Thèse de Doctorat

Présenté à l'Université Pierre et Marie Curie — Paris 6

Spécialité: Physique théorique

Structures intégrables dans les théories de jauge et les théories des cordes supersymmétriques

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Integrable structures in the gauge theories and in the supersymmetric string theories

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Soutenu le 3 Octobre 2012 devant le jury composé de:

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Abstract

In this thesis is given a review of the methods of integrability in the context of the AdS/CFT correspondence. We investigate integrable structures on both sides of the AdS/CFT duality using different methods.

On the string side of the duality we observe how the supersymmetry and automorphism of the symmetry group organize the model into integrable one. Then, using the consequences of the finite gap method for the integrable system we perform a one–loop quantization procedure which allows us to compute the one–loop spectrum of the model. We illustrate this method by computing the spectrum of a short string.

On the gauge side we review the method of the functional Y–system equations for computing the spectrum of the theory in the finite volume. Due to the existence of the two–particle S–matrix it is possible to use the Zamolodchikov's trick to setup a system of functional equations, which can be later recast as a Hirota equation defined on some domain. In the strong coupling limit these equations can be drastically simplified. This gives us a chance to have an analytic solution of them, which can be compared to the string side computation. These two results are in a perfect agreement.

Contents

Chapter 1

Introduction.

This thesis is devoted to the study of integrability and its application to Maldacena's AdS/CFT duality [1, 2, 3, 4]. This duality is one of many gauge/gravity correspondences between theories of quantum gravity and particle physics. Let us review this duality very briefly.

In string theory, contrary to the usual quantum field theory, particles are not point-like, but rather small vibrating strings. They can form a closed loop or have they ends attached to hypersurfaces, which are called *D-branes* [5]. They are classified by their spatial dimensions, so, for example, D2 is just a plane. The condition that open strings can end on D-branes (and nowhere else) means that there exists a particular spectrum of such strings. By quantizing these open strings we obtain all the fields that propagate between these D-branes.

This quantum propagation can be described by the usual path integral, and, in principle, by the corresponding Feynman diagrams. Since our fundamental objects have some size, these diagrams would not consist of usual lines and vertices, but rather of Riemann surfaces with h handles and n holes. The sum over all possible topologies will give us the propagator in the given conditions.

This general scheme leads to the very important duality when we consider the stack of N D3 branes in type IIB superstring theory in 10 dimensional space. The excitations of these branes are the open strings, and the closed strings are the excitations of the 10 dimensional empty space. The low energy effective action reads

$$S = S_{brane} + S_{SUGRA} + S_{int}, \tag{1.0.1}$$

where S_{brane} describes the massless string states by an $\mathcal{N} = 4$ SYM lagrangian. S_{SUGRA} is the bulk action, or, in other words, the type IIB supergravity effective action. S_{int} is the coupling of these two systems.

In the low energy limit when coupling of the two systems is small ($\alpha' \to 0$) we obtain a two decoupled systems: four dimensional $U(N)$ SYM and ten dimensional type IIB supergravity. The brane is therefore an heavy hyperplane and the geometry of the deformed bulk is

$$ds^2 = \frac{ds_4^2}{\sqrt{1+\frac{l^4}{r^4}}} + \sqrt{1+\frac{l^4}{r^4}}\left(dr^2 + r^2 d\Omega_{S^5}^2\right), \qquad l^4 = 4\pi g_s N \alpha'^2. \tag{1.0.2}$$

One can see that near horizon ($r = 0$) geometry is equivalent to the $AdS_5 \times S^5$ space:

$$ds^2 = l^2\left(\frac{dU^2}{U^2} + U^2 ds_4^2\right) + l^2 d\Omega_{S^5}^2, \quad U = r/l^2. \tag{1.0.3}$$

All string excitations survive in this limit. That's why it is rather natural to conjecture that there is an equivalence of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and string theory on the $AdS_5 \times S^5$.

The main parameters in these theories are the number of colours N, coupling constant λ (or g_s), string tension T which is related to the α' and the radius of the S^5 (or AdS_5 since they are equal). The anomalous dimensions of the operators in gauge theory Δ are dual to the energies of the corresponding strings solutions E. The duality itself relates classical free strings and the strong coupled SYM (or, highly quantum strings with the perturbative regime in SYM). So we can hope that some new ingredients would shed some light on the origins of this duality.

Luckily, in the last decade, enormous progress was made in computing the spectrum of conformal dimensions of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the planar limit (when $N \to \infty$). This progress was made possible by the discovery of integrability [6]. Usually the presence of integrability means that we can solve the system exactly, and this is extremely important for the understading the nature of AdS/CFT correspondence.

To understand this importance let us recall that the AdS/CFT correspondence between gauge and string theories is the duality of weak/strong coupling type. That means that except for the quantities protected by the symmetry and some special limiting regimes, comparison of gauge and string theory predictions requires essentially nonperturbative calculations.

One of the main properties of the AdS/CFT correspondence was that integrability was discovered on the both sides of the duality. On the side of the gauge theory integrability traces back to the high–energy QCD [7, 8], where it plays main role in the solution of the evolution equation. Then, almost 10 years later, it was discovered in the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory: firstly at one loop [6] and then it was conjectured to hold at all loops [9]. According to the integrability conjecture, the single trace local operators correspond to the states of an integrable spin chain in which the dilatation operator plays the role of a Hamiltonian. At one–loop level this is a spin chain with the nearest neighbors' interactions. It can be diagonalized for example by algebraic Bethe Ansatz (see, for example, [10] for a review). Here Bethe Ansatz played a bit non–standard role. It was known that it works for integrable 1+1 dimensional theory, but in our case it allows us to solve (i.e. find the spectrum) the four–dimensional theory. Moreover, it was possible at arbitrary coupling constant. The conjectured asymptotic Bethe ansatz equations [11, 12, 13] (also in the review cited above) interpolating between weak and strong coupling allowed to perform refined checks for operators with large charges. Thus the $\mathcal{N}=4$ SYM was the first four–dimensional theory solved by the means of the integrability.

Since we managed to reduce (in some sense) our four–dimensional problem to the two–dimensional one, it is not very suprpising that we would be able to apply another integrable methods to solve the system completely. Namely, one can perform the Zamolodchikov trick [14] to write down the Y-system equations. These equations allows us to compute the spectrum of the theory in the finite volume (or, having in mind the operator/spin chain correspondence, the anomalous dimension of operator of any length). It occurs, however, that analytic solution is possible only in a few situations. One of such situation — computation of the anomalous dimension of the Konishi operator [15] — we will describe in details.

On the string side of the theory the classical integrability was discovered in [16]. The equations of motion of the string sigma-model for $AdS_5 \times S^5$ background (which is dual to $\mathcal{N}=4$ SYM theory), admit a Lax representation. This Lax representation generates an infinite set of conserved charges [17]. Although there is no proof, but there is a huge evidence that integrability is still in place even in quantum case.

Due to integrability, the equations of motion of the strings can be integrated by the finite–gap technique [18]. This method gives rise to the multi–valued function $p(x)$ which is defined on a Riemann surfrace. This function has a physical meaning as quasimomenta, completely analagous to the Bohr–Sommerfeld quasimomenta in non–relativistic quanum mechanics. Suprisingly, it is possible to perform a one–loop quantization procedure based on the Bohr–Sommerfeld technique. We did for a concrete solution — folded string.

Then we perform a short string expansion. The semiclassical approach typically demands that the conserved charges scale as the coupling constant. For the case of the spinning folded string the two charges, the Lorentz spin S and R-charge J, should scale in such a way that the ratios $\mathcal{S}=S/\sqrt{\lambda}$, $\mathcal{J}=J/\sqrt{\lambda}$ remain fixed. The expansion of the energy is of the form

$$E \equiv \gamma + S + J = \sqrt{\lambda} E_0(\mathcal{S},\mathcal{J}) + E_1(\mathcal{S},\mathcal{J}) + \frac{1}{\sqrt{\lambda}} E_2(\mathcal{S},\mathcal{J}) + \dots\,. \tag{1.0.4}$$

Then, in order to approach the short operator regime we re-expand the result in the limit when $S, J \sim 1$ and thus $\mathcal{S} \sim \mathcal{J} \ll 1$. As it was pointed out in [19] the expansion above reorganizes into a power series of the type

$$E = \lambda^{1/4} a_0 + \frac{1}{\lambda^{1/4}} a_2 + \dots\,, \tag{1.0.5}$$

where only the classical energy E_0 contributes to the first coefficient a_0 and both E_1 and E_0 contribute to the coefficient a_2. Thus with some caution one may assume that the short strings with $S, J \sim 1$, which are in principle deeply quantum states, still can be reached using the quasi-classical methods. In this way, the complications with the direct treatment of the Y-system can be escaped. In this thesis we compute the first two coefficients in the expansion (1.0.4) using the algebraic curve quantization procedure for an arbitrary $\mathcal{S}$ and $\mathcal{J}$.

What is important is that in the result we do not get any logarithmic terms which would signal order-of-limits problems. From the Asymptotic Bethe Ansatz (ABA) we know that the Konishi state in the $sl(2)$ sector is given by $S = J = 2$ and $n = 1$. Substituting these values of the parameters we indeed obtain a result consistent with the numerical predictions and the predictions from the perturbative string computations [20].

Another example, which is treated in the section 3.5, is that of the long strings with large Lorentz spin S and small twist $J = \ell\, 4g \log S$. It is well understood that this case can be obtained from the generic two-cut solution in the $sl(2)$ sector [21, 22, 23, 24]

$$E(\ell, \mathscr{S}) = \left(\sqrt{\lambda} f_0(\ell) + f_1(\ell) + \frac{1}{\sqrt{\lambda}} f_2(\ell)\right) \log \mathscr{S} + \dots. \tag{1.0.6}$$

The leading logarithmic scaling is a generic feature in gauge theories [25, 26, 27, 28] and the coefficient of $\log \mathscr{S}$ is the so-called generalized scaling function. The functions $f_0(\ell)$, $f_1(\ell)$, $f_2(\ell)$ were derived explicitly in [21, 22, 23, 24, 29, 30].

It happens that all the three coefficients have a well-defined limit at $\ell = 0$ and that in this limit they reproduce correctly the strong coupling expansion of the cusp anomalous dimension. In particular the $\ell = 0$ result obtained in this order of limits coincides with the solution obtained [31, 32, 33, 34, 35, 36, 37] via the Beisert–Eden–Staudacher equation [13, 38], which supposes $J = 2$ and S large and then $g \to \infty$. A recent review of this subject appeared in [39].

In this thesis we also revisit the computation of the classical and one-loop energy for the long string both from the point of view of the algebraic curve and the Y-system, having in view the finite size corrections. We obtain results for all orders in $1/\log S$ and we neglect terms of the order $\log S/S$ and higher. At $\ell = 0$ the result is particularly simple

$$E_{\ell=0} = S + J + 4g\left(\log\frac{2S}{g} - 1\right) - \frac{3\log 2}{\pi}\log\frac{2S}{g} + \frac{6\log 2}{\pi} + 1 - \frac{5\pi}{12\log(2S/g)} + \mathscr{O}(1/g) \tag{1.0.7}$$

The subleading part in $\log S$ is the so-called virtual scaling function computed in [40, 41] while the $1/\log S$ part agrees with the results in [42, 43, 44]. In [44] the last term in (1.0.7) was given the simple interpretation of contribution of massless excitations propagating on a string of length $L = 2\log S$, with total result:

$$\delta E_1 = -\frac{\pi}{12\log S} \times (\text{number of massless modes}). \tag{1.0.8}$$

The massive modes lead to correction of the type e^{-mL}, where $m \sim \ell$ and $L = 2\log S$; these contributions have to be summed up properly in order to reproduce the massless limit.

In our computation, the four massive mode contributions come via the wrapping corrections. From the Y-system point of view, this part is constituted by two contributions (virtual particle contribution and back-reaction of the roots) which become separately divergent when $\ell \to 0$. The algebraic curve computation does not see any divergence, and this may be compared to the particularly smooth behavior of the algebraic curve prediction for the short strings.

Finally, the contribution of the massless mode comes via the asymptotic Bethe ansatz. This might seem surprising, since at finite coupling there are no $1/\log S$ corrections for the twist-two operator $J=2$, just the $(\log S)^0$ term [45, 46]. This is obviously due to the different order of limits which are considered and might be explained by the fact that the bosonic modes of the $O(6)$ sigma model [47, 48, 49] acquire a dynamically generated mass at finite coupling.

As the result we will be able to check our predictions for the short string energy with the anomalous dimension of the corresponding operator in the gauge theory, which will be computed also using integrability.

In the chapter 4 we give a review of the Thermodynamical Bethe Ansatz (TBA) exemplifying it on the $O(4)$ σ–model. It is one of the simplest models for illustrating the nested Bethe ansatz technique, but all the main properties are still true for more complicated models, like $PSU(2,2|4)$ σ–model which we will discuss in the chapter 5. We will derive from the TBA the system of coupled equations (Y–system), which in principle could be solved.

There is very powerful method for solving these equations in the general setup — Backlund transformations [50, 51, 52]. We will illustrate this on the example of the $GL(N|M)$ group and then move to our case. It occurs that some group structure (the existence of the $\mathbb{Z}_4$ automorphism) is crucial for the Y–system construction — the same structure which was very important for the integrability in the string side of the duality.

In the last chapter we compare two sides of the duality and two quantities which were obtained by the very different techniques. It occurs despite of the different origins the two expressions from different side of duality can be more or less easily recasted from one to another, giving, of course, the same result for the spectrum of the theory.

The main results of this thesis are published in the paper [53]. There are also some unpublished results concerning string theory on arbitrary background which will appear in the paper with K. Zarembo.

I would like to express my gratitude to my advisors — Ivan Kostov and Didina Serban, for many interesting discussions during all these 3 years from which I benefited a lot. Every day I felt their kindness and deep support in all my studies.

It is pleasure for me to acknowledge Iosif Bena, Gregory Korchemsky, Vladimir Kazakov and Vincent Pasquier for the stimulating discussions at the Institute or around. During my two-months visit to Nordita in 2011 I was lucky to discuss with Konstantin Zarembo, whom I am very grateful to. I would like to thank my collaborators, Dmytro Volin and Nikolay Gromov for the great experience of working together.

This thesis could not be defended without jury, so my sincere thanks goes to Arkady Tseytlin, Charlotte Kristjansen, Joe Minahan and Jean-Bernard Zuber for the very quick reports in the shortage of time and very insightful comments and questions during the defence.

Also I would like to thank all the stuff of the Institute for the hospitability and their readiness to help.

Finally I am very thankful to my father and my wife for their love, support and patience.

Chapter 2

The algebraic curve setup

In this chapter we will present the scheme of the quasi-classical quantization based on the algebraic curve formalism.

First we will review the action of the theory. Starting from the classical bosonic action ([54]) we will explore integrability of the model [16]. We will show how one can define the flat connection for the σ–model on the curved background. Then, using the fact that this connection depends on the spectral parameter $x \in \mathbb{C}$, we will argue that this system is classically integrable, because it will be possible to construct the infinte tower of the conserved charges.

Of course, this construction is quite general, but this can be applied to the concrete classical solutions (see, for example, [55, 56] for reviews). We will consider the two-cut folded string solution. Based on the quasi-classical quantization scheme [57] we will quantize it and then proceed to the short string limit to compare it with the gauge theory computation.

2.1 The Green–Schwarz action

Now we will give a short review of the Metsaev–Tseytlin construction. The type IIB superstring theory in a curved space with a 5-form Ramond-Ramond background is defined by the Green–Schwarz action [58] (the fermion part the of the action taken from [59]):

$$\mathscr{L} = \frac{1}{2}\sqrt{-h}h^{ab}\eta_{\hat{M}\hat{N}}E_a^{\hat{M}}E_b^{\hat{N}} + \bar{\theta}^I\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\not{E}_a\mathscr{D}_b^{JK}\theta^K, \tag{2.1.1}$$

where θ^I are 10d Majorana spinors ($\bar{\theta} = \theta^t C$) satisfying

$$\Gamma^{11}\theta^I = \begin{cases} \sigma_3^{IJ}\theta^J & \text{in type IIA} \\ \theta^I & \text{in type IIB.} \end{cases} \tag{2.1.2}$$

The worldsheet projections of the local frame $E_M^{\hat{M}}$ ($\eta_{\hat{M}\hat{N}}E_M^{\hat{M}}E_N^{\hat{N}} = G_{MN}$), and their Dirac contractions are defined as

$$E_a^{\hat{M}} = \partial_a X^M E_M^{\hat{M}}, \qquad \not{E}_a = E_a^{\hat{M}}\Gamma_{\hat{M}}. \tag{2.1.3}$$

The covariant derivative acts on the worldsheet fermions as

$$\mathscr{D}_b^{JK} = D_b \delta^{JK} + \frac{1}{8} \mathscr{F}^{JK} \not{E}_b, \tag{2.1.4}$$

where D_b is the ordinary covariant derivative projected onto the worldsheet:

$$D_b = \partial_b + \frac{1}{4} \partial_b X^M \Omega_M^{\hat{M}\hat{N}} \Gamma_{\hat{M}\hat{N}}, \tag{2.1.5}$$

$\Omega_M^{\hat{M}\hat{N}}$ is the spin connection and $\mathscr{F}^{JK}$ contains the coupling to the RR fields:

$$\mathscr{F}^{JK} = \sum_{n=1}^{5} \frac{1}{n!} F_{\hat{M}_1 \ldots \hat{M}_n} \Gamma^{\hat{M}_1 \ldots \hat{M}_n} \sigma_{(n)}^{JK}, \tag{2.1.6}$$

where $F_{\hat{M}_1 \ldots \hat{M}_n}$ is the field strength of the $(n-1)$-form RR field, projected onto the local frame, and

$$\sigma_{(1)} = -i\sigma_2, \qquad \sigma_{(2)} = \sigma_3, \qquad \sigma_{(3)} = \sigma_1, \qquad \sigma_{(4)} = -\mathbb{1}, \qquad \sigma_{(5)} = -\frac{i}{2}\sigma_2. \tag{2.1.7}$$

2.2 Bosonic string in the $AdS_5 \times S^5$

As we saw in the previous section, one can in principle write down the gauge–fixed full lagrangian for bosons and fermions. However, it will be a bit complicated expression. It is suitable for studying, for example, worldsheet scattering of the excitations [60, 61], but for our purposes we can boil this construction down to the classical level.

On the classical level all the fermion fields vanish, so we are left with the pure bosonic action. In the conformal gauge the AdS_5 and S^5 parts of the theory are decoupled, and the only interaction between them is only due to the Virasoro conditions. We consider the bosonic part of the full action below.

Firstly, let us define the background rigorously. The AdS_d space is a hypersurface described by the equation

$$X_{-1}^2 + X_0^2 - X_1^2 - \ldots X_d^2 = 1. \tag{2.2.1}$$

Also there is a metric inherited from the $\mathbb{R}^d$, that is, the action of the $SO(d,2)$ group and AdS_{d+1} is a homogeneous space of $SO(d,2)$. Moreover, there is a *little group* for this space — $SO(d,1)$. That's why AdS_{d+1} has the coset structure: $AdS_{d+1} = SO(d,2)/SO(d,1)$. Equivalently, AdS can be defined as a set of equivalence classes of the right $SO(d,1)$ action on $SO(d,2)$.

One can define the string action on AdS_{d+1} starting with sigma–model on $SO(d,2)$ and then gauging the little group action $SO(d,1)$ by a non-dynamical gauge field. It is always possible to construct the gauge action using the little group formalism (see, for example, [62]). The gauge transformations are right multiplications from $SO(d,1)$: $g(x) \to g(x)h(x)$. In this setup gauge fixing is equivalent to picking one representative in each class (i.e. embedding of AdS_{d+2} in $SO(d,2)$). For example in AdS_5, one can use Poincare coordinates (note that they cover only the part of AdS_5):

$$g=\begin{pmatrix} \frac{z}{2}\left(1+\frac{1+x^2}{z^2}\right) & \frac{x_\nu}{z} & \frac{z}{2}\left(1-\frac{1-x^2}{z^2}\right) \\ \frac{x_\mu}{z} & \eta_{\mu\nu}+\frac{2}{(z+1)^2+x^2}\frac{x_\mu x_\nu}{z} & \frac{z x_\mu}{(z+1)^2+x^2}\left(1-\frac{1-x^2}{z^2}\right) \\ \frac{z}{2}\left(1-\frac{1-x^2}{z^2}\right) & \frac{z x_\nu}{(z+1)^2+x^2}\left(1-\frac{1-x^2}{z^2}\right) & 1+\frac{1}{2}\frac{z^3}{(z+1)^2+x^2}\left(1-\frac{1-x^2}{z^2}\right)^2 \end{pmatrix}, \tag{2.2.2}$$

where $\mu,\nu=0,\dots,d-1$. Finally the metric in the $AdS_5\times S^5$ can be written in the conformally-flat form

$$ds^2=\frac{dz^2+dx^2}{z^2}. \tag{2.2.3}$$

However, for our purposes it is better to keep everything on the abstract level, without the concrete realization of the embeddings.

Let us discuss the transformation properties of the current

$$J_a=g^{-1}\partial_a g. \tag{2.2.4}$$

The current algebra is Lie algebra $\mathfrak{so}(d,2)$ and transforms like

$$J_a\to h^{-1}J_a h+h^{-1}\partial_a h, \tag{2.2.5}$$

where $h\in SO(d,1)$ and corresponds to the non–homogeneous part of the transformation. This suggests us to decompose the current into two parts:

$$J_a=J_{a0}+J_{a2}, \tag{2.2.6}$$

where $J_{a0}\in\mathfrak{so}(d,1)$ and J_{a2} belongs to the orthogonal complement to $\mathfrak{so}(d,2)$. In the standard notation of embedding $\mathfrak{so}(d,1)\subset\mathfrak{so}(d,2)$ this complement is the first row $(d+1)$–dimensional vector.

We denote this orthogonal element by $\mathfrak{f}$:

$$\mathfrak{so}(d,2)=\mathfrak{so}(d,1)\oplus\mathfrak{f}. \tag{2.2.7}$$

Under the gauge transformations the non–homogeneous term is absorbed naturally into J_{a0}, while the J_{a2} component of the current transforms as the matter field: $J_{a2}\to h^{-1}J_{a2}h$. So we can use this current to construct a gauge–invariant string action:

$$S=g\int d^2x\sqrt{-h}h^{ab}\,\mathrm{tr}\,J_{a2}J_{b2} \tag{2.2.8}$$

This allows us to immediately construct the supersymmetric completion of the sigma–model, which will be done below (see chapter 2.2.2) and prove (at least clasically) integrability (chapter 2.2.1). This parametrization is especially useful since it does not employ the partricular embedding. All the equations can be thus written entirely in terms of currents.

The variation of the action gives the conservation law:

$$2D_a\left(\sqrt{-h}h^{ab}J_{b2}\right)=0, \tag{2.2.9}$$

where D_a is a gauge derivative built out the gauge current J_{a0}:

$$D_a = \partial_a + [J_{a0}, \cdot]. \tag{2.2.10}$$

The flatness condition of J_2 projected onto $\mathfrak{so}(d,1)$ and orthogonal completion looks like

$$\begin{aligned} D_a J_{b2} - D_b J_{a2} &= 0, & (2.2.11)\\ F_{ab} + [J_{a2}, J_{b2}] &= 0. & (2.2.12)\end{aligned}$$

Here the F_{ab} is naturally connection: $F_{ab} = \partial_a J_{b0} - \partial_b J_{a0} + [J_{a0}, J_{b0}]$. Also we can derive the Virasoro constraints from the equation of motion for the metric:

$$h^{ab} \mathrm{tr}\, J_{\pm a2} J_{\pm b2} = 0. \tag{2.2.13}$$

Here $\pm$ superscript denotes the worldsheet light-cone projections:

$$J_{\pm a2} = \left(\delta_a^b \pm \frac{1}{\sqrt{-h}} h_{ac} \epsilon^{cb} \right) J_{b2}. \tag{2.2.14}$$

In the next section we will study the remarkable property of these equations, namely, integrability.

2.2.1 Integrability

The geometric origin of integrability of sigma–model on AdS_{d+1} is an extra $\mathbb{Z}_2$ symmetry of the AdS metric. It is invariant under the reflection $X_A \to -X_A$. The AdS_{d+1} hypersurface is a symmetric space.

In the coset construction the $\mathbb{Z}_2$ symmetry acts by changing the sign of the J_2 component of the current:

$$J_{a0} \to J_{a0}, \quad J_{a2} \to -J_{a2}. \tag{2.2.15}$$

The action (2.2.8) and the equations of motion (2.2.9), (2.2.11) are invariant under this transformation. On the more formal level (which will be useful in the supersymmetric case), the $\mathbb{Z}_2$ symmetry can be defined as an automorphism of the algebra $\mathfrak{so}(d,2)$ which preserves the coset decomposition. The automorphism acts trivially on $\mathfrak{so}(d,1)$ but changes the sign of all the orthogonal elements (which reflects in changing the sign in the corresponding current).

The crucial point for integrability is that this transformation is consistent with the commutation relations of $\mathfrak{so}(d,2)$, so that the reflection of the orthogonal element $\mathfrak{f}$ is a symmetry of $\mathfrak{so}(d,2)$. Let us write down the following relations:

$$[\mathfrak{so}(d,1), \mathfrak{so}(d,1)] \in \mathfrak{so}(d,1), \quad [\mathfrak{so}(d,1), \mathfrak{f}] \in \mathfrak{f}, \quad [\mathfrak{f}, \mathfrak{f}] \in \mathfrak{so}(d,1). \tag{2.2.16}$$

Because of this, the flatness condition decomposes into the two equations (2.2.11), which admits the Lax representation:

$$L_a = J_{a0} + \frac{x^2+1}{x^2-1} J_{a2} - \frac{2x}{x^2-1} \frac{1}{\sqrt{-h}} h_{ab} \epsilon^{bc} J_{c2}. \tag{2.2.17}$$

If the currents satisfy the equations of motion, the Lax connection is flat:

$$\partial_a L_b - \partial_b L_a + [L_a, L_b] = 0. \tag{2.2.18}$$

The Virasoro constraints do not follow from the Lax representation, but are very natural for integrability [63].

Using this Lax representation one can immediately construct the tower of conserved charges. To do this, we write down the monodromy matrix

$$\Omega(x) = P\exp\oint_\gamma L(x). \tag{2.2.19}$$

Here γ is a path of equal time on the worldsheet. However, due to the flatness condition we can deform our contour without changing the integral.

This monodromy is a group element of the $SO(d,2)$ group and hence eigenvalues of the monodromy matrix is gauge-invariant. Also they are time-independent, so they can be used to define the desired tower of conserved charges. More precisely, we construct the matrix $T(x,z)$

$$T(x,z) \equiv \mathrm{tr}(z\mathbb{1} - \Omega(x)), \tag{2.2.20}$$

and expand it in x and z. The coefficients in this expansion are the conserved charges of our model.

2.2.2 Switching on the supersymmetry

The self-consistent AdS_{d+1} backgrounds are supersymmetric, so we will need to couple σ–model on AdS with fermions. In the case of $AdS_5 \times S^5$ the coset looks as follows [64]

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}, \tag{2.2.21}$$

and is the coset of the $PSU(2,2|4)$, superconformal group of the dual $\mathcal{N}=4$ SYM theory. Green–Schwarz action should contain usual metric coupling term $g_{MN}\partial X^M \partial X^M$ and a fermionic Wess-Zumino term. The coset construction (2.2.21) provides a natural way because of the $\mathbb{Z}_4$ automorphism – an extension of the $\mathbb{Z}_2$ symmetry discussed in the previous section.

Full classification of all $\mathbb{Z}_4$ cosets was given by Zarembo [65]. They happen to be integrable and contain AdS as part of their supergeometry. We review briefly the construction of these cosets.

A coset G/H_0 of the supergroup G posseses a $\mathbb{Z}_4$ supersymmetry if $\mathfrak{h}_0$ is invariant under linear automorphism $\mathcal{M}$ of order 4 that acts on $\mathfrak{g}$. An automorphism $\mathcal{M}$ is a linear map from $\mathfrak{g}$ to $\mathfrak{g}$ that preserves the Lie bracket. The diagonalization of the $\mathbb{Z}_4$ charge

$$\mathcal{M}(\mathfrak{h}_n) = i^n \mathfrak{h}_n, \qquad n = 0,1,2,3 \tag{2.2.22}$$

defines a $\mathbb{Z}_4$ decomposition of $\mathfrak{g}$ (grading):

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3. \tag{2.2.23}$$

This decomposition is consistent with the commutation relations:

$$\{\mathfrak{h}_n, \mathfrak{h}_m\} \in \mathfrak{h}_{n+m (\mod 4)}. \tag{2.2.24}$$

Taking in the account the fact that $\mathscr{M}^2 = (-1)^F$ (F is the number of fermions) we conclude that $\mathfrak{h}_0 \oplus \mathfrak{h}_2$ is the bosonic superalgebra of $\mathfrak{g}$.

The coset representative, $g(x)$, is subject to gauge transformations $g(x) \to g(x)h(x)$, where $h(x) \in H_0$. The decomposition of the current J_a now contains 4 terms due to the $\mathbb{Z}_4$ grading:

$$J_a = g^{-1}\partial_a g = J_{a0} + J_{a1} + J_{a2} + J_{a3}. \tag{2.2.25}$$

So we can write the general form of the Green–Schwarz action in terms of currents. Bosonic currents are coupled with metric, and fermionic currents couples with ϵ^{ab}.

$$S = g \int d^2x \, \mathrm{str} \left(\sqrt{-h} h^{ab} J_{a2} J_{b2} + \epsilon^{ab} J_{a1} J_{a3} \right). \tag{2.2.26}$$

Now we can write down the equation of motion and flatness condition, analogously to the $\mathbb{Z}_2$ case (see, for example, [66]):

$$\begin{aligned}
2D_a\left(\sqrt{-h}h^{ab}J_{b2}\right) - \epsilon^{ab}[J_{a1}, J_{b1}] + \epsilon^{ab}[J_{a3}, J_{b3}] &= 0,\\
\left(\sqrt{-h}h^{ab} + \epsilon^{ab}\right)[J_{a2}, J_{b1}] &= 0,\\
\left(\sqrt{-h}h^{ab} - \epsilon^{ab}\right)[J_{a2}, J_{b3}] &= 0,\\
\epsilon^{ab}\left(2D_a J_{b2} + [J_{a1}, J_{b1}] + [J_{a3}, J_{b3}]\right) &= 0,\\
\epsilon^{ab}\left(D_a J_{b1} + [J_{a2}, J_{b3}]\right) &= 0,\\
\epsilon^{ab}\left(D_a J_{b3} + [J_{a2}, J_{b1}]\right) &= 0,\\
F_{ab} + [J_{a2}, J_{b2}] + [J_{a1}, J_{b3}] + [J_{a3}, J_{b1}] &= 0.
\end{aligned} \tag{2.2.27}$$

Here $D_a = \partial_a + [J_{a0}, \cdot]$ and $F_{ab} = \partial_a J_{b0} - \partial_b J_{a0} + [J_{a0}, J_{b0}]$.

There exists a Lax representation which again can be expressed through the currents:

$$L_a = J_{a0} + \frac{x^2+1}{x^2-1} J_{a2} - \frac{2x}{x^2-1} \frac{1}{\sqrt{-h}} h_{ab} \epsilon^{bc} J_{c2} + \sqrt{\frac{x+1}{x-1}} J_{a1} + \sqrt{\frac{x-1}{x+1}} J_{a3}. \tag{2.2.28}$$

The equations of motion follows from the flatness condition for L_a.

2.2.3 Flat connection

Let us write down the Metsaev–Tseytlin action for the Green–Schwarz superstring in $AdS_5 \times S^5$. It is given in terms of the algebra current $J = -g^{-1}dg$, where $g(\sigma,\tau)$ is an element of $psu(2,2|4)$:

$$S = \frac{\sqrt{\lambda}}{4\pi}\int \mathrm{str}\left(J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)}\right) + \Lambda \wedge \mathrm{str}\, J^{(2)}. \tag{2.2.29}$$

The last term is the Lagrange multiplier to ensure that $J^{(2)}$ is super-traceless.

Let us say a few words about $psu(2,2|4)$. It is a particular real form of $psl(4|4)$. The Cartan basis of this Lie superalgebra is described in [67]. The Dynkin diagram looks as follows

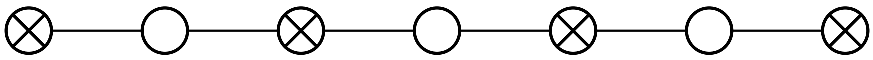

Figure 2.1: One of the possible Dynkin diagrams for the $psu(2,2|4)$.

This action has a local symmetry $g \to gh$, where $h \in sp(2,2)\times sp(4)$:

$$J^{(i)} \to h^{-1}J^{(i)}h, \quad i = 1,2,3. \tag{2.2.30}$$

For a purely bosonic representative g we can write

$$g = \begin{pmatrix} \mathscr{Q} & 0 \\ 0 & \mathscr{R} \end{pmatrix}, \tag{2.2.31}$$

where $\mathscr{R} \in SU(4)$ and $\mathscr{Q} \in SU(2,2)$. Let us note that $\mathscr{R}E\mathscr{R}^T$ is a parametrisation of S^5 since is invariant under $\mathscr{R} \to \mathscr{R}k$ with $k \in SP(4)$. In the same manner $\mathscr{Q}E\mathscr{Q}^T$ parametrises the AdS_5 space. So we can define embedding coordinates u and v:

$$u_j\Gamma_j^{SO(6)} = \mathscr{R}E\mathscr{R}^T, \quad v_j\Gamma_j^{SO(4,2)} = \mathscr{Q}E\mathscr{Q}^T. \tag{2.2.32}$$

This coordinates satisfy

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 = 1, \tag{2.2.33}$$

$$-v_1^2 - v_1^2 - v_1^2 - v_1^2 + v_1^2 + v_1^2 = 1. \tag{2.2.34}$$

In terms of these coordinates the bosonic part of Metsaev–Tseytlin action can be expressed in the form of a σ–model:

$$S = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \sqrt{h}\left(h^{\mu\nu}\partial_\mu u \cdot \partial_\nu u + \lambda_u(u\cdot u - 1) - (u \leftrightarrow v)\right). \tag{2.2.35}$$

Since the action is invariant under the $SO(4,2)$ and $SO(6)$ transformations, it will be natural to choose Noether charges corresponding to the described 3+3 linear isometries of the $AdS_5 \times S^5$: the time t and two angles φ_a from the AdS_5 part and three angles ϕ_i from the S^5 part:

$$S_0 \equiv E = \sqrt{\lambda}\mathrm{E}, \quad S_1 = \sqrt{\lambda}\mathrm{S}_1, \quad S_2 = \sqrt{\lambda}\mathrm{S}_2, \tag{2.2.36}$$
(2.2.37)

Now let us discuss the flatness condition. It reads

$$dJ - J \wedge J = 0. \tag{2.2.38}$$

It follows that we can construct the connection depending on the spectral parameter x which is flat for any x:

$$L(x) = J^{(0)} + \frac{x^2+1}{x^2-1}J^{(2)} - \frac{2x}{x^2-1} * J^{(2)} + \sqrt{\frac{x+1}{x-1}}J^{(1)} + \sqrt{\frac{x-1}{x+1}}J^{(3)}. \tag{2.2.39}$$

This connection generates an infinite tower of conserved charges, and this guarantees the classical integrability of the model. We can define the monodromy matrix

$$\Omega(x) = P\exp\oint_\gamma L(x), \tag{2.2.40}$$

where γ is any path wrapping the worldsheet cylinder once. The flatness condition (2.2.38) implies that $\Omega(x)$ will be path-independent, so we can choose γ to be the $\tau-$constant. Thus the eigenvalues of $\Omega(x)$ are time independent too and the only dependence is the dependence on a spectral parameter x, which can be arbitrary complex value. Expanding $\Omega(x)$ in x we obtain the infinite series of integrals of motion, and that proofs integrability.

Now let us move to the eigenvalues themselves. Since $\Omega(x)$ is unitary, we can denote the eigenvalues by

$$(e^{i\hat{p}_1(x)}, e^{i\hat{p}_2(x)}, e^{i\hat{p}_3(x)}, e^{i\hat{p}_4(x)} | e^{i\tilde{p}_1(x)}, e^{i\tilde{p}_2(x)}, e^{i\tilde{p}_3(x)}, e^{i\tilde{p}_4(x)}). \tag{2.2.41}$$

Here $\hat{p}_i$ corresponds to the AdS_5 part and $\tilde{p}_i$ — to the S^5 part. The set of p_i is called *quasimomenta*.

These eigenvalues are the roots of the characteristic equation

$$\mathrm{sdet}\,(y\mathbb{1} - \Omega(x)) = 0. \tag{2.2.42}$$

Of course it is not necessary for this equation to be a polynomial — in principle it can have singularities and infinite genus. But there is an important class of configurations, the so called "finite gap solutions". They have finite genus and in this instance equation (2.2.42) defines *algebraic curve*. It was shown in [18] that all the classical solutions of this model can be equivalently characterized in terms of theis algebraic curves (or, the same, in terms of quasimomenta).

Since the $\mathbb{Z}_4$ acts on the flat connection as (2.2.22), it is easy to see that it is equivalent to the inversion of the spectral parameter:

$$\mathcal{M}(L(x)) = L(1/x). \tag{2.2.43}$$

Obviously, the same holds for the monodromy matrix, since the $\mathbb{Z}_4$ action on the Lie algebra can be lifted to the group action with the exponent.

How it will reflect on the quasiomenta? Let's consider the $\mathbb{Z}_4$ action on the Cartan generators of our superalgebra:

$$\mathcal{M}(H_l) \equiv H_m S_{lm}. \tag{2.2.44}$$

So one can see that for quasimomenta

$$p_l(1/x) = S_{lm} p_m(x). \tag{2.2.45}$$

Thus we conclude that the knowledge of the quasimomenta in the physical region $|x| > 1$ is sufficient to reconstruct them on the full complex plane.

From super-tracelessness of the connection it follows that the monodromy matrix should be unimodular: $\mathrm{sdet}\,\Omega(x) = 1$. On the level of quasimomenta it means that

$$\sum_{i=1}^{4} (\hat{p}_i(x) - \tilde{p}_i(x)) = 2\pi n, \quad n \in \mathbb{Z}. \tag{2.2.46}$$

This means, in turn, that $p(x)$ can be defined on a 8–sheet Riemann surface. These sheets are connected by several cuts, which emerge from the solution of algebraic equation (2.2.42). The branch points of these cuts are the points where two eigenvalues of $\Omega(x)$ become equal.

2.3 Solutions and their quasimomenta

Now let us make the bridge between classical solutions and the set of quasimomenta. Although we will need an explicit classical solution, let us discuss first the constraints on the behaviour near the special points like poles, infinity etc.

- **Cuts.** There is a Riemann surface, and all eight sheets of this surface are connected with each other by cuts. Consider the cut $\mathscr{C}^{ij}$ which connects sheets i and j. On each sheet there is a definite single-valued quasimomentum (p_i and p_j). These quasimomenta have discontinuities:

  $$p_i^+ - p_j^- = 2\pi n_{ij}, \quad n_{ij} \in \mathbb{Z}, \quad x \in C^{ij}. \tag{2.3.1}$$

  Here + and − denotes that we take values of $p(x)$ above and below the cut correspondingly.

  Not all the combinations of i and j are accepted to get the physical excitations. We can use for i the set $\{\tilde{1},\tilde{2},\hat{1},\hat{2}\}$ and for j the set $\{\tilde{3},\tilde{4},\hat{3},\hat{4}\}$.

  We can see that there are sixteen combinations which lead to the sixteen physical polarizations (8 bosonic and 8 fermionic) of the superstring in $AdS_5 \times S^5$:

  $$S^5 \;:\; (\tilde{1},\tilde{3}),(\tilde{1},\tilde{4}),(\tilde{2},\tilde{3}),(\tilde{2},\tilde{4}); \tag{2.3.2}$$
  $$AdS_5 \;:\; (\hat{1},\hat{3}),(\hat{1},\hat{4}),(\hat{2},\hat{3}),(\hat{2},\hat{4}); \tag{2.3.3}$$
  $$\text{Fermions} \;:\; (\tilde{1},\hat{3}),(\tilde{1},\hat{4}),(\tilde{2},\hat{3}),(\tilde{2},\hat{4}),(\hat{1},\tilde{3}),(\hat{1},\tilde{4}),(\hat{2},\tilde{3}),(\hat{2},\tilde{4}). \tag{2.3.4}$$

- **Inversion symmetry.** As we have shown in the previous section, the $\mathbb{Z}_4$ automorphism of the algebra $psu(2,2|4)$ imposes the following relations (see Appendix B for the derivation) for the quasimomenta (see also [68] for the first derivation of them):

$$
\begin{aligned}
\tilde{p}_{1,2}(x) &= -\tilde{p}_{2,1}(1/x) - 2\pi m, && (2.3.5)\\
\tilde{p}_{3,4}(x) &= -\tilde{p}_{4,3}(1/x) + 2\pi m, && (2.3.6)\\
\hat{p}_{1,2,3,4}(x) &= -\hat{p}_{2,1,4,3}(1/x). && (2.3.7)
\end{aligned}
$$

- **Poles.** One can see directly from the Lax connection that the monodromy matrix and therefore quasimomenta have poles at $x = \pm 1$. But they are not completely independent as one can expect from the naive point of view. First the inversion symmetry leaves only 4 independent values of the poles (instead of 8). The unimodularity condition (2.2.46) reduces 4 to 3, and now this is time to use the Virasoro constraints.

  In terms of the currents they can be formulated very simple. Procedure is the same as in the case of $\mathbb{Z}_2$ (2.2.13), so we can just lift up everything to the case of $\mathbb{Z}_4$:

$$
\mathrm{str}\left(J^{(2)}\right)^2 = 0. \qquad (2.3.8)
$$

  So we are eventually left with only two independent poles. Generally they can be written as

$$
\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4;\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4\} \sim \frac{\{\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm;\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm\}}{x\pm 1}. \qquad (2.3.9)
$$

- **Asymptotics for $x \to \infty$.** In the limit when $x \to \infty$ Lax connection reduces to the Noether current. That's why asymptotics of the quasimomenta gives global charges of the classical solution. We can define $\mathcal{Q} = Q/\sqrt{\lambda}$, with this definition we have, for example,

$$
\mathcal{E} = \frac{1}{4\pi}\lim_{x\to\infty} x\left(\hat{p}_1(x) + \hat{p}_2(x)\right). \qquad (2.3.10)
$$

  In more general view we can have for all the charges:

$$
\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} \sim \frac{2\pi}{x} \begin{pmatrix} +\mathcal{E} - \mathcal{S}_1 + \mathcal{S}_2 \\ +\mathcal{E} + \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} + \mathcal{S}_1 + \mathcal{S}_2 \\ \hline +\mathcal{J} + \mathcal{J}_2 - \mathcal{J}_3 \\ +\mathcal{J} - \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J} + \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J} - \mathcal{J}_2 - \mathcal{J}_3 \end{pmatrix} \qquad (2.3.11)
$$

- **Filling fraction.** There are action variables of the theory, which are associated with each cut between i and j polarisation:

$$S_{ij} = \pm \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{C_{ij}} \left(1 - \frac{1}{x^2} p_i(x)\right) dx \tag{2.3.12}$$

Here the contour C_{ij} encircles the corresponding square root cut. One can prove that these variables are indeed the action variables of the theory [69, 70]. Finally we can notice that under Zhukovsky map

$$u = x + \frac{1}{x} \tag{2.3.13}$$

the expression for the filling fraction simplifies as

$$S_{ij} = \pm \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{C_{ij}} p_i(u) du. \tag{2.3.14}$$

This shows that from the Bohr–Sommerfeld point of view this pair of variables (p, u) is more suitable for quantization.

Chapter 3

The two-cut solution and quantization

In the previous chapter we saw that the algebraic curve can provide all the essential information about the classical and quasi-classical properties of the solution. Of course, this technique can be applied to rather general configurations, but we will restrict ourselves to the case of two-cut solution. This means that on the complex plane of the spectral parameter x we have two symmetric cuts.

It was shown in [68] that soutions of this type are equivalent to the folded spinning string founded in [71]. Since this solution has two charges S and J, its energy can be compared to the anomalous dimension of the Konishi operator (which has the same number of Dynkin labels) upon the setting $S = J = 2$.

In this chapter we will describe in details the two-cut solution in the terms of the quasimomenta. Then we will apply all the machinery from the previuos chapter to get the one-loop quasiclassical corrections to the energy of this solution in the case of arbitrary S and J. The answer is posed of several integrals which could be derived analytically in some particular cases. Namely, we will discuss the short operator limit, when the ratio S/g is small. Taking then the limit when J is small we get the energy of the solution, dual to the Konishi operator.

Other interesting limit which can be reached using this technique is the limit of the long folded string when S is large and J scales as $\log S$. This limit was discussed in [43] using the standard string quantization procedure. However, we are able to reproduce these results using the method of the algebraic curve and make a bridge to the dual description of the solution (see section 5.3). The form of the answer could also provide the nice physical interpreation which we give in the of the chapter.

3.1 Classical solution

Let us now shift from the general description to the concrete example which we will be use further. There is a specific type of solutions, the solutions with the two-cut structure of the Riemann surface.

As explained in [18], a general K–cut solution must be of the form

$$p'(x) = \frac{g(x)}{(x^2-1)^2\sqrt{f(x)}}, \tag{3.1.1}$$

where

$$f(x) = \prod_{j=1}^{2K}(x-x_j), \qquad g(x) = \sum_{j=1}^{N} c_j x^{j-1}. \tag{3.1.2}$$

Indeed we can notice that close to the branch point x_k we have

$$p'(x) \sim \partial_x \sqrt{x-x_k} \sim \frac{1}{\sqrt{x-x_k}}, \tag{3.1.3}$$

and near the poles $x = \pm 1$ we have

$$p'(x) \sim \partial_x \frac{1}{x \pm 1} \sim \frac{1}{(x\pm 1)^2}. \tag{3.1.4}$$

To construct the quasimomenta $p(x)$ we should integrate the meromorphic form $p'(x)dx$.

For the concrete case of the symmetric two-cut solution we can write our ansatz in the form

$$p'(x) = -\frac{\pi}{f(x)}\left(E\left(\frac{f(1)}{(x-1)^2} + \frac{f'(1)}{x-1} + \frac{f(-1)}{(x+1)^2} + \frac{f'(-1)}{x+1}\right) + 2(J_1 - J_2)\right), \tag{3.1.5}$$

where

$$f(x) = \sqrt{(x-a)(x-b)(x+a)(x+b)}, \qquad 1 < a < b. \tag{3.1.6}$$

The ansatz is constructed like this: the first four terms ensure the behaviour near the simple poles $x = \pm 1$ and the last two terms are adjusted to give the correct behaviour when $x \to \infty$.

Our symmetric two-cut solution describes the classical folded string in $AdS_5 \times S^5$ with twist J and Lorentz spin S. These conserved charges can be expressed in terms of the position of the branch points a, b (the integer n (the mode number) is related to the number of spikes):

$$S = 2ng\frac{ab+1}{ab}\left(bE\left(1-\frac{a^2}{b^2}\right) - aK\left(1-\frac{a^2}{b^2}\right)\right), \tag{3.1.7}$$

$$J = \frac{4ng}{b}K\left(1-\frac{a^2}{b^2}\right)\sqrt{(a^2-1)(b^2-1)}. \tag{3.1.8}$$

Then the classical energy can be computed as

$$E = 2ng\frac{ab-1}{ab}\left(bE\left(1-\frac{a^2}{b^2}\right)+aK\left(1-\frac{a^2}{b^2}\right)\right). \tag{3.1.9}$$

Here $E(x)$ and $K(x)$ are the elliptic integrals.

Now we can proceed to the general expression for the quasimomenta. Since we have the meromoprhic differential form $p'(x)dx$ we can integrate it and get

$$\begin{aligned}
\hat{p}_2 &= \pi n - \frac{J}{2g}\left(\frac{a}{a^2-1}-\frac{x}{x^2-1}\right)\sqrt{\frac{(a^2-1)(b^2-x^2)}{(b^2-1)(a^2-x^2)}} \\
&+ \frac{2abSF_1(x)}{g(b-a)(ab+1)} + \frac{J(a-b)F_2(x)}{2g\sqrt{(a^2-1)(b^2-1)}}, \\
\tilde{p}_2 &= \frac{Jx}{2g(x^2-1)}.
\end{aligned} \tag{3.1.10}$$

All the other quasi-momenta can be found from the standard symmetry relations for the $sl(2)$ sector

$$p_{\hat{2}}(x) = -p_{\hat{3}}(x) = -p_{\hat{1}}(1/x) = p_{\hat{4}}(1/x)\,, \tag{3.1.11}$$

$$p_{\tilde{2}}(x) = -p_{\tilde{3}}(x) = p_{\tilde{1}}(x) = -p_{\tilde{4}}(x)\,. \tag{3.1.12}$$

The functions $F_1(x)$ and $F_2(x)$ can be expressed in terms of the elliptic integrals:

$$\begin{aligned}
F_1(x) &= iF\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,|\frac{(a+b)^2}{(a-b)^2}\right), \\
F_2(x) &= iE\left(i\sinh^{-1}\sqrt{\frac{(b-a)(a-x)}{(b+a)(a+x)}}\,|\frac{(a+b)^2}{(a-b)^2}\right).
\end{aligned}$$

3.2 Fluctuation frequencies

An important feature of the algebraic curve quantization is that one can work with the off-shell fluctuation as it is described in detail in [72]. The off-shell fuctuation energies as functions of the spectral parameter x are much simpler than the usual fluctuation energies, usually obtained in the world-sheet quantization procedure, which are functions of mode numbers. The former should coincide with the later when evaluated at the special points of the curve given by

$$p_i(x_k^{ij}) - p_j(x_k^{ij}) = 2\pi k\,. \tag{3.2.1}$$

The fluctuations of the quasi-momenta with AdS–type excitations $(\tilde{2}\tilde{3})$ at z and S–type $(\hat{2}\hat{3})$ at y are given by

$$\delta\hat{p}_2 = \frac{\alpha(z)}{x-z} + \frac{\delta\alpha_-}{x-1} + \frac{\delta\alpha_+}{x+1}, \tag{3.2.2}$$

$$\delta\tilde{p}_2 = \frac{1}{f(x)}\left(-\frac{f(y)\alpha(y)}{x-y} + \frac{\delta\alpha_- f(1)}{x-1} + \frac{\delta\alpha_+ f(-1)}{x+1} - \frac{4\pi x}{\sqrt{\lambda}} + A\right), \tag{3.2.3}$$

where $\delta\alpha_\pm$ and A are the constants.

In general one has to compute $8+8$ different off-shell energies corresponding to the number of the physical world-sheet degrees of freedom. However as it was shown in [73] in the rank one sectors due to the inversion relations one can express all of them in terms of just two of them $\Omega^{\hat{2}\hat{3}}$ and $\Omega^{\tilde{2}\tilde{3}}$:

$$
\begin{aligned}
\Omega^{\hat{1}\hat{4}}(x) &= -\Omega^{\hat{2}\hat{3}}(1/x)-2\,,\\
\Omega^{\hat{1}\hat{3}}(x) &= \Omega^{\hat{2}\hat{4}}(x)=\frac{1}{2}\Omega^{\hat{1}\hat{4}}(x)-\frac{1}{2}\Omega^{\hat{1}\hat{4}}(1/x)-1\,,\\
\Omega^{\hat{1}\tilde{3}}(x) &= \Omega^{\hat{1}\tilde{4}}(x)=\Omega^{\hat{4}\tilde{1}}(x)=\Omega^{\hat{4}\tilde{2}}(x)=\frac{1}{2}\Omega^{\tilde{2}\tilde{3}}(x)+\frac{1}{2}\Omega^{\hat{2}\hat{3}}(x),\\
\Omega^{\hat{2}\tilde{3}}(x) &= \Omega^{\hat{2}\tilde{4}}(x)=\Omega^{\hat{3}\tilde{1}}(x)=\Omega^{\hat{3}\tilde{2}}(x)=\frac{1}{2}\Omega^{\tilde{2}\tilde{3}}(x)-\frac{1}{2}\Omega^{\hat{2}\hat{3}}(1/x)-1,\\
\Omega^{\tilde{2}\tilde{3}}(x) &= \Omega^{\tilde{2}\tilde{4}}(x)=\Omega^{\tilde{3}\tilde{1}}(x)=\Omega^{\tilde{3}\tilde{2}}(x)=\Omega^{\tilde{2}\tilde{3}}(x)\,.
\end{aligned}
\tag{3.2.4}
$$

Since we will consider only the $sl(2)$ sector, the fluctuation energies in S^5 should be trivial and can be written down immediately:

$$
\Omega^{\tilde{2}\tilde{3}}(x)=+\frac{2}{ab-1}\frac{\sqrt{a^2-1}\sqrt{b^2-1}}{x^2-1}\,. \tag{3.2.5}
$$

Calculation of $\Omega^{\hat{2}\hat{3}}(x)$ is a little bit more involved. However the steps one should follow are exactly the same as in [73] and we simply give the result here

$$
\Omega^{\hat{2}\hat{3}}(x)=+\frac{2}{ab-1}\left(1-\frac{f(x)}{x^2-1}\right), \tag{3.2.6}
$$

where $f(x)=\sqrt{x-a}\sqrt{a+x}\sqrt{x-b}\sqrt{b+x}$.

3.3 One–loop shift

In the previous sections we prepared all the necessary ingredients needed for the one-loop corrections to the classical energy. As we mentioned in the previous section the usual excitation energies, typically used in the worldsheet calculations, can be obtained from the off-shell fluctuation energies $\Omega^{ij}(x)$ by setting x to the value given be the equation (3.2.1), and then sum over all polarizations (ij) and all mode numbers k. Doing this explicitly is almost impossible for the given quasi-momenta. The standard way to overcome this difficulty is to rewrite the sum as an integral (see, for example, [57])

$$
\mathcal{E}=\frac{1}{2}\sum_{ij}(-1)^{F_{ij}}\oint\frac{dx}{2\pi i}\left(\Omega^{ij}(x)\partial_x\log\sin\frac{p_i-p_j}{2}\right). \tag{3.3.1}
$$

Here F_{ij} is the fermionic number: $F_{ij}=0$ for bosonic polarizations and $F_{ij}=1$ for fermionic. The term $\partial_x\log\sin\frac{p_i-p_j}{2}$ has the poles at the solutions of (3.2.1).

The contour of integration encircles all the possible fluctuations x_k^{ij}. This result is already explicit enough, however it is instructive to deform the contour into the unit circle (for each (ij)). During this contour deformation we can get two types of terms:

- Contribution from the integration on the unit circle for each polarization (ij).
- Additional contribution from the cuts of the classical solution. Only the term with $(ij) = (\hat{2}\hat{3})$ gets such a contribution.

It is also convenient to use the variable z instead of x:

$$x = z + \sqrt{z^2 - 1}, \tag{3.3.2}$$

which maps the unit circle $|x| = 1$ onto the interval $z \in [-1, 1]$. Also we can split the logarithm in two parts:

$$\log \sin \frac{p_i - p_j}{2} = \frac{i(p_i - p_j)}{2} + \log\left(1 - e^{-i(p_i - p_j)}\right), \tag{3.3.3}$$

which holds up to some irrelevant constant. In this way we split the finite size effects from the asymptotic contribution. Indeed, for $z \in [-1, 1]$ $e^{-i(p_i - p_j)}$ is exponentially suppressed for large J. Substituting this into (3.3.1), we get two terms, δE_1 and δE_2:

$$\delta E_1 = \sum_{ij} (-1)^{F_{ij}} \int_{-1}^{1} \frac{dz}{2\pi i} \left(\Omega^{ij}(z) \partial_z \frac{i(p_i - p_j)}{2}\right), \tag{3.3.4}$$

$$\delta E_2 = \sum_{ij} (-1)^{F_{ij}} \int_{-1}^{1} \frac{dz}{2\pi i} \left(\Omega^{ij}(z) \partial_z \log(1 - e^{-i(p_i - p_j)})\right). \tag{3.3.5}$$

One should take in account the contribution which we get by deforming the contour, which encircles the cuts $[-b, -a]$ and $[a, b]$. This contribution can be written as

$$\delta E_3 = -\frac{4}{ab - 1} \int_a^b \frac{dx}{2\pi i} \frac{f(x)}{x^2 - 1} \partial_x \log \sin p_{\hat{2}}. \tag{3.3.6}$$

where we use (3.1.11), (3.1.12).

The one-loop shift is then given by

$$E^{1-\text{loop}} = \delta E_1 + \delta E_2 + \delta E_3\,. \tag{3.3.7}$$

Using the symmetry relations (3.1.11), (3.1.12) (3.2.4), one can rewrite the sums δE_1 and δE_2 through the functions $p_{\hat{2}}(x)$, $p_{\tilde{2}}(x)$, $\Omega^{\hat{2}\hat{3}}(x)$, $\Omega^{\tilde{2}\tilde{3}}(x)$ defined above.

Let us consider for example the set of polarizations which belongs to the S^5. As we already have seen, all the frequencies are equal to $\Omega^{\tilde{2}\tilde{3}}(x)$. So our sum in δE_2 simplifies drastically and gives

$$\delta E_2^{S^5} = 4 \int_0^1 \frac{dz}{\pi} \text{Im}\left[\Omega^{\tilde{2}\tilde{3}}(z) \partial_z \log\left(1 - e^{-2ip_{\tilde{2}}(z)}\right)\right]. \tag{3.3.8}$$

One can easily show that all the contributions in $\delta E_{1,2}$ can be very naturally summed up in a pretty nice way:

$$
\begin{aligned}
\delta E_1 &= 2\int_0^1 \frac{dz}{\pi}\mathrm{Im}(p_{\hat{2}}-p_{\tilde{2}})\partial_z \mathrm{Im}(\Omega^{\hat{2}\hat{3}}-\Omega^{\tilde{2}\tilde{3}}),\\
\delta E_2 &= 2\int_0^1 \frac{dz}{\pi}\mathrm{Im}\left(\partial_z\Omega^{\tilde{2}\tilde{3}}\log\frac{(1-e^{-ip_{\tilde{2}}-i\bar{p}_{\hat{2}}})(1-e^{-ip_{\tilde{2}}+ip_{\hat{2}}})}{(1-e^{-2ip_{\tilde{2}}})^2}-\right.\\
&- \left.\partial_z\Omega^{\hat{2}\hat{3}}\log\frac{(1-e^{-2ip_{\hat{2}}})(1-e^{-ip_{\hat{2}}+i\bar{p}_{\hat{2}}})}{(1-e^{-ip_{\hat{2}}-ip_{\tilde{2}}})^2}\right).
\end{aligned}
\tag{3.3.9}
$$

Here for shorteness we denote $\bar{p}(z)=p(1/z)$.

3.4 The short operator limit

In this section we will exploit the explicit exact result for one-loop applicable for arbitrary $J, S \sim g$ which was derived in the previous section. These formulae involve a single integration and they can be evaluated numerically for various values of parameters.

The analytical evaluation of these integrals in general is not straightforward. In some limits, however, the integrands could simplify considerably so that the integration can be performed analytically. In this section we will consider one of such limits, namely we fix the ratio $r=J/S$ and then expand the result for small S/g. We will then motivate the relevance of this limit for the Konishi operator as well as for the similar type of operators with very few fields.

This limit is not completely trivial, the reason being that the algebraic curve becomes singular in this case: both positive branch points a and b approach the pole at $x=1$ as it can be easily seen from (3.1.7) and (3.1.8).

In the limit of small S and J charges are given by

$$S = 2n\pi g s^2+\mathcal{O}(s^6), \tag{3.4.1}$$
$$J = 2n\pi g r s^2+\mathcal{O}(s^7), \tag{3.4.2}$$
$$E = 4n\pi g s+\frac{1}{4}\pi g n\left(2r^2+3\right)s^3-\frac{1}{128}s^5\left(\pi g n\left(4r^4-20r^2+21\right)\right)+\mathcal{O}(s^6). \tag{3.4.3}$$

Here $r=J/S$ since we want to keep it ratio as arbitrary parameter.

One can see that, since $4\pi g=\sqrt{\lambda}$, we have

$$s=\frac{\sqrt{2S/n}}{\lambda^{1/4}}. \tag{3.4.4}$$

In this limit one can simplify the expressions for a and b:

$$a = 1+\frac{r^2 s^3}{8}+\frac{1}{128}\left(r^2-r^4\right)s^5+\frac{r^4 s^6}{128}+\mathcal{O}(s^7), \tag{3.4.5}$$

$$b = 1+2s+2s^2+\frac{1}{8}\left(r^2+7\right)s^3+\frac{1}{4}\left(r^2-1\right)s^4+\frac{1}{256}\left(-2r^4+34r^2-85\right)s^5+\mathcal{O}(s^6). \tag{3.4.6}$$

And the energy of our "short string" is

$$\frac{E}{n\sqrt{\lambda}} = s+\frac{1}{16}\left(2r^2+3\right)s^3+\frac{1}{512}\left(-4r^4+20r^2-21\right)s^5+\mathcal{O}(s^6). \tag{3.4.7}$$

So finally we got the classical energy of the generalized–folded string as a function of S and J.

Now we can proceed to the one–loop computation. We consider in some detail only the evaluation of δE_1 and then give the result for the others integrals. In general, the expression for δE_1 can be written conveniently as it is shown in (3.3.9):

$$\delta E_1 = -\frac{2}{\pi}\int_0^1 \mathrm{Im}\left(\Omega^{\hat{2}\hat{3}}(z)-\Omega^{\tilde{2}\tilde{3}}(z)\right)\mathrm{Im}\left(p'_{\hat{2}}(z)-p'_{\tilde{2}}(z)\right)dz\,. \tag{3.4.8}$$

First, assuming $z-1\sim 0$ we expand the integrand to get

$$-\int_0^1 \frac{2z^2 s}{(z^2-1)^2}dz+\dots\,. \tag{3.4.9}$$

Apparently the integral is divergent close to $z=1$. This divergence should be canceled when the integrand is treated more accurately for small $z-1$. There are two important scales when z approaches 1: when one zooms close to the branch point b which scales as s^2 then $z=1-s^2\zeta$ and when one further zooms so that we can distinguish the smallest branch point a from 1 i.e. $z=1-s^6\xi$. For each of these scales the integral is divergent, however, when all the three regions are combined together the divergences must cancel. What we get for δE_1 is

$$\delta E_1 \simeq -s\log\frac{rs^2}{2}-\frac{s}{2}. \tag{3.4.10}$$

The contributions δE_2 and δE_3 can be computed similarly. The results for these contributions are:

$$\delta E_2 \simeq s\log s+c_1 s\,, \tag{3.4.11}$$

$$\delta E_3 \simeq s\log\frac{rs}{2}+\frac{s}{4}-c_1 s\,, \tag{3.4.12}$$

where the numerical constant c_1 is $c_1\simeq 0.0203628454$. Notice that there are various log divergences which all cancel when the terms are combined together and the final result is very simple[1]

[1] In order to get the ABA result with Hernandez-Lopez phase one should drop the δE_2 contribution. In this case one would get $-s\log s\simeq\frac{\sqrt{2S}}{4\lambda^{1/4}}\log\lambda$ divergence. Exactly this divergence was indeed observed in [74] for the Konishi anomalous dimension ($S=2$) computed in the ABA framework.

$$\Delta^{1-\text{loop}} = \delta E_1 + \delta E_2 + \delta E_3 \simeq -\frac{s}{4} = -\frac{\sqrt{2S}}{4\lambda^{1/4}} \,. \tag{3.4.13}$$

In fact this result was previously obtained in [75] numerically with only two digits precision. This unpublished result was already used in [19] for the Konishi operator. The main difference with [19] is that we do not assume that $J = 0$, instead from the point of view of algebraic curve and its relation to the ABA it is rather obvious that one should take $J = 2$ for Konishi operator instead. Here we follow the approach of [19] with this small modification, which, however, changes the result considerably. Now we simply combine the classical energy (3.4.7) and the one-loop result (3.4.13) to get

$$\Delta^{\text{classical}} + \Delta^{1-\text{loop}} = \lambda^{1/4}\sqrt{2S} + \frac{1}{\lambda^{1/4}}\frac{2J^2 + S(3S-2)}{4\sqrt{2S}} \,. \tag{3.4.14}$$

The contribution to the first term comes solely from the classical energy, whereas both classical energy and the one-loop energy contribute to the second term. It is very tempting to assume that this pattern will continue further and in order to find the contribution to the next term one should also compute a two-loop correction. Strictly speaking this result holds assuming no non-perturbative terms contribute and that at each loop level the contribution can be represented as a regular series in s vanishing at $s = 0$.

For the Konishi state the numerical prediction is already available [76]. To compare one should substitute $S = J = 2$ as we discussed above. Equation (3.4.14) produces

$$\Delta^{\text{classical}} + \Delta^{1-\text{loop}}\Big|_{S=2,J=2,n=1} = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} \tag{3.4.15}$$

in the perfect agreement with the Y-system numerical prediction of [76] (see also [77]), string theory computation [20] and with computation in the pure spinor formalism [78].

A natural question one can ask is whether this prediction is going to be correct for short operators other than Konishi. To address this question we consider an operator similar to Konishi, with $J = 3$, $S = 2$ and $n = 1$, which we denote as $(3,2,1)$. From (3.4.14) we see that our prediction produces

$$\Delta^{(3,2,1)} = 2\lambda^{1/4} + \frac{13}{4\lambda^{1/4}} \,. \tag{3.4.16}$$

We compared this result with the numerical data from [79].

As one can see from Fig. 3.1, our analytical results perfectly match these numerical points.

3.5 The $1/\log S$ corrections for the long folded string

In this section we derive the finite size corrections for the regime when S is large and $J = 4g\ell\log S$ with ℓ finite. The corrections are obtained by computing the three integrals (3.3.4)-(3.3.6) and in an alternative way by using the Y–system at one loop derived in [80] (section 5.3). We obtain the corrections at arbitrary order in $1/\log S$,

Table 3.1: Konishi-like operator with $J=3$. The full dimension $\Delta=\gamma^{\text{anom}}+S+J$ for various values of g. Numerical data by [79]. The numerical absolute error is about $\pm 3\times 10^{-4}$.

g	Δ	g	Δ
0	5	0.9	7.6632
0.1	5.0777	1.0	7.9794
0.2	5.2883	1.1	8.2848
0.3	5.5854	1.2	8.5801
0.4	5.9275	1.3	8.8661
0.5	6.2868	1.4	9.1436
0.6	6.6456	1.5	9.4129
0.7	7.0023	1.6	9.6752
0.8	7.3354	1.7	9.9308

and we neglect all the inverse powers of S, as well as the $\log S/S$ terms. As a byproduct, we are rederiving the known results for the generalized scaling function up to one loop [22, 23], as well as the virtual scaling function [40] to the same order. The computations are done for arbitary ℓ, but of course the formulas greatly simplify for the GKP [3] limit $\ell=0$. In this limit, the energy is given by

$$E_{\ell=0}=S+J+4g\left(\log\frac{2S}{g}-1\right)-\frac{3\log 2}{\pi}\log\frac{2S}{g}+\frac{6\log 2}{\pi}+1-\frac{5\pi}{12\log(2S/g)}+\mathcal{O}(1/g)\,. \quad (3.5.1)$$

The $(\log S)^0$ part is in agreement with [40, 41, 81, 43], while the $1/\log S$ part agrees with the results in [42, 43, 44]. The $-5\pi/12\log S$ term can be interpreted as coming from the finite size corrections associated to 5 massless bosonic fields [44, 82], and the $\log S$ plays the role of the effective length of the string. At $\ell\neq 0$, four of these bosonic modes are massive, and their contribution is captured by the wrapping corrections. The fifth mode is massless, and it contributes via the anomaly term in the asymptotic Bethe ansatz equations. Although at weak coupling the asymptotic Bethe ansatz yields no $1/\log S$ corrections for the twist-two operator $J=2$, [46], at strong coupling the situation is different. This can be attributed to the different order in which the limits $S\to\infty$ and $g\to\infty$ are taken.

In the limit of the long string $S\to\infty$, the endpoints $\pm b$ of the cuts of the algebraic curve go to infinity and the solution becomes effectively one-cut. The expression for the charges (3.1.7)-(3.1.9) simplify and, up to negative powers in S, we have

$$\frac{S}{2g}=b,\qquad \frac{J}{4g}=\sqrt{a^2-1}\log\frac{2S}{ag}\,,\qquad \frac{\Delta}{2g}=\frac{S}{2g}+a\log\frac{2S}{ag}. \quad (3.5.2)$$

In particular, we notice that the parameter $\ell\equiv J/4g\log S$ is related to the position of the endpoint a of the cut by

$$\ell=\sqrt{a^2-1}\left(1-\frac{\log(ag/2)}{\log S}\right). \quad (3.5.3)$$

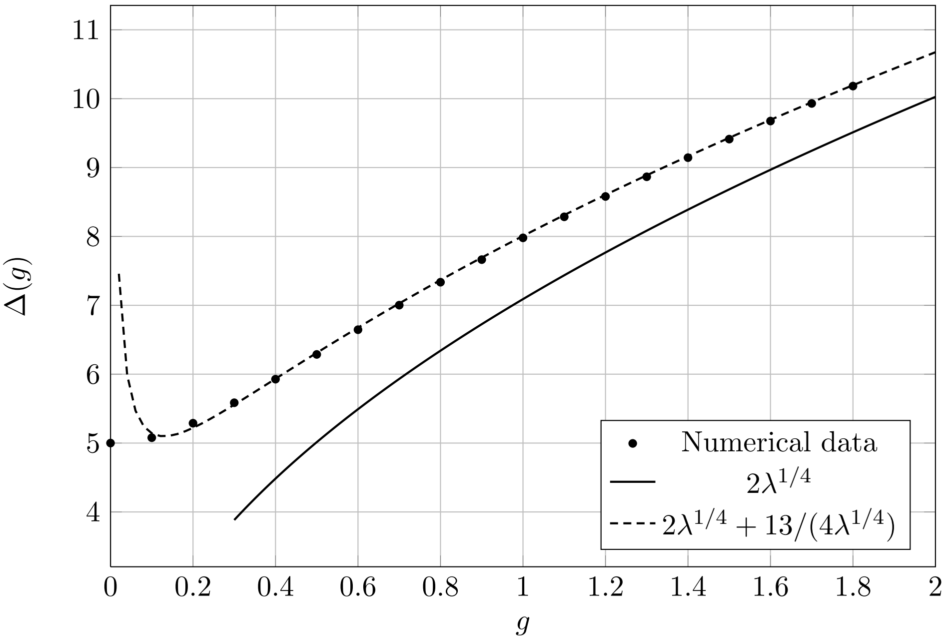

Figure 3.1: Numerical results from the Y-system for $J = 3$, $S = 2$, $n = 1$ compared with our analytical strong coupling expansion (3.4.14). Here as everywhere $\lambda = 16\pi^2 g^2$.

There will be trivial $1/\log S$ terms coming from this relation between ℓ and a. After some manipulation, the elliptic functions reduce in the large spin limit to simpler functions and the quasi-momenta (3.1.10) become

$$p_{\hat 2}(x) = \frac{J}{2g\sqrt{a^2-1}}\frac{x\sqrt{a^2-x^2}}{x^2-1} - 4\arctan\sqrt{\frac{a-x}{a+x}}, \tag{3.5.4}$$
$$p_{\tilde 2}(x) = \frac{J}{2g}\frac{x}{x^2-1}\,.$$

The off-shell frequencies (3.2.5) and (3.2.6) are given in this limit by

$$\Omega^{\tilde 2\tilde 3} = \frac{2}{a}\frac{\sqrt{a^2-1}}{x^2-1}\,, \qquad \Omega^{\hat 2\hat 3} = \frac{2}{a}\frac{\sqrt{a^2-x^2}}{x^2-1}\,. \tag{3.5.5}$$

With these data in hand we are able to proceed to the computation of the three integrals giving the complete one-loop contribution to the energy. The easiest part to compute is δE_2, which can be reduced to

$$\delta E_2 = \frac{4\sqrt{a^2-1}}{a\pi}\int_1^\infty dt\,\frac{\log(1-e^{-Jt/2g})}{\sqrt{1-1/t^2}} \equiv \frac{\sqrt{a^2-1}}{a}\mathscr{I}(2\ell\log S)\,. \tag{3.5.6}$$

The following two representations of $\mathscr{I}(\alpha)$ are particularly useful:

$$\mathscr{I}(\alpha) = -\sum_{n=1}^{\infty}\frac{4}{n\pi}K_1(n\alpha) \tag{3.5.7}$$
$$= -\frac{2\pi}{3\alpha} + 2 + \frac{\alpha}{2\pi}\left(2\gamma_E - 1 + 2\log\left(\frac{\alpha}{4\pi}\right)\right) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k+1)\Gamma(2k+1)}{\Gamma(k+1)\Gamma(k+2)(4\pi)^{2k}}2k+1\,.$$

From the first representation (a similar representation was obtained in [82]) we deduce that at large α, $\mathscr{I}(\alpha) \sim e^{-\alpha}/\sqrt{\alpha}$, so for finite ℓ the associated finite-size corrections vanish exponentially. The second representation in (3.5.7) is useful in the small ℓ regime, where it gives

$$\delta E_2 \simeq \frac{\ell \mathscr{I}(2\ell \log S)}{\sqrt{1+\ell^2}} = -\frac{4\pi}{12\log S} + \mathcal{O}(\ell)\,. \tag{3.5.8}$$

Note that (3.5.8) appears only for $\ell \log S \ll 1$ which practially corresponds to $\mathscr{J} \to 0$ limit prior to the large $\mathscr{S}$ limit.

In the $O(6)$ language, [47],[48], this is the correction coming from four of five bosonic modes which are massive at finite $\mathscr{J}$ (hence exponential suppression at large α) but become perturbatively massless in the limit $\mathscr{J} \to 0$.

The other two contributions to the one-loop energy are

$$\begin{aligned} \delta E_1 &= -\frac{4}{a}\int_{U_+} \frac{dy}{2\pi} \,\mathrm{Im}\, \frac{\sqrt{a^2-y^2}-\sqrt{a^2-1}}{y^2-1}\partial_y \,\mathrm{Im}\, G_0(y)\,, \\ \delta E_3 &= -\frac{4}{a}\int_a^\infty \frac{dy}{2\pi}\frac{\sqrt{y^2-a^2}}{y^2-1} p' \coth p\,, \end{aligned} \tag{3.5.9}$$

where we have denoted $p \equiv i p_{\hat{2}}(y+i0)$, $G_0(y) \equiv p_{\hat{2}}(y) - p_{\tilde{2}}(y)$ and the contour U_+ is the upper half of the unit circle running clockwise. The last term can be split naturally into two parts

$$\delta E_3 = \delta E_{3,\mathrm{an}} + \delta E_{3,\mathrm{m}}\,. \tag{3.5.10}$$

The anomaly-like term $\delta E_{3,\mathrm{an}}$ contains the finite-size corrections associated to the fifth bosonic mode, which remains massless for arbitrary ℓ

$$\delta E_{3,\mathrm{an}} \equiv -\frac{4}{a}\int_a^\infty \frac{dy}{2\pi}\frac{\sqrt{y^2-a^2}}{y^2-1} p'(\coth p - 1) = -\frac{\pi}{12(1+\ell^2)\log S} + \mathcal{O}(1/\log^2 S)\,. \tag{3.5.11}$$

This is exactly the contribution of the only at $\mathscr{J} \neq 0$ massless mode identified by Giombi, Ricci, Roiban and Tseytlin [44]. A more refined evaluation of the anomaly part, up to $\log S/S$ terms, can be straightforwardly done using

$$\delta E_{3,\mathrm{an}} = \sum_{n=1}^{\infty} f^{(n)}(0)\frac{\zeta(n+1)}{2^n} + \mathcal{O}(\log S/S) \quad \text{with} \quad f(p) = -\frac{2}{a\pi}\frac{\sqrt{y^2(p)-a^2}}{y^2(p)-1}. \tag{3.5.12}$$

The remaining two contributions δE_1 and $\delta E_{3,\mathrm{m}}$ reproduce the results already existing in the literature [21, 22, 23, 24, 29] with

$$\delta E_{3,\mathrm{m}} = -\frac{4}{a}\int_a^\infty \frac{dy}{2\pi}\frac{\sqrt{y^2-a^2}}{y^2-1} p' = \frac{a-\left(a^2+1\right)\mathrm{arccoth}\, a}{a\pi}\log\frac{2S}{ag} + \frac{4\,\mathrm{arccoth}\, a}{a\pi} \tag{3.5.13}$$

and

$$\begin{aligned} \delta E_1 &= -\frac{4}{a}\int_{U_+} \frac{dy}{2\pi} \,\mathrm{Im}\, \frac{\sqrt{a^2-y^2}-\sqrt{a^2-1}}{y^2-1}\partial_y \,\mathrm{Im}\, G_0(y) \\ &= -\frac{(a^2+1)\mathrm{arccoth}\, a^2 + 2a^2\log(1-a^{-2})+1}{a\pi}\log\frac{2S}{ag} \\ &+\frac{1}{a\pi}\left(4a\,\mathrm{arccot} a - 4\sqrt{a^2-1}\,\mathrm{arccot}\sqrt{a^2-1} + 2\log(1-a^{-4})\right) \end{aligned} \tag{3.5.14}$$

At $\ell = 0$ we get as expected

$$\delta E_1 + \delta E_{3,\mathrm{m}} = -\frac{3\log 2}{\pi}\log\frac{2S}{g} + \frac{6\log 2}{\pi} + 1\,. \tag{3.5.15}$$

Chapter 4

Thermodynamical Bethe Ansatz and Y–system for the $O(4)$ model

In this chapter we will formulate a quite general method for solving integrable systems at any finite size through the Y–system.

The study of such systems has a long story, starting from the seminal paper of Matsubara [83]. In this paper he proposed the so-called *Matsubara's trick*: instead of considering our system at finite temperature T, one can recast it as quantum system in the periodic imaginary time t. We will use this trick as a key component to solution of our finite–size system.

Lüscher found the leading finite–size corrections to the mass gap in relativistic two–dimensional quantum field theories [84, 85]. It occurs that these corrections depends only on the (asymptotic) S–matrix of the theory. Recently this result was generalized to the multi–particle states in integrable two–dimensional theories [86].

The first method for the systematical computation of the finite–size corrections was proposed in [87] (see also [88] for comprehensive review). It was based on the light-cone discrete regularization of the integrable theory. Once we discretize the theory, it is possible to build euclidean transfer matrices and then study it using the Bethe ansatz technique. The Bethe ansatz can be rewritten through the counting function (which counts the quantum numbers of the configuration) and the resulting non-linear integral equation (the so-called *Destri–de Vega equation*) can be used for studying the vacuum energy. The treatment of the excited states was done in [89]. Also one should notice that the same equation was obtained in the context of the condensed matter theory [90].

However, there is no general recipe to obtain the discrete regulatization for a general integrable QFT. That's why we will use another scheme, proposed by Zamolodchikov [14]. Namely, he proposed to use double Wick rotation: using the Matsubara's trick we can first find the free energy in the infinite volume but finite temperature. Secondly we swap the euclidean time and space interpreting the free energy as the ground state of the system in finite volume $L = 1/T$. This scheme is known as Thermodynamical Bethe Ansatz.

These TBA equations usually can be rewritten in the form of the *Y–system functional equations* [91]. They can be, in their turn, rewritten as non-linear integral equation as in the Destri–de Vega scheme.

So we will develop this TBA scheme in the next sections. Firstly we will study one of the simplest examples — $O(4)$ σ–model. We will observe that the main ingredient in the TBA is the factorizable two–particles S–matrix. Starting from this S–matrix, we will write down the nested Bethe ansatz equations. Then, using the double Wick rotation, we will reduce them to the TBA equations, putting the whole theory on the torus with one circumference, R, very large and another one, L, arbitrary. Our goal will be to compute the ground state energy for this finite radius (so the ground state of the system in a finite volume).

More formally, we can compute euclidean path integral Z

$$Z = e^{-RE_0(L)}. \tag{4.0.1}$$

We can compute this quantity swapping the roles of L and R. Since $R \to \infty$ the spectrum corresponding to the new Hamiltonian can be computed from the asymptotic Bethe ansatz with the finite "temperature" $1/L$. Thus we will have

$$E_0(L) = f(L), \tag{4.0.2}$$

where $f(L)$ is a free energy per unit length of the $O(4)$ σ–model at the temperature $1/L$ in the almost infinite volume $R \to \infty$.

Then we will rewrite TBA equations as Y–system assuming that this Y–system describes not only the ground state but the excited states too.

This scheme is valid for every σ–model, so using the $O(4)$ σ–model as a warm-up we will proceed to the Metsaev–Tseytlin σ–model.

4.1 $O(4)$ model and Bethe Ansatz

Here we review nested Bethe ansatz and TBA equations for $O(4)$ model, or, equivalently, for $SU(2)$ Principal Chiral Field (PCF). We will closely follow the paper [92].

The action of the $O(4)$ model is given by the usual expression

$$S = g \int dt\, dx\, (\partial_\alpha X_a)^2, \quad \sum_{a=1}^{4} X_a^2 = 1. \tag{4.1.1}$$

It is equivalent to the $SU(2) \otimes SU(2)$ PCF with the X_i fields packed into a algebra element

$$h = X_4 + i \sum_{j=1}^{3} X_j \sigma_j. \tag{4.1.2}$$

This theory in infinite volume is asymptotically free and the spectrum consists of a single particle of mass $m = \Lambda \exp(-2\pi g)$, where Λ is a cut-off ([93], [94],[95]).

Let us consider the scattering of two particles with momenta and energies

$$p_j = m \sinh(\pi\theta_j), \quad E_j = m \cosh(\pi\theta_j). \tag{4.1.3}$$

The exact S–matrix (proposed by Alexander and Alexey Zamolodchikov in [96]) is Lorentz–invariant and thus depends only on the difference of rapidities $\theta = \theta_1 - \theta_2$:

$$\hat{S}_{12}(\theta) = S_0(\theta)\frac{\hat{R}(\theta)}{\theta - i} \otimes S_0(\theta)\frac{\hat{R}(\theta)}{\theta - i}, \quad S_0(\theta) = i\frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2}\right)\Gamma\left(\frac{i\theta}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\theta}{2}\right)\Gamma\left(-\frac{i\theta}{2}\right)}. \tag{4.1.4}$$

Here $\hat{R}(\theta)$ is the $SU(2)$ R–matrix (see, for example, [10]). In the fundamental representation it has standard form

$$\hat{R}(\theta) = \theta + i\hat{P}. \tag{4.1.5}$$

Here $\hat{P}$ is the permutation operator that exchange the spins of scattered particles.

This S–matrix has several important features:

- analyticity
- unitarity
- crossing

For the reasons of crossing it was established dressing factor $S_0(\theta)$, which obeys crossing relation:

$$S_0(\theta + i/2)S_0(\theta - i/2) = \frac{\theta - i/2}{\theta + i/2}. \tag{4.1.6}$$

One can use this S–matrix to study the scattering of N particles in infinite volume. Here we consider infinite volume as a periodic space of big circumference. In this theory we have only one scale: the mass of particle, m. So all the distances should be compared to this natural scale. In other words, infinite volume means $L \gg m^{-1}$. Let us now set $m = 1$ and measure everything in m.

The periodicity condition for the wave function of N–particle state on a large circle of the length R can be written as follows:

$$\hat{\mathcal{T}}(\theta_j)e^{iR\sinh(\pi\theta_j)}\Psi = \Psi, \tag{4.1.7}$$

where $\mathcal{T}$ is the transfer matrix which is trace along the auxiliary space (we mark it as “0”) which we scatter against all physical particles (for a comprehensive review see [10]):

$$\mathcal{T}(\theta) = \mathrm{tr}_0\left(S_{01}(\theta - \theta_1)\ldots S_{0N}(\theta - \theta_N)\right). \tag{4.1.8}$$

This periodicity condition simply means that if we pick the particle j and carry it along the circle, the phase of it’s wave function will consist of trivial term Rp_j (for free propagation) and some phase shifts due to the scattering with other particles. But this full phase should be a multiple of 2π, so we obtain (4.1.7).

It is possible to this $\mathcal{T}(\theta)$ to get a full set of equations on rapidities $\{\theta_i\}$. This procedure will lead us to the set of *Bethe ansatz equations*, namely, to the *nested* Bethe ansatz. Let us introduce some useful polynomials, associated with rapidities:

$$\phi(\theta)=\prod_{j=1}^{N}(\theta-\theta_j),\quad S(\theta)=\prod_{j=1}^{N}S_0(\theta-\theta_j),\quad Q_w(\theta)=\prod_{j=1}^{J_w}(\theta-w_j). \tag{4.1.9}$$

Let us mention that we should diagonalize simultaneously the two parts of the $SU(2)\otimes SU(2)$ transfer matrix. We treat each part as independent problem of diagonalization $SU(2)$ transfer matrix, keeping in mind that they are related.

We introduce states with J_u spins down (and $N-J_u$ spins up) for the left $SU(2)$ and J_v spins down (and $N-J_v$) for the right $SU(2)$. For these states we have J_u Bethe roots u_j and J_v Bethe roots v_j. In the end of the day we should obtain a set of equations on $\{u_j\},\{v_j\},\{\theta_j\}$. One can show that, if $T_1^w(\theta)$ is the transfer matrix for the roots of type w, then

$$\mathcal{T}(\theta)\Psi=\frac{S^2(\theta)}{\phi^2(\theta-i)}T_1^w(\theta-i/2)T_1^v(\theta-i/2)\Psi, \tag{4.1.10}$$

and

$$T_1^w=\frac{Q_w(\theta+i)\phi(\theta-i/2)+Q_w(\theta-i)\phi(\theta+i/2)}{Q_w(\theta)}. \tag{4.1.11}$$

Having these expressions in hand, we can write down explicitly the periodicity condition (4.1.7):

$$e^{-imR\sinh(\pi\theta_j)}=-S^2(\theta_j)\frac{Q_u(\theta_j+i/2)}{Q_u(\theta-i/2)}\frac{Q_v(\theta_j+i/2)}{Q_v(\theta_j-i/2)}. \tag{4.1.12}$$

Also we have equations for auxiliary Bethe roots which comes from the condition of the cancellation of the poles in T_1^w:

$$-\frac{Q_u(w_j+i)}{Q_u(w_j-i)}=\frac{\phi(w_j+i/2)}{\phi(w_j-i/2)}. \tag{4.1.13}$$

In principle, these equations can be solved and for the given state we obtain the energy in the form

$$E=\sum_{j=1}^{N}\cosh(\pi\theta_j). \tag{4.1.14}$$

Let us consider how these equations could be simplified in the limit $L\to\infty$. It occurs that the roots organise themselves into “strings” on the complex plane.

For example, when $\mathrm{Re}u_j>0$, the r.h.s. of the auxiliary Bethe equations (4.1.13) diverges:

$$\frac{\phi(u_j+i/2)}{\phi(u_j-i/2)}\to\infty \tag{4.1.15}$$

so the l.h.s. should diverge as well. This means that we should have a pole in the denominator, at a point $u_j-u_k=i$. In thermodynamic limit $N\to\infty$ it means that we will have a strings on a complex plane, which can be parametrized as follows:

$$u^n_{j,a} = j^n_j + \frac{i}{2}(n+1) - ia, \quad a = 1, \dots n. \tag{4.1.16}$$

For example, the real Bethe root corresponds to the string of the length 1, i.e. $n = 1$.

Now we can use this special type of root distribution to simplify Bethe equations. Let us multiply them for u_j belonging to a same string (we will call this procedure as *fusion*). We will have a set of Bethe equations for the centers of the strings (which are real numbers):

$$e^{-iRp(\theta_\alpha)} = \prod_{\beta \neq \alpha} S_0^2(\theta_\alpha - \theta_\beta) \prod_{j,n} \frac{\theta_\alpha - u^n_j + \frac{in}{2}}{\theta_\alpha - u^n_j - \frac{in}{2}}, \tag{4.1.17}$$

$$\prod_\beta \frac{u^n_j - \theta_\beta + \frac{in}{2}}{u^n_j - \theta_\beta - \frac{in}{2}} = \frac{u^n_j - u^m_k - i\frac{n+m}{2}}{u^n_j - u^m_k + i\frac{n+m}{2}} \cdot \frac{u^n_j - u^m_k - i\frac{|n-m|}{2}}{u^n_j - u^m_k + i\frac{|n-m|}{2}} \cdot \prod_{s=\frac{|n-m|}{2}}^{\frac{n+m}{2}} \frac{u^n_j - u^m_k + is}{u^n_j - u^m_k - is}. \tag{4.1.18}$$

In the thermodynamic limit the distribution of the roots on the complex plane is almost continuous, so we can rewrite all the Bethe equations in terms of density of the roots. Let us introduce the three type of densities since we have three type of roots.

Say, we can use the subindex 0 for the θ–density, the subindex $n > 0$ for the u–density and subindex $n < 0$ for the v–density. Here n corresponds to the number of roots in the string. Also we will have the unfilled solutions – holes – which can be obtained by removing excitations from the Dirac sea. There are also three types of holes, which are in one-to-one correspondence with particles. So it is very natural to introduce the hole density $\bar{\rho}_n$.

We can obtain the equations for the densities. Let us take the logarithmic derivative of both sides. The next step is to change the sum over roots to the integral over real axis with density. We end up with an integral equation on ρ_n and $\bar{\rho}_n$ with the kernel, which depends only on scalar factor $S_0(\theta)$:

$$\rho_n + \bar{\rho}_n = \frac{R}{2}\cosh(\pi\theta)\delta_{n0} - \sum K_{n,m} * \rho_m. \tag{4.1.19}$$

Here $*$ is the standard convolution

$$f * g = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta'), \tag{4.1.20}$$

and $K_{n,m}$ is the kernel for scattering of two Bethe strings with lengths n and m.

As for kernel it is very hard to get an explicit expression since we have effectively scattering between strings of different lengths. But we can separate this scattering into three parts: scattering of pure physical excitations, scattering of physical excitations and Bethe strings and scattering of Bethe strings only.

The simplest is the scattering of physical excitations:

$$K_{0,0}(\theta) \equiv \frac{1}{2\pi i}\frac{d}{d\theta}\log S_0^2(\theta). \tag{4.1.21}$$

A little bit complicated is the scattering "physical–strings":

$$K_{0,n}(\theta) \equiv \frac{1}{2\pi i}\frac{d}{d\theta}\log\frac{\theta - i|n|/2}{\theta + i|n|/2} = \frac{1}{\pi}\frac{2|n|}{4\theta^2 + |n|^2} \equiv K_n. \quad (4.1.22)$$

And for interaction "strings–string" the best we can have is such a implicit sum:

$$K_{n,m}(\theta) = \sum_{i=\frac{|n-m|}{2}}^{\frac{n+m}{2}} 2K_{2i}(\theta) - K_{n+m}(\theta) + K_{|n-m|}(\theta)\delta_{n\neq m}. \quad (4.1.23)$$

Following the general recipe, we can proceed to the Fourier transform of these kernels to simplify the solution of the density equation. One can check that

$$\hat{K}_0 = \frac{\exp(-|\omega|/2)}{\cosh\omega/2}, \qquad \hat{K}_n(\omega) = \exp(-|n|\omega/2). \quad (4.1.24)$$

Having this in mind we can explicitly calculate the Fourier image for $K_{n,m}$:

$$\hat{K}_{n,m} = \coth\left(\frac{|\omega|}{2}\right)(\exp(-|\omega|/2|m-n|) - \exp(-|\omega|/2(m+n))) - \delta_{n,m}. \quad (4.1.25)$$

Also let us mention the useful formula which we will use for obtaining the set of the local equations. One can check that

$$\left(\hat{K}_{nm} + \delta_{nm}\right)^{-1} = \delta_{nm} - \hat{s}\left(\delta_{n,m+1} + \delta_{n,m-1}\right), \qquad n,m > 0, \quad (4.1.26)$$

where the operator $\hat{s}$ has the following form:

$$\hat{s}(\omega) = \frac{1}{2\cosh\omega/2}. \quad (4.1.27)$$

For example, we can check that $K_0 = 2s * K_1$.

Now let us construct the free energy. As described, for example, in [97], one should construct the functional

$$f(L) = \int d\theta\left(\rho_0 L\cosh\pi\theta - \sum_{n=-\infty}^{n=\infty}\rho_n\log\left(1+\frac{\bar{\rho}_n}{\rho_n}\right) + \bar{\rho}_n\log\left(1+\frac{\rho_n}{\bar{\rho}_n}\right)\right), \quad (4.1.28)$$

and minimize it keeping satisfied the relation (4.1.19). The term "functional" reflects the fact that the integral depends on the concrete distribution of ρ_n and $\bar{\rho}_n$. Here the quantity $1/L$ plays the role of the temperature, as we described in the beginning of this section.

Taking the variation of the equation (4.1.19), we have

$$\delta f = \delta\bar{\rho}_n + \delta\rho_n + \sum_{-\infty}^{\infty} K_{nm} * \delta\rho_m. \quad (4.1.29)$$

Minimum condition for $f(L)$ yields $\delta f = 0$, so after little algebra we get a set of TBA-like equations

$$\epsilon_n = L\cosh(\pi\theta)\delta_{n,0} + \sum_{-\infty}^{\infty} K_{mn} * \log(1+\exp(-\epsilon_m)), \qquad \frac{\rho_n}{\bar{\rho}_n} = \exp(-\epsilon_n). \quad (4.1.30)$$

In the literature these equations often called as *Yang–Yang equations* [98]. One can prove that solutions of these equations always exist.

Substituing this equation for the ratio $\rho_n\ \bar{\rho}_n$ into the equation (4.1.28), we get finally the expression for the free energy $E_0(L)$:

$$E_0(L) = -L \int \frac{d\theta}{2} \cosh(\pi\theta) \log(1+\exp(-\epsilon_0(\theta))). \tag{4.1.31}$$

This is the desired finite size ground energy of the $SU(2)$ PCF model.

Let us convert these equations into a local set of integral equations. To do this we define $Y_m = \exp(\epsilon_m)$, $Y_0 = \exp(-\epsilon_0)$.

Now we consider the equation (4.1.30) for $n \neq 0$. Under this condition we can apply the operator (4.1.26) to the equation (4.1.30), and immediately get

$$\log Y_n + L\cosh(\pi\theta)\delta_{n,0} = \sum_{m=-\infty}^{\infty} I_{nm} s * \log(1+Y_m). \tag{4.1.32}$$

Now we should verify this equation for the case $n = 0$. Let us examine some relations between kernels entering in the equation (4.1.30) for $n = -1,0,1$. For example,

$$K_{0,0} = -K_0 = -2s * K_1, \quad K_{0,\pm1} = K_1, \quad K_{\pm m,0} = -K_m, \quad m > 0. \tag{4.1.33}$$

Also we have the relation

$$K_{\pm m,\pm1} = K_{m+1} + K_{m-1}\delta_{m-1}. \tag{4.1.34}$$

Now we apply the operator s to the (4.1.30) with $n = 1$. One can use the fact that

$$s * (K_{m+1} + K_{m-1}) = K_m. \tag{4.1.35}$$

Applying this, we get

$$s * K_1 * \log(1+\exp(-\epsilon_0)) = \sum_{m=1}^{\infty} K_m * \log(1+\exp(-\epsilon_m)) - s * \log(1+\exp(\epsilon_1)). \tag{4.1.36}$$

The last term is separated because the last term ($m = 1$) is separated in the equation (4.1.34). Similarly, for $n = -1$ we have

$$s * K_1 * \log(1+\exp(-\epsilon_0)) = \sum_{m=-\infty}^{-1} K_{|m|} * \log(1+\exp(-\epsilon_m)) - s * \log(1+\exp(\epsilon_{-1})). \tag{4.1.37}$$

And for the $n = 0$

$$\epsilon_0 = L\cosh\pi\theta - K_0 * \log(1+\exp(-\epsilon_0)) + \sum_{|m|\neq 0} K_{|m|} * \log(1+\exp(-\epsilon_m)). \tag{4.1.38}$$

Now let us sum up all these three equations. Firstly, we observe that the infinite sums completely cancel out. Secondly, using the identity $K_0 = 2s * K_1$, we see that all the convolutions with $\log(1+\exp-\epsilon_0)$ cancel out too. So the whole sum is completely equal to the (4.1.32). Thus we prove the equation (4.1.32) for all n.

We can rewrite it in even simpler form if we use some properties of the analytic functions. Let us consider the function $g(\theta)$ which is analytic inside the strip $Im(\theta) < 1/2$. We have

$$\int_{-\infty}^{+\infty} d\theta \, \frac{g(\theta+i/2)+g(\theta-i/2)}{2\cosh(\pi(\theta-x))} = \frac{1}{2i}\oint d\theta \, \frac{g(\theta)}{\sinh(\pi(\theta-x))} = g(x). \tag{4.1.39}$$

This identity can be rewritten in terms of operator s:

$$s*(g(\theta+i/2)+g(\theta-i/2)) = g(\theta). \tag{4.1.40}$$

So using this formula we can rewrite everything as a set of functional equations (so called *Y–system*) at a finite temperature $1/L$:

$$Y_n^+ Y_n^- = (1+Y_{n-1}(\theta))(1+Y_{n+1}(\theta)). \tag{4.1.41}$$

For the convenience we denoted here

$$f^{\pm} = f(\theta \pm i/2), f^{\pm\pm} = f(\theta \pm i). \tag{4.1.42}$$

We can notice that any information about size of the system has disappeared completely from the equations (4.1.41). One should pose himself the question — how could we use solve the system of the equations without any reference to the size of the system?

The answer is lying in the boundary conditions. Obviously, the system of equations (4.1.41) has more than one solution, but only one of them leads to the physical ground state energy. The only way to fix this ambiguity is to impose boundary conditions, i.e. the asymptotics of Y_n at large θ. More precisely, one can retrieve the Bethe equations (4.1.17) from Y–system with the condition

$$Y_n \sim e^{-L\cosh\pi\theta\delta_{n,0}}. \tag{4.1.43}$$

What about the excited states? We know from many examples 2D [99, 100, 101] that there are other solutions there which describe the excited states. The excited state is characterized by N–particle excitation of the vacuum, so the final formula for the energy of the N–particle excited state is modified:

$$E(L) = -\frac{1}{2}\int d\theta \cosh(\pi\theta)\log(1+Y_0) + \sum_{i=1}^{N} m\cosh(\pi\theta_i), \tag{4.1.44}$$

where the points θ_i are the singularities of the first term:

$$Y_0(\theta_i \pm i/2) = -1, \qquad i = 1,2,\ldots,N. \tag{4.1.45}$$

It is worth to mention that last term in (4.1.44) is generated by analytic continuation in L of the first term (integral of Y_0) and picking up its logarithmic poles.

Moreover, as we will check, these condition on the logarithmic poles render the Bethe equations for θ, i.e. for physical rapidities.

Now we can proceed to solution of these $Y-$system equations.

4.2 Hirota equation for the $O(4)$ model

4.2.1 General properties of the Hirota equation

The $Y-$system discussed in the previous section can be rewritten in the form of the *Hirota equation* [102]. To do this, let us denote

$$Y_n(\theta) = \frac{T_{n+1}(\theta)T_{n-1}(\theta)}{g\left(\theta+\frac{in}{2}\right)\bar{g}\left(\theta-\frac{in}{2}\right)}, \tag{4.2.1}$$

where $g(\theta)$ is just an arbitrary function ("gauge"). Under this substituion, our $Y-$system transforms to

$$T_n^+T_n^- - T_{n-1}T_{n+1} = g(\theta+in/2)g(\theta-in/2). \tag{4.2.2}$$

We will omit the argument if it is not confusing.

This equation has a number of remarkable properties. Firstly, let us mention that it is integrable with the Lax representation. Consider a vector of two functions $\{Q(x),\bar{Q}(x)\}$. One can write the system of two equations — compatibility condition of them will give the initial Hirota equation:

$$T_{n+1}Q\left(\theta+\frac{in}{2}\right)-T_n^-Q\left(\theta+\frac{i(n+2)}{2}\right) = g\left(\theta+\frac{in}{2}\right)\bar{Q}\left(\theta-\frac{i(n+2)}{2}\right), \tag{4.2.3}$$

$$T_{n-1}\bar{Q}\left(\theta-\frac{i(n+2)}{2}\right)-T_n^-\bar{Q}\left(\theta-\frac{in}{2}\right) = -\bar{g}\left(\theta-\frac{in}{2}\right)Q\left(\theta+\frac{in}{2}\right). \tag{4.2.4}$$

Setting $n=0$ we observe that equations (4.2.3) take the form of the usual Baxter equations for the spin chains,

$$T_1(\theta)Q(\theta) = g(\theta)\bar{Q}(\theta-i)+T_0^-Q(\theta+i), \tag{4.2.5}$$

$$T_{-1}(\theta)\bar{Q}(\theta-i) = -\bar{g}(\theta)Q(\theta)+T_0^-\bar{Q}(\theta). \tag{4.2.6}$$

This Lax representation guarantees the integrability of the Hirota equation. We also can use equations (4.2.3) for expressing T_1, T_{-1}:

$$T_1 = T_0^-\frac{Q(\theta+i)}{Q(\theta)}+g(\theta)\frac{\bar{Q}(\theta-i)}{Q(\theta)}, \tag{4.2.7}$$

$$T_{-1} = T_0^+\frac{Q(\theta)}{Q(\theta+i)}-g(\theta)\frac{\bar{Q}(\theta)}{Q(\theta+i)}. \tag{4.2.8}$$

Moreover, these Lax relations are linear in T_n and because of that they can be used to obtain a very general and explicit solution in terms of $T_0, g(\theta), Q(\theta)$:

$$T_n = \frac{Q\left(\theta+\frac{i(n+1)}{2}\right)}{Q\left(\theta-\frac{i(n+1)}{2}\right)} T_0\left(\theta-\frac{in}{2}\right) + \tag{4.2.9}$$

$$+ \quad Q\left(\theta+\frac{i(n+1)}{2}\right)\bar{Q}\left(\theta-\frac{i(n+1)}{2}\right)\sum_{j=1}^{n}\frac{g\left(x-\frac{i(n+1)}{2}+ij\right)}{Q\left(\theta-\frac{i(n-1)}{2}+ij\right)Q\left(\theta-\frac{i(n+1)}{2}+ij\right)}. \tag{4.2.10}$$

With the equation (4.2.1) it leads to the solution of initial Y–system.

Second interesting property of the Hirota equation is the symmetry. Namely, there is a symmetry which corresponds to the exchange between u and v wings of the symmetry group $SU(2)\otimes SU(2)$. This symmetry is guaranteed by exchanging $Y_k \Leftrightarrow Y_{-k}$, or, on the level of T–function, $T_k \Leftrightarrow T_{-k}$ with the simultaneous exchanging $g \Leftrightarrow -\bar{g}$, $Q \Leftrightarrow \bar{Q}(\theta-i)$, $\bar{Q} \Leftrightarrow Q(\theta+i)$.

Moreover, one can observe the analogue of the gauge symmetry in the Hirota equation. Namely, it's remain invariant under such transformation:

$$T_n(\theta) \rightarrow h\left(\theta+\frac{in}{2}\right)\bar{h}\left(\theta-\frac{in}{2}\right)T_n(\theta), \tag{4.2.11}$$

$$g(\theta) \rightarrow h^- h^+ g(\theta), \tag{4.2.12}$$

$$\bar{g}(\theta) \rightarrow \bar{h}^- \bar{h}^+ \bar{g}(\theta), \tag{4.2.13}$$

$$Q(\theta) \rightarrow h^- Q(\theta). \tag{4.2.14}$$

The condition that $\bar{h}$ is complex conjugated to h should be assumed to preserve reality of the $T_n(\theta)$.

Following [51] we can write the general solution of the Hirota equation in the determinant form:

$$T_n(\theta) = f\left(\theta+\frac{in}{2}\right)\begin{vmatrix} Q\left(\theta+\frac{i(n+1)}{2}\right) & R\left(\theta+\frac{i(n+1)}{2}\right) \\ \bar{Q}\left(\theta-\frac{i(n+1)}{2}\right) & \bar{R}\left(\theta-\frac{i(n+1)}{2}\right)\end{vmatrix}. \tag{4.2.15}$$

Here $f(\theta)$ is an arbitrary periodic function, $f(\theta+i) = f(\theta)$. Q and R are two linearly independent solutions of the Lax equations (4.2.3) related by the Wronskian relation

$$g(\theta) = f\left(\theta+\frac{in}{2}\right)\begin{vmatrix} R(\theta) & Q(\theta) \\ R(\theta+i) & Q(\theta+i)\end{vmatrix}. \tag{4.2.16}$$

The last remark which can be done in the context of the general properties of the Y–system is that it can be generalized to the two–dimensional lattice. Since it will be useful in the context of the AdS/CFT correspondence, we describe this generalisation here.

The Y–function on the two–dimensionall lattice depends on the two discrete coordinates: $Y_{a,s}$. The Y–system for this lattice can be written as

$$Y^+_{a,s}Y^-_{a,s} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+1/Y_{a+1,s})(1+1/Y_{a-1,s})}. \tag{4.2.17}$$

These equations also should be accompanied by the boundary conditions. Again, it can be reformulated as the Hirota equation via the transform

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}. \tag{4.2.18}$$

The Hirota equation in this case take the form

$$T^+_{a,s}T^-_{a,s} = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}. \tag{4.2.19}$$

In the case of two–dimensional Hirota equation the gauge transformation is richer. Namely, one can define the T–function up to the

$$T_{a,s} \to g_1\left(\theta + \frac{i(a+s)}{2}\right) g_2\left(\theta + \frac{i(a-s)}{2}\right) g_3\left(\theta - \frac{i(a+s)}{2}\right) g_4\left(\theta - \frac{i(a-s)}{2}\right) T_{a,s}. \tag{4.2.20}$$

One can notice that this Hirota equation is equivalent to the equation obeyed by the transform matrices of spin chain with auxiliary spaces corresponding to the rectangular Young tableaux of size (a,s). For the more details on this subject see the seminal paper [103].

Finally, we see that on the boundary of the domain (i.e. when a or s is equal to zero) our two–dimensional Y–system reduces to the discrete version of the d'Alembert equation with the solution

$$T_{0,s}(\theta) = g_1\left(\theta + \frac{is}{2}\right) g_2\left(\theta - \frac{is}{2}\right). \tag{4.2.21}$$

4.2.2 Large volume solution

In the previous chapter we obtained a general solution of the Hirota equation for $O(n)$ model (or of the Y – system equations). As we noticed, they should be accompanied by the boundary conditions in order to get well-defined solution with the physical meaning. We will demonstrate how our equations work if taken with the boundary conditions for the large volume L.

The main problem is that one should take in account not only the physical excitations of the $U(1)$ sector, but also the excitations from the left and right wings of the $SU(2) \otimes SU(2)$. The classification of all the solutions of the Y – system is very complicated and thus the main problem will be to identify the large L solutions in our Y – system.

From the definition we have that

$$Y_0(\theta) = \frac{T_1(\theta)T_{-1}(\theta)}{g(\theta)\bar{g}(\theta)} \sim e^{-L\cosh(\pi\theta)}, \qquad L \to \infty. \tag{4.2.22}$$

Looking at this equation we see that our Y – system is completely decoupled: its wings are independent and we can treat them separately. Moreover, since we have a discrete symmetry between the wings (see the paragraph after the equation (4.2.9)) we can easily get the solution for the right wing from the solution for the left.

From this and from the equation (4.2.22) we conclude that

$$T^u_{-1}(\theta) \to 0, \qquad T^v_1(\theta) \to 0. \tag{4.2.23}$$

Now we can make the simple assumtion about the functions $T_n(\theta)$ and $Q(\theta)$. First, let's assume that $T_n(\theta)$ is polynomial in θ. Second, we make the same assumption for the $Q(\theta)$. For the naturality of these assumptions one can refer to the (4.1.9).

Now from the equations (4.2.7) and from the assumptions above one can conclude that

$$T^{u+}_0 = g^u(\theta), \tag{4.2.24}$$

$$T^u_1(\theta) = \frac{T^{u+}_0 Q^u(\theta - i) + T^{u-}_0 Q^u(\theta + i)}{Q^u(\theta)} \tag{4.2.25}$$

And it is completely equivalent to the Baxter equations for the "magnon" rapidities (4.1.11). So the polynomiality condition for T^u_1 will give us precisely the Bethe equations for these rapidities. Notice that in this limit we have $T^u_0(\theta) = \phi(\theta)$ (for the definition of $\phi(\theta)$ see (4.1.9)).

What about the physical rapidities? They can be obtained from setting $n = 0$. Under this substitution we have from the definition (4.2.1)

$$Y_0(\theta_j \pm i/2) = -1, \tag{4.2.26}$$

where θ_j is the zero of the $T_0(\theta)$. At $n = 0$ it follows from the $Y-$system that

$$Y^+_0 Y^-_0 = \frac{T^{u+}_1 T^{v+}_{-1} T^{u-}_1 T^{v-}_{-1}}{\big(\phi(\theta + i)\phi(\theta - i)\big)^2}. \tag{4.2.27}$$

From the initial crossing relation (4.1.6) one can immediately have

$$S(\theta + i)S(\theta) = \frac{\phi(\theta)}{\phi(\theta + i)}, \tag{4.2.28}$$

so we can rewrite (4.2.27) as

$$Y^+_0 Y^-_0 = \left(\frac{T^u_1 T^v_{-1}(S^+)^2}{(\phi^-)^2}\right)^+ \left(\frac{T^u_1 T^v_{-1}(S^+)^2}{(\phi^-)^2}\right)^- . \tag{4.2.29}$$

From which equation it is obvious to see that

$$Y_0 \sim T^u_1 T_{-1} \frac{(S^+)^2}{(\phi^-)^2}. \tag{4.2.30}$$

It is true up to a zero mode factor. This factor should guarantee the proper asymptotics and cancels in the equation (4.2.29). So we can write the simplest form of this factor as

$$Y_0 = e^{-L\cosh(\pi\theta)} T^u_1 T_{-1} \frac{(S^+)^2}{(\phi^-)^2}. \tag{4.2.31}$$

Now it is time to evaluate this Y_0 function at the point θ_j. At first sight Y_0 is exponentially small since it contains the exponent of the large volume L. But on the boundary of the physical strip it is almost 1 (we can notice also that the exponent is purely imaginary). So we get

$$-1 \sim e^{iL\sinh(\pi\theta_j)} \frac{Q_u(\theta_j + i/2) Q_v(\theta_j + i/2)}{Q_u(\theta_j - i/2) Q_v(\theta_j - i/2)} \prod_k S_0^2(\theta_k - \theta_j). \tag{4.2.32}$$

This equation is compeletely identical to the Bethe equation for the physical rapidity (4.1.12), as we expected in the beginning of this chapter.

4.2.3 Vacuum solution at finite volume

From the previous section we see that the simplest vacuum solution for large L corresponds to the $Q^u(\theta) = 1$, $\phi(\theta) = 1$. The general expression for the T – function (4.2.9) gives us that in the leading order $T_{s-1} = s$. Of course, it is true only in the leading order. It is obvious that T^u_{-1} should not be strictly zero, so we need to compute T^u_{-1} in the next order.

To do this, we rembember that there is a gauge transform which relates left and right wings:

$$T^u_{-1} = T^v_{-1} |h(\theta - i/2)|^2. \tag{4.2.33}$$

As we saw in the previous section this gauge can be related to the zero mode of the Y – function and thus we can write that

$$T^u_{-1} \sim e^{-L\cosh(\pi\theta)}. \tag{4.2.34}$$

Now we can omit the v wing since it is obvious how to relate everything from one wing to another. We can improve now our equation for T_s with the following ansatz:

$$\begin{aligned} T_{s-1}(\theta) &= s + G(\theta - is/2) - G(\theta + is/2), & (4.2.35)\\ g(\theta) &= 1 + G(\theta + i0) - G(\theta + i), & (4.2.36)\\ \bar{g}(\theta) &= 1 + G(\theta - i) - G(\theta - i0), & (4.2.37) \end{aligned}$$

where $G(\theta)$ is the resolvent

$$G(\theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\theta'}{\theta - \theta'} T_{-1}(\theta'). \tag{4.2.38}$$

For example, we have $T_0(\theta) = g(\theta - i/2)$ as we already see in the previous chapters. This function, as we discussed, is analytic in the physical strip $|\text{Im}\theta| < 1/2$ so the functions $g(\theta), \bar{g}(\theta)$ can be treated as the analytic continuation of T_0 beyond the cuts (up to the shift).

As we can see, the solution of the (4.2.35) is determined by only one function $T_{-1}(\theta)$. But let us remember that this function is not completely independent — it is constrained with the condition (4.2.1) for $n = 0$. Moreover, we can write again the Hirota equation for the $n = 0$ and get

$$(1+Y_1)(1+Y_{-1}) = \left(\frac{T_1(\theta+i/2)T_1(\theta-i/2)}{g(\theta+i/2)g(\theta-i/2)}\right)^2. \quad (4.2.39)$$

From these two equations we immediately get

$$T_{-1}(\theta) = T_1(\theta)\frac{g(\theta)\bar{g}(\theta)}{[g(\theta+i/2)\bar{g}(\theta-i/2)]^{*2s}}e^{-L\cosh(\pi\theta)}. \quad (4.2.40)$$

We recall that s is an inverse shift operator, and can be alternatively written as

$$s = \frac{1}{D+D^{-1}}. \quad (4.2.41)$$

So finally we got a closed equation for $T_{-1}(\theta)$ (or for Y_0). It can be solved numerically basically for all L giving the ground energy with the help of the equation (4.1.31). One can find the results of this iterative procedure in [92].

4.2.4 Physical excitations in $U(1)$ sector

Now we can consider the $U(1)$ sector of the theory, which is characterized by N physical particles without magnon excitations. Since we have no magnon rapidities u_j, v_j we can set all the Q's to 1. But now we will have the roots of physical excitations $\theta_j, \quad j=1,\dots,N$ so the roots of T_0 on the real axis.

From the Hirota equation we have (for the infinite volume)

$$T_{s-1} = f(\theta+is/2) - f(\theta-is/2), \quad (4.2.42)$$

where $f(\theta)$ is a polynomial satisfying

$$f^+ - f^- = \phi(\theta). \quad (4.2.43)$$

An improved solution of the Hirota equation can be obtained through the same trick as in previous chapter, namely,

$$\begin{aligned} T_0(\theta) &= \phi(\theta)+\bar{G}(\theta-i/2)+G(\theta+i/2), && (4.2.44)\\ g(\theta) &= \phi(\theta+i/2)+\bar{G}(\theta+i0)+G(\theta+i), && (4.2.45)\\ \bar{g}(\theta) &= \phi(\theta-i/2)+\bar{G}(\theta-i)+G(\theta-i0). && (4.2.46)\end{aligned}$$

Here the resolvent $G(\theta)$ is again defined via the $T_{-1}(\theta)$:

$$G(\theta) = \frac{\phi(\theta-i/2)}{2\pi i}\int_{-\infty}^{\infty}\frac{d\theta'}{\theta-\theta'}\frac{T_{-1}(\theta')}{\phi(\theta'-i/2)}. \quad (4.2.47)$$

Recalling that $Q = 1$ and using the determinant formula (4.2.15) for the general solution of Hirota equation, we get for T_s

$$\begin{aligned} T_s(\theta) &= R\left(\theta+\frac{i(s+1)}{2}\right) - R\left(\theta-\frac{i(s+1)}{2}\right), && (4.2.48)\\ R(\theta) &= P(\theta)+s*\left(G(\theta+i/2)+\bar{G}(\theta-i/2)\right). && (4.2.49)\end{aligned}$$

Chapter 5

The Y–system for AdS/CFT σ–model

5.1 Hirota equation for the $GL(K|M)$ symmetry

As we saw in the previous chapter the solution of the $O(4)$ spin chain (or $O(4)$ σ–model) with the help of nested Bethe ansatz worked like this. We continuosly deduce the rank of the symmetry group applying Bethe procedure several times. For example, in our $O(4)$ case we divide our system in three sectors (left and right wing and the central node) and wrote for each sector it's own Bethe equation. That's how one can go in general from $GL(N)$ to $GL(N-1)$ unless one reach the trivial case $N = 0$. During these steps one introduces T and Q functions which obey the Baxter relations. The zeroes of the T-functions gives us the Bethe equations. This quantum technique has a classical interpretation in terms of the *Backlund transform* [104, 50], and Baxter relations play the role of auxiliary problem for the Hirota equation. The rank of the group becomes an additional parameter. Moreover, dependance on this parameter is desribed again by Hirota equation. The solutions are polynomials of the spectral parameter, and their zeroes obeys Bethe equations.

One can say that we have the Backlund flow which comes from the highest rank to the trivial rank undressing the Bethe equations to the simplest ones.

Here we will review this procedure (following [52]) for the supersymmetric case $GL(K|M)$. In this case we will have two discrete flows of the Backlund transform, which correspond to the bosonic and fermionic ranks K and M. Consistency condition for these flows leads us to the bi-linear equation on the eigenvalues of the Baxter operators.

For the sake of simplicity we restrict ourselves to the case $K = M = 2$ (it corresponds to the concrete situation of the AdS_5/CFT_4 duality).

It occurs that there is a domain where the functions $T^{K,M}_{a,s}(u) \neq 0$. Namely, it is a *"fat hook"*, i.e. $0 \le a \le K$ or $0 \le s \le M$ and $a, s \ge 0$ [50].

On this lattice we can define the Y–system (4.2.17) and the Hirota equation (4.2.19). As in the one–dimensional case, on the boundary it becomes the discrete d'Alembert equation, with the solution

$$T^{K,M}_{0,s}(u) = g_-(u - is/2)g_+(u + is/2). \tag{5.1.1}$$

Since we have the gauge freedom (see (4.2.20)) we can fix the boundary condition with

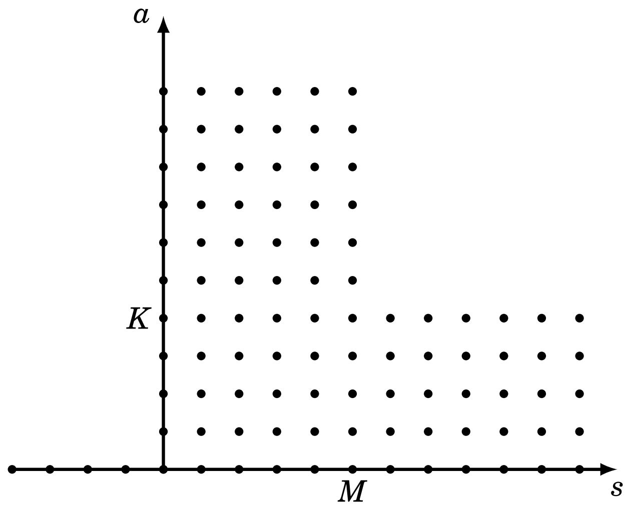

Figure 5.1: Fat hook on the (a,s) lattice for $GL(K|M)$.

$$T^{K,M}_{0,s}(u) = Q_{K,M}(u - is/2), \qquad T^{K,M}_{a,0}(u) = Q_{K,M}(u + ia/2). \tag{5.1.2}$$

Here $Q_{K,M}(u)$ is a polynomial.

The Backlund transform can be formulated in the operator formalism. Namely, let us consider the difference operator of infinite order

$$W(u) = \sum_{s\geq 0} \frac{T_{1,s}(u + i(s-1)/2)}{Q(u)} D^{2s}, \tag{5.1.3}$$

where $D \equiv e^{i\partial_u/2}$. It is a non–commutative generating functional for the T–functions $T_{1,s}(u)$. Generally, one can introduce this object at any level

$$W_{k,m}(u) = \sum_{s\geq 0} \frac{T^{k,m}_{1,s}(u + i(s-1)/2)}{Q_{k,m}(u)} D^{2s}. \tag{5.1.4}$$

Also one can show formally that

$$W^{-1}_{k,m}(u) = \sum_{a\geq 0} D^{2a} \frac{T^{k,m}_{a,1}(u - i(a+1)/2}{Q_{k,m}(u-i)}. \tag{5.1.5}$$

Now we define such the functions

$$X_{k,m}(u) = \frac{Q_{k,m}(u+i)Q_{k-1,m}(u-i)}{Q_{k,m}(u)Q_{k-1,m}(u)}, \tag{5.1.6}$$

$$Y_{k,m}(u) = \frac{Q_{k,m-1}(u+i)Q_{k,m}(u-i)}{Q_{k,m-1}(u)Q_{k,m}(u)}. \tag{5.1.7}$$

Using the Lax representation for the Hirota equation at $a = 0$ one can prove two recurrence relations for the operators $W_{k,m}(u)$:

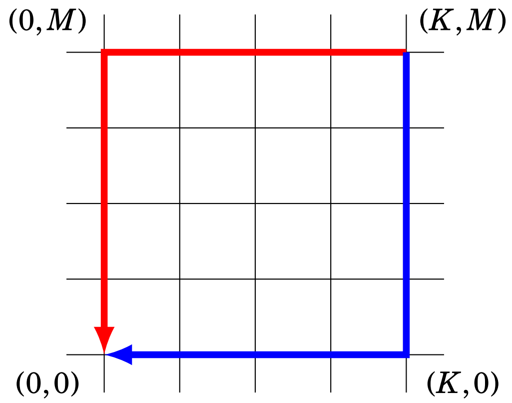

Figure 5.2: Two possible paths for the Backlund transformation.

$$W_{k-1,m}(u) = \left(1 - X_{k,m}(u)D^2\right) W_{k,m}(u), \quad (5.1.8)$$

$$W_{k,m+1}(u) = \left(1 - Y_{k,m+1}(u)D^2\right) W_{k,m}(u). \quad (5.1.9)$$

So now we can go from the point (k,m) to the point $(k-1,m-1)$ on two different ways. Or, reformulating, $W_{k,m}(u)$ could be obtained from $W_{0,0}(u) = 1$ applying recurrent relations (5.1.8). Moving firstly in m–direction from $(0,0)$ to $(0,M)$ and secondly in k–direction from $(0,M)$ to (K,M) we get

$$W_{k,m}(u) = \prod_{K \geq k \geq 1} \left(1 - X_{k,M} D^2\right)^{-1} \cdot \prod_{M \geq m \geq 1} \left(1 - Y_{0,m} D^2\right). \quad (5.1.10)$$

Now we can start from $(0,0)$ firstly in k–direction and then in m–direction, and we get

$$W_{k,m}(u) = \prod_{M \geq m \geq 1} \left(1 - Y_{K,m} D^2\right) \cdot \prod_{K \geq k \geq 1} \left(1 - Y_{k,0} D^2\right)^{-1}. \quad (5.1.11)$$

These two operations should be equivalent and we should demand

$$W_{k-1,m-1} = \left(1 - X_{k,m-1} D^2\right)\left(1 - Y_{k,m} D^2\right)^{-1} W_{k,m} = \quad (5.1.12)$$

$$= \left(1 - Y_{k-1,m} D^2\right)^{-1} (1 - X_{k,m} D^2) W_{k,m}. \quad (5.1.13)$$

So we have the discrete version of the zero-curvature condition on (k,m) lattice which can be written in the form

$$\left(1 - Y_{k-1,m} D^2\right)\left(1 - X_{k,m-1} D^2\right) = (1 - X_{k,m} D^2)(1 - Y_{k,m} D^2). \quad (5.1.14)$$

We have the series in D^2 in r.h.s. and l.h.s., so it will generate a series of the Baxter relations. For example, the simples one looks like

$$\frac{T_{1,1}^{K,M}(u)}{Q_{K,M}(u)} = \sum_{k=1}^{K} X_{k,M}(u) - \sum_{m=1}^{M} Y_{0,M}(u), \quad (5.1.15)$$

where $X_{k,M}$ and $Y_{0,M}$ should be expressed through Q as in (5.1.6). It works in the same manner for all $T_{a,s}$: they are expressed through different sums of X and Y, which could be expressed only in terms of Q's. Finally we will have a series of TQ–relations. The zeroes of Q's should obey the Bethe equations.

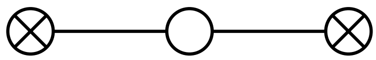

Figure 5.3: On of the possible Dynkin diagrams for the $su(2|2)$ group.

Let us exemplify this technology on the simplest case, namely $su(2|2)$. Here $K = M = 2$ and we should start with $W_{2,2}$ to reach $W_{0,0}$. The consistency condition (5.1.14) gives

$$W_{2,2}(u) = \left(1 - Y_{2,2}D^2\right)\left(1 - X_{2,1}D^2\right)^{-1}\left(1 - X_{1,1}D^2\right)^{-1}\left(1 - Y_{0,1}D^2\right)W_{0,0}(u). \tag{5.1.16}$$

It means that $T_{1,1}$ can be written in the form

$$T^{2,2}_{1,1} = Q_{2,2}(X_{1,1} + X_{2,1} - Y_{2,2} - Y_{0,1}) \tag{5.1.17}$$

$$Q_{2,2} = \left(\frac{Q^{++}_{1,1}Q^{--}_{0,1}}{Q_{1,1}Q_{1,0}} + \frac{Q^{++}_{2,1}Q^{--}_{1,1}}{Q_{2,1}Q_{1,1}} - \frac{Q^{++}_{2,1}Q^{--}_{2,2}}{Q_{2,1}Q_{2,2}} - \frac{Q^{++}_{0,0}Q^{--}_{0,1}}{Q_{0,0}Q_{0,1}}\right). \tag{5.1.18}$$

Generally, $T_{1,1}$ could have poles at zeroes of the polynomials $Q_{k,m}$. The Bethe ansatz equations follow immediately from the condition of the regularity of $T^{2,2}_{1,1}$ at these zeroes, as in case of usual $SU(n)$ spin chains. Each equation which one can get from this condition corresponds to the "undressing" path from the point $(2,2)$ to the poing $(0,0)$. For example, going from $(2,2)$ to the $(0,2)$ (or, more generally, from (K,M) to $(0,M)$) we get precisely the same Bethe equations like in [105] (historically the nested Bethe ansatz equations were obtained in [98]) for the bosonic case: each Backlund movement will be corresponding to the embedding of the type $GL(K) \supset GL(K-1)$.

5.2 The $PSU(2,2|4)$ Y–system

To apply the Y–system machinery to the AdS/CFT problem one should consider a non-compact symmetry group $PSU(2,2|4)$ since it is a symmetry of the Metsaev–Tseytlin σ–model. So we should build a Hirota equation with such a symmetry. The main difference from the $SU(2|2)$ case is that $PSU(2,2|4)$ is non-compact and that's why we will have the different boundary conditions.

In order to work with the version of Hirota equation for the non-compact group we will start from the theoretical–group interpretation of the Hirota equation in the classical version. Then we will proceed to the quantum version of the Hirota equation based on the intuition from the algebraic curve and ABA for the $PSU(2,2|4)$.

5.2.1 Classical Hirota equation and characters of $PSU(2,2|4)$

Let us recall that the main object which guaranteed integrability of the Metsaev–Tseytlin σ–model was the monodromy matrix $\Omega(x)$ (see (2.2.40)). It is the group element of the $SU(2,2|4)$. In the fundamental representation it has $4+4$ eigenvalues:

$$\Omega(x) = \text{diag}(x_1, x_2, x_3, x_4 | y_1, y_2, y_3, y_4). \tag{5.2.1}$$

Here x_i corresponds to the S^5 part of the σ–model, y_i — to the AdS_5 part. Supertrace of the monodromy matrix can be defined in any unitary highest weight irrep λ:

$$T_\lambda = \text{str}_\lambda \Omega(x)^V. \tag{5.2.2}$$

The highest weight irreps of $u(2,2|4)$ can be parametrized by the Young tableaux. There is a special type of irreps (rectangular irreps) $[a,s]$ for which $\lambda_i = s+2, i = 1,\dots,a$. One can check that they obey such a relation

$$[a,s] \otimes [a,s] = [a+1,s] \otimes [a-1,s] \oplus [a,s+1] \otimes [a,s-1]. \tag{5.2.3}$$

Taking the trace along both sides of this equality, we obtain

$$T_{a,s}T_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}, \tag{5.2.4}$$

which is nothing but Hirota equation for the characters ([106, 107]). Moreover, it occurs that all the representations can be generated from $T_{1,s}$ by the simple consequence of the Hirota equation, which is called *Jacobi–Trudi formula*:

$$T_{a,s} = \det_{1\le j,j\le a} T_{1,s+i-j}. \tag{5.2.5}$$

It is possible also to write down the generating function of the $T_{1,s}$:

$$w(z) = \text{sdet}\,(\mathbb{1} - z\Omega(x))^{-1} = \frac{(1-y_1z)(1-y_2z)(1-y_3z)(1-y_4z)}{(1-x_1z)(1-x_2z)(1-x_3z)(1-x_4z)}. \tag{5.2.6}$$

With this function one can write for the $T_{1,s}$

$$T_{1,s} = \frac{1}{2\pi i}\oint \frac{dz}{z^{s+1}} w(z). \tag{5.2.7}$$

We should note that the characters of compact $su(4|4)$ and non-compact $su(2,2|4)$ both obey equations (5.2.4), (5.2.7), but the boundary conditions for the Hirota equations and the contour are different. If the contour encircles the origin but not the poles of the denominator, then the corresponding $T_{a,s}$ will be non-zero only inside the fat hook (see the figure 5.1) and it corresponds to the compact unitary representations of the $U(4|4)$.

If the contour encircles the poles of the denominator x_3^{-1}, x_4^{-1}, then the corresponding characters will be non-zero outside of the *T–hook* (see [108, 109] for further explanations).

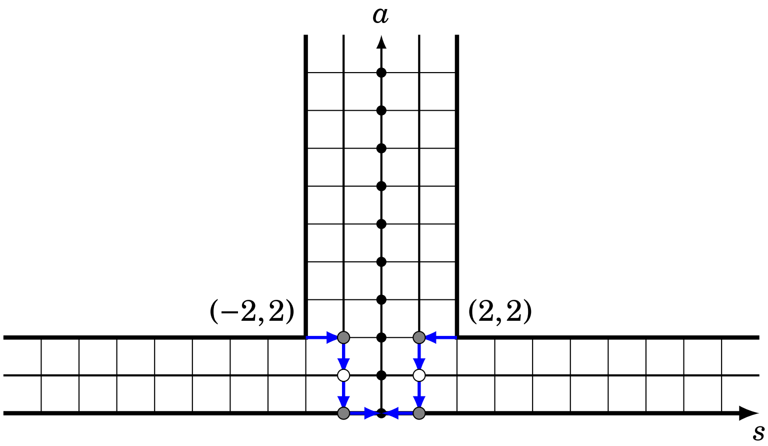

Figure 5.4: T–hook for the $PSU(2,2|4)$. It can be obtained by gluing of the two $SU(2|2)$ wings along the the black nodes. We can define a Backlund flow directly on this picture – from the points $(2,2)$ and $(-2,2)$ to the point $(0,0)$.

5.2.2 Symmetries of the characters

The characters defined in the previous section has some symmetries. First of all, there is a discrete symmetry between two wings of the T–hook:

$$T_{a,s}(x_1,\dots,x_4|y_1,\dots y_4) = T_{a,-s}\left(\frac{1}{x_1},\dots,\frac{1}{x_4}\middle|\frac{1}{y_1},\dots,\frac{1}{y_4}\right). \tag{5.2.8}$$

Also there is a permutational symmetry (which occurs instead of usual Weyl symmetry for the compact groups):

$$x_1,x_2 \leftrightarrow x_2,x_1; \quad x_3,x_4 \leftrightarrow x_4,x_3; \quad \{y_1,\dots,y_4\} \leftrightarrow \mathrm{Perm}\{y_1,\dots,y_4\}. \tag{5.2.9}$$

Second, there is a symmetry inherited from the Z_4 symmetry of the coset. As we saw in the section 2.3, the Z_4 automorphism of the coset implies

$$x_{1,2,3,4}(1/x) = \frac{1}{x_{2,1,3,4}(x)}, \quad y_{1,2,3,4}(1/x) = \frac{1}{y_{2,1,4,3}(x)}. \tag{5.2.10}$$

From the unitarity of the monodromy matrix $\Omega(x)$ we can also conclude that

$$\overline{x_i(x)} = \frac{1}{x_i(\bar{x})}, \quad \overline{y_i(x)} = \frac{1}{y_i(\bar{x})}. \tag{5.2.11}$$

On the unit circle $|x| = 1$ and we have $\bar{x} = 1/x$, so we get

$$\overline{x_{1,2,3,4}(x)} = x_{2,1,4,3}(x), \quad \overline{y_{1,2,3,4}(x)} = y_{2,1,4,3}(x). \tag{5.2.12}$$

Now we can collect all the information about symmetries of the characters. From (5.2.8) and (5.2.12) we obtain that on the unit circle $|x| = 1$

$$\overline{T_{a,s}} = T_{a,s}. \tag{5.2.13}$$

The Y–functions, which can be expressed through $T_{a,s}$ via (4.2.18) are also real, $\overline{Y_{a,s}} = Y_{a,s}$.

5.2.3 Towards the quantum Hirota equation

Although there is no rigorous proof of quantum integrability of the Metsaev–Tseytlin superstring σ–model, there is a wide evidence that it is true. This evidence is based on the classical integrability of the model (which was discussed in the section 2.2.3), numerous perturbative computations and comparsions with the corresponding CFT theory. Moreover, the Metsaev–Tseytlin σ–model is not well defined as a quantum theory, but these computations and the AdS/CFT hypothesis itself can shed some light on the rigorous definition of the superstring theory.

In the section 4.1 we have reviewed the problem of the energy spectrum for the massive $O(4)$ σ–model in a finite volume (i.e. on the cylinder of the some finite radius). The Hirota equation which we got is rather universal: the only thing which differs from model to model is the boundary conditions and, surely, concrete type of interaction, which is encoded in the S–matrix. The general form, however, stays the same for all the $gl(N|M)$ algebras (see [92, 110] for the table of integrable models and their Y–systems). As we saw in the previous section, the Y–system has a deep mathematical sense in a classical limit – namely, the solutions of the Y–system are the characters of the corresponding symmetry group, which is an additional argument to the universality of the Y–system.

The Metsaev–Tseytlin σ–model is in the same class of integrable theories. It is massive σ–model with the $gl(n|m)$ type of the symmetry (up to the non–compactness of the $PSU(2,2|4)$). The only serious difference is that this σ–model is not explicitly relativistic theory, in contrast to the $O(4)$ σ–model. However, it is not a really big problem to incorporate the property into the Y–system machinery.

We already have some intuition about the boundary conditions on the (a,s) lattice — the full quantum Hirota equation should have the same boundary conditions as the classical one.

The next step is to identify the spectral parameter u which is a bit more involved than in $O(4)$ σ–model. Luckily, we have a well-developed general recipe [69, 70, 111] which proposes us to take the pair (p,u) where p is a quasimomenta and u is a corresponding spectral parameter entering the Bohr–Sommerfeld integral (see section 2.3 for the rigorous expression).

We will assume that this spectral parameter u is the same as in the full quantum AdS/CFT Y–system. Then the initial parameter x is a double–valued function of the new parameter u. This implies some additional analyticity features of the $Y_{a,s}$. Namely, we expect that $Y_{a,s}$ will have cuts parallel to the real axis with the branchpoints at $\pm 2g + in/2$. To fix the cut structure we distinguish two kinematics: the physical and the mirror. In the mirror kinematics the role of time and space is swapped.

$$x^{ph}(u) = \frac{1}{2}\left(\frac{u}{g} + \sqrt{\frac{u}{g}-2}\sqrt{\frac{u}{g}+2}\right), \qquad x^{mir}(u) = \frac{1}{2}\left(\frac{u}{g} + i\sqrt{4-\frac{u^2}{g^2}}\right). \tag{5.2.14}$$

These functions have branch cuts at $(-2g,2g)$ and $(-\infty,-2g)\cup(2g,\infty)$ respectively.

There is one thing left in our discussion. When one tries to truncate the Y–system from the full (a,s) lattice to the T–hook one has to set Y–functions to zero on the vertical boundaries and to ∞ at the horizontal boundaries. What can we say about the Y–

functions in the corner, $Y_{2,\pm 2}$? They contains an uncertainty 0/0, so we should use some additional information to resolve this uncertainty.

In order to fix the functions $Y_{2,\pm 2}$ we can use a fact from the TBA [112]: in the mirror kinematics $Y_{2,\pm 2}(u)$ and $Y_{1,\pm 1}(u)$ are related on two sides of the cut $(-\infty,2g)\cup(2g,\infty)$ by

$$Y_{2,\pm 2}(u+i0)=\frac{1}{Y_{1,\pm 1}(u-i0)}. \tag{5.2.15}$$

Once we have a solution of the Y–system, the corresponding energy of a string state (i.e. anomalous dimension of a SYM operator) can be obtained from the expression very similar to (4.1.44):

$$E=\sum_{j}^{K}\epsilon_1^{ph}(u_{4,j})+\sum_{a=1}^{\infty}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\frac{\partial\epsilon_a^{mir}}{\partial u}\log\left(1+Y_{a,0}(u)\right). \tag{5.2.16}$$

Here $\epsilon_a^{ph,mir}$ is defined via $x^{ph,mir}(u)$ with such a formula:

$$\epsilon_a(u)=a+\frac{2ig}{x(u+ia)}-\frac{2ig}{x(u-ia)}. \tag{5.2.17}$$

The physical roots (corresponds to the middle node) are subject to exact finite size Bethe equations

$$Y_{1,0}^{ph}(u_{4,j})=-1. \tag{5.2.18}$$

The last important condition on Y–function is that at large L the Y–function of the middle node should be exponentially supressed on the real axis in the mirror sheet

$$Y_{a,0}(u)\sim e^{-ip_a^{mir}(u)L},\qquad p_a(u)=-i\log\left(\frac{x(u+ia)}{x(u-ia)}\right)^{L}. \tag{5.2.19}$$

5.2.4 Full quantum Hirota equation for the AdS/CFT σ–model

It is known [113] that all-loop result for the gauge theory could be obtained from the general $PSU(2,2|4)$ Y–system by redefining the polynomials Q in such a way that

$$Q_{2,2}(u)=R^{-(+)},\quad Q_{2,2}^{--}(u)=R^{-(-)},\quad Q_{0,0}=B^{+(-)}(u),\quad Q_{0,0}^{++}(u)=B^{+(+)}(u), \tag{5.2.20}$$

where

$$R_n^{(\pm)}(u)=\prod_j^{K_n}\frac{x(u)-x^{\mp}(u_{n,j})}{\left(x^{\mp}(u_{n,j})\right)^{1/2}},\qquad B_n^{(\pm)}(u)=\prod_j^{K_n}=\frac{1/x(u)-x^{\mp}(u_{n,j})}{\left(x^{\mp}(u_{n,j})\right)^{1/2}}. \tag{5.2.21}$$

With this definition we have $Q_n^{\pm}(u)=(-g)^{K_n}R_n^{\pm}B_n^{\pm}(u)$ and for the T–function

$$T_{1,1}=R^{-(+)}\left(\frac{Q_2^{++}Q_1^{-}}{Q_2Q_1^{+}}+\frac{Q_3^{+}Q_2^{--}}{Q_3^{-}Q_2}-\frac{Q_3^{+}R^{-(-)}}{Q_3^{-}R^{-(+)}}-\frac{B^{+(+)}Q_1^{+}}{B^{+(-)}Q_1^{+}}\right). \tag{5.2.22}$$

From the symmetry relations in the section 5.2.2 it follows that

$$\overline{T_{1,1}(1/x)} = \frac{B^{+(-)}Q_1^+Q_3^-}{R^{-(+)}Q_1^-Q_3^+}T_{1,1}(x). \tag{5.2.23}$$

As in the section 4.2.2 now we can glue the two wings of our T–hook via the condition similar to the (4.2.39):

$$Y_{a,0}(x) = \left(\frac{x(u-ia/2)}{x(u+ia/2)}\right)^L \frac{f(u-ia/2)}{f(u+ia/2)} T^l_{a,-1}T^r_{a,1}. \tag{5.2.24}$$

In this equation the first two factors are zero modes, i.e. the solutions of the equation

$$\frac{f_a^+ f_a^-}{f_{a+1}f_{a-1}} = 1. \tag{5.2.25}$$

The dependence on L is fixed by the asymptotical condition (5.2.19) at large u. The function $f(u)$ can be determined from the condition (5.2.18).

Thus we obtained the Y–system for our quantum σ–model problem. In the next section we will solve this Y–system for the Konishi operator in a strong coupling limit and compare the results with what we got from the algebraic curve computation.

5.3 The $1/\log S$ corrections from the Y–system

In the strong coupling limit $g \to \infty$ Y–system drastically simplifies. In this limit the CFT side of the AdS/CFT hypothesis should be equivalent to the quasi-classical strings, so all the spectrum computations from the Y–system can be compared to the algebraic curve computation which was done in the section 3.4. So it will be possible to identify the origin of different corrections from the point of view of the Y–system, and this exercise may shed some light on the relation between the sides of the duality.

One can notice [80] that in this scaling limit

$$\frac{R^{(+)}B^{(-)}}{R^{(-)}B^{(+)}} = \prod_{j=1}^{M} \frac{x(z)-x_j^-}{x(z)-x_j^+}\frac{1/x(z)-x_j^+}{1/x(z)-x_j^-} \approx \frac{1}{f(z)\bar{f}(z)}, \tag{5.3.1}$$

where

$$f(z) = \exp\left(-iG(x(z)\right)\,, \quad \bar{f}(z) = \exp\left(+iG(1/x(z)\right)\,, \quad \Delta = \exp\left(-\frac{J}{2g\sqrt{1-z^2}}\right) \tag{5.3.2}$$

and $G(x)$ is the resolvent

$$G(x) = \frac{1}{g}\sum_{j=1}^{S}\frac{1}{x-x_j}\frac{x_j^2}{x_j^2-1}\,. \tag{5.3.3}$$

From the general solution it is possible to obtain the expression for the functions $Y_{a,0}$ and $Y_{a,1}$ in terms of two yet unknown functions. Gluing them together (i.e. matching wings) and using general Y–system equations we can deduce that

$$Y_{1,1}Y_{2,2}=\frac{1}{f(z)\bar{f}(z)}\prod_{m=1}^{\infty}\left(1+Y_{a,m}\right). \tag{5.3.4}$$

Now one can recall that due to the Hirota equation

$$1+Y_{a,s}=\frac{T_{a,s}^2}{T_{a+1,s}T_{a-1,s}}, \tag{5.3.5}$$

so we can rewrite equation (5.3.4) without infinite product:

$$Y_{1,1}Y_{2,2}=\frac{T_{1,0}}{T_{0,0}}\Delta. \tag{5.3.6}$$

Applying now all this information to the energy equation (5.2.16) and having in mind that in strong coupling

$$\epsilon_1^{ph}(z)=\frac{x^2+1}{x^2-1}+\mathcal{O}\left(\frac{1}{g^2}\right),\qquad \epsilon_a^{mir}(z)=-\frac{iaz}{\sqrt{1-z^2}}, \tag{5.3.7}$$

we get

The expression of the energy at one loop, including the finite-size correction, is

$$E=\sum_{j=1}^{S}\frac{x_j^2+1}{x_j^2-1}+\int_{-1}^{1}\frac{dz}{2\pi}\frac{z}{\sqrt{1-z^2}}\partial_z\mathcal{M}_0=\sum_{j=1}^{S}\frac{x_j^2+1}{x_j^2-1}-\int_{-1}^{1}\frac{dz}{2\pi}\frac{1}{(1-z^2)^{3/2}}\mathcal{M}_0, \tag{5.3.8}$$

where

$$\mathcal{M}_0=\log\frac{(f\Delta-1)^4(\bar{f}\Delta-1)^4}{(\Delta-1)^4(f\bar{f}\Delta-1)^2(f^2\Delta-1)(\bar{f}^2\Delta-1)}, \tag{5.3.9}$$

The integration is done in the mirror regime, with $x(z)=z+i\sqrt{1-z^2}$. The second term in (5.3.8) is given by the contribution of the virtual particles circulating along the circumference of the system and which scatter with the magnons with rapidity x_j. We are therefore going to call this term the virtual particle contribution. In finite volume, the positions of the Bethe roots x_j are slightly shifted from their infinite volume positions due to their interaction with the virtual particles; we are going to call this effect backreaction. In the one-loop limit, the backreaction can be taken into account [80] by adding an extra potential term to the Bethe ansatz equations, which become

$$\begin{aligned}2\pi n &= p(x+i0)+p(x-i0)+\alpha(x)p'(x)\cot p(x)+\mathcal{V}(x)\\ &- 2i\sum_{k=1}^{M}\int_{-1}^{1}dz\,(r(x,z)\mathcal{M}_+-r(1/x,z)\mathcal{M}_-+u(x,z)\mathcal{M}_0)\end{aligned} \tag{5.3.10}$$

with

$$p(x)=\frac{J}{2g}\frac{x}{x^2-1}+G(x)\quad\text{and}\quad \alpha(x)=\frac{1}{g}\frac{x^2}{x^2-1}. \tag{5.3.11}$$

The effective potential in the second line of (5.3.10) is given in term of the kernels

$$r(x,z)=\frac{x}{x^2-1}\frac{\partial_z}{2\pi g}\frac{1}{x-x(z)},\qquad u(x,z)=\frac{x}{x^2-1}\frac{\partial_z}{2\pi g}\frac{1}{x^2(z)-1}, \tag{5.3.12}$$

and the functions

$$\mathcal{M}_{+}=\log\frac{(f\Delta-1)^2}{(f^2\Delta-1)(f\bar{f}\Delta-1)}\,,\qquad \mathcal{M}_{-}=\log\frac{(\bar{f}\Delta-1)^2}{(\bar{f}^2\Delta-1)(f\bar{f}\Delta-1)}\,. \tag{5.3.13}$$

By inspection, the imaginary part of the resolvent $G(x)$ in the mirror regime is always negative with $\mathrm{Im}\,G(x)\sim-\log S$, so that we have

$$\mathcal{M}_0\simeq-4\log(1-\Delta)-\Delta(f^2+\bar{f}^2+2f\bar{f}-4f-4\bar{f})=-4\log(1-\Delta)-4\Delta R(R-2)\,, \tag{5.3.14}$$

where $R=\exp(\mathrm{Im}\,G(x))\cos(\mathrm{Re}\,G(x))$. The last term in $\mathcal{M}_0$ is suppressed by a negative power of S. The only region where R can be close to 1 is $x\simeq1$, but in this region it is Δ which is exponentially suppressed. We conclude that the correction to the energy due to the virtual particles is

$$\delta E_v=4\int_{-1}^{1}\frac{dz}{2\pi}\frac{1}{(1-z^2)^{3/2}}\log(1-\Delta)=\mathcal{I}(2\ell\log S)\,. \tag{5.3.15}$$

with $\mathcal{I}(\alpha)$ defined in (3.5.6) and (3.5.7). It is interesting to note that the virtual particle correction is singular when $\ell\to0$, and that this divergence will be compensated by the backreaction of the roots. It is likely that such a phenomenon happens whenever the endpoint a of the cut approaches the singularity $x=1$. In particular (logarithmic) singularities appear for separate E_i terms in the small $\mathcal{S},\mathcal{J}$ limit, see section 3.4. A similar effect is observed when $f_2(\ell)$ is expanded at small ℓ [24]. This partially reflects the complicated analytical structure of the Y-system.

Let us now compute the backreaction term, *i.e.* the second line in the BES equation (5.3.10). The contribution from $\mathcal{M}_\pm$ is vanishing again as a negative power of S. The term containing $\mathcal{M}_0$ is simply

$$8i\frac{x}{x^2-1}\int_{-1}^{1}\frac{dz}{2\pi g}\partial_z\left(\frac{1}{x(z)^2-1}\right)\log(1-\Delta)=\frac{\mathcal{I}(\alpha)}{g}\frac{x}{x^2-1}\,. \tag{5.3.16}$$

Let us remind that at the leading order the asymptotic Bethe ansatz equations are written as

$$2\pi n=G_0(x+i0)+G_0(x-i0)+2V_0(x) \tag{5.3.17}$$

with

$$2V_0(x)=\frac{J}{g}\frac{x}{x^2-1}=4\ell\log S\frac{x}{x^2-1} \tag{5.3.18}$$

and $G_0(x)$ a function analytic everywhere except of the cuts on the intervals $(-\infty,-a)\cup(a,\infty)$. We conclude that the only effect of the backreaction at one loop is to renormalize the coefficient of the potential term and therefore to renormalize ℓ by a one-loop quantity

$$\ell\to\tilde{\ell}=\ell+\frac{\mathcal{I}(\alpha)}{4g\log S}\simeq\ell-\frac{\pi}{12\ell g\log^2 S}\,. \tag{5.3.19}$$

To fix uniquely the solution of the leading order equation (5.3.17) we have to supply the asymptotics at infinity for $G_0(x)$. This is a little bit tricky if we have sent the endpoints of the two cuts $(-b,-a)$ and (a,b) to infinity first. A straighforward procedure is to solve the equation (5.3.17) for finite b (see [22]), imposing that $G_0(x) \sim 1/x$ at $x\to\infty$ and then take the limit of infinite b. The result of this procedure would give $G_0(x) = p_{\hat 2}(x) - p_{\tilde 2}(x)$ with $p_{\hat 2}(x)$, $p_{\tilde 2}(x)$ from (3.1.10). An alternative is to work directly with $b\to\infty$ and impose the same asymptotics for $G_0(x)$ as in the weak-coupling, one loop limit [114]

$$G_0(x) \sim 2i\log(S/gx) \quad \text{for} \quad x\to\infty - i0\,. \tag{5.3.20}$$

The solution to the equation (5.3.17) supplemented with this condition at infinity reads

$$\begin{aligned} G_0(x) &= \sqrt{a^2-x^2}\oint_{\mathscr{C}}\frac{dy}{2\pi i}\frac{V_0(y)}{(x-y)\sqrt{a^2-y^2}} - 4\arctan\sqrt{\frac{a-x}{a+x}} &(5.3.21)\\ &= \frac{J}{2g}\frac{x}{x^2-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-1}}-1\right) - 4\arctan\sqrt{\frac{a-x}{a+x}}\,, &(5.3.22)\end{aligned}$$

where in the first line the contour of integration $\mathscr{C}$ encircles the cuts $(-\infty,-a)\cup(a,\infty)$ and can be closed at infinity counterclockwise. The value of a is fixed by the asymptotics at infinity (5.3.20) and it yields the same condition as (3.5.3)

$$\sqrt{a^2-1} = \frac{J}{4g\log(2S/ag)} = \ell + \mathscr{O}(1/\log S)\,. \tag{5.3.23}$$

The anomalous dimension at leading order is given by

$$\begin{aligned} E_0 &= -\oint_{\mathscr{C}}\frac{dx}{2\pi i}\frac{2}{x^2-1}\frac{G_0(x)}{\alpha(x)} = \frac{J}{\sqrt{a^2-1}}\left(a-\sqrt{a^2-1}\right) - \frac{4g}{a}\\ &= 4g\log S\left(\sqrt{1+\ell^2}-\ell\right) + \mathscr{O}((\log S)^0) &(5.3.24)\end{aligned}$$

with the integration contour running counterclockwise around $x=0$. Now we can estimate the one-loop correction from the backreaction due to the shift $\ell\to\ell+\delta\ell$ from equation (5.3.19),

$$\delta E_b = 4g\log S\,\delta\left(\sqrt{1+\ell^2}-\ell\right) = -\mathscr{I}(2\ell\log S) + \frac{\ell\mathscr{I}(2\ell\log S)}{\sqrt{1+\ell^2}}. \tag{5.3.25}$$

The one-loop wrapping corrections are then given by:

$$\delta E_w = \delta E_v + \delta E_b = \frac{\ell\mathscr{I}(2\ell\log S)}{\sqrt{1+\ell^2}} = -\frac{4\pi}{12\log S} + \mathscr{O}((\ell\log S)\log(\ell\log S)) \tag{5.3.26}$$

and they coincide with the contribution of the four massive modes δE_2 (3.5.8).

The wrapping corrections give the $1/\log S$ corrections corresponding to only four of the five bosonic modes. To find the fifth one we are going to solve the one-loop equation for the resolvent

$$0 = G_1(x+i0) + G_1(x-i0) + 2V_1(x) \tag{5.3.27}$$

with

$$\frac{2V_1(x)}{g} = \mathscr{V}(x) + \alpha(x)\, p_0' \cot p_0 \,, \qquad p_0(x) = G_0(x) + V_0(x)\,. \tag{5.3.28}$$

Here $\alpha(x)\, p_0' \cot p_0$ is the so-called anomaly term and $\mathscr{V}(x)$ is the Hernandez-Lopez phase with integral representation [115]

$$\mathscr{V}(x) = \int_{U^+} \frac{dy}{2\pi} \left(\frac{\alpha(x)}{x-y} - \frac{\alpha(1/x)}{1/x - y} \right) \partial_y (G_0(y) - G_0(1/y))\,, \tag{5.3.29}$$

where the integral is taken clockwise on the upper half of the unit circle U_+. The solution to the one-loop equation can be again written in an integral form [22, 23]

$$G_1(x) = \oint_{\mathscr{C}} \frac{dy}{2\pi i} \frac{V_1(y)}{(x-y)} \frac{\sqrt{a^2 - y^2}}{\sqrt{a^2 - x^2}}\,. \tag{5.3.30}$$

The one-loop correction to the energy is given, similarly to the leading order, by

$$E_{1,\text{ABA}} = -\oint_{\mathscr{C}} \frac{dx}{2\pi i} \frac{2G_1(x)}{x^2} = \frac{2}{a} \oint_{\mathscr{C}} \frac{dy}{2\pi i} \frac{\sqrt{a^2 - y^2}}{y^2} V_1(y)\,. \tag{5.3.31}$$

Substituting the value of the potential $V_1(x)$ we retrieve the contributions (3.5.9) from the algebraic curve computation

$$\begin{aligned} E_{1,\text{ABA}} &= -\frac{4}{a} \int_a^\infty \frac{dy}{2\pi} \frac{\sqrt{y^2 - a^2}}{y^2 - 1} p' \coth p - \\ &- \frac{4}{a} \int_{U_+} \frac{dy}{2\pi} \operatorname{Im} \frac{\sqrt{a^2 - y^2} - \sqrt{a^2 - 1}}{y^2 - 1} \partial_y \operatorname{Im} G_0(y) = \\ &= \delta E_1 + \delta E_3\,. \end{aligned} \tag{5.3.32}$$

This result confirms that the asymptotic Bethe ansatz contribution is captured by $\delta E_1 + \delta E_3$ and that the wrapping corrections are reproduced by δE_2.

Chapter 6

Conclusions and future directions

In this thesis we studied the different aspects of integrability on the both sides of AdS/CFT correspondence.

It is now clear that the technique of integrability, which originally was used to solve the two dimensional models, can play an important role in the complete solving the spectral problem in higher dimensional gauge theory. Presumably, there are some important features of the string and gauge theories which automatically imply the (classical) integrability. There is a classification of all integrable AdS/CFT backgrounds [65] which shows that AdS_5/CFT_4 is not a unique integrable duality. Indeed, one can explore the integrable structure in AdS_4/CFT_3 correspondence (see [116] for the review), in AdS_3/CFT_2 [66] and in AdS_2/CFT_1 [117]. The geometry of the background are very different (nevertheless they all have the form $AdS_d \times \mathscr{M}$), but behind them we see the $\mathbb{Z}_4$ automorphism and hence the classical integrable structures analogously to the construction proposed in [16], as we discussed it in the chapter 2.2. There is a plenty of things to be done here: first of all, for some of these dualities their CFT dual is not known. There is a hope that integrability can shed some light on this subject [118], but we are still far from the final answer. Second, once the dual will be constructed it has to be solved, apparently, again with the help of integrability. We can even construct the Bethe ansatz for some of these duals (see [119] for such a construction in AdS_3/CFT_2 case), but immediately there are some new questions to solve. It is known that AdS_3/CFT_2 duality contains some massless modes — and it is a challenge to incorporate them into the duality, since the Bethe ansatz for them could be singular.

But for the AdS_5/CFT_4 duality we do not have such a questions. Moreover, one can say that in principle the spectral problem is solved. For the moment we have all the equations desribing the spectral problem at any coupling constant (Y–system). We have shown how to solve them analytically in one particular case, but numerically they can be solved with the given precision for the large class of operators.

But the main problem of the Y–system is the complexity of analytic properties of the Y–functions. That's why analytical solution of these equations in general situation is still nearly impossible. There are some attempts to avoid this difficulties reformulating the Y–system in a Destri–de Vega form, i.e. finite system of nonlinear integral equations (FiNLIE) [79]. In a recent paper [120] the power of FiNLIE was demonstrated by the analytical computation of the six–loop anomalous dimension of the Konishi operator (see also derivation of the same dimension from the Lüscher corrections [121]). Since this computation is based on the recursive procedure, one can expect that the next terms could be obtained in the same manner. But the main ingredient — direct derivation of the Y–system (or FiNLIE) from the σ–model or the gauge theory is still missing.

However, in general, the spectral problem is more or less solved, at least on the technical level. We still do not know exactly the spectrum of which operator we are looking for, because we do not know the full hamiltonian of the theory, but the computation itself is here. To completely solve the gauge theory one should also compute the three–point function. Since the $\mathcal{N}=4$ SYM is conformal it is sufficient to compute two–point function (i.e. solve the spectral problem) and three–point function. The recent progress in this field was made again with the help of integrability [122, 123, 124, 125, 126, 127].

If we compute some three-point functions in the gauge theory, there should be dual quantity in the string theory. Namely, one can try to compute the correlation function of the three vertex operators of particular massive string states in $AdS_5 \times S^5$ theory (see [128, 129, 130] for the first computations of these quantities). Most of the work here should be done in the near future, but there is no doubts that integrability will continue to play the key role in the investigation of the AdS/CFT correspondence.

Appendix A

Strings on arbitrary background

Here we review the procedure of the gauge-fixing in the Green–Schwarz action for arbitrary curved background.

To fix the gauge we will use the following strategy [131, 60, 61]: first we transform into the reference frame moving in the φ direction with the speed v. The velocity v will serve as a regularization parameter, which we will set to one at the end of the calculation. At $v<1$, the resulting gauge is equivalent to the interpolating a-gauge introduced in [132]. The gauge-fixed Lagrangian is substantially more complicated in this family of gauges, compared to the light-cone gauge, obtained in the limit $v\to 1$. Then we T-dualize in the φ direction, integrate the worldsheet metric, and fix the static gauge $\tilde{\varphi}=\sigma$, $t=\tau$ in the resulting Nambu-Goto action. Expanding the Nambu action to the quartic order in the fields, we get the desired light-cone Lagrangian.

Let us discuss the general procedure of T–duality for the σ–model on any background. Let us consider the simplest example of T–duality. The action of σ–model looks like

$$S=\int d\tau d\sigma\sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j g_{ij}-\epsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^j b_{ij}+2\partial_\alpha\phi^i(h^{\alpha\beta}u_{\beta i}-\epsilon^{\alpha\beta}v_{\beta i})\right). \qquad \text{(A.0.1)}$$

T–duality occurs when one starts to transform field ϕ^i with some shift depending on (τ,σ):

$$\phi^i\to\phi^i+\delta\phi(\tau,\sigma). \qquad \text{(A.0.2)}$$

For the sake of simplicity we will consider the case when only ϕ^1 changes. The rest of the fields remain the same.

To make the action S gauge–invariant, we should introduce the covariant derivative:

$$D_\alpha\phi^1=\partial_\alpha\phi^1+A_\alpha. \qquad \text{(A.0.3)}$$

Also one should add the term $\tilde{\phi}^1\epsilon^{\alpha\beta}F_{\alpha\beta}$ to be sure that the gauge field A_α has the correct dynamics. So we can rewrite action in the form

$$S=\int d\tau d\sigma\sqrt{-h}\Big(h^{\alpha\beta}D_\alpha\phi^1 D_\beta\phi^1 g_{11}+2D_\alpha\phi^1 D_\beta\phi^j g_{1j}+D_\alpha\phi^i D_\beta\phi^j g_{ij}- \\ -\epsilon^{\alpha\beta}D_\alpha\phi^i D_\beta\phi^j b_{ij}-2\epsilon^{\alpha\beta}D_\alpha\phi^1 D_\beta\phi^j b_{1j}+ \\ 2D_\alpha\phi^1(h^{\alpha\beta}u_{\beta 1}-\epsilon^{\alpha\beta}v_{\beta 1})+2\partial_\alpha\phi^a(h^{\alpha\beta}u_{\beta a}-\epsilon^{\alpha\beta}v_{\beta a})+\tilde{\phi}^1\epsilon^{\alpha\beta}F_{\alpha\beta}\Big). \tag{A.0.4}$$

This action is gauge invariant, and we can fix the gauge just setting $\phi^1=0$. So our covariant derivative becomes a multiplication by A_α. Now we can act in two ways.

1. Integrate out $\tilde{\phi}^1$. From the equations of motion for $\tilde{\phi}^1$ we obtain $F_{\alpha\beta}=0$. So we can conclude that $A_\alpha=\partial_\alpha\theta$. It is obvious that $\theta=\phi^1$.
2. Integrate out A_α. From the equations of motion for A_α we obtain

$$A_\alpha=\frac{1}{g_{11}}\Big(-\partial_\alpha\phi^a g_{1a}+h_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi^j b_{1j}-u_{\alpha 1}+h_{\alpha\beta}\epsilon^{\beta\rho}v_{\rho 1}-h_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\phi}^1\Big). \tag{A.0.5}$$

Comparing this expression with $A_\alpha=\partial_\alpha\phi^1$ we obtain following relation between original and dual variables:

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{\phi}^1=-h^{\alpha\beta}g_{11}\partial_\beta\phi^1-h^{\alpha\beta}\partial_\beta\phi^j g_{1j}+\epsilon^{\alpha\beta}\partial_\beta\phi^j b_{1j}-h^{\alpha\rho}u_{\rho 1}+\epsilon^{\alpha\rho}v_{\rho 1}. \tag{A.0.6}$$

Plugging this into the original action (A.0.1), we obtain the action in terms of dual variable $\tilde{\phi}^1$ and dual metric:

$$S=\int d\tau d\sigma\sqrt{-h}\Big(h^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\tilde{g}_{ij}-\epsilon^{\alpha\beta}\partial_\alpha\tilde{\phi}^i\partial_\beta\tilde{\phi}^j\tilde{b}_{ij}+2\partial_\alpha\phi^i(h^{\alpha\beta}\tilde{u}_{\beta i}-\epsilon^{\alpha\beta}\tilde{v}_{\beta i})\Big). \tag{A.0.7}$$

We can write down explicitly expressions for the dual metric (generalization of the Buscher's rules [133, 134]):

$$\begin{aligned} \tilde{g}_{11}=\frac{1}{g_{11}},\quad \tilde{g}_{ij}&=g_{ij}-\frac{g_{1i}g_{1j}-b_{1i}b_{1j}}{g_{11}},\quad \tilde{g}_{1i}=\frac{b_{1i}}{g_{11}};\\ \tilde{b}_{1i}=\frac{g_{1i}}{g_{11}},\quad \tilde{b}_{ij}&=b_{ij}-\frac{g_{1i}b_{1j}-b_{1i}g_{1j}}{g_{11}},\quad \tilde{b}_{i1}=-\frac{g_{1i}}{g_{11}}\\ \tilde{u}_{\alpha 1}=\frac{v_{\alpha 1}}{g_{11}},\quad \tilde{v}_{\alpha 1}=\frac{u_{\alpha 1}}{g_{11}},\quad \tilde{u}_{\alpha i}&=u_{\alpha i}-\frac{g_{1i}u_{\alpha 1}-b_{1i}v_{\alpha 1}}{g_{11}},\quad \tilde{v}_{\alpha i}=v_{\alpha i}-\frac{g_{1i}v_{\alpha 1}-b_{1i}u_{\alpha 1}}{g_{11}}. \end{aligned} \tag{A.0.8}$$

Now we can proceed to the light–cone gauge. We want to express this action through the variable x^+:

$$x^+=(1-a)t+\phi a=(1-a)x^0+ax^3. \tag{A.0.9}$$

After a little algebra one can get for the effective metric

$$
\begin{aligned}
g_{++} &= \frac{1}{(1-a)^2} g_{00}, \\
g_{33} &= g_{33} + \frac{a^2}{(1-a)^2} g_{00} - \frac{2a}{1-a} g_{30}, \\
g_{ij} &= g_{ij}, \\
g_{3+} &= -\frac{a}{(1-a)^2} g_{00} + \frac{1}{1-a} g_{30}, \\
g_{+i} &= \frac{1}{1-a} g_{0i}, \\
g_{3i} &= g_{3i} - \frac{a}{1-a} g_{0i}.
\end{aligned}
\tag{A.0.10}
$$

One can obtain the same transformation rules for $b_{ij}, u_{\beta i}, v_{\beta i}$.

Now we are perfoming T–duality in ϕ (x^3) direction. According to the results of the previous section, we obtain for the dual metric

$$
\begin{aligned}
\tilde{g}_{++} &= \frac{g_{33}g_{00} - g_{03}^2 + \left(b_{30} - \frac{a}{1-a} b_{00}\right)^2}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{03}}, \\
\tilde{g}_{33} &= \frac{(1-a)^2}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{30}}, \\
\tilde{g}_{ij} &= g_{ij} - \frac{(1-a)^2 g_{3i} g_{3j} - a(1-a)(g_{3i} g_{0j} + g_{3j} g_{0j}) + a^2 g_{0i} g_{0j} + (g \leftrightarrow b)}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{30}}, \\
\tilde{g}_{3i} &= \frac{(1-a)^2 b_{3i} - a(1-a) b_{0i}}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{30}}, \\
\tilde{g}_{3+} &= \frac{(1-a) b_{30} - a b_{00}}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{30}}, \\
\tilde{g}_{+i} &= \frac{(1-a)(g_{0i} g_{33} - g_{03} g_{3i}) + a(g_{00} g_{3i} - g_{0i} g_{03}) + (1-a)^{-1}(a b_{00} - (1-a) b_{30})(a b_{0i} - (1-a) b_{3i})}{(1-a)^2 g_{33} + a^2 g_{00} - 2a(1-a) g_{30}}.
\end{aligned}
\tag{A.0.11}
$$

For the anti-symmetric part of the action (and the part which is linear in field derivatives) we can also obtain the similar expressions.

Now we can integrate out the world–sheet metric:

$$
h_{\alpha\beta} = \tilde{g}_{MN} \partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N. \tag{A.0.12}
$$

So the T–dual Nambu-Goto action looks loke this:

$$
S = g \int d\tau d\sigma \left(\sqrt{-\det \tilde{g}_{MN} \partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N} + \epsilon^{\alpha\beta} \tilde{b}_{MN} \partial_\alpha \tilde{x}^M \partial_\beta \tilde{x}^N \right). \tag{A.0.13}
$$

We can finally fix the gauge with the conditions

$$
\tilde{x}^+ = \frac{\tau}{1-a}, \quad \tilde{\phi} = \sigma. \tag{A.0.14}
$$

Now let us compute the Nambu–Goto action. For that one needs to calculate the determinant and take a square root.

$$h \equiv \det_{\alpha\beta} h_{\alpha\beta} = \tilde{g}_{MN}\partial_0\tilde{X}^M\partial_0\tilde{X}^N \cdot \tilde{g}_{MN}\partial_1\tilde{X}^M\partial_1\tilde{X}^N - \left(\tilde{g}_{MN}\partial_0\tilde{X}^M\partial_1\tilde{X}^N\right)^2. \tag{A.0.15}$$

In our gauge (A.0.14) this determinant looks like

$$\begin{aligned} h &= \left(\frac{1}{(1-a)^2}\tilde{g}_{++} + \frac{1}{1-a}\tilde{g}_{+i}\partial_0\tilde{X}^i + \tilde{g}_{ij}\partial_0 X^i\partial_0 X^j\right)\cdot\left(\tilde{g}_{33} + \tilde{g}_{3i}\partial_1 X^i + \tilde{g}_{ij}\partial_1 X^i\partial_1 X^j\right) - \\ &- \left(\frac{1}{1-a}\tilde{g}_{+i}\partial_1\tilde{X}^i + \frac{1}{1-a}\tilde{g}_{+3} + \tilde{g}_{i3}\partial_0\tilde{X}^i + \tilde{g}_{ij}\partial_0 X^i\partial_1 X^j\right)^2. \end{aligned} \tag{A.0.16}$$

To obtain the BMN limit [135] one should expand this action in powers of $\tilde{X}$. To do this let us emphasize that $\tilde{g}_{ij}$ are the functions of the vielbein E_M^A, so we can control the order of each term in expansion.

We assume that the metric has two isometries, a timelike and a spacelike, that can be used to fix the light-cone gauge. The coordinates along the isometry directions will be denoted by $t \equiv X^0$ and $\varphi \equiv X^9$. We assume that the metric depends only on the transverse coordinates X^i, $i = 1,\dots,8$, and has the following form:

$$ds^2 = -G_{tt}dt^2 + G_{\varphi\varphi}\left(d\varphi + A_i dX^i\right)^2 + G_{ij}dX^i dX^j, \tag{A.0.17}$$

In the light-cone gauge, the string sigma-model becomes a complicated, very non-linear quantum field theory of the transverse string coordinates. However, if the string tension is small, this field theory is weakly-coupled and one can develop systematic perturbation theory in α' by expanding the metric near the light-cone geodesic $t = \tau = \varphi$. This is known as the Penrose expansion. Our goal will be to expand the Green-Schwarz action to the quartic in the transverse fields, and then fix the light-cone gauge. To this end, we assume the following scaling of the metric components:

$$\begin{aligned} G_{tt} &= 1 + \mathcal{O}\left(X^2\right), \qquad G_{\varphi\varphi} = 1 + \mathcal{O}\left(X^2\right), \\ A_i &= \mathcal{O}\left(X^2\right), \qquad G_{ij} = \delta_{ij} + \mathcal{O}\left(X^2\right). \end{aligned} \tag{A.0.18}$$

We will systematically drop terms of order higher than $\mathcal{O}(X^4)$. This way we will construct the action that contains all cubic terms and quartic terms of the form X^4 and $X^2\theta^2$. We will not study the $\mathcal{O}(\theta^4)$ terms, since the Lagrangian (2.1.1) is already truncated at the quadratic order in fermions. In principle the quartic fermion terms are also known for any supergravity background [136], and one can include those terms too, more or less reading off the four-fermion terms from [136], since at this order in the expansion they will not be affected by the gauge-fixing.

We can readily expand of the local frame and the spin connection around the light-cone geodesic:

$$\begin{aligned} E_0^{\hat{0}} &= \sqrt{G_{tt}} + \mathcal{O}\left(X^4\right), \qquad E_9^{\hat{9}} = \sqrt{G_{\varphi\varphi}} + \mathcal{O}\left(X^4\right), \\ E_i^{\hat{9}} &= A_i + \mathcal{O}\left(X^4\right), \qquad E_j^{\hat{i}} = \frac{1}{2}\left(\delta_{\hat{i}j} + G_{\hat{i}j}\right) + \mathcal{O}\left(X^4\right) = \delta_{\hat{i}j} + \mathcal{O}\left(X^2\right), \\ E_{\hat{9}}^i &= -A_i + \mathcal{O}\left(X^4\right), \qquad E_{\hat{j}}^i = \frac{1}{2}\left(\delta_{i\hat{j}} + G_{i\hat{j}}\right) + \mathcal{O}\left(X^4\right) = \delta_{i\hat{j}} + \mathcal{O}\left(X^2\right), \\ E_{\hat{0}}^0 &= \frac{1}{\sqrt{G_{tt}}} + \mathcal{O}\left(X^4\right), \qquad E_{\hat{9}}^9 = \frac{1}{\sqrt{G_{\varphi\varphi}}} + \mathcal{O}\left(X^4\right). \end{aligned} \tag{A.0.19}$$

The spin connection is needed only to the linear order in X^i:

$$\Omega_0^{\hat{0}\hat{i}} = \frac{1}{2}\partial_{\hat{i}} G_{tt} + \mathscr{O}\left(X^3\right), \qquad \Omega_9^{\hat{9}\hat{i}} = \frac{1}{2}\partial_{\hat{i}} G_{\varphi\varphi} + \mathscr{O}\left(X^3\right),$$
$$\Omega_9^{\hat{i}\hat{j}} = -\frac{1}{2}F_{\hat{i}\hat{j}} + \mathscr{O}\left(X^3\right), \qquad \Omega_j^{\hat{9}\hat{i}} = \frac{1}{2}F_{\hat{i}j} + \mathscr{O}\left(X^3\right),$$
$$\Omega_k^{\hat{i}\hat{j}} = \frac{1}{2}\partial_{\hat{j}} G_{\hat{i}k} - \frac{1}{2}\partial_{\hat{i}} G_{\hat{j}k} + \mathscr{O}\left(X^3\right), \tag{A.0.20}$$

where we have introduced the notation:

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{A.0.21}$$

Substituting these expression into (2.1.1) , and redefining

$$\begin{aligned} \mathbb{E}_a &= \partial_a X^i \Gamma_i \\ D_a &= \partial_a - \frac{1}{4}\partial_a X^i \left(F_{ij}\Gamma_{9j} + \partial_j G_{ki}\Gamma_{jk}\right), \end{aligned} \tag{A.0.22}$$

we find for the fermion part of the Lagrangian:

$$\begin{aligned} \mathscr{L}_{\mathrm{f}} = & \bar{\theta}^I \left(\sqrt{-h} h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\mathbb{E}_a \mathscr{D}_b^{JK}\theta^K \\ & + \bar{\theta}^I \left[\sqrt{G_{tt}}\left(\sqrt{-h}h^{ab}\delta^{IJ} + \varepsilon^{ab}\sigma_3^{IJ}\right)\Gamma_0 \mathscr{D}_a^{JK} \right. \\ & \left. + \frac{1}{8}\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\mathbb{E}_a\left(\mathscr{F}^{JK}\Gamma_0 + 2\partial_i G_{tt}\Gamma_{0i}\delta^{JK}\right)\right]\theta^K \partial_b t \\ & + \bar{\theta}^I \left\{\sqrt{G_{\varphi\varphi}}\left(\sqrt{-h}h^{ab}\delta^{IJ} + \varepsilon^{ab}\sigma_3^{IJ}\right)\Gamma_9 \mathscr{D}_a^{JK} \right. \\ & + \frac{1}{8}\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\mathbb{E}_a\left[\mathscr{F}^{JK}\Gamma_9 \right. \\ & \left.\left. + \left(2\partial_i G_{tt}\Gamma_{9i} - F_{ij}\Gamma_{ij}\right)\delta^{JK}\right]\right\}\theta^K \partial_b \varphi \\ & + \frac{1}{8}\bar{\theta}^I\left(G_{tt}\Gamma_0 \mathscr{F}^{IJ}\Gamma_0 + 2\partial_i G_{tt}\Gamma_i \delta^{IJ}\right)\theta^J \sqrt{-h}h^{ab}\partial_a t \partial_b t \\ & + \frac{1}{8}\bar{\theta}^I\left[G_{\varphi\varphi}\Gamma_9 \mathscr{F}^{IJ}\Gamma_9 - \left(2\partial_i G_{\varphi\varphi}\Gamma_i + F_{ij}\Gamma_{9ij}\right)\delta^{IJ}\right]\theta^J \sqrt{-h}h^{ab}\partial_a\varphi\partial_b\varphi \\ & + \frac{1}{8}\bar{\theta}^I\left\{\left(\sqrt{-h}h^{ab}\delta^{IJ} - \varepsilon^{ab}\sigma_3^{IJ}\right)\left[\sqrt{G_{tt}G_{\varphi\varphi}}\Gamma_0 \mathscr{F}^{JK}\Gamma_9 \right.\right. \\ & \left. + \left(2\partial_i G_{\varphi\varphi}\Gamma_{09i} - F_{ij}\Gamma_{0ij}\right)\delta^{JK}\right] \\ & + \left(\sqrt{-h}h^{ab}\delta^{IJ} + \varepsilon^{ab}\sigma_3^{IJ}\right)\left(\sqrt{G_{tt}G_{\varphi\varphi}}\Gamma_9 \mathscr{F}^{JK}\Gamma_0 \right. \\ & \left.\left. - 2\partial_i G_{tt}\Gamma_{09i}\delta^{JK}\right)\right\}\theta^K \partial_a t \partial_b \varphi. \end{aligned} \tag{A.0.23}$$

We can also write the bosonic part order by order:

$$
\begin{aligned}
\mathscr{L}_{X^2} &= \frac{g_{ij}}{2}\partial_\mu X^i \partial^\mu X^j + b_{ij}\left(X^{i'}\dot{X}^j - \dot{X}^i X^{j'}\right) + \frac{g_{\phi\phi}}{2} - \frac{g_{tt}}{2} + \frac{g_{\phi t}}{2}, \\
\mathscr{L}_{X^3} &= g_{\phi i}\partial_0 X^i - b_{ti}\partial_1 X^i, \\
\mathscr{L}_{X^4} &= \frac{1}{4}(1-g_{tt})^2 - \frac{1}{4}\left(1-g_{\phi\phi}\right)^2 + \frac{1}{4}g_{\phi t}^2 + \\
&+ \frac{1}{4}\left(1-g_{\phi\phi}g_{tt}\right)g_{ij}\left(\dot{X}^i\dot{X}^j + X^{i'}X^{j'}\right) - b_{\phi t}g_{ij}\dot{X}^i X^{j'}.
\end{aligned} \tag{A.0.24}
$$

For the part with fermion derivatives ($\tilde{u}_{\beta M}$, $\tilde{v}_{\beta M}$) we can split the terms into three groups: $\tilde{u}_{\beta 0}$, $\tilde{u}_{\beta 3}$ and the rest (transverse) terms. It occurs that after expansion in X there are no terms proportional to $\partial_\alpha \tilde{\phi}$ — one should keep in mind that we are interested in the lagrangian only up to the quartic order. The rest terms are very simple and have the “naive” form:

$$
\mathscr{L}_{(\theta,\partial\theta)} = 2\partial_\alpha X^+\left(h^{\alpha\beta}u_{\beta+} - \epsilon^{\alpha\beta}v_{\beta+}\right) + 2\partial_\alpha X^i\left(h^{\alpha\beta}u_{\beta i} - \epsilon^{\alpha\beta}v_{\beta i}\right). \tag{A.0.25}
$$

Here $u_{\beta+}, v_{\beta+}, u_{\beta i}, v_{\beta i}$ can be easily expressed in terms of fermions $(\bar{\theta}, \partial_\beta\theta)$ and vielbein E_M^A. One can see that these terms are of quadratic order in fermions and after gauge fixing become the standard kinetic terms in the Dirac form. Roughly speaking they can be obtained from the original action (2.1.1) just setting $\phi = t = \tau$.

Appendix B

$\mathbb{Z}_4$ automorphism and $psu(2,2|4)$ superalgebra.

In the section 2.2.3 we observe that there is a special matrix S_{lm} which defines the inversion symmetry for the quasimomenta. In this section we will compute S_{lm} explicitly.

Since $psu(2,2|4)$ is a real form of $psl(4|4)$ we will start with $sl(4|4)$. It is an algebra of complex $(4|4)\times(4|4)$ matrices with zero supertrace:

$$\mathrm{str}\, M = \sum_i (-1)^{|i|} M_{ii} = 0, \tag{B.0.1}$$

where $|i|$ is the parity. Obviously, we can place bosonic and fermionic parts in this matrix in the several ways. In the standard choise the diagonal 4×4 blocks are bosonic, the off–diagonal are fermionic so that

$$su(4)\oplus su(4) = so(6)\oplus so(6)\oplus u(1). \tag{B.0.2}$$

Here the $u(1)$ term is central and can be factorized out, and we will left with the $psl(4|4)$.

The standard choice for the basis of Cartan generators is

$$(H_l)_{ij} = (-1)^{|i|}\delta_{ij}(\delta_{i,l}-\delta_{i,l+1}), \quad l = 1,\ldots,7, \tag{B.0.3}$$

and the Cartan matrix is

$$A_{lm} = \mathrm{str}\, H_l H_m = \left[(-1)^{|l|}+(-1)^{|l|}\right]\delta_{lm} - (-1)^{|l|}\delta_{l,m+1} - (-1)^{|l+1|}\delta_{l,m-1}. \tag{B.0.4}$$

The Cartan matrix depends on the parity of the rows and columns, which we can choose in many ways. Let us fix it like this (“B” and “F” stands for the bosonic or fermionic parity):

H_1	H_2	H_3	H_4	H_5	H_6	H_7	
1							B
1	-1						F
	1	-1					F
		-1	1				B
			-1	1			B
				1	-1		F
					1	-1	F
						-1	B

With this choice of H_i the Cartan matrix is

$$A = \begin{pmatrix} & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & \end{pmatrix}. \tag{B.0.5}$$

Following [137] we find the $\mathbb{Z}_4$ automorphism $\mathscr{M}$ acts on the supermatrix like this:

$$\mathscr{M} \circ \begin{pmatrix} A & \Theta \\ \Psi & B \end{pmatrix} = \begin{pmatrix} JA^tJ & -J\Psi^tJ \\ J\Theta^tJ & JB^tJ \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -\mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}. \tag{B.0.6}$$

From the last three equations we can compute the matrix S_{lm} defined in (2.2.44):

$$S = \begin{pmatrix} & & 1 & -1 & & & \\ & 1 & & -1 & & & \\ 1 & & & -1 & & & \\ & & & -1 & & & \\ & & & -1 & & & 1 \\ & & & -1 & & 1 & \\ & & & -1 & 1 & & \end{pmatrix}. \tag{B.0.7}$$

We can use these expressions for the building the set of the quasimomenta with respect to the inversion symmetry.

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