In the following we set up some conventions for embedding the five-dimensional Hodge star in ten dimensions, relevant for the calculations of Chapter 5. Considering a Lorentzian signature manifold $M_d$, the Hodge star squares as in (A.9). So in $d = 5$

$$*_5 *_5 \omega_r = -\omega_r \tag{A.26}$$

while in $d = 10$

$$*_{10} *_{10} \omega_r = (-1)^{1+r}\omega_r. \tag{A.27}$$

We thus have

$$\begin{aligned}
*_{10}\,\omega_p &= (-1)^p e^{(2-2p)(u_3-u_1)}(*_5\omega_p) \wedge J_1' \wedge J_2' \wedge (g_5 + A_1)\\
*_{10}\Big(\omega_p \wedge (g_5 + A_1)\Big) &= e^{2pu_1+(8-2p)u_3}(*_5\omega_p) \wedge J_1' \wedge J_2'\\
*_{10}\Big(\omega_p \wedge J_1'\Big) &= (-1)^p e^{(-6+2p)u_1-4u_2+(2-2p)u_3}(*_5\omega_p) \wedge J_2' \wedge (g_5 + A_1)\\
*_{10}\Big(\omega_p \wedge J_2'\Big) &= (-1)^p e^{(-6+2p)u_1+4u_2+(2-2p)u_3}(*_5\omega_p) \wedge J_1' \wedge (g_5 + A_1)\\
*_{10}\Big(\omega_p \wedge J_1' \wedge (g_5 + A_1)\Big) &= e^{(-4+2p)u_1-4u_2+(8-2p)u_3}(*_5\omega_p) \wedge J_2'\\
*_{10}\Big(\omega_p \wedge J_2' \wedge (g_5 + A_1)\Big) &= e^{(-4+2p)u_1+4u_2+(8-2p)u_3}(*_5\omega_p) \wedge J_1'\\
*_{10}\Big(\omega_p \wedge \Omega'\Big) &= (-1)^p e^{(-6+2p)u_1+(2-2p)u_3}(*_5\omega_p) \wedge \Omega' \wedge (g_5 + A_1)\\
*_{10}\Big(\omega_p \wedge \Omega' \wedge (g_5 + A_1)\Big) &= e^{(-4+2p)u_1+(8-2p)u_3}(*_5\omega_p) \wedge \Omega'\\
*_{10}\Big(\omega_p \wedge J_1' \wedge J_2'\Big) &= (-1)^p e^{(-10+2p)u_1+(2-2p)u_3}(*_5\omega_p) \wedge (g_5 + A_1)
\end{aligned} \tag{A.28}$$

# Appendix B

# EGG details

We collect in this Appendix elements of the $E_{7(7)}$ theory, as the representations that are used in the main text as well as the relevant tensor products, in the group decompositions $SL(2,\mathbb{R})\times O(6,6)$ and $SL(8,\mathbb{R})$. Then the complete calculations necessary for the analysis presented in Chapter 4 are listed for each of the EGG equations both for the $\mathcal{N}=1$ and for the $\mathcal{N}=2$ case.

## B.1 $E_{7(7)}$ representations and tensor products

### B.1.1 $SL(2,\mathbb{R})\times O(6,6)$

The fundamental **56** representation decomposes as

$$\nu = (\nu^{iA}, \nu^{+})$$
$$\mathbf{56} = (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32})$$

For the adjoint **133** of $E_{7(7)}$ we have

$$\mu = (\mu^{i}{}_{j}, \mu^{A}{}_{B}, \mu^{i-})$$
$$\mathbf{133} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32'})$$

where $\mu^{i}{}_{i}=0$ and $\mu^{AB}=\mu^{A}{}_{C}\,\eta^{CB}$ is antisymmetric. The **912** decomposes as

$$\phi = (\phi^{iA}, \phi^{i}{}_{j}{}^{+}, \phi^{A-}, \phi^{iABC})$$
$$\mathbf{912} = (\mathbf{2},\mathbf{12}) + (\mathbf{3},\mathbf{32}) + (\mathbf{1},\mathbf{352}) + (\mathbf{2},\mathbf{220})$$

where $\Gamma_A\Phi^{A-}=0$ and $\phi^{iABC}$ is fully antisymmetric in $ABC$. There are various tensor products projected on some particular representation that are used throughout in Chapter 4 These are:

$\mathbf{56}\times\mathbf{56}\big|_{\mathbf{1}}$ (*i.e.* the symplectic invariant)

$$\mathcal{S}(\nu,\hat{\nu}) = \epsilon_{ij}\eta_{AB}\nu^{iA}\hat{\nu}^{jB} + \langle\nu^{+},\hat{\nu}^{+}\rangle \tag{B.1}$$

$\mathbf{56}\times\mathbf{56}\big|_{\mathbf{133}}$

$$
\begin{aligned}
(\nu\cdot\hat\nu)^i{}_j &= 2\epsilon_{jk}\eta_{AB}\nu^{iA}\hat\nu^{kB}\\
(\nu\cdot\hat\nu)^A{}_B &= 2\epsilon_{ij}(\nu^{iA}\hat\nu^j{}_B+\hat\nu^{iA}\nu^j{}_B)+\langle\nu^+,\Gamma^A{}_B\hat\nu^+\rangle\\
(\nu\cdot\hat\nu)^{i-} &= \nu^{iA}\Gamma_A\hat\nu^+ + \hat\nu^{iA}\Gamma_A\nu^+;
\end{aligned}
\tag{B.2}
$$

$\mathbf{56}\times\mathbf{133}\big|_{\mathbf{56}}$

$$
\begin{aligned}
(\nu\cdot\mu)^{iA} &= \mu^i{}_j\nu^{jA}+\mu^A{}_B\nu^{iB}+\langle\mu^{i-},\Gamma^A\nu^+\rangle\\
(\nu\cdot\mu)^{+} &= \frac{1}{4}\mu_{AB}\Gamma^{AB}\nu^+ + \epsilon_{ij}\nu^{iA}\Gamma_A\mu^{j-}\ ;
\end{aligned}
\tag{B.3}
$$

the adjoint action on the adjoint, *i.e.* $\mathbf{133}\times\mathbf{133}\big|_{\mathbf{133}}$ ;

$$
\begin{aligned}
(\mu\cdot\hat\mu)^i{}_j &= \hat\mu^i{}_k\mu^k{}_j-\mu^i{}_k\hat\mu^k{}_j+\epsilon_{jk}(\langle\hat\mu^{i-},\mu^{k-}\rangle-\langle\mu^{i-},\hat\mu^{k-}\rangle)\\
(\mu\cdot\hat\mu)^A{}_B &= \hat\mu^A{}_C\mu^C{}_B-\mu^A{}_C\hat\mu^C{}_B+\epsilon_{ij}\langle\hat\mu^{i-},\Gamma^A{}_B\mu^{j-}\rangle\\
(\mu\cdot\hat\mu)^{i-} &= \hat\mu^i{}_j\mu^{j-}-\mu^i{}_j\hat\mu^{j-}+\frac{1}{4}\hat\mu_{AB}\Gamma^{AB}\mu^{i-}-\frac{1}{4}\mu_{AB}\Gamma^{AB}\hat\mu^{i-}
\end{aligned}
\tag{B.4}
$$

and $\mathbf{56}\times\mathbf{133}\big|_{\mathbf{912}}$

$$
\begin{aligned}
(\nu\cdot\mu)^{iA} &= \mu^i{}_j\nu^{jA}+\mu^A{}_B\nu^{iB}+\langle\nu^+,\Gamma^A\mu^{i-}\rangle\\
(\nu\cdot\mu)^i{}_j{}^+ &= \mu^i{}_j\nu^+-\epsilon_{jk}\nu^{(i|A}\Gamma_A\mu^{k)-}\\
(\nu\cdot\mu)^{A-} &= -\mu^A{}_B\Gamma^B\nu^+ + \frac{1}{10}\mu_{BC}\Gamma^{ABC}\nu^+ + \epsilon_{ij}\nu^{iA}\mu^{j-}-\frac{1}{11}\epsilon_{ij}\nu^{iB}\Gamma_B{}^A\mu^{j-}\\
(\nu\cdot\mu)^{iABC} &= 3\nu^{i[A}\mu^{BC]}+\langle\nu^+,\Gamma^{ABC}\mu^{i-}\rangle\ .
\end{aligned}
\tag{B.5}
$$

### B.1.2 $SL(8,\mathbb{R})$

For the **912** we have

$$
\begin{aligned}
\phi &= (\phi^{ab},\phi^{abc}{}_d,\tilde\phi_{ab},\tilde\phi_{abc}{}^d)\\
\mathbf{912} &= \mathbf{36}+\mathbf{420}+\mathbf{36'}+\mathbf{420'}
\end{aligned}
\tag{B.6}
$$

with $\phi^{ab}=\phi^{ba}$, $\phi^{abc}{}_d=\phi^{[abc]}{}_d$ and $\phi^{abc}{}_c=0$ and similarly for the tided objects.
The $SL(8,\mathbb{R})$ decomposition of the tensor products is the following.
The adjoint action on the fundamental, $\mathbf{56}\times\mathbf{133}\big|_{\mathbf{56}}$ is[1].

$$
\begin{aligned}
(\nu\cdot\mu)^{ab} &= \mu^a{}_c\nu^{cb}+\mu^b{}_c\nu^{ac}+\star\mu^{abcd}\tilde\nu_{cd}\\
(\nu\cdot\mu)_{ab} &= -\mu^c{}_a\tilde\nu_{cb}-\mu^c{}_b\tilde\nu_{ac}-\mu_{abcd}\nu^{cd}
\end{aligned}
\tag{B.7}
$$

[1]Note that this convention differs by a sign in the $\star\mu$ term than the one used in [52,176]. This choice is correlated with the choice in (B.36), and affects a few signs in the equations that follow (those in the terms involving $\star\mu$).

where $\star\mu$ is the 8-dimensional Hodge dual. The symplectic invariant $\mathbf{56}\times\mathbf{56}\big|_{\mathbf{1}}$ reads

$$\mathcal{S}(\nu,\hat{\nu}) = \nu^{ab}\tilde{\hat{\nu}}_{ab} - \tilde{\nu}_{ab}\hat{\nu}^{ab} \tag{B.8}$$

The $\mathbf{56}\times\mathbf{56}\big|_{\mathbf{133}}$ reads

$$\begin{aligned}(\nu\cdot\hat{\nu})^a{}_b &= (\nu^{ca}\tilde{\hat{\nu}}_{cb} - \frac{1}{8}\delta^a{}_b\nu^{cd}\tilde{\hat{\nu}}_{cd}) + (\hat{\nu}^{ca}\tilde{\nu}_{cb} - \frac{1}{8}\delta^a{}_b\hat{\nu}^{cd}\tilde{\nu}_{cd}) \\ (\nu\cdot\hat{\nu})_{abcd} &= -3(\tilde{\nu}_{[ab}\tilde{\hat{\nu}}_{cd]} + \frac{1}{4!}\epsilon_{abcdefgh}\nu^{ef}\hat{\nu}^{gh})\end{aligned} \tag{B.9}$$

The adjoint action on the adjoint $\mathbf{133}\times\mathbf{133}\big|_{\mathbf{133}}$ gives

$$\begin{aligned}(\mu\cdot\hat{\mu})^a{}_b &= (\mu^a{}_c\hat{\mu}^c{}_b - \hat{\mu}^a{}_c\mu^c{}_b) - \frac{1}{3}(\star\mu^{acde}\hat{\mu}_{bcde} - \star\hat{\mu}^{acde}\mu_{bcde}) \\ (\mu\cdot\hat{\mu})_{abcd} &= 4(\mu^e{}_{[a}\hat{\mu}_{bcd]e} - \hat{\mu}^e{}_{[a}\mu_{bcd]e})\end{aligned} \tag{B.10}$$

The $\mathbf{56}\times\mathbf{133}\big|_{\mathbf{912}}$ is

$$\begin{aligned}(\nu\cdot\mu)^{ab} &= (\nu^{ac}\mu^b{}_c + \nu^{bc}\mu^a{}_c) \\ (\nu\cdot\mu)_{ab} &= -(\tilde{\nu}_{ac}\mu^c{}_b + \tilde{\nu}_{bc}\mu^c{}_a) \\ (\nu\cdot\mu)^{abc}{}_d &= -3(\nu^{[ab}\mu^{c]}{}_b - \frac{1}{3}\nu^{e[a}\mu^b{}_e\delta^{c]}{}_d) + 2(\tilde{\nu}_{ed}\star\mu^{abce} + \frac{1}{2}\tilde{\nu}_{ef}\star\mu^{ef[ab}\delta^{c]}{}_d) \\ (\nu\cdot\mu)_{abc}{}^d &= -3(\tilde{\nu}_{[ab}\mu^d{}_{c]} - \frac{1}{3}\tilde{\nu}_{e[a}\mu^e{}_b\delta^d{}_{c]}) + 2(\nu^{ed}\mu_{abce} + \frac{1}{2}\nu^{ef}\mu_{ef[ab}\delta^d{}_{c]})\end{aligned} \tag{B.11}$$

The $\mathbf{912}\times\mathbf{56}\big|_{\mathbf{133}}$ gives

$$\begin{aligned}(\phi\cdot\nu)^a{}_b &= (\nu^{ca}\tilde{\phi}_{cb} + \tilde{\nu}_{cb}\phi^{ca}) + (\tilde{\nu}_{cd}\phi^{cda}{}_b - \nu^{cd}\tilde{\phi}_{cdb}{}^a) \\ (\phi\cdot\nu)_{abcd} &= -4(\tilde{\phi}_{[abc}{}^e\tilde{\nu}_{d]e} - \frac{1}{4!}\epsilon_{abcdm_1m_2m_3m_4}\phi^{m_1m_2m_3}{}_e\nu^{m_4e})\end{aligned} \tag{B.12}$$

and finally $\mathbf{912}\times\mathbf{133}\big|_{\mathbf{56}}$ is

$$\begin{aligned}(\phi\cdot\mu)^{ab} &= -(\phi^{ac}\mu^b{}_c - \phi^{bc}\mu^a{}_c) - 2\phi^{abc}{}_d\mu^d{}_c \\ &\quad + \frac{2}{3}(\tilde{\phi}_{m_1m_2m_3}{}^a\star\mu^{m_1m_2m_3b} - \tilde{\phi}_{m_1m_2m_3}{}^b\star\mu^{m_1m_2m_3a})\end{aligned} \tag{B.13}$$

$$\begin{aligned}(\phi\cdot\mu)_{ab} &= (\tilde{\phi}_{ac}\mu^c{}_b - \tilde{\phi}_{bc}\mu^c{}_a) - 2\tilde{\phi}_{abc}{}^d\mu^c{}_d \\ &\quad - \frac{2}{3}(\phi^{m_1m_2m_3}{}_b\,\mu_{m_1m_2m_3a} - \phi^{m_1m_2m_3}{}_a\,\mu_{m_1m_2m_3b})\end{aligned} \tag{B.14}$$

## B.2 $GL(6,\mathbb{R})$ embedding in $SL(8,\mathbb{R})$

The $GL(6,\mathbb{R})$ weights of the different $O(6,6)\times SL(2,\mathbb{R})$ representations is worked out in [97]. It turns out that the two components of an $SL(2,\mathbb{R})$ doublet have different

$GL(6,\mathbb{R})$ weights. To find the $GL(6,\mathbb{R})$ weight in the $SL(8,\mathbb{R})$ decomposition, we use that $SL(8,\mathbb{R}) \supset SL(2,\mathbb{R}) \times GL(6,\mathbb{R}) \subset O(6,6) \times SL(2,\mathbb{R})$, where the common $GL(6,\mathbb{R})$ piece corresponds to the diffeomorphisms. Decomposing $a = (m,i)$ with $m = 1,..,6$ a $GL(6)$ index and $i = 1,2$ an $SL(2)$ index, the embedding of $SL(2,\mathbb{R}) \times GL(6,\mathbb{R}) \subset SL(8,\mathbb{R})$ is the following

$$\begin{aligned} M^a{}_b &= \begin{pmatrix} (\det a)^{-1/4} a^m{}_n & 0 \\ 0 & (\det a)^{1/4} \begin{pmatrix} (\det a)^{-1/2} e^{\phi} & 0 \\ 0 & (\det a)^{1/2} e^{-\phi} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} (\det a)^{-1/4} a^m{}_n & 0 & 0 \\ 0 & (\det a)^{-1/4} e^{\phi} & 0 \\ 0 & 0 & (\det a)^{3/4} e^{-\phi} \end{pmatrix} \end{aligned} \tag{B.15}$$

where $M \in SL(8,\mathbb{R}), a \in GL(6,\mathbb{R})$, and we have added explicit factors of the dilaton that are needed in order to get the right transformation properties of the connection. Since a six-form transforms by a factor $(\det g)^{1/2}$ (or equivalently $1/\det a$), we can write the 8-dimensional metric as

$$\hat{g}_{ab} = \begin{pmatrix} (\det g)^{-1/4} g_{mn} & 0 & 0 \\ 0 & (\det g)^{-1/4} e^{-2\phi} & 0 \\ 0 & 0 & (\det g)^{3/4} e^{2\phi} \end{pmatrix} \tag{B.16}$$

The different $SL(8,\mathbb{R})$ components of **56** representation $\nu = (\nu^{ab}, \tilde{\nu}_{ab})$ transform therefore according to

$$\begin{aligned} &\tilde{\nu}_{mn} \in \left(\Lambda^6 T^*M\right)^{-1/2} \otimes \Lambda^2 T^*M \ , && \nu^{mn} \in \left(\Lambda^6 T^*M\right)^{-1/2} \otimes \Lambda^4 T^*M \\ &\tilde{\nu}_{1m} \in \mathcal{L} \otimes \left(\Lambda^6 T^*M\right)^{-1/2} \otimes T^*M \ , && \nu^{1m} \in \mathcal{L}^{-1} \otimes \left(\Lambda^6 T^*M\right)^{-1/2} \otimes \Lambda^5 T^*M \\ &\tilde{\nu}_{2m} \in \mathcal{L}^{-1} \otimes \left(\Lambda^6 T^*M\right)^{-1/2} \otimes (T^*M \otimes \Lambda^6 T^*M) \ , && \nu^{2m} \in \mathcal{L} \otimes \left(\Lambda^6 T^*M\right)^{-1/2} \otimes TM \\ &\tilde{\nu}_{12} \in \left(\Lambda^6 T^*M\right)^{-1/2} \otimes \Lambda^6 T^*M \ , && \nu^{12} \in \left(\Lambda^6 T^*M\right)^{-1/2} \end{aligned} \tag{B.17}$$

where we have introduced a trivial real line bundle $\mathcal{L}$ with sections $e^{-\phi} \in \mathcal{L}$ to account for factors of the dilaton. The adjoint $\mu = (\mu^a{}_b, \mu_{abcd})$ has the following $GL(6,\mathbb{R})$ and dilaton assignments

$$\begin{gathered} \mu^1{}_1 = -\mu^2{}_2 \in \mathbb{R} \ , \quad \mu^1{}_2 \in \mathcal{L}^{-2} \otimes \Lambda^6 T^*M \ , \quad \mu^2{}_1 \in \mathcal{L}^2 \otimes \Lambda^6 TM \ , \quad \mu^m{}_n \in TM \otimes T^*M \\ \mu^1{}_m \in \mathcal{L}^{-1} \otimes T^*M, \quad \mu^2{}_m \in \mathcal{L} \otimes \Lambda^5 TM \ , \quad \mu^m{}_1 \in \mathcal{L} \otimes TM \ , \quad \mu^m{}_2 \in \mathcal{L}^{-1} \otimes \Lambda^5 T^*M \ , \\ \mu_{mnpq} \in \Lambda^2 TM \ , \quad \mu_{mnp1} = \mathcal{L} \otimes \Lambda^3 TM \ , \quad \mu_{mnp2} = \mathcal{L}^{-1} \otimes \Lambda^3 T^*M \ , \quad \mu_{mn12} \in \Lambda^2 T^*M \end{gathered} \tag{B.18}$$

Finally, the **912** multiplied by $\mathcal{L}\otimes(\Lambda^6 T^*M)^{-1/2}$ (a T-duality invariant factor), transforms as

$$
\begin{aligned}
\phi^{11} &\in \mathcal{L}^{-1}\otimes\mathbb{R}\ , & \phi'_{11} &\in \mathcal{L}^{3}\otimes\Lambda^6 TM \\
\phi^{12} &\in \Lambda^6 TM\ , & \phi'_{12} &\in \mathbb{R} \\
\phi^{22} &\in \mathcal{L}^{3}\otimes(\Lambda^6 TM)^2\ , & \phi'_{22} &\in \mathcal{L}^{-1}\otimes\Lambda^6 T^*M \\
\phi^{mnp}{}_q &\in \Lambda^3 TM\otimes T^*M\ , & \phi'_{mnp}{}^q &\in \Lambda^3 TM\otimes TM \\
\phi^{mnp}{}_1 &\in \mathcal{L}^{2}\otimes\Lambda^3 TM\ , & \phi'_{mnp}{}^1 &\in \Lambda^3 TM \\
\phi^{mnp}{}_2 &\in \Lambda^3 T^*M\ , & \phi'_{mnp}{}^2 &\in \mathcal{L}^{2}\otimes\Lambda^3 TM\otimes\Lambda^6 TM \\
\phi^{mn1}{}_2 &\in \mathcal{L}^{-1}\otimes\Lambda^4 T^*M\ , & \phi'_{mn1}{}^2 &\in \mathcal{L}^{3}\otimes\Lambda^4 TM\otimes\Lambda^6 TM \\
\phi^{mn2}{}_1 &\in \mathcal{L}^{3}\otimes\Lambda^2 TM\otimes\Lambda^6 TM\ , & \phi'_{mn2}{}^1 &\in \mathcal{L}^{-1}\otimes\Lambda^2 T^*M
\end{aligned}
\tag{B.19}
$$

## B.3 EGG twisted equations

We present in this section the twisted equation for an object $L \in \mathbf{56}$ as well for $K \in \mathbf{133}$ using the prescriptions of (4.66). All of our results will be presented in the $SL(8,\mathbb{R})$ decomposition of the derivative and of the fluxes, given respectively in (4.60) and (4.64), and the corresponding $SL(8,\mathbb{R})$ components of the tensor products given in Appendix B.1.2. Using this decomposition the distinction between spinor and vector representations becomes less straightforward, as it is for the picture of the algebraic structures being thought as a generalization of the pure spinors (which we previously see as an embedding in $E_{7(7)}$ representations for instance in (4.32)-(4.33)). In the following equations $*$ refers to a six-dimensional Hodge operator. By using equations (B.9) and (B.12), we get the following expressions for the twisted derivative of $\lambda = (\lambda^{ab}, \tilde{\lambda}_{ab})$, projected onto the $\mathbf{133}$

$$(\mathcal{D}\lambda)^1{}_1 = -\frac{1}{4}\nabla_p \lambda^{p2} \tag{B.20}$$

$$(\mathcal{D}\lambda)^2{}_2 = \frac{3}{4}\nabla_m \lambda^{m2} \tag{B.21}$$

$$(\mathcal{D}\lambda)^1{}_2 = -\nabla_m \lambda^{1m} - e^{\phi}(*F_6)\lambda^{12} + e^{\phi}F_0\tilde{\lambda}_{12} + \frac{e^{\phi}}{2}F_{mn}\lambda^{mn} - \frac{e^{\phi}}{2}(*F_4)^{np}\tilde{\lambda}_{np} \tag{B.22}$$

$$(\mathcal{D}\lambda)^m{}_2 = -\nabla_p \lambda^{mp} - \frac{1}{2}(*H)^{mnp}\tilde{\lambda}_{np} - e^{\phi}(*F_6)\lambda^{m2} - e^{\phi}(*F_4)^{mn}\tilde{\lambda}_{n1} \tag{B.23}$$

$$(\mathcal{D}\lambda)^1{}_m = \nabla_m \lambda^{12} + e^{\phi}F_0\tilde{\lambda}_{1m} + e^{\phi}F_{mn}\lambda^{n2} \tag{B.24}$$

$$(\mathcal{D}\lambda)^n{}_m = \nabla_m \lambda^{n2} - \frac{1}{4}g^n{}_m \nabla_p \lambda^{p2} \tag{B.25}$$

$$(\mathcal{D}\lambda)_{mnp2} = -\frac{3}{2}\nabla_{[m}\tilde{\lambda}_{np]} + \frac{1}{2}H_{mnp}\lambda^{12} - \frac{3}{2}e^{\phi}F_{[mn|}\tilde{\lambda}_{|p]1} - \frac{e^{\phi}}{2}F_{mnpq}\lambda^{2q} \tag{B.26}$$

$$(\mathcal{D}\lambda)_{mn12} = -\nabla_{[m}\tilde{\lambda}_{n]1} + \frac{1}{2}H_{mnp}\lambda^{p2}\ . \tag{B.27}$$

On the other hand to get the twisted derivative of (a generic) $K$ projected on the $\mathbf{56}$, we make use of the tensor products (B.7) and (B.13) to explicitly recover

$$(\mathcal{D}K)^{mn} = -2\nabla_p K^{mnp2} + (*H)^{mnp}K^2{}_p + e^{\phi}(*F_4)^{mn}K^2{}_1 \tag{B.28}$$

$$\widetilde{(\mathcal{D}K)}_{mn} = -2\nabla_{[m}K^2{}_{n]} + e^{\phi}F_{mn}K^2{}_1 \tag{B.29}$$

$$(\mathcal{D}K)^{m1} = 2\nabla_p K^{mp12} + e^{\phi}F_0 K^m{}_1 - e^{\phi}(*F_4)^{mn}K^2{}_n - e^{\phi}F_{np}K^{2npm} \tag{B.30}$$

$$\widetilde{(\mathcal{D}K)}_{m1} = -\nabla_m K^2{}_1 \tag{B.31}$$

$$(\mathcal{D}K)^{m2} = 0 \tag{B.32}$$

$$\begin{aligned}\widetilde{(\mathcal{D}K)}_{m2} = &-\nabla_p K^p{}_m - H_{mpq}K^{pq12} - e^{\phi}(*F_6)K^2{}_m - e^{\phi}F_{mp}K^p{}_1 \\ &+ e^{\phi}(*F_4)^{pq}K_{1pqm}\end{aligned} \tag{B.33}$$

$$(\mathcal{D}K)^{12} = -e^{\phi}F_0 K^2{}_1 \tag{B.34}$$

$$\widetilde{(\mathcal{D}K)}_{12} = -\nabla_n K^n{}_1 - \frac{1}{3}H_{npq}K^{2npq} - e^{\phi}(*F_6)K^2{}_1 \tag{B.35}$$

where we have used that

$$\star K^{abcd} = -K^{abcd} \tag{B.36}$$

which is a consequence of fact that $K$ is purely in the **63** of $SU(8)$.

# B.4 $\mathcal{N}=1$ supersymmetry

## B.4.1 Supersymmetric variations

We can specialize the supersymmetry transformations of the fermionic fields in the democratic formulation presented in Chapter 1 for type IIA by taking (A.19), with the following explicit choice for the $\mathcal{P}$ matrices is

$$\begin{aligned} \mathcal{P} = i\Gamma^{12}\,, \quad \mathcal{P}_0 = \mathcal{P}_4 = \Gamma^1\,, \quad \mathcal{P}_2 = \mathcal{P}_6 = -i\Gamma^2, \\ \gamma^m\mathcal{P}_0 = -i\Gamma^{2m}\,, \gamma^m\mathcal{P}_2 = \Gamma^{1m}. \end{aligned} \tag{B.37}$$

By using the normed restricted ansatz (4.67), imposing the gravitino variations (2.29) to vanish, and looking to its internal components we get that $\mathcal{N}=1$ supersymmetry requires

$$\nabla_m\theta^1 + \frac{i}{8}H_{mnp}\Gamma^{np12}\theta^1 - \frac{e^\phi}{8}\not{F}_{\rm i}\Gamma_m\theta^2 = 0\ , \tag{B.38}$$

and the same exchaging $1 \leftrightarrow 2$, where we have defined

$$\not{F}_i = -i\not{F}_h\Gamma^2 + \not{F}_a\Gamma^1 \tag{B.39}$$

in terms of the “hermitean" and “antihermitean" pieces of $F$, namely

$$F_h = \frac{1}{2}(F + s(F)) = F_0 + F_4\ , \quad F_a = \frac{1}{2}(F - s(F)) = F_2 + F_6 \tag{B.40}$$

and finally we use the slash convention

$$\not{F}_{(n)} = \frac{1}{n!}F_{i_1\ldots i_n}\Gamma^{i_1\ldots i_n}\ . \tag{B.41}$$

We can deduce a similar equations involving $\bar\theta$, which reads

$$\nabla_m\bar\theta^1 - \frac{i}{8}H_{mnp}\bar\theta^1\Gamma^{np12} + \frac{e^\phi}{8}\bar\theta^2\Gamma_m\not{F}_{\rm i} = 0\ , \tag{B.42}$$

as well valid when exchaging $1 \leftrightarrow 2$.
Looking instead at the external components of (2.29), we get that $\mathcal{N}=1$ vacua should satisfy

$$\delta\psi_\mu = 0 \ \Leftrightarrow\ i\not{\partial}_{\rm e}A\,\theta^1 + \frac{e^\phi}{4}\not{F}_{\rm e}\theta^2 = 0, \tag{B.43}$$

and similarly exchanging 1 and 2, where

$$\not{F}_{\rm e} = \not{F}_h\Gamma^1 - i\not{F}_a\Gamma^2 \tag{B.44}$$

and

$$\partial\!\!\!/_{\rm e} A = \partial_m A\,\Gamma^{m12} \ . \tag{B.45}$$

The hermitean conjugate equation reads

$$i\,\bar{\theta}^1 \partial\!\!\!/_{\rm e} A + \frac{e^{\phi}}{4}\bar{\theta}^2 F\!\!\!/_{\rm e} = 0, \tag{B.46}$$

Demanding dilatino variation (2.30) to be equal to zero, we get

$$i\partial\!\!\!/_{\rm e}\phi\,\theta^1 + \frac{1}{12}H_{mnp}\Gamma^{mnp}\theta^1 + \frac{e^{\phi}}{4}F\!\!\!/_{\rm d}\theta^2 = 0 \tag{B.47}$$

where we have defined

$$F\!\!\!/_{\rm d} = (5-n)F\!\!\!/_{\rm e} \ . \tag{B.48}$$

The hermitean conjugate equation reads

$$i\bar{\theta}^1\partial\!\!\!/_{\rm e}\phi - \frac{1}{12}H_{mnp}\bar{\theta}^1\Gamma^{mnp} + \frac{e^{\phi}}{4}\bar{\theta}^2 F\!\!\!/_{\rm d} = 0 \tag{B.49}$$

### B.4.2 $\mathcal{N}=1$ supersymmetry and $\mathcal{D}L$

Multiplying Eq.(B.38) (coming from the internal gravitino variation) for the covariant derivative of $\theta^1$ ($\theta^2$), on the right by $e^{2A-\phi}\theta^2$ ($e^{2A-\phi}\theta^1$), and substracting the two equations, we get the following equation for the covariant derivative of $L'$

$$(\Delta_m L')^{\alpha\beta} \equiv \nabla_m L'^{\alpha\beta} - \partial_m(2A-\phi)L'^{\alpha\beta} + \frac{1}{4}(iH_{mnp}\Gamma^{np12}L')^{\alpha\beta} - \frac{e^{\phi}}{4}(F\!\!\!/_{\rm i}\Gamma_m\pi')^{\alpha\beta} = 0 \ . \tag{B.50}$$

where we have defined

$$\pi'^{\alpha\beta} \equiv e^{2A-\phi}(\theta^2\theta^2 - \theta^1\theta^1)^{\alpha\beta} \equiv e^{2A-\phi}\pi^{\alpha\beta} \ . \tag{B.51}$$

We will also need the $SL(8)$ object $\pi^{abcd}$, which we define to be

$$\pi'^{abcd} = \frac{\sqrt{2}}{4}\pi'^{\alpha\beta}\Gamma^{abcd}{}_{\beta\alpha} \tag{B.52}$$

On the other hand, multiplying (B.43) (coming from external gravitino variation on $\theta^1$) by $\theta^2$, and substracting to the equation with $\theta^1$ and $\theta^2$ exchanged, we get the following equation

$$(\Delta_e L)^{\alpha\beta} \equiv i\partial_m A\,(\Gamma^{m12}L)^{\alpha\beta} + \frac{e^{\phi}}{4}(F\!\!\!/_{\rm e}\pi)^{\alpha\beta} = 0 \ . \tag{B.53}$$

If instead we multiply (B.43) by $\theta^1$ and substract the corresponding equation for $\theta^2$ multiplied by $\theta^2$, we get

$$(\Delta_e \pi)^{\alpha\beta} \equiv i\partial_m A\,(\Gamma^{m12}\pi)^{\alpha\beta} + \frac{e^{\phi}}{4}(\not{F}_{\rm e} L)^{\alpha\beta} = 0 \ . \tag{B.54}$$

Doing the same on the dilatino (B.47) we get

$$(\Delta_d L)^{\alpha\beta} \equiv i\partial_m \phi\,(\Gamma^{m12}L)^{\alpha\beta} + \frac{1}{12}H_{mnp}(\Gamma^{mnp}L)^{\alpha\beta} + \frac{e^{\phi}}{4}(\not{F}_{\rm d}\pi)^{\alpha\beta} = 0 \ , \tag{B.55}$$

and a similar equations with $L$ and $\pi$ exchanged, that will not be used.
We show here how supersymmetry requires equations (4.76)-(4.78) to vanish. For each of them, we use (B.50) plus $l_e$ times (B.53) and $l_d$ times (B.55), and take in the one to last step

$$l_e = -2 \ , \quad l_d = 1 \ . \tag{B.56}$$

We show that susy requires Eq. (4.76) to vanish by

$$\begin{aligned}
0 =& \frac{\sqrt{2}}{4}\mathrm{Tr}\left(\Gamma^{12}\Delta_m L' + i\Gamma_m(l_e\Delta_e + l_d\Delta_d)L'\right) \\
=& \nabla_m L'^{12} - \partial_m(2A-\phi)L'^{12} - \partial_m(l_e A + l_d\phi)L'^{12} + \frac{i}{4}(-1+l_d)H_{mpq}L'^{pq} \\
&- \frac{e^{\phi}}{8}[F_{pq}(-1+l_e+3l_d) - i(*F_4)(1+l_e+l_d)]\pi'^{2pq}{}_m \\
=& \nabla_m L'^{12} \\
=& (\mathcal{D}L')^1{}_m \ ,
\end{aligned} \tag{B.57}$$

To get (4.77) we do

$$\begin{aligned}
0 =& \frac{\sqrt{2}}{4}\mathrm{Tr}\left(-\Gamma^{mn}\Delta_n L' + i\Gamma^{m12}(l_d\Delta_d L' + l_e\Delta_e L')\right) \\
=& -\nabla_p L'^{mp} + \partial_n(2A-\phi)L'^{mn} + \partial_n(l_e A + l_d\phi)L'^{mn} + \frac{i}{4}(3-l_d)(*H)^{mpq}L'_{pq} \\
&- \frac{e^{\phi}}{8}[F_{pq}(-1+l_e+3l_d) - i(*F_4)_{pq}(1+l_e+l_d)]\pi'^{1pq}{}_m \\
=& -\nabla_p L'^{mp} + \frac{i}{2}(*H)^{mnp}L'_{np} \\
=& (\mathcal{D}L')^m{}_2 \ ,
\end{aligned}$$

while for (4.78) we use

$$
\begin{aligned}
0 =&\frac{\sqrt{2}}{8}\mathrm{Tr}\left(3i\Gamma_{[mn|}\Delta_{p]}L' - \Gamma_{mnp12}(l_d\Delta_d L' + l_e\Delta_e L')\right)\\
=&\frac{3i}{2}\nabla_{[m}L'_{np]} - \frac{3}{2}i\partial_{[m}(2A-\phi)L'_{|np]} - \frac{3}{2}i\partial_{[m|}(l_eA + l_d\phi)L'_{|np]}\\
&+\frac{1}{4}(3-l_d)H_{mnp}L'_{12} + \frac{3}{4}(-1+l_d)(*H)_{[mn|q}L'^{q}{}_{|p]}\\
&+\frac{e^{\phi}}{8}[F_0(-3+l_e+5l_d) - i(*F_6)(3+l_e-l_d)]\pi'_{2mnp}\\
&+3\frac{e^{\phi}}{8}[iF_{[m|q}(-1+l_e+3l_d) + (*F_4)_{[m|q}(1+l_e+l_d)]\pi'^{1q}{}_{|np]}\\
=&\frac{3i}{2}\nabla_{[m}L'_{np]} + \frac{1}{2}H_{mnp}L'^{12}\\
=&(\mathcal{D}L')_{mnp2}\ .
\end{aligned}
$$

### B.4.3 $\mathcal{N}=1$ supersymmetry and $\mathcal{D}K$

We define the following quantities

$$K_0' = e^A K_0 \ , \qquad K_1' = e^A K_1 \ , \qquad K_2' = e^{3A} K_2 \ , \qquad K_3' = e^{3A} K_3 \ . \tag{B.58}$$

Combining (B.38) multiplied by $\bar{\theta}$ with (B.42) multiplied by $\theta$, we obtain

$$\Delta_m K_0 \equiv e^{-\phi}\nabla_m (e^{\phi} K_0)^{\alpha}{}_{\beta} + \frac{i}{8} H_{mnp}[\Gamma^{np12}K_0' - K_0'\Gamma^{np12}]^{\alpha}{}_{\beta} - \frac{e^{\phi}}{8}[\not{F}_i \Gamma_m K_1' - K_1'\Gamma_m \not{F}_i]^{\alpha}{}_{\beta} = 0 \tag{B.59}$$

$$\Delta_m K_1 \equiv e^{-\phi}\nabla_m (e^{\phi} K_1)^{\alpha}{}_{\beta} + \frac{i}{8} H_{mnp}[\Gamma^{np12}K_1' - K_1'\Gamma^{np12}]^{\alpha}{}_{\beta} - \frac{e^{\phi}}{8}[\not{F}_i \Gamma_m K_0' - K_0'\Gamma_m \not{F}_i]^{\alpha}{}_{\beta} = 0 \tag{B.60}$$

$$\Delta_m K_2 \equiv e^{-\phi}\nabla_m (e^{\phi} K_2)^{\alpha}{}_{\beta} + \frac{i}{8} H_{mnp}[\Gamma^{np12}K_2' - K_2'\Gamma^{np12}]^{\alpha}{}_{\beta} - i\frac{e^{\phi}}{8}[\not{F}_i \Gamma_m K_3' + K_3'\Gamma_m \not{F}_i]^{\alpha}{}_{\beta} = 0 \tag{B.61}$$

$$\Delta_m K_3 \equiv e^{-\phi}\nabla_m (e^{\phi} K_3)^{\alpha}{}_{\beta} + \frac{i}{8} H_{mnp}[\Gamma^{np12}K_3' - K_3'\Gamma^{np12}]^{\alpha}{}_{\beta} + i\frac{e^{\phi}}{8}[\not{F}_i \Gamma_m K_2' + K_2'\Gamma_m \not{F}_i]^{\alpha}{}_{\beta} = 0 \tag{B.62}$$

where the factors of the dilaton inside the covariant derivatives are there to cancel the explicit dilaton dependence of $K$ (see (4.38)).
Multiplying the external gavitino or dilatino equation, Eqs. (B.43) and (B.47) by $\bar{\theta}^2$ on the right, and adding it to the same equation with $\theta^1$ and $\theta^2$ exchanged, we get

$$(\Delta_e K_1)^{\alpha}{}_{\beta} \equiv i\partial_m A[\Gamma^{m12}K_1]^{\alpha}{}_{\beta} + \frac{e^{\phi}}{4}[\not{F}_e K_0]^{\alpha}{}_{\beta} = 0 \ , \tag{B.63}$$

$$(\Delta_d K_1)^{\alpha}{}_{\beta} \equiv i\partial_m \phi[\Gamma^{m12}K_1]^{\alpha}{}_{\beta} + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_1]^{\alpha}{}_{\beta} + \frac{e^{\phi}}{4}[\not{F}_d K_0]^{\alpha}{}_{\beta} = 0 \ . \tag{B.64}$$

We can also use the complex conjugate equations (B.46), (B.49) multiplied on the left by $\theta^2$. This gives

$$(K_1 \Delta_e)^{\alpha}{}_{\beta} \equiv i\partial_m A[K_1\Gamma^{m12}]^{\alpha}{}_{\beta} + \frac{e^{\phi}}{4}[K_0 \not{F}_e]^{\alpha}{}_{\beta} = 0 \ , \tag{B.65}$$

$$(K_1 \Delta_d)^{\alpha}{}_{\beta} \equiv i\partial_m \phi[K_1\Gamma^{m12}]^{\alpha}{}_{\beta} - \frac{1}{12}H_{mpq}[K_1\Gamma^{mpq}]^{\alpha}{}_{\beta} + \frac{e^{\phi}}{4}[K_0 \not{F}_d]^{\alpha}{}_{\beta} = 0 \tag{B.66}$$

(B.67)

We will also need the corresponding equations mixing $K_3$ and $K_2$

$$(\Delta_e K_3)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_3]^\alpha{}_\beta - i\frac{e^\phi}{4}[\not{F}_e K_2]^\alpha{}_\beta = 0 \tag{B.68}$$

$$(K_3\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_3\Gamma^{m12}]^\alpha{}_\beta + i\frac{e^\phi}{4}[K_2\not{F}_e]^\alpha{}_\beta = 0 \tag{B.69}$$

$$(\Delta_d K_3)^\alpha{}_\beta \equiv i\partial_m\phi[\Gamma^{m12}K_3]^\alpha{}_\beta + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_3]^\alpha{}_\beta - i\frac{e^\phi}{4}[\not{F}_d K_2]^\alpha{}_\beta = 0 \tag{B.70}$$

$$(K_3\Delta_d)^\alpha{}_\beta \equiv i\partial_m\phi[K_3\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K_3\Gamma^{mpq}]^\alpha{}_\beta + i\frac{e^\phi}{4}[K_2\not{F}_d]^\alpha{}_\beta = 0 \tag{B.71}$$

$$(\Delta_e K_2)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_2]^\alpha{}_\beta + i\frac{e^\phi}{4}[\not{F}_e K_3]^\alpha{}_\beta = 0 \tag{B.72}$$

$$(K_2\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_2\Gamma^{m12}]^\alpha{}_\beta - i\frac{e^\phi}{4}[K_3\not{F}_e]^\alpha{}_\beta = 0 \tag{B.73}$$

$$(\Delta_d K_2)^\alpha{}_\beta \equiv i\partial_m\phi[\Gamma^{m12}K_1]^\alpha{}_\beta + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_1]^\alpha{}_\beta + i\frac{e^\phi}{4}[\not{F}_d K_3]^\alpha{}_\beta = 0 \tag{B.74}$$

$$(K_2\Delta_d)^\alpha{}_\beta \equiv i\partial_m\phi[K_1\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K_1\Gamma^{mpq}]^\alpha{}_\beta - i\frac{e^\phi}{4}[K_3\not{F}_d]^\alpha{}_\beta = 0 \tag{B.75}$$

and the following ones involving $K_0$ and $K_1$

$$(\Delta_e K_0)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_0]^\alpha{}_\beta + \frac{e^\phi}{4}[\not{F}_e K_1]^\alpha{}_\beta = 0\ , \tag{B.76}$$

$$(K_0\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_0\Gamma^{m12}]^\alpha{}_\beta + \frac{e^\phi}{4}[K_1\not{F}_e]^\alpha{}_\beta = 0 \tag{B.77}$$

Given a generic $K$ and product of gamma matrices $\Gamma^{a_1\ldots a_i}$ we will make use of the following type of combinations

$$\mathrm{Tr}\left([\Gamma^{a_1\ldots a_i},\Delta_d]K\right) \equiv \mathrm{Tr}\left((\Gamma^{a_1\ldots a_i}\Delta_d - \Delta_d\Gamma^{a_1\ldots a_i})K\right) = \mathrm{Tr}\left(\Gamma^{a_1\ldots a_i}\Delta_d K - K\Delta_d\Gamma^{a_1\ldots a_i}\right) \tag{B.78}$$

and similarly for the anticommutator.
We then distinguish the matching with supersymmetry for $K_1'$ and $K_+'$.

### B.4.4 $\mathcal{D}K_1'$

We want to show that susy requires (4.71) and (4.73). We recall that as shown in (4.68), $K_1$ has only nonzero components with an odd number of internal indices. The idea is to reconstruct the twisted derivative of the corresponding $K'$ appearing in each of the equations by summing an equation coming from internal gravitino (which gives a covariant derivative of $K$ with no dilaton or warp factors) together with equations coming from external gravitino plus dilatino, which contribute the required derivatives of dilaton and warp factor. We start by showing that susy requires (4.79) to vanish. We use the following combination of equations: (B.60) coming from internal gravitino, (B.115) and (B.117) from external gravitino, and (B.116), (B.118) from dilatino (the last four multiplied by arbitrary coefficients $n_e$ and $n_d$, that will be set to $n_e = 1, n_d = -1$).

$$\begin{aligned}
0 =& -\frac{i}{4}\mathrm{Tr}\left(\Gamma^{mnp2}(e^A\Delta_p K_1) + \{\Gamma^{mn1}, (n_e\Delta_e + n_d\Delta_d)\}K_1'\right) \\
=& -2e^{A-\phi}\nabla_p(e^\phi K_1{}^{mnp2}) - 2\partial_p(n_e A + n_d\phi)K_1'^{\,mnp2} \\
&+ \frac{1}{2}(1+n_d)H^{mn}{}_p K_1'^{\,1p} + \frac{1}{2}(3+n_d)(*H)^{mn}{}_p K_1'^{\,2}{}_p \\
&- \frac{1}{2}e^{-2A+\phi}F_0(4+n_e+5n_d)K_+'^{\,mn12} - \frac{1}{4}e^{-2A+\phi}(*F_4)_{pq}(n_e+n_d)K_+'^{\,pqmn} \\
&- \frac{1}{2}e^{-2A+\phi}F^{[m|p}(2+n_e+3n_d)K_{+p}'{}^{|n]} \\
=& -2\nabla_p K_1'^{\,mnp2} + (*H)^{mnp}K_1'^{\,2}{}_p \\
=& (\mathcal{D}K_1')^{mn}
\end{aligned} \tag{B.79}$$

where in the third equality we have used the values $n_e = 1, n_d = -1$.
To show that (4.80) vanishes, we use

$$\begin{aligned}
0 =& -\frac{1}{4}\mathrm{Tr}\left(2\Gamma^2{}_{[m}(e^A\Delta_{n]}K_1) - i[\Gamma^{mn1}, n_e\Delta_e + n_d\Delta_d]K_1'\right) \\
=& -2e^{A-\phi}\nabla_{[m}(e^\phi K_1{}^2{}_{n]}) - 2\partial_{[m}(n_e A + n_d\phi)K_1'^{\,2}{}_{n]} \\
&- H_{pq[m}K'_1{}^{1pq}{}_{n]}(1+n_d) + \frac{1}{2}e^{-2A+\phi} * F_6\,(-2+n_e-n_d)K_{+mn}'{}^{12} \\
&+ \frac{1}{4}e^{-2A+\phi}F_{pq}(2+n_e+3n_d)K_+'^{\,pq}{}_{mn} + \frac{1}{2}e^{-2A+\phi}(*F_4)_{[m|}{}^p(n_e+n_d)K_{+p|n]}' \\
=& -2\nabla_{[m}K_1'^{\,2}{}_{n]} \\
=& (\widetilde{\mathcal{D}K_1'})_{mn}
\end{aligned} \tag{B.80}$$

where we have chosen again $n_e = 1, n_d = -1$.
To show that (4.81) vanishes, we use

$$\begin{aligned}
0 =& -\frac{i}{4}\mathrm{Tr}\left(i\Gamma^n{}_1(e^A\Delta_n K_1) + \Gamma^2(n_d\Delta_d + n_e\Delta_e)K_1'\right) \\
=& -e^{A-\phi}\nabla_n(e^\phi K_1{}^n{}_1) - \partial_p(n_e A + n_d\phi)K_1'^{\,p}{}_1 - \frac{1}{6}H_{pqr}(3+n_d)K_1'^{\,2pqr} \\
&+ \frac{1}{4}e^{-2A+\phi}\Big[F_{pq}(2+n_e+3n_d) + i(*F_4)_{pq}(n_e+n_d)\Big]K_+'^{\,pq12} \\
=& -\nabla_n K_1'^{\,n}{}_1 - \frac{1}{3}H_{pqr}K_1'^{\,2pqr} \\
=& (\widetilde{\mathcal{D}K_1'})_{12}
\end{aligned} \tag{B.81, B.82}$$

where we have used again $n_e = 1, n_d = -1$.
For the vectorial equation (4.82), we use (B.62) and (B.69) to get

$$\begin{aligned} 0 &= \mathrm{Tr}\left(-e^A \Delta_m K_3 + K_0' \Delta_e \Gamma^m\right) \\ &= -4e^A \partial_p A K_3^{mp} + e^\phi F_0 K_1'{}^m{}_1 - e^\phi (*F_4)^{mn} K_1'{}^2{}_n - e^\phi F_{np} K_1'{}^{2npm} \\ &= -4e^A \partial_p A K_3^{mp} + (\mathcal{D}K_1')^{m1} \end{aligned} \tag{B.83}$$

where we have used $K_2 = K_1 \Gamma^{12}$ and $K_0 = -iK_3\Gamma^{12}$, and in the last line we have used (4.82). For the last equation (4.83) we use (B.59) and (B.68)

$$\begin{aligned} 0 &= \mathrm{Tr}\left(e^A \Delta_m K_0 + iK_3' \Delta_e \Gamma^m\right) \\ &= 4ie^{A-\phi} \nabla_m (e^\phi K_3{}^1{}_2) - 8e^A \partial_p A K_{3m}{}^{p12} - e^\phi * F_6 K_1'{}^2{}_m - e^\phi F_{mn} K_1'{}^n{}_1 + e^\phi (*F_4)^{np} K_1'{}_{1npm} \\ &= 4ie^A \partial_m A\, K_3{}^1{}_2 - 8e^A \partial_p A K_{3m}{}^{p12} + (\widetilde{\mathcal{D}K_1'})_{m2} \end{aligned} \tag{B.84}$$

where in the second equality we have used again $K_0 = -iK_3\Gamma^{12}$, and in the third equality we have used (4.74) (which will be shown to hold in the next sub appendix).

### B.4.5 $\mathcal{D}K_+'$

The other set of equations involves

$$K_+' = K_3' + iK_2' = e^{3A}(K_3 + iK_2)\ . \tag{B.85}$$

From (4.68), we see that $K_+$ with an odd number of internal indices is proportional to $iK_2$, while for an even number of internal indices, $K_+$ is proportional to $K_3$.
To show the first equation in (4.72), we use (B.61), (B.72) and (B.74) to get

$$\begin{aligned} 0 =& \frac{1}{4} \mathrm{Tr}\left(\Gamma^{mnp2}(e^{3A}\Delta_p K_2) + i\Gamma^{mn1}(n_e \Delta_e K_2' + n_d \Delta_d K_2')\right) \\ =& -2e^{3A-\phi} \nabla_p (e^\phi K_+{}^{mnp2}) - 2\partial_p (n_e A + n_d \phi) K_+'{}^{mnp2} + 2i\partial_{[m}(n_e A + n_d \phi) K_{+n]}'^2 \\ &+ \frac{1}{2}(1+n_d) H_{mnp} K_+'^{1p} + in_d H_{pq[m} K_+'^{1pq}{}_{|n]} + \frac{1}{2}(3+n_d)(*H)^{mnp} K_{+p}'^2 \\ &+ \frac{e^\phi}{4}\left(F_0(n_e + 5n_d) - i(*F_6)(-4 + n_e - n_d)\right) K_+'^{mn} \\ &+ \frac{e^\phi}{4}\left(iF^{mn}(4 + n_e + 3n_d) - (*F_4)^{mn}(n_e + n_d)\right) K_+'^{12} \\ &+ \frac{e^\phi}{8}\left(i(*F_2)^{mn}{}_{pq}(n_e + 3n_d) - F^{mn}{}_{pq}(n_e + n_d)\right) K_+'^{pq} \\ &+ e^\phi \left(F^{[m|}{}_p (n + 3n_d) - i(*F_4)^{[m|}{}_p(-2 + n_e + n_d)\right) K_+'^{p12|n]}\ . \end{aligned} \tag{B.86}$$

and

$$0 =\frac{1}{4}\mathrm{Tr}\left(2i\Gamma_{2[n}(e^{3A}\Delta_{m]}K_2) - \Gamma_{mn1}(n_e\Delta_e K_2' + n_d\Delta_d K_2')\right) \tag{B.87}$$
$$= - 2e^{3A-\phi}\nabla_{[m}(e^{\phi}K_+{}^2{}_{n]}) - 2i\partial_p(n_eA + n_d\phi)K_+^{\prime mnp2} - 2\partial_{[m}(n_eA + n_d\phi)K_{+|n]}^{\prime 2}$$
$$+ i\frac{n_d}{2}H_{mnp}K_+^{\prime 1p} - (1 + n_d)H_{pq[m}K'^{1pq}{}_{|n]} + i\frac{n_d}{2}(*H)^{mnp}K_{+p}^{\prime 2}$$
$$+ \frac{e^{\phi}}{4}\left(iF_0(2 + n_e + 5n_d) + (*F_6)(n_e - n_d)\right)K'_{+mn}$$
$$- \frac{e^{\phi}}{4}\left(F_{mn}(n_e + 3n_d) + i(*F_4)_{mn}(2 + n_e + n_d)\right)K_+^{\prime 12}$$
$$- \frac{e^{\phi}}{8}\left((*F_2)_{mnpq}(n_e + 3n_d) + iF_{mnpq}(-2 + n_e + n_d)\right)K'_+{}^{pq}$$
$$+ e^{\phi}\left(iF_{[m|p}(n_e + 3n_d) + (*F_4)_{[m|p}(n_e + n_d)\right)K_+{}'^{p12}{}_{|n]}\ .$$

Note that in the NS sector $K_+$ reduces to $K_2$, while in the R-R sector it is proportional to $K_3$. We combine these two, choosing $n_e = \frac{3}{2}, n_d = -\frac{1}{2}$, and we get

$$0 =(\mathrm{B.86}) - i(\mathrm{B.87})$$
$$= - 2\nabla_p K'_+{}^{mnp2} + 2i\nabla_{[m}K'_+{}^2{}_{n]} + (*H)_{mnp}K'_+{}^2{}_p - e^{\phi}(*F_4 - iF_2)_{mn}K'_+{}^{12}$$
$$= (\mathcal{D}K'_+)_{mn} - i(\widetilde{\mathcal{D}K'_+})_{mn}.$$

For the 12 components we use

$$0 =\frac{1}{4}\mathrm{Tr}\left(i\Gamma^{n1}(e^{3A}\Delta_n K_2) - i\Gamma^2(n_e\Delta_e K_2' + n_d\Delta_d K_2')\right)$$
$$=ie^{3A-\phi}\nabla_n(e^{\phi}K_+^{n1}) + i\partial_n(n_eA + n_d\phi)K_+^{\prime n1} + \frac{i}{2}(1 + \frac{n_d}{3})H_{pqr}K_+^{\prime 2pqr}$$
$$+ \frac{e^{\phi}}{4}\left(-F_0(6 + n_e + 5n_d) + i * F_6(n_e - n_d)\right)K_+^{\prime 12}$$
$$- \frac{e^{\phi}}{8}\left(iF_{mn}(n_e + 3n_d) - (*F_4)_{mn}(-2 + n_e + n_d)\right)K_+^{\prime mn}$$
$$=i\nabla_n K_+^{\prime n1} + \frac{i}{3}H_{pqr}K_+^{\prime 2pqr} + e^{\phi}(-F_0 + i * F_6)K_+^{\prime 12}$$
$$=(\mathcal{D}K'_+)_{12} - i(\widetilde{\mathcal{D}K'_+})_{12}\ . \tag{B.88}$$

where we have chosen $n_e = 3, n_d = -1$.

We are left with the vectorial components. The last equation in (4.72) is trivial (see (B.32)). To show the $m1$ component, we use

$$0 = -\frac{1}{4}\mathrm{Tr}\left(\Gamma^{12}(e^{3A}\Delta_m K_3) + in_e\{\Delta_e, \Gamma_m\}K_3'\right)$$
$$= e^{3A-\phi}\nabla_m(e^{\phi}K_+^{12}) - n_e\partial_m AK_+^{\prime 12} + i\frac{e^{\phi}}{4}(n_e - 1)[-F_0K'_{+m1} + (*F_4)_{mp}K_+^{\prime 2p} + F_{pq}K_+^{\prime 2pq}{}_m]$$
$$= (\widetilde{\mathcal{D}K'_+})_{m1} - \partial_m(4A - \phi)K_+^{\prime 12}$$

where we have taken $n_e = 1$.
For the $(\widetilde{\mathcal{D}K'})_{m2}$ equation, we first note that supersymmetry requires their R-R pieces to vanish by itself, namely

$$0 = \text{Tr}\left(\Delta_m K_3'\right) = e^{\phi}\left((*F_6)(K_+')_{m2} + F_{mp}K_+'^{1p} + (*F_4)^{pq}(K_+')_{1pqm}\right) = \mathcal{F}_{RR}\big|_{m2} ,$$

while in the $m1$ equation, the R-R piece is proportional to a derivative of the warp factor, *i.e.*

$$\begin{aligned}0 =&\text{Tr}\left(e^{3A}\Delta^m K_0\right) = 4ie^{3A}\nabla_m K_+{}^1{}_2 + e^{\phi}\left(F_0 K_+'^{m1} - (*F_4)^{mp}K_+'{}^2{}_p - F_{pq}K_+'^{2pqm}\right)\\ =&4i\partial_m A K_+'{}^1{}_2 + \mathcal{F}_{RR}\big|^{m1} .\end{aligned}$$

Then we use

$$\begin{aligned}0 =&\frac{1}{4}\text{Tr}\left(i\Gamma^{mp12}(e^{3A}\Delta_p K_3) + [\Gamma^m, n_e\Delta_e + n_d\Delta_d]K_3'\right) + \mathcal{F}_{RR}\big|^{m1} + 4i\partial_m A K_+'{}^1{}_2\\ =& + 2e^{3A}\nabla_p K_+^{mp12} + 2\partial_p(n_e A + n_d\phi)K_+'^{mp12} - \frac{1}{4}(n_d+2)H^m{}_{pq}K_+'^{mp12} + i\frac{e^{\phi}}{4}\Big[(*F_6)(5 - n_e + n_d)K_+'^{m1}\\ &+ F_{mp}(3 + 3n_d + n_e)K_+'^{2p} + (*F_4)_{pq}(-1 + n_e + n_d)K_+'^{2pqm}\Big] + \mathcal{F}_{RR}\big|^{m1} + 4i\partial_m K_+'{}^1{}_2\\ =& \;(\mathcal{D}K_+')^{m1} - 2\partial_p\phi K_+'^{mp12} + 4i\partial_m A K_+'{}^1{}_2\end{aligned}$$

where in the last equality we have chosen $n_e = 3, n_d = -2$. For the $m2$ component, we use

$$\begin{aligned}0 =&\frac{1}{4}\text{Tr}\left(\Gamma^p{}_m e^{3A}\Delta_p K_3 - i[\Gamma_{m12}, n_e\Delta_e + n_d\Delta_d]K_3'\right) + \mathcal{F}_{RR}\big|_{m2}\\ =& - e^{3A}\nabla_p K_+{}^p{}_m - \partial_p(n_e A + n_d\phi)K_+'^p{}_m - \frac{1}{2}H_{mpq}K_+'^{pq12}(2 + n_d) + i\frac{e^{\phi}}{4}\Big[F_0(5 + n_e + 5n_d)K_{+m2}'\\ &+ F_{pq}(1 + n_e + 3n_d)K_+'^{1pq}{}_m + (*F_{(4)})_{mp}(-3 + n_e + n_d)K_+'^{1p}\Big] + \mathcal{F}_{RR}\big|_{m2}\\ =& \;(\widetilde{\mathcal{D}K_+'})_{m2} - \partial_p(2A - \phi)K_+'^p{}_m + H_{mpq}K_+'^{pq12}\end{aligned} \tag{B.89}$$

where here we have inserted $n_e = 5, n_d = -2$.

## B.5 $\mathcal{N} = 2$ supersymmetry

### B.5.1 Supersymmetric variations

In a totally equivalent way to what we did in Appendix B.4.1, we can specialize the supersymmetric variations for type IIA to the spinor ansatz (4.34). We thus use once more (B.37): from the internal components of the gravitino variation (2.29) we find that $\mathcal{N} = 2$ supersymmetry requires

$$\delta\psi_m = 0 \;\Leftrightarrow\; \nabla_m\theta^I = -\frac{i}{8}H_{mnp}\Gamma^{np12}\theta^I + \frac{e^{\phi}}{8}\not{F}_{\text{i}}\Gamma_m\theta^I , \tag{B.90}$$

where $\not{F}_i$ is defined in (B.39). In the same way, looking at the internal components of the same equation, we recover while from the external gravitino variation, we get

$$\delta\psi_\mu = 0 \;\Leftrightarrow\; i\not{\partial}_{\rm e} A\, \theta^I + \frac{e^\phi}{4}\not{F}_{\rm e}\theta^I = 0, \quad , \tag{B.91}$$

where we defined $\not{F}_e$ in (B.44). Finally, imposing dilatino (2.30) variation to vanish, we get

$$\delta\lambda^{(10)} = 0 \;\Leftrightarrow i\not{\partial}_{\rm e}\phi\, \theta^I + \frac{1}{12}H_{mnp}\Gamma^{mnp}\theta^I + \frac{e^\phi}{4}\not{F}_{\rm d}\theta^I = 0\ . \tag{B.92}$$

where $\not{F}_d$ is as in (B.48).

### B.5.2 $\mathcal{N} = 2$ supersymmetry and $\mathcal{D}L$

We define in this subsection

$$L' \equiv e^{-\phi}L\,. \tag{B.93}$$

and, in a total equivalent way to $\mathcal{N} = 1$ supersymmetry, we define

$$(\Delta_m L')^{\alpha\beta} = \nabla_m L'^{\alpha\beta} + \partial_m\phi L'^{\alpha\beta} + \frac{i}{4}H_{mnp}(\Gamma^{np12}L')^{\alpha\beta} - \frac{e^\phi}{4}(\not{F}_i\Gamma_m L')^{\alpha\beta} = 0\,, \tag{B.94}$$

$$(\Delta_e L')^{\alpha\beta} = i\partial_p A(\Gamma^{p12}L')^{\alpha\beta} + \frac{e^\phi}{4}(\not{F}_e L')^{\alpha\beta} = 0\,, \tag{B.95}$$

$$(\Delta_d L')^{\alpha\beta} = i\partial_p\phi(\Gamma^{p12}L')^{\alpha\beta} + \frac{1}{12}H_{pqr}(\Gamma^{pqr}L')^{\alpha\beta} + \frac{e^\phi}{4}(\not{F}_d L')^{\alpha\beta} = 0\,, \tag{B.96}$$

$$(L'\Delta_e)^{\alpha\beta} = i\partial_p A(L'\Gamma^{p12})^{\alpha\beta} - \frac{e^\phi}{4}(L'\not{F}_e)^{\alpha\beta} = 0\,, \tag{B.97}$$

$$(L'\Delta_d)^{\alpha\beta} = i\partial_p\phi(L'\Gamma^{p12})^{\alpha\beta} - \frac{1}{12}H_{pqr}(L'\Gamma^{pqr})^{\alpha\beta} - \frac{e^\phi}{4}(L'\not{F}_d)^{\alpha\beta} = 0\,, \tag{B.98}$$

$$\tag{B.99}$$

We would also need a sort of transposed of internal gravitino relation

$$(L'\Delta_m)^{\alpha\beta} = \nabla_m L' + \partial_m\phi L' - \frac{i}{4}H_{mnp}L'\Gamma^{np12} + \frac{e^\phi}{4}L'\not{F}_i\Gamma_m = 0\,. \tag{B.100}$$

We then begin to recover the relation between supersymmetry and each of the non-trivial twisted equations. Unless otherwise specified, we will take explicitly

$$l_d = 1\,. \tag{B.101}$$

in the very last step of the following equations. We start from

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left[\Gamma^{12}\Delta_m L' + i\Gamma_m l_d \Delta_d L'\right] \\
&= \nabla_m\phi L'^{12} + \partial_m\phi L'^{12} - l_d\partial_m\phi L'^{12} \\
&+ \frac{i}{4}H_{mnp}L'^{np}(-1+l_d) \\
&+ \frac{e^\phi}{4}\left[iF_0(1-5l_d) + (*F_6)(1-l_d)\right]L'^1{}_m \\
&+ \frac{e^\phi}{4}\left[F_{mp}(-1-3l_d) + i(*F_4)_{mp}(1-l_d)\right]L'^{2p} \\
&= \nabla_m L'^{12} - ie^\phi F_0 L'^1{}_m - e^\phi F_{mp}L'^{2p} \\
&= (\mathcal{D}L')^1{}_m\,. \qquad \text{(B.102)}
\end{aligned}$$

Then,

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left[-\Gamma^{mp}\Delta_p L' + i\Gamma^{m12}l_d\Delta_d L'\right] \\
&= -\nabla_p\phi L'^{mp} + (l_d-1)\partial_p\phi L'^{mp} + \\
&+ \frac{i}{4}(3-l_d)(*H)^{mpq}L'_{pq} \\
&+ \frac{e^\phi}{4}\left(iF_0(5-5l_d) - (*F_6)(5-l_d)\right)L'^m{}_2 \\
&+ \frac{e^\phi}{4}\left(F^{mp}(3-3l_d) - i(*F_4)^{mp}(3-l_e-l_d)\right)L'^1{}_p \\
&- \nabla_p L'^{mp} + \frac{i}{2}(*H)^{mpq}L'_{pq} - e^\phi(*F_6)L'^m{}_2 - e^\phi(*F_4)_{mp}L'^{1p}\,, \\
&= (\mathcal{D}L')^m{}_2\,. \qquad \text{(B.103)}
\end{aligned}$$

Look then at

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left[\frac{3i}{2}\Gamma_{[mn|}\Delta_{|p]}L' + \frac{1}{2}\Gamma_{mnp}{}^{12}l_d\Delta_d L'\right] \\
&= +\frac{3}{2}i\nabla_{[m|}L'_{|np]} + \frac{3}{2}i\partial_{[m|}(1-l_d)L'_{|np]} \\
&+ \frac{1}{4}(3-l_d)H_{mnp}L'^{12} + \frac{3}{4}(1-l_d)(*H)_{q[mn|}L'^q{}_{|p]} \\
&- \frac{3}{8}e^\phi\left(iF_{[mn|}(1+3l_d) + (*F_4)_{[mn|}(1-l_d)\right)L'^1{}_{|p]} \\
&- \frac{e^\phi}{8}\left(i(*F_2)_{mnpq}(3-3l_d) + F_{mnpq}(3+l_d)\right)L'^{2q} \\
&= \frac{3}{2}i\nabla_{[m|}L'_{|np]} + \frac{1}{2}H_{mnp}L'^{12} - \frac{3}{2}ie^\phi F_{[mn|}L'^1{}_{|p]} - \frac{e^\phi}{2}F_{mnpq}L'^{2q} \\
&= (\mathcal{D}L')_{mnp2}\,. \qquad \text{(B.104)}
\end{aligned}$$

Consider now

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left[-\Gamma^{m1}\Delta_m L' + i\Gamma^2 l_d \Delta_d L'\right] \\
&= -\nabla_m\phi L'^{m1} - \partial_m(1-l_d)L'^{1m} \\
&- \frac{e^\phi}{4}\left(iF_0(-6+5l_d) + (*F_6)(6-l_d)\right)L'^{12} \\
&- \frac{e^\phi}{8}\left(F^{mn}(2-3l_d) + i(*F_4)^{mn}(-2-l_d)\right)L'_{mn} \qquad \text{(B.105)}
\end{aligned}$$

Choosing this time $l_d = +2$ we recover

$$\begin{aligned}
0 &= -\nabla_m L'^{1m} + \partial_m\phi L'^{1m} - e^\phi(iF_0 + (*F_6))L'^{12} + e^\phi(F_{mn} + i(*F_4)_{mn})L'^{mn} \\
&= (\mathcal{D}(e^{-\phi}L'))^1{}_2 \qquad \text{(B.106)}
\end{aligned}$$

We are then left with the two equations which were trivially vanishing in the $\mathcal{N}=1$ case. For the following, we use only the internal gravitino constraint

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\mathrm{Tr}\left[-\Gamma_{[n|1}\Delta_{|m]}L'\right] \\
&= -i\nabla_{[m|}\phi L'_{|n]1} - i\partial_{[m|}L'_{|n]1} + \frac{1}{2}H_{mnp}L'^{2p} \\
&= (\mathcal{D}L')_{mn12} - i\partial_{[m|}L'_{|n]1} \\
&= (\mathcal{D}(e^\phi L'))_{mn12} \qquad \text{(B.107)}
\end{aligned}$$

We are then left with $(\mathcal{D}L')^n{}_m$. On one hand, we have

$$\begin{aligned}
0 = \frac{\sqrt{2}}{4}\mathrm{Tr}\left[\Delta_m L'\Gamma^{n2}\right] &= 2\nabla_{[m|}L'^{|n]2} + 2\partial_{[m|}\phi L'^{|n]2} - \frac{i}{2}H^n{}_{mp}L'^{1p} \\
&+ \frac{e^\phi}{4}\left[iF_0 - (*F_6)\right]L'^n{}_m \\
&+ \frac{e^\phi}{4}\left[F^n{}_m - i(*F_4)^n{}_m\right]L'^n{}_m \\
&+ \frac{e^\phi}{4}\left[(*F_2)^n{}_{mpq} - iF^n{}_{mpq}\right]L'^{pq} \qquad \text{(B.108)}
\end{aligned}$$

but on the other hand we can use

$$\begin{aligned}
0 = \frac{\sqrt{2}}{4}\mathrm{Tr}\left[L'\Delta_m\Gamma^{n2}\right] &= 2\nabla_{[m|}L'^{|n]2} + 2\partial_{[m|}\phi L'^{|n]2} + \frac{i}{2}H^n{}_{mp}L'^{1p} \\
&+ \frac{e^\phi}{4}\left[iF_0 + (*F_6)\right]L'^n{}_m \\
&+ \frac{e^\phi}{4}\left[-F^n{}_m - i(*F_4)^n{}_m\right]L'^n{}_m \\
&+ \frac{e^\phi}{4}\left[-(*F_2)^n{}_{mpq} - iF^n{}_{mpq}\right]L'^{pq} \qquad \text{(B.109)}
\end{aligned}$$

By comparing the two, one recover the following combination to vanish

$$\frac{i}{2}H^{n}{}_{mp}L'^{1p}+\frac{e^{\phi}}{4}\left[(*F_6)L'^{n}{}_{m}-F^{n}{}_{m}L'^{12}+\frac{1}{2}(*F_2)^{n}{}_{mpq}L'^{pq}\right]=0 \tag{B.110}$$

Consider then

$$\begin{aligned}0&=\frac{\sqrt{2}}{4}\mathrm{Tr}\left[\Delta_m\Gamma^{n2}L'+i[\Delta_d l_d+\Delta_e l_e,\Gamma_{mn}{}^{1}]L'\right]\\&=2\nabla_{[m|}L'^{|n]2}+2\partial_{[m|}(1+l_d\phi+l_eA)L'^{|n]2}\\&-i\frac{e^{\phi}}{4}\left[F_0(-1+5l_d+l_e)L'^{n}{}_{m}+(*F_4)^{n}{}_{m}(-1+l_d+l_e)L'^{12}+\frac{1}{2}(1+l_d+l_e)L'^{pq}\right]\end{aligned}$$

Therefore, to simplify at best we choose

$$l_d=\frac{1}{2}\,,\quad l_e=-\frac{3}{2}\,. \tag{B.111}$$

we obtain

$$\begin{aligned}0&=2\nabla_{[m|}L'^{|n]2}+3\partial_{[m|}(\phi-A)L'^{|n]2}-i\frac{e^{\phi}}{2}(*F_4)^{n}{}_{m}L'^{12}\\&=(\mathcal{D}(e^{\frac{3}{2}(\phi-A)}L'))^{n}{}_{m}-i\frac{e^{\phi}}{2}(*F_4)^{n}{}_{m}L'^{12}\,.\end{aligned} \tag{B.112}$$

### B.5.3 $\mathcal{N}=2$ supersymmetry and $\mathcal{D}K$

As we commented in the main text, the non-trivial structure of the SU(8) spinors (4.34) in this case does not allow to isolate only some nonvanishing components of the triplet $K_a$ similarly to the what we recovered in the $\mathcal{N}=1$ case (4.68), which in turn made possible to recover different supersymmetry equations for $K'_3$ and $K'_+$ respectively. In this case we are led to work out supersymmetry constraints for a generic element of the triplet $K_a$. Using the coniugate of (B.90), namely

$$\nabla_m\bar{\theta}_I=\frac{i}{8}H_{mnp}\bar{\theta}_I\Gamma^{np12}-\frac{e^{\phi}}{8}\bar{\theta}_I\Gamma_m\not{F}_{\rm i}\;, \tag{B.113}$$

we compute the internal gravitino variation as

$$\Delta_mK'\equiv\nabla_mK'^{\alpha}{}_{\beta}-\partial(A-\phi)K'^{\alpha}{}_{\beta}+\frac{i}{8}H_{mnp}[\Gamma^{np12}K'-K'\Gamma^{np12}]^{\alpha}{}_{\beta}-\frac{e^{\phi}}{8}[\not{F}_i\Gamma_mK'-K'\Gamma_m\not{F}_i]^{\alpha}{}_{\beta}=0\,, \tag{B.114}$$

we also have

$$(\Delta_eK')^{\alpha}{}_{\beta}\equiv i\partial_mA[\Gamma^{m12}K_1]^{\alpha}{}_{\beta}+\frac{e^{\phi}}{4}[\not{F}_eK']^{\alpha}{}_{\beta}=0\;, \tag{B.115}$$

$$(\Delta_dK')^{\alpha}{}_{\beta}\equiv i\partial_m\phi[\Gamma^{m12}K']^{\alpha}{}_{\beta}+\frac{1}{12}H_{mpq}[\Gamma^{mpq}K']^{\alpha}{}_{\beta}+\frac{e^{\phi}}{4}[\not{F}_dK_0]^{\alpha}{}_{\beta}=0\;. \tag{B.116}$$

and the transposed versions

$$(K'\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K'\Gamma^{m12}]^\alpha{}_\beta - \frac{e^\phi}{4}[K'\not{F}_e]^\alpha{}_\beta = 0 \ , \tag{B.117}$$

$$(K'\Delta_d)^\alpha{}_\beta \equiv i\partial_m\phi[K'\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K'\Gamma^{mpq}]^\alpha{}_\beta - \frac{e^\phi}{4}[K'\not{F}_d]^\alpha{}_\beta = 0\,. \tag{B.118}$$

We sketch in the following how supersymmetry implies the twisted differential equations in the $\mathcal{N}=2$. We look first the $ab = mn, 12$ components. In order, we have

$$\begin{aligned}
0 &= -\frac{i}{4}\mathrm{Tr}[-\Gamma^{mnp2}\Delta_p L' + i\Gamma^{mn1}(n_e\Delta_e + n_d\Delta_d)K'] \\
&= -2\nabla_p K'^{mnp2} + -2\partial_p(A-\phi)K'^{mnp2} - 2i\partial_r(n_eA + n_d\phi)g^{r[m}K'^{2|n]} - 2\partial_p(n_d\phi + n_eA)K'^{2mnp} \\
&+ \frac{1}{2}(1-n_d)H^{mn}{}_p K'^{1p} - in_dH_{pq}{}^{[m|}K'^{1pq|n]} + \frac{1}{2}(3-n_d)(*H)^{mnp}K'^2{}_p \\
&+ \frac{e^\phi}{4}\left(F_0(-4+n_e+5n_d) - i(*F_6)(n_e-n_d)\right)K'^{mn} \\
&+ \frac{e^\phi}{4}\left(iF^{mn}(n_e+3n_d) - (*F_4)^{mn}(4+n_e+n_d)\right)K'^{12} \\
&+ \frac{e^\phi}{8}\left(i(*F_2)^{mn}{}_{pq}(n_e+3n_d) - F^{mn}{}_{pq}(n_e+n_d)\right)K'^{pq} \\
&+ e^\phi\left(F^{[m|}{}_p(-2+n_e+3n_d) - i(*F_4)^{[m|}{}_p(n_e+n_d)\right)K'^{p12|n]}
\end{aligned} \tag{B.119}$$

$$\begin{aligned}
0 &= -\frac{1}{4}\mathrm{Tr}[-2\Gamma^2{}_{[n|}\Delta_{|m]}K' + i\Gamma_{mn1}(n_e\Delta_e + n_d\Delta_d)K')] \\
&= -2\nabla_{[m|}K'^2{}_{|n]} - 2\nabla_{[m|}(A-\phi)K'^2{}_{|n]} + 2\partial_{[m|}(n_eA+n_d\phi)K'^2{}_{|n]} - 2i\partial_p(n_d\phi+n_eA)K'^{2mnp} \\
&- i\frac{n_d}{2}H_{mnp}K'^{1p} - (1-n_d)H_{[m|pq}K'^{1pq}{}_{|n]} - i\frac{n_d}{2}(*H)_{mnp}K'^{2p} \\
&+ \frac{e^\phi}{4}\left(iF_0(n_e+5n_d) + (*F_6)(2+n_e-n_d)\right)K'_{mn} \\
&+ \frac{e^\phi}{4}\left(-F_{mn}(+2+n_e+3n_d) - i(*F_4)_{mn}(n_e+n_d)\right)K'^{12} \\
&+ \frac{e^\phi}{8}\left(-(*F_2)_{mnpq}(-2+n_e+3n_d) - iF_{mnpq}(n_e+n_d)\right)K'^{pq} \\
&+ e^\phi\left(iF_{[m|p}(n_e+3n_d) + (*F_4)_{[m|p}(n_e+n_d)\right)K'^{q12}{}_{n]}
\end{aligned} \tag{B.120}$$

so, by choosing $n_d = \frac{1}{2}, n_e = -\frac{1}{2}$, we recover the following combination:

$$\begin{aligned}
0 &= (\text{B.119}) - i(\text{B.120}) = -2\nabla_p K'^{mnp2} + (*H)^{mnp}K'^{2p} - e^\phi(*F_4)^{mn}K'^{12} \\
&- i[-2\nabla_{[m|}K'^2{}_{|n]} - e^\phi F_{mn}K'^{12}] \\
&= (\mathcal{D}K')^{mn} - i(\widetilde{\mathcal{D}K'})^{mn}
\end{aligned} \tag{B.121}$$

In a very similar fashion, we consider

$$
\begin{aligned}
0 = &-\frac{1}{4}\text{Tr}\left[-\Delta_p K'\Gamma^{p1} - i\Gamma^2(n_d\Delta_d + n_e\Delta_e)L'\right] \\
&- \nabla_p K'^{p1} + \partial_p(A-\phi)K'^{p1} - \partial_p(n_eA + n_d\phi)K'^{1p} - \frac{1}{2}(1-\frac{n_d}{3})H_{mnp}K'^{2mnp} \\
&+ \frac{e^\phi}{4}\left(iF_0(5n_d+n_e) + (*F_6)(6+n_e-n_d)\right) \\
&- \frac{e^\phi}{4}\left(F_{mn}(-2+3n_d+n_e) + i(*F_4)_{mn}(n_e+n_d)\right)
\end{aligned} \tag{B.122}
$$

by choosing this time $n_d = 1$, $n_e = -1$ we recover

$$
\begin{aligned}
0 &= -\nabla_p K'^{p1} - \frac{1}{3}H_{mnp}K'^{2mnp} + e^\phi(iF_0 + (*F_6))K'^{12} \\
&= i\left[e^\phi F_0 - i(-\nabla_p K'^{p1} - \frac{1}{3}H_{mnp}K'^{2mnp} + e^\phi(*F_6)K'^{12})\right] \\
&= i\left[(\mathcal{D}K')^{12} - i(\widetilde{\mathcal{D}K'})^{12}\right]
\end{aligned} \tag{B.123}
$$

where in the last passage we used the R-R connection appearing in (B.34) for a generic $K'$.
We then analyze the components which even in the $\mathcal{N}=1$ case differ from zero. Consider

$$
\begin{aligned}
0 = -\frac{1}{4}\text{Tr}\left[\Delta_m K'^{12} - i\{\Delta_e n_e, \Gamma_m\}K'\right] &= -\nabla_m K'^2{}_1 + \partial_m(A-\phi)K'^2{}_1 + n_e\partial_p AK'^{12} \\
\frac{e^\phi}{4}\left[F_{mp}K'^{2p} + (*F_4)_{pq}K'^{2pq}{}_m - (*F_6)K'_{m1}\right]
\end{aligned} \tag{B.124}
$$

which by considering $n_e = 1$ obviously simplifies to

$$
0 \ = -\nabla_m K'^2{}_1 - \partial_m\phi K'^2{}_1 = (\widetilde{\mathcal{D}(e^\phi K'})_{m1}\,. \tag{B.125}
$$

For the components $(\mathcal{D}K')^{m1}$ and $(\widetilde{\mathcal{D}K})_m2$ we should separate teh R-R part of the connection in the comparison with supersymmetry, precisely as we already seen in the $\mathcal{N}=1$ case. For the first one, notice indeed that

$$
\begin{aligned}
0 = -\text{Tr}\left[[n_d\Delta_d + n_e\Delta_e, \Gamma_m]K'\right] &= e^\phi\left(F_0K'_{m1} - (*F_4)K'^{2p} - F_{pq}K'^{2pq}_m\right) - 8\partial_p AK'^{mp12} \\
= \mathcal{F}_R R\big|^{m1} - 8\partial_p AK'^{mp12}\,.
\end{aligned} \tag{B.126}
$$

We thus have

$$
\begin{aligned}
0 &= +\frac{i}{4}\text{Tr}\left[\Gamma^{mp12}\Delta_p K' + i[2\Delta_d - 5\Delta_e, \Gamma^m]K'\right] \\
&= -2\partial_p(-5A+2\phi)K'^{mp12} - 2\partial_p(A-\phi)K'^{mp12} + 2\nabla_p K'^{mp12} = +2\nabla_p K'^{mp12} - 2\partial_p(-4A+\phi)K'^{mp12} \\
&= 2\nabla_p K'^{mp12} - (\mathcal{F}_{RR}\big|^{m1} - 8\partial_p AK'^{mp12}) - 2\partial_p(-4A+\phi)K'^{mp12} \\
&= (\mathcal{D}K')^{m1} - 2\partial_p\phi K'^{mp12} = (\mathcal{D}(e^{-\phi}K'))^{m1}\,.
\end{aligned} \tag{B.127}
$$

where in the third line we made explicit use of (B.126).
Then for $(\widetilde{\mathcal{D}K})_{m2}$ the argument is similar. We first express the R-R connection in the following way

$$\begin{aligned}0 &= -\mathrm{Tr}\left[i\Gamma^{m21}\Delta_e K' + \Delta_m K'\right] = -4\partial_p A K'^{mp} + (*F_6)K'_{m2} + F_{mp}K'^{1p} + (*F_4)_{pq}K'^{1pq}{}_m \\ &= -4\partial_p A K'^{mp} + \mathcal{F}_{RR}\big|_{m2} \,.\end{aligned} \tag{B.128}$$

where we used the trace of the internal gravitino by itself.
Consider then

$$\begin{aligned}0 &= \frac{1}{4}\mathrm{Tr}\left[\Gamma^{pm}\Delta_p K' + i[3\Delta_e - 2\Delta_d, \Gamma^{m12}K']\right] \\ &= -\partial_p(3A - 2\phi)K'^{pm} + \partial_p(A-\phi)K'^{pm} - \nabla_p K'^{pm} = -\nabla_p K'^{pm} - \partial_p(2A-\phi)K'^{pm} \\ &= -\nabla_p K'^{pm} - H_{mpq}K'^{12pq} + (-4\partial_p A K'^{mp} + \mathcal{F}_{RR}\big|_{m2}) - \partial_p(2A-\phi)K'^{pm} \\ &= (\mathcal{D}K')_{m2} + \partial_p(2A+\phi)K'^{pm} + H_{mpq}K'^{12pq} \\ &= (\mathcal{D}(\widetilde{e^{-(2A+\phi)}}K'))_{m2} + e^{-(2A+\phi)}H_{mpq}K'^{12pq} \,.\end{aligned} \tag{B.129}$$

In order to math the twisted EGG equation, we are forced to introduce an explicit $H$-term, as it happened in the $\mathcal{N}=1$ case (B.89).

# Appendix C

# Conifold geometry

In this appendix we collect a review of the $T^{1,1}$ geometry, which is standard in the literature [39, 117, 173], together with the dictionary of the notation used in the various papers which we are using in Chapter 5.

## C.1 Introducing $T^{1,1}$

We introduce the (singular) conifold using the following surface embedded in $\mathbb{C}^4$ described by the following quadric

$$\sum_{a=1}^{4} z_a^2 = 0\,. \tag{C.1}$$

being $z_a$ complex numbers

$$z_a = x_a + iy_a\,, \qquad a = 1, \ldots, 4\,. \tag{C.2}$$

This space turns out to be a cone, as this equation is invariant under an overall real rescaling of the coordinates: indeed, if $z_a$ solves (C.1), so it does $\lambda z_a$ for any $\lambda$. The base of the cone is known as $T^{1,1}$, and the metric on the conifold may be rewritten as

$$ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}}\,. \tag{C.3}$$

Obviously this metric features a singularity at $r = 0$.
The base $T^{1,1}$ can also be pictured as the intersection of the conifold (C.1) and the seven sphere $\sum_{a=1}^{4} |z_a|^2 = 0$. An explicit parameterization uses the angles $0 \leq \psi \leq 4\pi\,,\ 0 \leq \theta_i \leq \pi\,, 0 \leq \phi_i \leq 2\pi\ i = 1, 2\,.$ in the combination

$$\theta_\pm = (\theta_1 \pm \theta_2)/2\,,$$
$$\phi_\pm = (\phi \pm \phi_2)/2\,.$$

The $T^{1,1}$ coordinates $z_a$ are then

$$z_1 = \frac{e^{i\psi/2}}{\sqrt{2}} \left(\cos\theta_+ \cos\phi_+ + i\cos\theta_- \sin\phi_+\right) \,, \tag{C.4}$$

$$z_2 = \frac{e^{i\psi/2}}{\sqrt{2}} \left(-\cos\theta_+ \sin\phi_+ + i\cos\theta_- \cos\phi_+\right) \,, \tag{C.5}$$

$$z_3 = \frac{e^{i\psi/2}}{\sqrt{2}} \left(-\sin\theta_+ \cos\phi_+ + i\sin\theta_- \sin\phi_+\right) \,, \tag{C.6}$$

$$z_4 = \frac{e^{i\psi/2}}{\sqrt{2}} \left(-\sin\theta_+ \sin\phi_+ - i\sin\theta_- \cos\phi_+\right) \,, \tag{C.7}$$

We now look for Ricci-flat Kähler metrics on the conifold. Imposing Ricci-flatness amounts to require that the base of the cone admits an Einstein metric. So far there exists two possible metrics which represent different geometries on $S^2 \times S^3$ compatible with the Einstein metric requirement, however just one of the two is compatible with the Kähler condition [39], which lead the following Kähler-Einstein metric on $T^{1,1}$

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \sum_{a=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{a=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \tag{C.8}$$

We thus see from this that $T^{1,1}$ is an $S^1$ bundle over $S^2 \times S^2$. Therefore, the compact and homogeneous space $T^{1,1}$ can also be defined as a coset space

$$T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)} = \frac{\mathbb{CP}^1 \times \mathbb{CP}^1}{U(1)} \,. \tag{C.9}$$

Topologically, this space is $S^2 \times S^3$. There is a very simple way to show this. Indeed if one "neglects" in the numerator (C.9) one of the two $SU(2)$ would get a coset $SU(2)/U(1) \simeq S^2$, then in this particular sense $T^{1,1}$ can be seen as an $S^2$ bundle. On the other hand this bundle is fibered on the missing $SU(2) \simeq S^3$, thus we have indeed an $S^2 \times S^3$ topology.
In the construction of our ansatz in Chapter 5 (see section 5.6.1) we will use the following standard invariant one-forms on $T^{1,1}$ [117, 173]:

$$\begin{aligned}
\sigma_1 &= \cos\frac{\psi}{2} d\theta_1 + \sin\frac{\psi}{2}\sin\theta_1 d\phi_1 \\
\sigma_2 &= \sin\frac{\psi}{2} d\theta_1 - \cos\frac{\psi}{2}\sin\theta_1 d\phi_1 \\
\sigma_3 &= \frac{1}{2} d\psi + \cos\theta_1 d\phi_1 \\
\Sigma_1 &= \cos\frac{\psi}{2} d\theta_2 + \sin\frac{\psi}{2}\sin\theta_2 d\phi_2 \\
\Sigma_2 &= \sin\frac{\psi}{2} d\theta_2 - \cos\frac{\psi}{2}\sin\theta_2 d\phi_2 \\
\Sigma_3 &= \frac{1}{2} d\psi + \cos\theta_2 d\phi_2,
\end{aligned} \tag{C.10}$$

which are both left- invariant

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k\,, \quad d\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma_j \wedge \Sigma_k \tag{C.11}$$

As recalled in the main text, the authors of [178] adopt instead a set of left - invariant forms ($e_i$) together with a set of right - invariant forms ($\epsilon_i$).

## C.2 Alternative one-form conventions

In the literature there are several other parameterizations of the possible forms on $T^{1,1}$. The one adopted in the Klebanov-Strassler solution [144] use the following set $(b^1, b^2, b^3, b^4, b^5)$ of one-forms

$$
\begin{aligned}
& b^1 \equiv -\sin\theta_1 \,, b^2 \equiv d\theta_1 \,, \\
& b^3 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \,, \\
& b^4 \equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \,, \\
& b^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \,.
\end{aligned} \tag{C.12}
$$

from which the following combinations can be constructed

$$
\begin{aligned}
g^1 = -\frac{b^1 - b^3}{\sqrt{2}} \,, \quad g^2 = -\frac{b^2 - b^4}{\sqrt{2}} \,, \\
g^3 = -\frac{b^1 + b_2}{\sqrt{2}} \,, \quad g^4 = -\frac{b^2 + b^4}{\sqrt{2}} \,, \\
g^5 = e^5 \,.
\end{aligned} \tag{C.13}
$$

Notice that in the Papadopoulos-Tseytlin ansatz [178] the forms $(e_i, \epsilon_i)$ indeed correspond to the set $(b^1, b^2, b^3, b^4, b^5)$ as

$$
e_1 = b^2 \,, \quad e_2 = b^1 \,, \quad \epsilon_1 = b^4 \,, \quad \epsilon_2 = b^3 \,, \quad \tilde{\epsilon}_3 = b^5. \tag{C.14}
$$

The Einstein metric on $T^{1,1}$ can be written by means of (C.13) as

$$
ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2 \,. \tag{C.15}
$$

The volume of $T^{1,1}$ can be easily be computed with these vielbeins, as

$$
\begin{aligned}
\mathrm{vol}_{T^{1,1}} &= \int_{T^{1,1}} \frac{1}{6^2} g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge \frac{1}{3} g^5 = \frac{1}{3^3 \cdot 2^2} \int_{T^{1,1}} b^1 \wedge b^2 \wedge b^3 \wedge b^4 \wedge b^5 \,. \\
&= \frac{1}{3^3 \cdot 2^2} \left( \int_0^{4\pi} d\psi \right) \cdot \left( \int_0^{2\pi} d\phi \right)^2 \cdot \left( \int_0^{\pi} d(\cos\theta) \right)^2 \\
&= \frac{4\pi \cdot (2\pi)^2 \cdot 2^2}{3^3 \cdot 2^2} = \frac{16}{27}\pi^3 \,.
\end{aligned} \tag{C.16}
$$

We then list the one-form conventions which translates from the various solution we are interested in. The dictionary between Klebanov-Strassler [144] and Papadopoulos-Tseytlin [178] is the following

$$
\begin{aligned}
e^1 = \frac{g^2 - g^4}{\sqrt{2}} \,, \quad e_2 = \frac{g^1 - g^3}{\sqrt{2}} \,, \\
\epsilon_1 = -\frac{g^2 + g^4}{\sqrt{2}} \,, \quad \epsilon_2 = -\frac{g^1 + g^3}{\sqrt{2}} \,, \\
\tilde{\epsilon}_3 = g^5 \,.
\end{aligned} \tag{C.17}
$$

On the other hand, to go from Papadopoulos-Tseytlin [178] to Maldacena-Nuñez [165], one needs to identify the forms of the first with the ones of the last as

$$\begin{aligned} e_1 = \left(\frac{A_1}{a}\right)_{MN} , \quad e_2 = \left(\frac{A_2}{a}\right)_{MN} \\ \epsilon_1 = (\omega_1)_{MN} , \quad \epsilon_2 = (\omega_2)_{MN} , \\ \tilde{\epsilon}_3 = (\omega_3 - A_3)_{MN} \end{aligned} \tag{C.18}$$

# Appendix D

# Truncation details

We collect in this Appendix details of the truncation worked out in Chapter 5, as the the five form reduction, the complete equations of motion for the three forms and for the dilation-axion of the reduced theory, part of which are relevant for the subtruncation to Papadopoulos-Tseytlin, as well as the dictionary of the scalars between our effective theory and the ones used for the Papadopoulos-Tseytlin ansatz in [178].

## D.1 The Five-Form

Due to the self-duality in ten dimensions the most involved part of reconstructing the five-dimensional action is the equations of motion for components of the five-form. Here we summarize various steps we have taken. To facilitate computing the exterior derivative, we first write the ansatz (5.107) in terms of untwisted fundamental forms

$$\begin{aligned} F_5 &= e^Z e^{8(u_3-u_1)}\mathrm{vol}_5 + e^Z J_1\wedge J_2\wedge(g_5+A_1) \\ &\quad +K_1\wedge J_1\wedge J_2 - e^{-8u_1}(*_5\tilde{K}_1)\wedge(g_5+A_1) \\ &\quad +K_{21}\wedge J_1\wedge(g_5+A_1) + e^{-4u_2+4u_3}(*_5\tilde{K}_{21})\wedge J_2 \\ &\quad +K_{22}\wedge J_2\wedge(g_5+A_1) + e^{4u_2+4u_3}(*_5\tilde{K}_{22})\wedge J_1 \\ &\quad +\big(L_2\wedge\Omega + c.c\big)\wedge(g_5+A_1) + e^{4u_3}\big((*_5\tilde{L}_2)\wedge\Omega + c.c\big) \end{aligned} \tag{D.1}$$

where the unprimed forms are given in terms of the primed ones by

$$K_1 = K_1' \tag{D.2}$$

$$K_{21} = K_{21}' - |v|^2 K_{22}' + 4\,\mathrm{Im}\,v L_2' \tag{D.3}$$

$$K_{22} = K_{22}' \tag{D.4}$$

$$L_2 = L_2' - \frac{i\overline{v}}{2}K_{22}'. \tag{D.5}$$

The Hodge star creats a bit of a mess as well so we have defined some new fields

$$\begin{aligned}
\tilde{K}_1 &= K_1' \\
&= K_1 && \text{(D.6)} \\
\tilde{K}_{21} &= K_{21}' \\
&= K_{21} - |v|^2 K_{22} - 4\mathrm{Im}\,(vL_2) && \text{(D.7)} \\
\tilde{K}_{22} &= K_{22}' - |v|^2 e^{-8u_2} K_{21}' + 4e^{-4u_2}\,\mathrm{Im}\,(vL_2') \\
&= \big(1+|v|^2 e^{-4u_2}\big)^2 K_{22} - |v|^2 e^{-8u_2} K_{21} + 4e^{-4u_2}\big(1+|v|^2 e^{-4u_2}\big)\mathrm{Im}\,(vL_2) && \text{(D.8)} \\
\tilde{L}_2 &= L_2' - \frac{i\overline{v}}{2} e^{-4u_2} K_{21}' \\
&= L_2\big(1+|v|^2 e^{-4u_2}\big) + \frac{i\overline{v}}{2}\big(1+|v|^2 e^{-4u_2}\big)K_{22} - \frac{i\overline{v}}{2} e^{-4u_2} K_{21} - \overline{v}^2 e^{-4u_2}\overline{L}_2 && \text{(D.9)}
\end{aligned}$$

and then the five-dimensional equations of motion are

$$\begin{aligned}
DL_2 - 3ie^{4u_3} *_5 \tilde{L}_2 &= -M_0 G_3 + N_0 H_3 - H_2 \wedge N_1 + M_1 \wedge G_2 \\
K_{21} \wedge F_2 + d\big(e^{4(u_2+u_3)} *_5 \tilde{K}_{22}\big) - 2e^{-8u_1} *_5 \tilde{K}_1 &= H_{11} \wedge G_3 + H_3 \wedge G_{11} && \text{(D.10)} \\
K_{22} \wedge F_2 + d\big(e^{4(-u_2+u_3)} *_5 \tilde{K}_{21}\big) - 2e^{-8u_1} *_5 \tilde{K}_1 &= H_{12} \wedge G_3 + H_3 \wedge G_{12} && \text{(D.11)} \\
L_2 \wedge F_2 + D(e^{4u_3} *_5 \tilde{L}_2) &= M_1 \wedge G_3 + H_3 \wedge N_1 && \text{(D.12)} \\
-d\big(e^{-8u_1} *_5 \tilde{K}_1\big) &= H_3 \wedge G_2 - H_2 \wedge G_3 && \text{(D.13)}
\end{aligned}$$

where the two-form $L_2$ is charged

$$DL_2 = dL_2 - 3iA_1 \wedge L_2. \qquad \text{(D.14)}$$

The five-dimensional kinetic terms which we get from these equations are rather off-diagonal

$$\begin{aligned}
\mathcal{L}_{F_{(5)},kin} &= -4e^{4u_3}\big(1+|v|^2 e^{-4u_2}\big) L_2 \wedge *_5 \overline{L}_2 + 4e^{-4u_2+4u_3}\Big(v^2 L_2 \wedge *_5 L_2 + c.c\Big) \\
&\quad -\frac{1}{2} e^{4u_2+4u_3}\big(1+|v|^2 e^{-4u_2}\big)^2 K_{22} \wedge *_5 K_{22} - \frac{1}{2} e^{-4u_2+4u_3} K_{21} \wedge *_5 K_{21} \\
&\quad +|v|^2 e^{-4u_2+4u_3} K_{22} \wedge *_5 K_{21} + 2e^{4u_3}\big(1+|v|^2 e^{-4u_2}\big)\Big(ivK_{22} \wedge *_5 L_2 + c.c\Big) \\
&\quad -2e^{-4u_2+4u_3}\Big(ivK_{21} \wedge *_5 L_2 + c.c\Big). && \text{(D.15)}
\end{aligned}$$

The Bianchi identities are

$$\begin{aligned}
de^Z &= P(H_{12} - H_{11}) + 4\big(M_1 \overline{N}_0 - M_0 \overline{N}_1 + c.c\big) && \text{(D.16)} \\
e^Z F_2 + dK_1 + 2K_{21} + 2K_{22} &= H_{11} \wedge G_{12} + H_{12} \wedge G_{11} + 4\big(M_1 \overline{N}_1 + c.c.\big) && \text{(D.17)} \\
dK_{21} &= P\,H_3 + H_{11} \wedge G_2 - H_2 \wedge G_{11} && \text{(D.18)} \\
dK_{22} &= -P\,H_3 + H_{12} \wedge G_2 - H_2 \wedge G_{12} && \text{(D.19)}
\end{aligned}$$

which we have solved in (5.112)-(5.115).

## D.2 The Three-Forms

The three-forms are given in terms of twisted fundamental forms by

$$\begin{aligned}
H_{(3)} &= H_3 + H_2 \wedge (g_5 + A_1) + H'_{11} \wedge J'_1 + H_{12} \wedge J'_2 \\
&\quad + \Big( M'_1 \wedge \Omega' + M_0 \Omega' \wedge (g_5 + A_1) + c.c \Big) \\
&\quad - 4\,\mathrm{Im}\,(M_0 v) J'_1 \wedge (g_5 + A_1) \qquad \text{(D.20)} \\
F_{(3)} &= P(J'_1 - J'_2) \wedge (g_5 + A_1) + G_3 + G_2 \wedge (g_5 + A_1) + G'_{11} \wedge J'_1 + G_{12} \wedge J'_2 \\
&\quad + \Big( N'_1 \wedge \Omega' + N'_0 \Omega' \wedge (g_5 + A_1) + c.c \Big) \\
&\quad - \big( P|v|^2 + 4\,\mathrm{Im}\,(N'_0 v) \big) J'_1 \wedge (g_5 + A_1), \qquad \text{(D.21)}
\end{aligned}$$

where

$$\begin{aligned}
H'_{11} &= H_{11} - |v|^2 H_{12} - 4\,\mathrm{Im}\,(v M_1) \\
M'_1 &= M_1 + \frac{i}{2} \overline{v} H_{12} \\
G'_{11} &= G_{11} - |v|^2 G_{12} - 4\,\mathrm{Im}\,(v N_1) \qquad \text{(D.22)} \\
N'_1 &= N_1 + \frac{i}{2} \overline{v} G_{12} \\
N'_0 &= N_0 - \frac{i}{2} P \overline{v}.
\end{aligned}$$

The equations of motion are

$$
\begin{aligned}
d\big(e^{4(u_1-u_3)-\phi} *_5 H_3\big) &= -e^Z G_3 + G_2 \wedge K_1 - G_{12} \wedge K'_{21} - G'_{11} \wedge K'_{22} \\
&\quad -4\big(N'_1 \wedge \overline{L}'_2 + c.c\big) + 4e^{4u_3}\big(N'_0 *_5 \overline{L}'_2 + c.c.\big) \\
&\quad +P\big(1-|v|^2 - 4\text{Im }(N'_0 v)\big)e^{-4(u_2-u_3)} *_5 K'_{21} \\
&\quad -Pe^{4u_2+4u_3)} *_5 K'_{22} + e^{4(u_1-u_3)+\phi} da \wedge (*_5 G_3) \qquad \text{(D.23)}
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{4(u_1+u_3)-\phi} *_5 H_2\big) &= 2(1-|v|^2)e^{-4(u_1+u_2)-\phi} *_5 H'_{11} + 2e^{-4(u_1-u_2)-\phi} *_5 H_{12} \\
&\quad +e^{4(u_1-u_3)-\phi} *_5 H_3 \wedge F_2 + 8e^{-4u_1-\phi}\text{Im}\,(v *_5 M'_1) \\
&\quad +G_3 \wedge K_1 + e^{-4(u_2-u_3)} G'_{11} \wedge *_5 K'_{21} + e^{4(u_2+u_3)} G_{12} \wedge *_5 K'_{22} \\
&\quad +4e^{4u_3}\big(N'_1 \wedge *_5 \overline{L}'_2 + c.c.\big) + e^{4(u_1+u_3)+\phi} da \wedge *_5 G_2 \qquad \text{(D.24)}
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{-4(u_1-u_2)-\phi} *_5 H_{12}\big) &= -4e^{-4u_1-\phi}\text{Im}\,(*_5 M'_1 \wedge Dv) + 12e^{-4u_1+8u_3-\phi}\text{Re}\,(M_0 v)\text{vol}_5 \\
&\quad +\big(P(1-|v|^2) - 4\text{Im }(N'_0 v)\big)e^Z e^{8(u_3-u_1)}\text{vol}_5 + e^{-8u_1} G'_{11} \wedge *_5 K_1 \\
&\quad +e^{4(u_2+u_3)} G_2 \wedge *_5 K'_{22} - G_3 \wedge K'_{21} \\
&\quad +e^{-4(u_1-u_2)+\phi} da \wedge *_5 G_{12} \qquad \text{(D.25)}
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{-4(u_1+u_2)-\phi} *_5 H'_{11}\big) &= -Pe^Z e^{8(u_3-u_1)}\text{vol}_5 + e^{-8u_1} G_{12} \wedge *_5 K_1 + e^{-4(u_2-u_3)} G_2 \wedge *_5 K'_{21} \\
&\quad -G_3 \wedge K'_{22} + e^{-4(u_1+u_2)+\phi} da \wedge *_5 G'_{11} \qquad \text{(D.26)}
\end{aligned}
$$

$$
\begin{aligned}
D\big(e^{-4u_1-\phi} *_5 M'_1\big) &= -3ie^{-4u_1+8u_3-\phi} M_0 \text{vol}_5 \\
&\quad +6e^{-4u_1-4u_2+8u_3-\phi}\overline{v}\text{Im}\,(vM_0)\text{vol}_5 - \frac{1}{2i}e^{-4(u_1+u_2)-\phi} *_5 H'_{11} \wedge D\overline{v} \\
&\quad e^Z e^{8(u_3-u_1)} N'_0 \text{vol}_5 + e^{-8u_1} N'_1 \wedge *_5 K_1 \\
&\quad -G_3 \wedge L'_2 + e^{4u_3} G_2 \wedge *_5 L'_2 + e^{-4u_1+\phi} da \wedge *_5 N'_1 \qquad \text{(D.27)}
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{4(u_1-u_3)+\phi} *_5 G_3\big) &= e^Z H_3 - K_1 \wedge H_2 + K'_{22} \wedge H'_{11} + K'_{21} \wedge H_{12} \\
&\quad +4e^{-4u_2+4u_3}\text{Im}\,(vM_0) *_5 K'_{21} + 4\big(L'_2 \wedge \overline{M}'_1 + c.c.\big) \\
&\quad -4e^{4u_3}\big( *_5 L'_2 \overline{M}_0 + c.c.\big) \qquad \text{(D.28)}
\end{aligned}
$$

$$
\text{(D.29)}
$$

$$
\begin{aligned}
d\big(e^{4(u_1+u_3)+\phi} *_5 G_2\big) &= e^{4(u_1-u_3)+\phi} *_5 G_3 \wedge F_2 + 2(1-|v|^2)\big(e^{-4(u_1+u_2)+\phi} *_5 G'_{11}\big) \\
&\quad +2e^{-4(u_1-u_2)+\phi} *_5 G_{12} - 8e^{-4u_1}\mathrm{Im}\,(v *_5 N'_1) + K_1 \wedge H_3 \\
&\quad -4e^{4u_3}\big(M'_1 \wedge *_5\overline{L}'_2 + c.c.\big) + e^{-4u_2+4u_3} *_5 K'_{21} \wedge H'_{11} \\
&\quad +e^{4u_2+4u_3} *_5 K'_{22} \wedge H_{12} \qquad \text{(D.30)}
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{-4(u_1-u_2)+\phi} *_5 G_{12}\big) &= -4e^{-4u_1+\phi}\mathrm{Im}\,(*_5 N'_1 \wedge Dv) + 12e^{-4u_1+8u_3+\phi}\mathrm{Re}\,(N'_0 v)\mathrm{vol}_5 \\
&\quad K'_{21} \wedge H_3 - e^{4(u_2+u_3)} *_5 K'_{22} \wedge H'_2 - e^{-8u_1} *_5 K_1 \wedge H'_{11} \\
&\quad +4\mathrm{Im}\,(M_0 v)e^Z e^{8(u_3-u_1)} \qquad \text{(D.31)}
\end{aligned}
$$

$$
d\big(e^{-4(u_1+u_2)+\phi} *_5 G'_{11}\big) = K'_{22} \wedge H_3 - e^{-4u_2+4u_3} *_5 K'_{21} \wedge H_2 - e^{-8u_1} *_5 K_1 \wedge H_{12} \qquad \text{(D.32)}
$$

$$
\begin{aligned}
D\big(e^{-4u_1+\phi} *_5 N'_1\big) &= -3ie^{-4u_1+8u_3+\phi}N'_0\mathrm{vol}_5 - \frac{1}{2i}e^{-4u_1-4u_2}(*_5 G'_{11}) \wedge D\overline{v} \\
&\quad +\frac{3\overline{v}}{2}e^{-4u_1-4u_2+8u_3+\phi}\big(P|v|^2 + 4\mathrm{Im}\,(N'_0 v)\big)\mathrm{vol}_5 \\
&\quad +L'_2 \wedge H_3 - e^{4u_3} *_5 L'_2 \wedge H_2 - e^{-8u_1} *_5 K_1 \wedge M'_1 \\
&\quad -M_0 e^Z e^{-8(u_1-u_3)}\mathrm{vol}_5 \qquad \text{(D.33)}
\end{aligned}
$$

## D.3 Dilaton-Axion

For the dilaton axion equations of motion is necessary to construct the follows quantities:

$$
\begin{aligned}
F_{(1)} \wedge *_{10} F_{(1)} &= J_1' \wedge J_2' \wedge (g_5 + A_1)\Big[da \wedge *_5 da\Big] && \text{(D.34)}\\
H_{(3)} \wedge *_{10} H_{(3)} &= J_1' \wedge J_2' \wedge g_5 \wedge \Big[e^{4(u_1-u_3)} H_3 \wedge *_5 H_3 + e^{4(u_1+u_3)} H_2 \wedge *_5 H_2 \\
&\quad + e^{-4(u_1+u_2)} H_{11}' \wedge *_5 H_{11}' + e^{-4(u_1-u_2)} H_{12} \wedge *_5 H_{12} \\
&\quad + 8e^{-4u_1} M_1' \wedge *_5 \overline{M}_1' + 8e^{-4u_1+8u_3} |M_0|^2 \mathrm{vol}_5 \\
&\quad + 16 e^{-4u_1-4u_2+8u_3} [\mathrm{Im}\,(M_0 v)]^2 \mathrm{vol}_5\Big] && \text{(D.35)}\\
F_{(3)} \wedge *_{10} F_{(3)} &= J_1' \wedge J_2' \wedge (g_5 + A_1) \wedge \Big[e^{4(u_1-u_3)} G_3 \wedge *_5 G_3 + e^{4(u_1+u_3)} G_2 \wedge *_5 G_2 \\
&\quad + e^{-4u_1-4u_2} G_{11}' \wedge *_5 G_{11}' + e^{-4u_1+4u_2} G_{12} \wedge *_5 G_{12} + 8e^{-4u_1} N_1' \wedge *_5 \overline{N}_1' \\
&\quad + e^{-4u_1+8u_3}\Big(8|N_0'|^2 + e^{4u_2} P^2 + e^{-4u_2}\big(P(|v|^2-1) + 4\,\mathrm{Im}\,(N_0' v)\big)^2\Big)\mathrm{vol}_5\Big] \\
H_{(3)} \wedge *_{10} F_{(3)} &= J_1' \wedge J_2' \wedge (g_5 + A_1) \wedge \Big[e^{4(u_1-u_3)} H_3 \wedge *_5 G_3 + e^{4(u_1+u_3)} H_2 \wedge *_5 G_2 \\
&\quad + e^{-4(u_1+u_2)} H_{11}' \wedge *_5 G_{11}' + e^{-4(u_1-u_2)} H_{12} \wedge *_5 G_{12} \\
&\quad + 4e^{-4u_1}\big(M_1' \wedge *_5 \overline{N}_1' + c.c.\big) + \Big(4e^{-4u_1+8u_3}\big(M_0 \overline{N}_0' + c.c.\big) \\
&\quad - 4e^{-4u_1-4u_2+8u_3}\,\mathrm{Im}(M_0 v)\big(P(1-|v|^2) - 4\mathrm{Im}(N_0' v)\big)\Big)\mathrm{vol}_5\Big] && \text{(D.36)}
\end{aligned}
$$

then (2.17) and (2.16) give the relevant equations of motion.

## D.4 The PT truncation

The dictionary for the scalar fields between the supersymmetric truncation of Chapter 5 and the Papadopoulos-Tseytlin ansatz [178] is the following

$$
\begin{aligned}
& & & P_{\text{truncation}} = 18 P_{\text{PT}}\,, & & && \text{(D.37)}\\
& e^{2u_1} = e^x\,, & & e^{2u_2} = e^g\,, & & 2^{1/3} 3 e^{2u_3} = e^{2p}\,, && \text{(D.38)}\\
& v_1 = a\,, & & v_2 = 0 & & , && \text{(D.39)}\\
& \frac{1}{3}\mathrm{Im}\,(\xi) = -\frac{1}{9}\mathrm{Re}\,(M_0) = -h_2\,, & & \mathrm{Re}\,(\xi) \sim \mathrm{Im}\,(M_0) = 0, & & && \text{(D.40)}\\
& \frac{\tilde{b}}{6} = h_1\,, & & \frac{b}{6} = \chi_{PT}\,, & & && \text{(D.41)}\\
& \frac{1}{3}\mathrm{Re}\,(\xi) = \frac{1}{9}\mathrm{Im}\,(N_0) = b_{PT}\,, & & \mathrm{Re}\,(\chi) \sim \mathrm{Re}\,(N_0) = 0, & & && \text{(D.42)}\\
& c = 0\,, & & \tilde{c} = 0 & & . && \text{(D.43)}
\end{aligned}
$$

Furthermore, due to the explicit factors of 1/6 and 1/9 taken in the $T^{1,1}$ metric, we have the following matching between the five dimensional metrics

$$(2^{1/3}3)g_{\mu\nu}^{\rm here} = g_{\mu\nu}^{\rm PT}\,. \tag{D.44}$$

# Bibliography


[1] M. Graña, F. Orsi, "N=1 vacua in Exceptional Generalized Geometry," JHEP **1108**, (2011) 109. [arXiv:1105.4855 [hep-th]].

[2] M. Graña, F. Orsi, "N=2 vacua in Exceptional Generalized Geometry," to appear.

[3] I. Bena, G. Giecold, M. Graña, N. Halmagyi, F. Orsi, "Supersymmetric Consistent Truncations of IIB on $T^{1,1}$," JHEP **1104**, 021 (2011) [arXiv:1008.0983 [hep-th]].

[4] G. Giecold, E. Goi, F. Orsi, "Assessing a candidate IIA dual to metastable supersymmetry-breaking," [arXiv:1108.1789 [hep-th]].

[5] F. Orsi, "String vacua in exceptional generalized geometry," Nucl. Phys. Proc. Suppl. **216**, 257-259 (2011).

---

[6] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, B. J. Spence, "Branes at conical singularities and holography," Adv. Theor. Math. Phys. **2**, 1249-1286 (1999). [arXiv:9808014 [hep-th]].

[7] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183-386 (2000). [hep-th/9905111].

[8] G. Aldazabal, E. Andres, P. G. Camara and M. Graña, "U-dual fluxes and Generalized Geometry," [arXiv:1007.5509 [hep-th]].

[9] D. Andriot, "Heterotic string from a higher dimensional perspective," Nucl. Phys. **B855**, 222-267 (2012). [arXiv:1102.1434 [hep-th]].

[10] D. Andriot, E. Goi, R. Minasian, M. Petrini, "Supersymmetry breaking branes on solvmanifolds and de Sitter vacua in string theory," JHEP **1105**, 028 (2011). [arXiv:1003.3774 [hep-th]].

[11] M. Awada and P. K. Townsend, "N=4 Maxwell-Einstein Supergravity in five-dimensions and its SU(2) gauging," *Nucl. Phys.* **B255** (1985) 617.

[12] D. Bailin and A. Love, "Kaluza-Klein theories," Rep. Prog. Phys. **50** (1987) 1087-1170.

[13] I. Bena, G. Giecold, N. Halmagyi, "The Backreaction of Anti-M2 Branes on a Warped Stenzel Space," JHEP **1104**, 120 (2011) [arXiv:1011.2195 [hep-th]].

[14] I. Bena, G. Giecold, M. Graña, N. Halmagyi, "On The Inflaton Potential From Antibranes in Warped Throats," [arXiv:1011.2626 [hep-th]].

[15] I. Bena, G. Giecold, M. Graña, N. Halmagyi, S. Massai, "On Metastable Vacua and the Warped Deformed Conifold: Analytic Results," [arXiv:1102.2403 [hep-th]].

[16] I. Bena, G. Giecold, M. Graña, N. Halmagyi, S. Massai, "The backreaction of anti-D3 branes on the Klebanov-Strassler geometry," [arXiv:1106.6165 [hep-th]].

[17] I. Bena, M. Graña, N. Halmagyi, "On the Existence of Meta-stable Vacua in Klebanov-Strassler," JHEP **1009**, 087 (2010). [arXiv:0912.3519 [hep-th]].

[18] I. Bena, E. Gorbatov, S. Hellerman, N. Seiberg, D. Shih, "A Note on (Meta)stable Brane Configurations in MQCD," JHEP **0611** (2006) 088. [arXiv:hep-th/0608157 [hep-th]].

[19] M. K. Benna, A. Dymarsky, I. R. Klebanov, and A. Solovyov, "On Normal Modes of a Warped Throat," *JHEP* **06** (2008) 070, [arXiv:0712.4404 [hep-th]].

[20] D. Berenstein, C. P. Herzog, P. Ouyang and S. Pinansky, "Supersymmetry breaking from a Calabi-Yau singularity," JHEP **0509**, 084 (2005) [arXiv:hep-th/0505029 [hep-th]].

[21] M. Berg, M. Haack, and W. Mueck, "Bulk Dynamics in Confining Gauge Theories," *Nucl. Phys.* **B736** (2006) 82–132, [arXiv:0507285 [hep-th]].

[22] M. Berg, M. Haack, and W. Mueck, "Glueballs vs. Gluinoballs: Fluctuation Spectra in non- AdS/non-CFT," *Nucl. Phys.* **B789** (2008) 1–44,[hep-th/0612224 [hep-th]].

[23] E. Bergshoeff, H. J. Boonstra, T. Ortin, "S duality and dyonic p-brane solutions in type II string theory," Phys. Rev. **D53** (1996) 7206-7212. [hep-th/9508091].

[24] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, "New Formulations of D=10 Supersymmetry and D8-O8 Domain Walls," Class. Quant. Grav. **18**, 3359 (2001) [arXiv:0103233 [hep-th]].

[25] M. Bershadsky, C. Vafa, V. Sadov, "D-branes and topological field theories," Nucl. Phys. **B463**, 420-434 (1996). [hep-th/9511222].

[26] M. Bertolini, F. Bigazzi and A. L. Cotrone, "Supersymmetry breaking at the end of a cascade of Seiberg dualities," Phys. Rev. D **72**, 061902 (2005) [arXiv:hep-th/0505055].

[27] F. Bigazzi, A. L. Cotrone, M. Petrini, A. Zaffaroni, "Supergravity duals of supersymmetric four-dimensional gauge theories," Riv. Nuovo Cim. **25N12**, 1-70 (2002). [hep-th/0303191].

[28] J. Blaback, U. H. Danielsson, D. Junghans, T. Van Riet, T. Wrase, M. Zagermann, "Smeared versus localised sources in flux compactifications," JHEP **1012** (2010) 043. [arXiv:1009.1877 [hep-th]].

[29] J. Blaback, U. H. Danielsson, D. Junghans, T. Van Riet, T. Wrase, M. Zagermann, "The problematic backreaction of SUSY-breaking branes," [arXiv:1105.4879 [hep-th]].

[30] J. Blaback, U. H. Danielsson, D. Junghans, T. Van Riet, T. Wrase, M. Zagermann, "(Anti-)Brane backreaction beyond perturbation theory," [arXiv:1111.2605 [hep-th]].

[31] R. Blumenhagen, B. Kors, D. Lust, S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," Phys. Rept. **445** (2007) 1-193. [hep-th/0610327].

[32] N. Bobev, N. Halmagyi, K. Pilch, and N. P. Warner, "Supergravity Instabilities of Non-Supersymmetric Quantum Critical Points," *Class.Quant.Grav.* **27** (2010) 235013, [arXiv:1006.2546 [hep-th]].

[33] V. Borokhov and S. S. Gubser, "Nonsupersymmetric deformations of the dual of a confining gauge theory," JHEP **0305**, 034 (2003) [arXiv:hep-th/0206098].

[34] J. Bovy, D. Lust, D. Tsimpis, "N = 1,2 supersymmetric vacua of IIA supergravity and SU(2) structures," JHEP **0508**, 056 (2005). [arXiv:0506160 [hep-th]].

[35] D. Brace, B. Morariu, B. Zumino, "Dualities of the matrix model from T duality of the Type II string," Nucl. Phys. **B545**, 192-216 (1999). [hep-th/9810099].

[36] D. Brace, B. Morariu, B. Zumino, "T duality and Ramond-Ramond backgrounds in the matrix model," Nucl. Phys. **B549**, 181-193 (1999). [hep-th/9811213].

[37] A. Buchel, J. T. Liu, "Gauged supergravity from type IIB string theory on Y**p,q manifolds," Nucl. Phys. **B771** (2007) 93-112. [hep-th/0608002].

[38] A. Butti, M. Graña, R. Minasian, M. Petrini, and A. Zaffaroni, "The baryonic branch of Klebanov-Strassler solution: A supersymmetric family of SU(3) structure backgrounds," *JHEP* **03** (2005) 069, [ArXiv:0412187 [hep-th]].

[39] P. Candelas and X. C. de la Ossa, "Comments on conifolds," *Nucl. Phys.* **B342** (1990) 246–268.

[40] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations For Superstrings," Nucl. Phys. B **258** (1985) 46.

[41] D. Cassani and A. Bilal, ‘Effective actions and N=1 vacuum conditions from SU(3) x SU(3) compactifications,” JHEP **0709**, 076 (2007) [arXiv:0707.3125 [hep-th]].

[42] D. Cassani, G. Dall’Agata, and A. F. Faedo, “Type IIB supergravity on squashed Sasaki-Einstein manifolds,” *JHEP* **1005** (2010) 094, [arXiv:1003.4283 [hep-th]].

[43] D. Cassani and A. F. Faedo, “A supersymmetric consistent truncation for conifold solutions,” *Nucl.Phys.* **B843** (2011) 455–484, [arXiv:1008.0883 [hep-th]].

[44] D. Cassani, A. F. Faedo, “Constructing Lifshitz solutions from AdS,” JHEP **1105**, 013 (2011). [arXiv:1102.5344 [hep-th]].

[45] D. Cassani, P. Koerber, “Tri-Sasakian consistent reduction,” [arXiv:1110.5327 [hep-th]].

[46] G. Cavalcanti and M. Gualtieri, “Generalized complex structures on nilmanifolds,” J. Symplectic Geom. **2**, 393 (2004). [arXiv:math.DG/0404451].

[47] A. H. Chamseddine and M. S. Volkov, “Non-Abelian BPS monopoles in N = 4 gauged supergravity,” *Phys. Rev. Lett.* **79** (1997) 3343–3346, [arXiv:9707176 [hep-th]].

[48] S. Chiossi, S. Salamon, “The Intrinsic torsion of SU(3) and G(2) structures,” [math/0202282 [math-dg]].

[49] R. Corrado, M. Gunaydin, N. P. Warner, and M. Zagermann, “Orbifolds and flows from gauged supergravity,” *Phys. Rev.* **D65** (2002) 125024, [arXiv:0203057 [hep-th]].

[50] T. Courant, “Dirac manifolds" *Trans. Amer.Math.Soc.*, 319 631-661, (1990).

[51] T. Courant and A. Weinstein, “Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*," volume 27 of *Travaux en Cours*, pages 39-49. Hermann, Paris (1988).

[52] E. Cremmer and B. Julia, “The SO(8) Supergravity,” Nucl. Phys. B **159**, 141 (1979).

[53] M. Cvetic, G. W. Gibbons, H. Lu, C. N. Pope, “Supersymmetric nonsingular fractional D-2 branes and NS NS 2 branes,” Nucl. Phys. **B606** (2001) 18-44 [arXiv:0101096 [hep-th]].

[54] M. Cvetic, G. W. Gibbons, H. Lu, C. N. Pope, “New complete noncompact spin(7) manifolds,” Nucl. Phys. **B620**, 29-54 (2002). [hep-th/0103155].

[55] M. Cvetic, G. W. Gibbons, H. Lu, C. N. Pope, “Ricci flat metrics, harmonic forms and brane resolutions,” Commun. Math. Phys. **232**, 457-500 (2003). [hep-th/0012011].

[56] M. Cvetic, H. Lu, C. N. Pope, “Consistent Kaluza-Klein sphere reductions,” Phys. Rev. **D62** (2000) 064028. [hep-th/0003286].

[57] M. Cvetic, H. Lu and C. N. Pope, "Brane resolution through transgression," Nucl. Phys. B **600**, 103 (2001) [arXiv:0011023 [hep-th]].

[58] G. Dall'Agata, C. Herrmann, and M. Zagermann, "General matter coupled N = 4 gauged supergravity in five dimensions," *Nucl. Phys.* **B612** (2001) 123–150, [arXiv:0103106 [hep-th]].

[59] D. Dall'Agata, M. Zagermann "Supergravity (Lecture Notes)", LACES School 2009 (7 September, 2010).

[60] B. de Wit, H. Samtleben and M. Trigiante, "On Lagrangians and gaugings of maximal supergravities," Nucl. Phys. B **655**, 93 (2003) [arXiv:0212239 [hep-th]].

[61] I. Dorfman, " Dirac structures of integrable evolution equations," *Phys. Lett. A*, **125(5)** 240-246, (1987).

[62] M. R. Douglas, "Branes within branes," [hep-th/9512077].

[63] M. R. Douglas, "Enhanced gauge symmetry in M(atrix) theory," JHEP **9707**, 004 (1997). [hep-th/9612126].

[64] M. R. Douglas, S. Kachru, "Flux compactification," Rev. Mod. Phys. **79** (2007) 733-796. [hep-th/0610102].

[65] M. J. Duff, B. E. W. Nilsson and C. N. Pope, "Kaluza-Klein Supergravity," Phys. Rept. **130** (1986) 1.

[66] M. J. Duff, C. N. Pope, "Consistent truncations in Kaluza-Klein theories" Nucl. Phys. B **255**, (1985)

[67] A. Dymarsky, "On gravity dual of a metastable vacuum in Klebanov-Strassler theory," JHEP **1105**, 053 (2011) [arXiv:1102.1734 [hep-th]].

[68] A. Dymarsky and D. Melnikov, "Gravity Multiplet on KS and BB Backgrounds," *JHEP* **05** (2008) 035, [arXiv:0710.4517 [hep-th]].

[69] A. Dymarsky, D. Melnikov, and A. Solovyov, "I-odd Sector of the Klebanov-Strassler Theory," *JHEP* **05** (2009) 105, [arXiv:0810.5666 [hep-th]].

[70] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici, G. Sarkissian, "D-branes in the background of NS fivebranes," Int. J. Mod. Phys. **A16**, 880-890 (2001) [arXiv:0005052 [hep-th]].

[71] J. Evslin and L. Martucci, "D-brane networks in flux vacua, generalized cycles and calibrations," JHEP **0707**, 040 (2007) [hep-th/0703129 [HEP-TH]].

[72] S. Fidanza, R. Minasian, A. Tomasiello, "On mirror symmetry with fluxes," Fortsch. Phys. **52**, 618-623 (2004). [Check ArXiv number]

[73] S. Franco, I. Garcia-Etxebarria, A. M. Uranga, "Non-supersymmetric meta-stable vacua from brane configurations," JHEP **0701** (2007) 085. [arXiv:hep-th/0607218].

[74] S. Franco, A. Hanany, F. Saad and A. M. Uranga, "Fractional branes and dynamical supersymmetry breaking," JHEP **0601**, 011 (2006) [arXiv:hep-th/0505040].

[75] S. Franco, A. M .Uranga, "Dynamical SUSY breaking at meta-stable minima from D-branes at obstructed geometries," JHEP **0606**, 031 (2006). [arXiv:hep-th/0604136].

[76] M. Fukuma, T. Oota, H. Tanaka, "Comments on T dualities of Ramond-Ramond potentials," Prog. Theor. Phys. **103**, 425-446 (2000). [hep-th/9907132].

[77] M. K. Gaillard, B. Zumino, "Duality Rotations for Interacting Fields," Nucl. Phys. **B193**, 221 (1981)

[78] S. J. Gates, C. M. Hull and M. Rocek, "Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248**, 157 (1984).

[79] J. P. Gauntlett, N. W. Kim, D. Martelli and D. Waldram, "Fivebranes wrapped on SLAG three-cycles and related geometry," JHEP **0111** (2001) 018 [arXiv:hep-th/0110034].

[80] J. P. Gauntlett, S. Kim, O. Varela, and D. Waldram, "Consistent supersymmetric Kaluza–Klein truncations with massive modes," *JHEP* **04** (2009) 102, [arXiv:0901.0676 [hep-th]].

[81] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, "G-structures and wrapped NS5-branes," Commun. Math. Phys. **247** (2004) 421 [arXiv:hep-th/0205050].

[82] J. P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, "Supersymmetric AdS(5) solutions of type IIB supergravity," Class. Quant. Grav. **23**, 4693-4718 (2006). [hep-th/0510125].

[83] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on S**2 x S**3," Adv. Theor. Math. Phys. **8**, 711 (2004) [hep-th/0403002].

[84] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, "A New infinite class of Sasaki-Einstein manifolds," Adv. Theor. Math. Phys. **8** (2006) 987 [hep-th/0403038].

[85] J. P. Gauntlett, D. Martelli and D. Waldram, "Superstrings with intrinsic torsion," Phys. Rev. D **69**, 086002 (2004). [arXiv:0302158 [hep-th]]

[86] J. P. Gauntlett and O. Varela, "Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions," *Phys.Rev.* **D76** (2007) 126007. [arXiv:0707.2315 [hep-th]].

[87] J. P. Gauntlett and O. Varela, "Universal Kaluza-Klein reductions of type IIB to N=4 supergravity in five dimensions," *JHEP* **1006** (2010) 081, [arXiv:1003.5642 [hep-th]].

[88] G. W. Gibbons, D. N. Page, C. N. Pope, "Einstein Metrics on S**3 R**3 and R**4 Bundles," Commun. Math. Phys. **127** (1990) 529.

[89] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications," Phys. Rev. D **66**, 106006 (2002) [arXiv:0105097 [hep-th]].

[90] E. G. Gimon, J. Polchinski, "Consistency conditions for orientifolds and d manifolds," Phys. Rev. **D54**, 1667-1676 (1996). [arXiv:9601038 [hep-th]].

[91] A. Giveon and D. Kutasov, "Brane dynamics and gauge theory," Rev. Mod. Phys. **71**, 983 (1999) [arXiv:9802067 [hep-th]].

[92] A. Giveon, D. Kutasov, O. Lunin, "Spontaneous SUSY Breaking in Various Dimensions," Nucl. Phys. **B822**, 127-134 (2009). [arXiv:0904.2175 [hep-th]].

[93] I. Gordeli and D. Melnikov, "On I-even Singlet Glueballs in the Klebanov-Strassler Theory," [arXiv:0912.5517 [hep-th]].

[94] M. Graña, "Flux compactifications in string theory: A Comprehensive review," Phys. Rept. **423**, 91-158 (2006). [hep-th/0509003].

[95] M. Graña, J. Louis and D. Waldram, "Hitchin functionals in N = 2 supergravity," JHEP **0601**, 008 (2006) [arXiv:hep-th/0505264].

[96] M. Graña, J. Louis and D. Waldram, "SU(3) x SU(3) compactification and mirror duals of magnetic fluxes," JHEP **0704**, 101 (2007) [arXiv:hep-th/0612237].

[97] M. Graña, J. Louis, A. Sim and D. Waldram, "E7(7) formulation of N=2 backgrounds," JHEP **0907** (2009) 104 [arXiv:0904.2333 [hep-th]].

[98] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," JHEP **0408** (2004) 046 [arXiv:hep-th/0406137].

[99] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, "Generalized structures of N=1 vacua," JHEP **0511** (2005) 020 [arXiv:hep-th/0505212].

[100] M. Graña, R. Minasian, M. Petrini, A. Tomasiello, "A Scan for new N=1 vacua on twisted tori," JHEP **0705**, 031 (2007). [hep-th/0609124].

[101] M. Graña, R. Minasian, M. Petrini, D. Waldram, "T-duality, Generalized Geometry and Non-Geometric Backgrounds," JHEP **0904**, 075 (2009). [arXiv:0807.4527 [hep-th]].

[102] M. Graña, J. Polchinski, "Supersymmetric three form flux perturbations on AdS(5)," Phys. Rev. **D63**, 026001 (2001). [hep-th/0009211].

[103] M. Graña, H. Triendl, to appear.

[104] M. B. Green, J. H. Schwarz, E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. ( 1987) 596 P. ( Cambridge Monographs On Mathematical Physics).

[105] M. Gualtieri, "Generalized Complex Geometry," Oxford University DPhil thesis (2004) [arXiv:math.DG/0401221].

[106] S. S. Gubser, "Einstein manifolds and conformal field theories," Phys. Rev. **D59**, 025006 (1999). [hep-th/9807164].

[107] S. S. Gubser, C. P. Herzog, and I. R. Klebanov, "Symmetry breaking and axionic strings in the warped deformed conifold," *JHEP* **09** (2004) 036, [arXiv:0405282 [hep-th]].

[108] S. S. Gubser, C. P. Herzog, S. S. Pufu and T. Tesileanu, "Superconductors from Superstrings," Phys. Rev. Lett. **103**, 141601 (2009) [arXiv:0907.3510 [hep-th]].

[109] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, [arXiv:9802109 [hep-th]].

[110] S. S. Gubser, S. S. Pufu and F. D. Rocha, "Quantum critical superconductors in string theory and M-theory," Phys. Lett. B **683**, 201 (2010) [arXiv:0908.0011 [hep-th]].

[111] S. S. Gubser, I. R. Klebanov, "Baryons and domain walls in an N=1 superconformal gauge theory," Phys. Rev. **D58**, 125025 (1998). [hep-th/9808075].

[112] M. Haack, D. Lust, L. Martucci, A. Tomasiello, "Domain walls from ten dimensions," JHEP **0910**, 089 (2009). [arXiv:0905.1582 [hep-th]].

[113] N. Halmagyi, "Missing Mirrors: Type IIA Supergravity on the Resolved Conifold," [arXiv:1003.2121 [hep-th]].

[114] N. Halmagyi, M. Petrini, A. Zaffaroni, "Non-Relativistic Solutions of N=2 Gauged Supergravity," JHEP **1108**, 041 (2011). [arXiv:1102.5740 [hep-th]].

[115] N. Halmagyi, K. Pilch, C. Romelsberger, and N. P. Warner, "The Complex geometry of holographic flows of quiver gauge theories," *JHEP* **0609** (2006) 063, [arXiv:0406147 [hep-th]].

[116] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, "Emergent Quantum Near-Criticality from Baryonic Black Branes," *JHEP* **1003** (2010). [arXiv:0911.0400 [hep-th]].

[117] C. P. Herzog, I. R. Klebanov, and P. Ouyang, "Remarks on the warped deformed conifold," [arXiv:0108101 [hep-th]].

[118] C. P. Herzog, “String tensions and three-dimensional confining gauge theories,” Phys. Rev. **D66** (2002) 065009. [arXiv:hep-th/0205064].

[119] C. P. Herzog, I. R. Klebanov, “Gravity duals of fractional branes in various dimensions,” Phys. Rev. **D63**, 126005 (2001). [arXiv:hep-th/0101020].

[120] N. Hitchin, “The geometry of three-forms in six and seven dimensions,” J. Diff. Geom. **55** (2000), no.3 547 [arXiv: math.DG/0010054],

[121] N. Hitchin, “Stable forms and special metrics,” in “Global Differential Geometry: The Mathematical Legacy of Alfred Gray”, M.Fernandez and J.A.Wolf (eds.), Contemporary Mathematics **288**, American Mathematical Society, Providence (2001) [arXiv:math.DG/0107101],

[122] N. Hitchin, “Generalized Calabi-Yau manifolds,” Quart. J. Math. Oxford Ser. **54** (2003) 281 [arXiv:math.dg/0209099].

[123] N. Hitchin, “Lectures on generalized geometry,” [arXiv:1008.0973 [math.DG]].

[124] G. T. Horowitz, R. C. Myers, “The value of singularities,” Gen. Rel. Grav. **27** (1995) 915-919. [gr-qc/9503062].

[125] P. Hoxha, R. R. Martinez-Acosta, C. N. Pope, “Kaluza-Klein consistency, Killing vectors, and Kähler spaces,” Class. Quant. Grav. **17** (2000) 4207-4240. [ArXiv:hep-th/0005172].

[126] C. M. Hull, “Superstring Compactifications With Torsion And Space-Time Supersymmetry,” in *Turin 1985, Proceedings, Superunification and Extra Dimensions*, 347.

[127] C. M. Hull, “Compactifications Of The Heterotic Superstring,” Phys. Lett. B **178** (1986) 357.

[128] C. M. Hull, “Generalised geometry for M-theory,” JHEP **0707**, 079 (2007) [arXiv:hep-th/0701203].

[129] C. M. Hull, P. K. Townsend, “Unity of superstring dualities,” Nucl. Phys. **B438**, 109-137 (1995). [hep-th/9410167].

[130] K. A. Intriligator, N. Seiberg, “Lectures on Supersymmetry Breaking,” Class. Quant. Grav. **24**, S741-S772 (2007). [arXiv:hep-ph/0702069]

[131] K. A. Intriligator and N. Seiberg, “The Runaway quiver,” JHEP **0602**, 031 (2006) [arXiv:hep-th/0512347].

[132] K. A. Intriligator, N. Seiberg, D. Shih, “Dynamical SUSY breaking in meta-stable vacua,” JHEP **0604** (2006) 021. [arXiv:hep-th/0602239].

[133] K. A. Intriligator, N. Seiberg, D. Shih, "Supersymmetry breaking, R-symmetry breaking and metastable vacua," JHEP **0707**, 017 (2007). [arXiv:hep-th/0703281].

[134] S. Ivanov, G. Papadopoulos, "A No go theorem for string warped compactifications," Phys. Lett. **B497**, 309-316 (2001). [hep-th/0008232].

[135] S. Ivanov, G. Papadopoulos, "Vanishing theorems and string backgrounds," Class. Quant. Grav. **18**, 1089-1110 (2001). [math/0010038 [math-dg]].

[136] C. Jeschek and F. Witt, "Generalised G(2)-structures and type IIB superstrings," JHEP **0503**, 053 (2005) [arXiv:hep-th/0412280].

[137] D. D. Joyce "Compact manifolds with special holonomy", Oxford Mathematical Monographs series, **OUP**, July 2000.

[138] D. D. Joyce, "Compact Riemannian 7-Manifolds with Holonomy G2. I," J. Diff. Geom. **43** (1996) 291-328

[139] S. Kachru, R. Kallosh, A. D. Linde, S. P. Trivedi, "De Sitter vacua in string theory," Phys. Rev. **D68** (2003) 046005. [arXiv:hep-th/0301240].

[140] S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister, S. P. Trivedi, "Towards inflation in string theory," JCAP **0310** (2003) 013. [arXiv:hep-th/0308055].

[141] S. Kachru, J. Pearson, H. L. Verlinde, "Brane / flux annihilation and the string dual of a nonsupersymmetric field theory," JHEP **0206**, 021 (2002). [arXiv:hep-th/0112197].

[142] I. R. Klebanov, N. A. Nekrasov, "Gravity duals of fractional branes and logarithmic RG flow," Nucl. Phys. **B574**, 263-274 (2000). [hep-th/9911096].

[143] I. R. Klebanov, S. S. Pufu, "M-Branes and Metastable States," [arXiv:1006.3587].

[144] I. R. Klebanov, M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities," JHEP **0008**, 052 (2000). [arXiv:hep-th/0007191].

[145] I. R. Klebanov, A. A. Tseytlin, "Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories," Nucl. Phys. **B578**, 123-138 (2000). [hep-th/0002159].

[146] I. R. Klebanov, E. Witten, "Superconformal field theory on three-branes at a Calabi-Yau singularity," Nucl. Phys. **B536**, 199-218 (1998). [hep-th/9807080].

[147] P. Koerber, "Lectures on Generalized Complex Geometry for Physicists," Fortsch. Phys. **59** (2011) 169-242. [arXiv:1006.1536 [hep-th]].

[148] P. Koerber and L. Martucci, "Deformations of calibrated D-branes in flux generalized complex manifolds," JHEP **0612**, 062 (2006) [hep-th/0610044].

[149] P. Koerber and L. Martucci, "From ten to four and back again: how to generalize the geometry," JHEP **0708** (2007) 059 [arXiv:0707.1038 [hep-th]].

[150] P. Koerber and L. Martucci, "D-branes on AdS flux compactifications," JHEP **0801**, 047 (2008) [arXiv:0710.5530 [hep-th]].

[151] P. Koerber, D. Tsimpis, "Supersymmetric sources, integrability and generalized-structure compactifications," JHEP **0708**, 082 (2007). [arXiv:0706.1244 [hep-th]].

[152] Y. Kosmann-Schwarzbach, "Derived brackets," *Lett.Math.Phys.*, **69** 61-87, 2004, [arXiv:0312524 [math.dg]]

[153] R. G. Leigh, "Dirac-Born-Infeld Action from Dirichlet Sigma Model," Mod. Phys. Lett. **A4**, 2767 (1989).

[154] J. T. Liu, P. Szepietowski, "Supersymmetry of consistent massive truncations of IIB supergravity," [arXiv:1103.0029 [hep-th]].

[155] J. T. Liu, P. Szepietowski, and Z. Zhao, "Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds," *Phys.Rev.* **D81** (2010) 124028, [arXiv:1003.5374 [hep-th]].

[156] A. Loewy, Y. Oz, "Branes in special holonomy backgrounds," Phys. Lett. **B537** (2002) 147-154. [arXiv:hep-th/0203092].

[157] H. Lu, C. N. Pope, and K. S. Stelle, "M-theory/heterotic duality: A Kaluza-Klein perspective," *Nucl. Phys.* **B548** (1999) 87–138, [arXiv:9810159 [hep-th]].

[158] D. Lust, F. Marchesano, L. Martucci, D. Tsimpis, "Generalized non-supersymmetric flux vacua," JHEP **0811**, 021 (2008). [arXiv:0807.4540 [hep-th]].

[159] D. Lust, P. Patalong, D. Tsimpis, "Generalized geometry, calibrations and supersymmetry in diverse dimensions," JHEP **1101**, 063 (2011). [arXiv:1010.5789 [hep-th]].

[160] D. Lust, D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity," JHEP **0502**, 027 (2005). [hep-th/0412250].

[161] D. Lust, D. Tsimpis, "Classes of AdS(4) type IIA/IIB compactifications with SU(3) x SU(3) structure," JHEP **0904** (2009) 111. [arXiv:0901.4474 [hep-th]].

[162] D. Lust, D. Tsimpis, "New supersymmetric AdS(4) type II vacua," JHEP **0909** (2009) 098. [arXiv:0906.2561 [hep-th]].

[163] J. Maharana, J. H. Schwarz, "Noncompact symmetries in string theory," Nucl. Phys. **B390**, 3-32 (1993). [hep-th/9207016].

[164] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [arXiv:9711200 [hep-th]].

[165] J. M. Maldacena, C. Nuñez, "Towards the large N limit of pure N=1 superYang-Mills," Phys. Rev. Lett. **86**, 588-591 (2001). [hep-th/0008001].

[166] J. M. Maldacena, C. Nuñez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. **A16**, 822-855 (2001). [hep-th/0007018].

[167] J. Maldacena and D. Martelli, "The unwarped, resolved, deformed conifold: Five-branes and the baryonic branch of the Klebanov-Strassler theory,". [arXiv:0906.0591 [hep-th]].

[168] J. Maldacena, D. Martelli and Y. Tachikawa, "Comments on string theory backgrounds with non-relativistic conformal symmetry," JHEP **0810**, 072 (2008) [arXiv:0807.1100 [hep-th]].

[169] L. Martucci, "Supersymmetric D-branes on flux backgrounds," Fortsch. Phys. **55**, 771 (2007) [hep-th/0701093].

[170] L. Martucci and P. Smyth, "Supersymmetric D-branes and calibrations on general N=1 backgrounds," JHEP **0511**, 048 (2005) [hep-th/0507099].

[171] R. C. Myers, "Dielectric branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[172] P. McGuirk, G. Shiu, Y. Sumitomo, "Non-supersymmetric infrared perturbations to the warped deformed conifold," Nucl. Phys. **B842** (2010) 383-413. [arXiv:0910.4581 [hep-th]].

[173] R. Minasian and D. Tsimpis, "On the geometry of non-trivially embedded branes," *Nucl. Phys.* **B572** (2000) 499–513. [arXiv:9911042 [hep-th]].

[174] S. Mukai, "Symplectic Structure of the Moduli Space of Sheaves on an Abelian or K3 Surface," *InventMath.*, 77 101-116, (1984).

[175] H. Ooguri, Y. Ookouchi, "Meta-Stable Supersymmetry Breaking Vacua on Intersecting Branes," Phys. Lett. **B641** (2006) 323-328. [arXiv:hep-th/0607183].

[176] P. P. Pacheco and D. Waldram, "M-theory, exceptional generalised geometry and superpotentials," JHEP **0809** (2008) 123 [arXiv:0804.1362 [hep-th]].

[177] L. A. Pando Zayas, A. A. Tseytlin, "3-branes on resolved conifold," JHEP **0011**, 028 (2000). [hep-th/0010088].

[178] G. Papadopoulos and A. A. Tseytlin, "Complex geometry of conifolds and 5-brane wrapped on 2- sphere," *Class. Quant. Grav.* **18** (2001) 1333–1354, [arXiv:0012034 [hep-th]].

[179] J. Polchinski, M. J. Strassler, "The String dual of a confining four-dimensional gauge theory," [hep-th/0003136].

[180] J. Polchinski, A. Strominger, "New vacua for type II string theory," Phys. Lett. **B388** (1996) 736-742. [hep-th/9510227].

[181] C. N. Pope and N. P. Warner, "An SU(4) Invariant Compactification of d = 11 Supergravity on a Stretched Seven Sphere," *Phys. Lett.* **B150** (1985) 352. [arXiv:1008.0883 [hep-th]].

[182] L. Randall, R. Sundrum, "A Large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83**, 3370-3373 (1999). [hep-ph/9905221].

[183] L. J. Romans, "Massive N=2a Supergravity in Ten-Dimensions," Phys. Lett. **B169**, 374 (1986).

[184] H. Samtleben, "Lectures on Gauged Supergravity and Flux Compactifications," *Class. Quant. Grav.* **25** (2008) 214002, [arXiv:0808.4076 [hep-th]].

[185] J. Schon and M. Weidner, "Gauged N = 4 supergravities," *JHEP* **05** (2006) 034, [arXiv:0602024 [hep-th]].

[186] J. H. Schwarz, "Convariant Field Equations of Chiral N=2 D = 10 Supergravity," *Nucl. Phys.* **B226** (1983) 269. [arXiv:1008.0883 [hep-th]].

[187] A. Sen, C. Vafa, "Dual pairs of type II string compactification," Nucl. Phys. **B455**, 165-187 (1995). [hep-th/9508064].

[188] P. Sěvera and A. Weinstein, "Poisson geometry with a 3 form background," *Prog. Theor. Phys. Suppl.* **144** (2001) 145 [arXiv: 0107133 [math-sg]].

[189] Y. Shadmi, Y. Shirman, "Dynamical supersymmetry breaking," Rev. Mod. Phys. **72** (2000) 25-64. [arXiv:hep-th/9907225].

[190] K. Skenderis, M. Taylor, and D. Tsimpis, "A Consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure,". [arXiv: 1003.5657 [hep-th]].

[191] R. Slansky, "Group Theory for Unified Model Building," Phys. Rept. **79**, 1 (1981).

[192] J. Sparks, "Sasaki-Einstein Manifolds," arXiv:1004.2461 [math.DG].

[193] M.B. Stenzel, "Ricci-flat metrics on the complexification of a compact rank one symmetric space," Manuscripta Mathematica, **80**, 151 (1993).

[194] A. Strominger, "Superstrings with Torsion," Nucl. Phys. B **274** (1986) 253.

[195] T. R. Taylor, C. Vafa, "R-R flux on Calabi-Yau and partial supersymmetry breaking," Phys. Lett. **B474**, 130-137 (2000). [hep-th/9912152].

[196] A. Tomasiello, "Reformulating supersymmetry with a generalized Dolbeault operator," JHEP **0802**, 010 (2008). [arXiv:0704.2613 [hep-th]].

[197] A. Tomasiello, “Geometrical methods for string compactifications,” LACES 2009.

[198] C. Vafa, “The String landscape and the swampland,” [hep-th/0509212]

[199] S. Weinberg “Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity” John Wiley and Sons, Inc. (1972)

[200] E. Witten, “Anti-de sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [arXiv:9802150 [hep-th]].