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L'énergie noire et la formation des grandes structures de l'Univers

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Abbreviations

GR	**G**eneral **R**elativity
EFT of DE	**E**ffective **F**ield **T**heory of **D**ark **E**nergy
EOM	**E**quation(s) **O**f **M**otion
DOF	**D**egree(s) **O**f **F**reedom
FLRW	**F**riedmann-**L**emaître-**R**oberston-**W**alker
ADM	**A**rnowitt-**D**eser-**M**isner
EP	**E**quivalence **P**rinciple
WEP	**W**eak **E**quivalence **P**rinciple
EFTI	**E**ffective **F**ield **T**heory of **I**nflation

Notations

Φ	00 part of the metric
Ψ	trace of the spatial metric
$f_A \equiv \frac{\partial f}{\partial A}$	
$\phi_\mu \equiv \nabla_\mu \phi,$	
$\phi_{\mu\nu} \equiv \nabla_\nu \nabla_\mu \phi, \ \ldots$	
$X \equiv \phi_\mu \phi^\mu$	
$R \equiv {}^{(3)}R$	When not specified, the Ricci scalar is the spatial one.

Chapter 1

Introduction

I feel particularly lucky to have been working on my Ph.D at such an exciting time for cosmology. With the fantastic results of the Planck mission [1], our picture of the Universe and its history has become much clearer. The precision of these observations, as well as that of future large scale structure missions such as EUCLID [2] and LSST [3], has also highlighted the challenges that cosmology faces. One of them is that theorists need to build efficient ways to compare theory and observations to say anything quantitative about potential deviations from the standard model of cosmology, ΛCDM. Secondly, to make sure no theoretically consistent model is overlooked, the conditions to have stable theories need to be further investigated. These are the two ideas that fueled my research during my Ph.D.

In particular, this has led me to develop a way to parametrize deviations from ΛCDM at the level of linear perturbations, which is called the Effective Field Theory of Dark Energy (EFT of DE). While the background evolution of the Universe is now quite constrained by distance measurements, much less is known about the evolution of the inhomegeneities that give rise to the large scale structure. Studying their behavior in the linear regime, where theoretical control is still reachable, should prove very informative. I will show in Chapter 2 that the EFT of DE allows for a systematic and quantitative exploration of deviations from ΛCDM, because of its model independence and minimal number of parameters.

While working on the EFT of DE, we realized that what was thought to be the largest class of stable theories for gravity plus a scalar, Horndeski theories [4], could actually be extended. Usually, the stability of theories is obtained by imposing that the equations of motion do not contain terms with more than two derivatives. In Chapter 3, I will argue that this is actually not a necessary condition for scalar-tensor theories. This means that before discarding higher derivatives theories, a more careful analysis needs to be performed. As we shall see, this opens the gate to new models.

Although most of cosmology has been focused on scalar perturbations since they have been actually observed, the precision reached by BICEP2 [5] seems to indicate that detecting primordial gravitational waves might well be within our grasp. They are potentially a great source of information on the early universe, since the standard predictions for tensor modes from inflation give straightforward access to its energy scale.

In Chapter 4, I will present why, contrarily to the scalar case, the predictions for tensor modes are very robust. In particular, this implies that it is difficult to get a scale invariant power spectrum for gravitational waves without a period of inflation.

The final subject that I will discuss is the work I have done on consistency relations. These relations allow to express $(n+1)$-point correlation functions of the cosmic density fields in term of the n-point ones in the limit where one density field is slowly varying in space. As I will show in Chapter 5, their strength comes from the fact that very little information on the n others fields is needed: only that they have Gaussian initial conditions and that they obey the Equivalence Principle. This is a huge advantage since taking correlation functions in the large scale structure typically requires to deal with non linear evolutions that are hard to control theoretically. This control is further limited by the poorly known relation between the galaxy distribution, that we observe, and the underlying dark matter distribution, that we predict. The lack of an accurate understanding of these phenomena reduces the amount of information that can be extracted from galaxy surveys. Since consistency relations do not rely on the knowledge of short scales physics, they do not suffer from this problem. In particular, this gives access to new ways of testing the Equivalence Principle on very large scales, where gravity is less tested.

In this thesis, I have chosen to present the main results of my work and to emphasize the intuition behind those results, as well as their physical implications. This is why I have tried to keep things short in the main text. Any potential thirst for technical aspects can be quenched by the full articles A–G at the end of this thesis.

Chapter 2

The Effective Field Theory of Dark Energy

When looking at alternatives to the standard ΛCDM+GR model, the simplest and most common way is to introduce an extra scalar field (see [6] for a review). It can either act as an additional dark energy fluid, or as a modification of the laws of gravity themselves. It is the easiest modification one can make and is as such the first that should be explored: there is only one additional degree of freedom to consider, making it an informative step before looking at more complicated scenarios. Even in some cases where multiple degrees of freedom are added, such as in massive [7] or bimetric gravity [8] for example, one recovers the case of a single scalar field in relevant limits.

This universality is yet more manifest for a second reason. The goal of the modifications at hand are to try and explain the current accelerated expansion of the Universe [9, 10]. Thus, in general, any field added for this purpose will have a background value that is time dependent, since the homogeneous Universe evolves in time. This explicitly breaks the time diffeomorphism invariance, that can be restored as usual with Goldstone modes, which would be a single scalar in this case (see for example [11]). Therefore, the low energy perturbations around a time dependent background will generically be described by this scalar, regardless of the fundamental origin of the theory.

These ideas were first developed in the case of inflation in [12] under the name of the Effective Field Theory of Inflation and then used for example to compute higher order correlation functions, which allow to probe non-Gaussianities [13, 14]. Later, it was applied in the context of late time acceleration in the Effective Field Theory of Dark Energy (EFT of DE) in [15, 16] and also [17].

In this section, I will present the concepts behind such an approach as well as its many advantages, based on the work I did in [GLPV1], later summarized in a review [GLV].

2.1 The Unitary Gauge Action

The first thing I will assume is the Weak Equivalence Principle, namely that there exists a metric that universally couples to the matter sector, even if the formalism I am going to present would apply if species coupled to different metrics.

Next, the goal is to look for a generic action that would describe cosmological perturbations around a FLRW background when looking at cosmology beyond ΛCDM. By this I mean either dark energy and/or modifications of the actual laws of gravity. For concreteness, I will consider the case of an extra scalar field, ϕ. However, the idea is to be as model independent as possible considering these assumptions.

As I mentioned before, this scalar field, in a cosmological context, is naturally expected to be spacelike, i.e. to have a gradient such that $\nabla_\mu\phi\nabla^\mu\phi < 0$. In this case, the hypersurfaces of constant ϕ define a preferred foliation of time. It is convenient to use the gauge freedom in the theory to choose this specific time: this is called the unitary gauge.

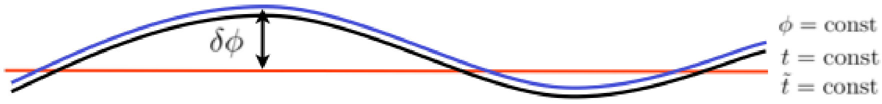

FIGURE 2.1: The original time $\tilde{t}$ hypersurface in red. In black, the new time in unitary gauge, that is chosen to match the ϕ hypersurfaces (blue).

By doing so, the perturbation in the scalar field are hidden, since now we have

$$\phi(\tilde{t},\vec{x}) = \phi_0(\tilde{t}) + \delta\phi(\tilde{t},\vec{x}) = \phi_0(t)\,, \tag{2.1}$$

where the last equality holds because of the choice of specific time t that is made. Of course, the perturbation $\delta\phi$ did not disappear, it is part of the perturbations of the metric. For example, the standard kinetic term for ϕ becomes in this gauge

$$X \equiv \nabla_\mu\phi\nabla^\mu\phi = g^{00}\dot{\phi}_0^2\,, \tag{2.2}$$

so that these quantities still contribute to the perturbative expansion through $g^{00} = -1 + \delta g^{00}$. The unitary gauge has therefore the advantage of having to deal only with the metric, however it has a minor inconvenient. Since a choice of time was made, the invariance under time reparametrization is lost (while leaving the spatial one intact). This means that the theory will not be manifestly covariant, as can be seen already from eq. (2.2). Indeed, tensors with upper indices set to 0 are allowed in this gauge (they correspond to contractions with the gradient of the scalar field, e.g. $\mathcal{P}^{00} \sim \mathcal{P}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$). This should not be worried over, as a simple redefinition of time

$$t \to t + \pi(t,\vec{x})\,, \tag{2.3}$$

allows to explicitly reintroduce the invariance under time reparametrization of the theory [11]. This is known as the Stueckelberg trick and the variable π is the field that non linearly realizes this invariance. This will be useful to change gauge. In particular, to go to Newtonian gauge, where the equations of motion (EOM) have an easier interpretation.

Nevertheless, the unitary gauge will enable us to write the most general action for a scalar-tensor theory, without reference to a specific model. Indeed, in this gauge, all the terms that are invariant under spatial diffeomorphisms are in principle allowed. Further conditions can be imposed, such as second-order EOM for example, but the basic ingredients can be obtained from the geometry of the hypersurfaces of Fig. 2.1 and are the following:

- The normal vector orthogonal to the surfaces, $n_\mu \equiv -\frac{\nabla_\mu \phi}{\sqrt{-X}}$. This term is the one responsible for the presence of tensors with 0 as upper indices.
- The extrinsic curvature, $K_{\mu\nu}$. It quantifies the variation of the normal vector

$$K_{\mu\nu} \equiv \left(g_{\mu\sigma} + n_\mu n_\sigma\right) \nabla^\sigma n_\nu \,. \tag{2.4}$$

  This quantity tells us how the hypersurfaces are embedded in the full 4-D space.
- The final ingredient is the intrinsic curvature, given be the 3-D Ricci tensor R_{ij} of the hypersurface. This is the equivalent[1] of the 4-D Riemann tensor ${}^{(4)}R_{\mu\nu\rho\sigma}$ for the full space. In what follows, unless specified explicitly with a (4), the Ricci tensor R_{ij} and scalar R will always be the 3-D ones.

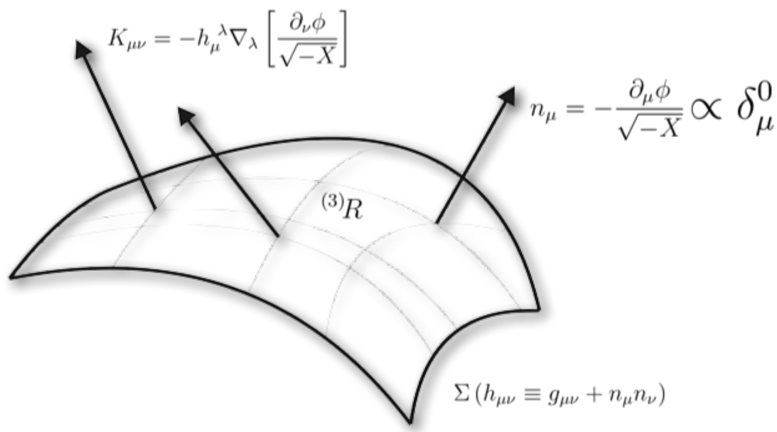

FIGURE 2.2: The ϕ hypersurface and its geometrical quantities.

The numbers of combinations of these terms is infinite. This is why in the following I will impose restrictions on the categories of action I will consider. To be more quantitative, I will discuss these restrictions in the formalism of Arnowitt-Deser-Misner (ADM) [18].

2.2 ADM formalism and the Effective Field Theory of Dark Energy

In order to be more specific about the action, I will go one step further in the distinction between space and time. To make more explicit the 3+1 decomposition, I will use the ADM form of the metric, namely write the line element as

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)\,, \tag{2.6}$$

where N is the lapse, N^i the shift and h_{ij} is the spatial metric on constant time hypersurfaces, which can be decomposed into a scalar part, ζ, and a tensorial one, γ_{ij} as

$$h_{ij} = a^2 e^{2\zeta}(\delta_{ij} + \gamma_{ij})\,, \quad \partial_i \gamma_{ij} = \gamma_{ii} = 0\,. \tag{2.7}$$

[1] In three dimensions, there is as much information in the Ricci tensor as in the Riemann tensor since

$$R_{\mu\nu\rho\sigma} = R_{\mu\rho}h_{\nu\sigma} - R_{\nu\rho}h_{\mu\sigma} - R_{\mu\sigma}g_{\nu\rho} + R_{\nu\sigma}h_{\mu\rho} - \frac{1}{2}R(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho})\,. \tag{2.5}$$

With this metric and in unitary gauge, the basic ingredients I mentioned above take the simpler form

$$n_\mu = -\delta^0_\mu N\,, \quad g^{00} = -\frac{1}{N^2}\,, \tag{2.8}$$

$$K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - D_i N_j - D_j N_i\right]\,. \tag{2.9}$$

The other components are not needed. Indeed, $K^{0i} = K^{00} = 0$ since by definition (2.4) the extrinsic curvature is orthogonal to the unit vector, $n_\mu K^{\mu\nu} = 0$. D_i is the covariant derivative associated with the spatial metric h_{ij}. The 3-D Ricci tensor R_{ij} is the standard one constructed from this metric. With this decomposition of the metric, any Lagrangian respecting the spatial diffeomorphisms invariance can be cast into the generic form

$$S_g = \int d^4x\sqrt{-g}\,L(N, K_{ij}, R_{ij}, h_{ij}, D_i, \partial^0; t)\,. \tag{2.10}$$

As an example, the Einstein-Hilbert action of standard GR,

$$S_{\rm GR} = \int d^4x\sqrt{-g}\,\frac{M_{\rm Pl}^2}{2}\,{}^{(4)}R\,, \tag{2.11}$$

can be rewritten in this form as

$$L_{\rm GR} = \frac{M_{\rm Pl}^2}{2}\left[K_{ij}K^{ij} - K^2 + R\right]\,, \tag{2.12}$$

using the Gauss Codazzi relation

$${}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_\mu(K n^\mu - n^\rho\nabla_\rho n^\mu)\,. \tag{2.13}$$

Virtually all known models of dark energy involving a single field can be mapped onto a specific form of the Lagrangian (2.10). However, the real strength of this approach is that it allows to generically look at modifications of ΛCDM, without the need to specify a model.

To be quantitative, I will only look at the linearized theory, which means the action will only contain perturbations up to second order. Secondly, I will discuss the case where the three DOF of the theory (the two tensor polarizations and the additional scalar) obey second-order dynamics, to ensure stability. Moreover, I will assume that the full theory is given by an action $S_{\rm full} = S_g + S_{\rm mat}$, where $S_{\rm mat}$ is an action that describes minimally coupled matter. Then, one expands eq. (2.10) in terms of the perturbative quantities

$$\delta N \equiv N - 1\,, \quad \delta K^i{}_j \equiv K^i{}_j - H\delta^i{}_j\,, \quad R^i{}_j\,. \tag{2.14}$$

Let me concentrate more particularly on the scalar sector, since this is were restrictions need to be imposed in order to keep second-order dynamics. I will use the further parametrization

$$N^i = \delta^{ij}\partial_j\psi\,, \tag{2.15}$$

for the scalar part of g^{0i}. Together with the form of the metric (2.7), the perturbations of the geometrical quantities read

$$\delta\sqrt{h} = 3a^3\zeta\,, \qquad \delta K^i{}_j = \left(\dot{\zeta} - H\delta N\right)\delta^i_j - \frac{1}{a^2}\delta^{ik}\partial_k\partial_j\psi\,, \tag{2.16}$$

and

$$\delta_1 R_{ij} = -\delta_{ij}\partial^2\zeta - \partial_i\partial_j\zeta \,, \qquad \delta_2 R = -\frac{2}{a^2}\left[(\partial\zeta)^2 - 4\zeta\partial^2\zeta\right] . \tag{2.17}$$

I will restrict to the case where no time derivatives ∂^0 appear explicitly in the Lagrangian, since it leads in general to extra DOF (see Article G for a discussion on including such derivatives). In this case, the variation with respect to δN and ψ gives constraint equations. They allow to express δN and ψ in terms of ζ and its derivatives, yielding an action only for this variable. It is on this action that conditions need to be imposed to get second-order dynamics[2].

An example of such conditions concerns the derivative with respect to the extrinsic curvature, which is of the form

$$\frac{\partial^2 L}{\partial K_i^j\,\partial K_k^l} = \hat{\mathcal{A}}_K\,\delta_j^i\,\delta_l^k + \mathcal{A}_K\left(\delta_l^i\,\delta_j^k + \delta^{ik}\delta_{jl}\right) , \tag{2.18}$$

because of the FLRW symmetries of the background. In order to prevent higher order derivatives, one has to prescribe $\hat{\mathcal{A}}_K = -2\mathcal{A}_K$. Two other conditions need to be imposed and then the most general action that abides by these criteria can be written as

$$\boxed{\begin{aligned} S_g = \int d^4x a^3 \frac{M^2}{2}\Bigg[&\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2 + (1+\alpha_T)\left(\delta_{(2)}R + \frac{\delta\sqrt{h}}{a^3}R\right) + H^2\alpha_K\delta N^2 \\ &+ 4H\alpha_B\delta N\delta K + (1+\alpha_H)R\,\delta N\Bigg] + \cdots \end{aligned}} , \tag{2.19}$$

where $h = \det h_{ij}$ and the $\cdots$ denotes terms that vanish when the background equations are enforced. The functions M and α_i are all in principle dependent on time, which is allowed by the presence of the extra scalar field. Additionally, one can define

$$\alpha_M \equiv \frac{2\dot{M}}{HM} , \tag{2.20}$$

which parametrizes the potential time dependence of the Planck mass. These coefficients, originally introduced in [19], are defined so that the standard case of ΛCDM+GR would correspond to setting all of them to zero.

They can be related to the original Lagrangian (2.10) and its derivatives with respect to the various quantities $N, K_{ij}, \ldots$ The starting point is to define the equivalent of the Planck mass, M, which is associated with the normalization of the tensor kinetic term, $\dot{\gamma}_{ij}^2$. Since $\dot{\gamma}_{ij}$ only appears in K_{ij}, the M is going to be given by the derivative of the Lagrangian with respect to the extrinsic curvature, eq. (2.18). More precisely,

$$M^2 \equiv 2\mathcal{A}_K . \tag{2.21}$$

Then, all the coefficients α_i follow almost algorithmically. For instance,

$$\alpha_K \equiv \frac{2L_N + L_{NN}}{H^2M^2} . \tag{2.22}$$

[2]Indeed, it is too restrictive to impose no higher derivatives in all of the equations before the constraint are solved. Indeed, such constraints might remove these higher derivatives so that the actual propagating DOF still obeys a second-order EOM. See Section 3.2 for more details.

The others, while being slightly more involved, are of the same form, as can be seen in Table 1 of Article G. In the next section, for concreteness, I will give examples on how to get these parameters in the case of specific models.

2.3 Going from models to the EFT of DE

Once a model is decomposed in 3+1 quantities, computing its parameters is completely automatic, making the link with possible constraints straightforward. Let me go through the functions α_a one at a time, increasing the complexity of the model needed to illustrate the parameter.

- α_K

  Taking the simplest case of GR plus quintessence, [20] i.e.

$$L = \frac{M_{\rm Pl}^2}{2}\left[K_{\mu\nu}K^{\mu\nu} - K^2 + R\right] - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\,. \tag{2.23}$$

  After going to unitary gauge, one finds

$$M = M_{\rm Pl}\,, \quad \alpha_K = \frac{\dot\phi_0^2}{H^2 M_{\rm Pl}^2}\,, \tag{2.24}$$

  while all the others coefficients vanish. One can indeed check that ΛCDM corresponds to all the α_i being zero: one recovers the cosmological constant for $\dot\phi_0 = 0$, which would set $\alpha_K = 0$.

  As a side note, it might seem odd that the potential V does not appear in eq. (2.24). The reason is that this parametrization is specifically designed to look at linear perturbations, while V is a background quantity in unitary gauge. More precisely, the Friedmann equations impose

$$V = \frac{M_{\rm Pl}^2}{2}\left[2\dot H + 3H^2\left(2 - \Omega_{\rm m}\right)\right]\,. \tag{2.25}$$

  Therefore, if the history of H and the matter content are known, V is fixed.

- α_B

  This example requires a more complicated model: kinetic braiding [21]. This theory is characterized by a Lagrangian of the form

$$L_3 = L_{\rm GR} + G_3(X)\,\Box\phi = L_{\rm GR} - \int G_{3X}\sqrt{-X}\,{\rm d}X\,K\,. \tag{2.26}$$

  Since the $\Box$ operator is made with covariant derivatives, $\Box\phi$ contains derivative couplings $(\partial g)(\partial\phi)$ between gravity and the scalar, hence its name kinetic gravity braiding.

  The last term is going to give a nonzero α_B in the EFT Lagrangian (2.19), and the whole set of coefficients is given by

$$M = M_{\rm Pl} \quad \alpha_K = 12\dot\phi_0^3\,\frac{G_{3X} - \dot\phi_0^2 G_{3XX}}{HM_{\rm Pl}^2}\,, \quad \alpha_B = -\frac{G_{3X}\dot\phi_0^3}{HM_{\rm Pl}^2}\,, \tag{2.27}$$

where I have used the fact that in unitary gauge $X = -\dot{\phi}_0^2/N^2$, so that a dependence on X can be seen as a dependence on N and vice versa.

- α_T

  To get a non zero α_T, one needs a model that does not preserve the relation between the intrinsic and the extrinsic curvatures in eq. (2.12). Since the extrinsic curvatures give terms in $\dot{\gamma}_{ij}^2$ while the intrinsic one gives $(\partial_k \gamma_{ij})^2$, changing the relation between them brings a change in the speed of sound of tensors. This happens for example for what is known as the quartic galileon [22], whose Lagrangian is

$$L_4 = G_4(X)\,{}^{(4)}R - 2G_{4X}(X)\left[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right] . \tag{2.28}$$

  The covariant second derivatives of the scalar field introduce first derivatives for the metric through the Christoffel symbols, which modifies the kinetic terms for gravity and gives a non zero α_T. In unitary gauge this Lagrangian reads

$$L_4 = G_4 R + (2XG_{4X} - G_4)(K^2 - K^{ij}K_{ij}) , \tag{2.29}$$

  so that the EFT coefficients are

$$M^2 = 2\left(G_4 + G_{4X}\dot{\phi}_0^2\right) , \quad \alpha_K = -12\dot{\phi}_0^2\,\frac{G_{4X} - 8\dot{\phi}_0^2 G_{4XX} + 4\dot{\phi}_0^4 G_{4XXX}}{M^2} , \tag{2.30}$$

$$\alpha_B = 4\dot{\phi}_0^2\,\frac{G_{4X} - 2\dot{\phi}_0^2 G_{4XX}}{M^2} , \quad \alpha_T = -4\dot{\phi}_0^2\,\frac{G_{4X}}{M^2} , \tag{2.31}$$

I will not discuss here the case of α_H, which parametrizes deviations from Horndeski theories, since the next chapter is specifically focused on theories beyond Horndeski. In particular, the effect of α_H will be explored in Section 3.6.

The theoretical origin of the parameters α_a of eq. (2.19) is summarized in the following Table

	M^2	α_M	α_K	α_B	α_T	α_H
Interpretation	Normalization of the tensor quadratic action $\equiv$ Planck mass	Planck mass rate of change	Kinetic term for the scalar	Kinetic braiding between gravity and scalar	Modification of tensor sound speed	Theories beyond Horndeski
Example	GR (when constant)	$f(R)$ [23] Brans-Dicke [24]	k-essence [25]	Cubic Galileon [21]	Quartic Galileon [22]	G^3 theories (see Chapter 3)

TABLE 2.1: In the first row, the parameters α_i introduced in eq. (2.19).

2.4 Stability and theoretical consistency

Even if the terms in eq. (2.19) passed the first condition of yielding second-order dynamics (which guarantees the absence of extra, ghost-like DOF), further restrictions need to be imposed on the EFT parameters. Indeed, before thinking about comparing the predictions of a theory to observations, stringent constraints must be imposed in order for the theory to be stable. This is where using a parametrization at the level of the action and not of the EOM has a clear advantage, since these stability conditions can in principle be read off directly from the action. The idea can be simplified thusly: in the case of two scalar fields[3] $\psi_1(t,\vec{x}),\ \psi_2(t,\vec{x})$ their quadratic Lagrangian is generically of the form:

$$L = \xi\dot{\psi}_1^2 - c_1\partial_i\psi_1^2 + \dot{\psi}_2^2 - c_2\partial_i\psi_2^2 + V_{\rm int}(\psi_1,\psi_2). \tag{2.32}$$

In this illustrative case, the stability of the theory requires the coefficient ξ to be positive. When this is not the case, the field ψ_1 is called a ghost and in general violent instabilities are present in the theory.

Let me give some intuition on why that is, by thinking of the Lagrangian as $L = T - V$, where T is the kinetic energy and V the potential one. If the two signs are not the same in T, kinetic energy can flow without limits from one field to the other without changing the total energy $E = T + V$, meaning that the ground state of the theory is not stable (see [26] for a discussion on classical and quantum ghosts).

On top of this, one needs to impose that the coefficients c_1 and c_2 (which represent the squared sound speeds) are positive, to avoid gradient instabilities. These instabilities can be understood very easily from the EOM: when varying (2.32) with respect to ψ_1 for example, one gets

$$\ddot{\psi}_1 - c_1\Delta\psi_1 = \frac{1}{2}\frac{\partial V_{\rm int}}{\partial\psi_1}\,. \tag{2.33}$$

If c_1 is negative, this equation admits in Fourier space a solution $\psi_{\vec{k}}$ proportional to $e^{\sqrt{|c_1|}kt}$, which is divergent.

The analysis in the case of the action (2.19) is more involved, since tensor modes are present on top of the scalar. Moreover, other non dynamical variables are present (scalar and vector), so that at first glance the form of the quadratic action is not as simple as (2.32). If we parametrize the unitary gauge metric as before

$$N = 1 + \delta N\,, \quad N^i = \partial_i\psi + N_V^i\,, \quad h_{ij} = a^2e^{2\zeta}\left(\delta_{ij} + \gamma_{ij}\right)\,, \tag{2.34}$$

with $\partial_i N_V^i = 0$ and $\gamma_{ii} = \partial_i\gamma_{ij} = 0$, only ζ and γ_{ij} are dynamical[4]. Once the constraints are solved, the quadratic part of the action can be rewritten in terms of dynamical DOF only, in a manner very similar to eq. (2.32):

$$S = \int d^4x\,\frac{M^2a^3}{2}\left\{\frac{\alpha}{(1+\alpha_B)^2}\left[\dot{\zeta}^2 - c_s^2\frac{\partial_i\zeta^2}{a^2}\right] + \frac{\dot{\gamma}_{ij}^2}{4} - (1+\alpha_T)\frac{\partial_k\gamma_{ij}^2}{4a^2} + \frac{(\partial_i N_j^V + \partial_j N_i^V)^2}{4a^4}\right\}. \tag{2.35}$$

I have used the following definitions

$$\alpha \equiv \alpha_K + 6\alpha_B^2\,, \tag{2.36}$$

[3] I will not treat the case of one field, as it present less interests. In particular, one cannot have a ghost field in this case: the sign of the kinetic term does not matter when there is nothing to compare it to. Moreover, in cosmology, the scalar field is always coupled to gravity.

[4] In general, the spatial metric contains also a (non-dynamical) vectorial part, which can be set to zero by using the spatial gauge freedom.

and

$$c_s^2 \equiv 2\left\{1+\alpha_T - \frac{1+\alpha_H}{1+\alpha_B}\Big(1+\alpha_M - \frac{\dot H}{H^2}\Big) - \frac{1}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{1+\alpha_B}\right)\right\}\,, \tag{2.37}$$

the latter being valid only in the absence of matter. The stability conditions discussed above can be stated as

$$\begin{aligned} &M^2 > 0\,, \quad \alpha_K + 6\alpha_B^2 > 0\,, \\ &c_T^2 \equiv (1+\alpha_T) > 0\,, \quad c_s^2 > 0\,, \end{aligned} \tag{2.38}$$

which defines the tensor sound speed.

The presence of matter, both at the background and perturbative levels, slightly complicates the situation. In the case $\alpha_H = 0$, one finds

$$c_s^2 = 2\frac{(1+\alpha_B)^2}{\alpha}\left\{\frac{1}{1+\alpha_B}\Big(1+\alpha_M - \frac{\dot H}{H^2}\Big) - (1+\alpha_T) - \frac{\dot{\alpha_B}}{H(1+\alpha_B)^2}\right\} - \frac{\rho_{\rm m}+p_{\rm m}}{\alpha\, M^2 H^2}\,, \tag{2.39}$$

while the speed of sound for matter and tensors are unchanged. In the case $\alpha_H \neq 0$, which will be treated in more details in Chapter 3, both the sound speed of matter and the extra scalar field are affected.

Of course, the conditions (2.38) can be translated into conditions on parameters of models, using for example Section 2.3. However, the advantage of the EFT of DE is that those conditions are really imposed on deviations from ΛCDM, not just on a specific model. It might well be that the regions of the parameter space they allow are not fully explored by any of the known theories (which lead us to the theories beyond Horndeski of Chapter 3). As we will see, the same kind of reasoning applies to the comparison with observations.

2.5 Evolution of cosmological perturbations

In this section I will discuss the effects of the deviations from ΛCDM on the evolution of perturbations, in the vector, tensor and scalar sectors, the latter being the richest–and most complicated–in term of phenomenology. The matter sector will be parametrized by its total stress energy tensor, decomposed at linear order as

$$T^0_{\ 0} \equiv -(\rho_{\rm m} + \delta\rho_{\rm m})\,, \tag{2.40}$$

$$T^0_{\ i} \equiv \partial_i q_{\rm m} + \big(T^0_{\ i}\big)^T \equiv (\rho_{\rm m}+p_{\rm m})\partial_i v_{\rm m} + \big(T^0_{\ i}\big)^V\,, \tag{2.41}$$

$$T^i_{\ j} \equiv (p_{\rm m}+\delta p_{\rm m})\delta^i_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\sigma_{\rm m} + \big(\partial^i C_j + \partial_j C^i\big)^V + \big(T^i_{\ j}\big)^{TT}\,, \tag{2.42}$$

where $\delta\rho_{\rm m}$ and $\delta p_{\rm m}$ are the energy density and pressure perturbations, $q_{\rm m}$ and $v_{\rm m}$ are respectively the 3-momentum and the 3-velocity potentials; $\sigma_{\rm m}$ is the anisotropic stress potential. $(T^0_{\ i})^V$ is the transverse part of the matter energy flux, $\big(\partial^i C_j + \partial_j C^i\big)^V$ and $(T_{ij})^{TT}$ are respectively the transverse and the transverse-traceless parts of the spatial matter stress tensor.

2.5.1 Vector sector

As we have seen from eq. (2.35), the vector sector is the simplest one as it does not contain propagating DOF. However, the presence of a time varying Planck mass, characterized by $\alpha_M \neq 0$ still affects the perturbations. Indeed, when considering the full action supplemented by matter, the vector equation reads:

$$\frac{1}{2}\nabla^2 N_i^V = \frac{a^2}{M^2}\left(T^0_{\ i}\right)^V . \tag{2.43}$$

For a perfect fluid where $C_i^V = 0$, the conservation of the matter stress-energy tensor implies that $(T^0_{\ i})^T \propto 1/a^3$ [27]. Thus, the metric vector perturbations scale as

$$N_V^i \propto \frac{1}{aM^2} = \frac{1}{a^{1+\alpha_M}} , \tag{2.44}$$

where the last equality holds for a constant α_M. It is therefore interesting to see that the evolution of the vector sector only depends on a single parameter.

Since they typically decay, vector modes are very difficult to observe. This very fact already signals that α_M cannot be too negative, i.e. the Planck mass cannot have been growing too strongly in time, otherwise they would not necessarily be negligible today. If vectors mode were to be detected, this would allow to constrain α_M without having to treat the other parameters.

2.5.2 Tensor sector

The tensor sector, slightly more complicated, leads to the evolution equation

$$\ddot{\gamma}_{ij} + H(3+\alpha_M)\dot{\gamma}_{ij} - (1+\alpha_T)\frac{\nabla^2}{a^2}\gamma_{ij} = \frac{2}{M^2}\left(T_{ij}\right)^{TT} . \tag{2.45}$$

Thus, even for a perfect fluid where the anisotropic stress is zero, the propagation of tensor modes is affected both by an additional friction term proportional to α_M, as well as a different speed of propagation. In principle, the combined observation of vector and tensor modes could therefore provide constraints on α_M and α_T independently of each other and of the other α_i.

2.5.3 Scalar sector

2.5.3.1 Obtaining the equations

In principle, five (non independent) scalar equations can be derived from the action (2.19). Four are the Einstein scalar equations (00, $0i$, ii and ij traceless), where one needs to further introduce the scalar part of the traceless component of the spatial metric, χ

$$h_{ij} = a^2(1+2\zeta)\left[\delta_{ij} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\partial^2\right)\chi\right] . \tag{2.46}$$

Then, the action needs to be varied with respect to ζ, δN, ψ and χ, giving the four Einstein equations.

The fifth equation is the one for the scalar field ϕ. However, in unitary gauge this field is not explicit. One can still derive what would be the unitary gauge version of this equation (that will depend only on metric quantities) by imposing the invariance under time reparametrization of the action. Indeed, by definition of the unitary gauge,

$$\left.\frac{\delta S[\phi, g_{\mu\nu}]}{\delta\phi(x)}\right|_{\phi=t} = \frac{\delta S_{\text{u.g.}}[t, g_{\mu\nu}]}{\delta t}, \tag{2.47}$$

where the time derivative is understood as a partial one (that is to say, not taking into account the time dependence of the metric).

For a general infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$, the metric changes as $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. Therefore,

$$\delta S_{\text{u.g.}} = \int d^4x \frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}(x)} (\nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x)) + \frac{\delta S_{\text{u.g.}}}{\delta t} \xi^0 = 0\,. \tag{2.48}$$

After integrating by parts and combining this with eq. (2.47), one obtains that the equation of the scalar field in unitary gauge is simply the zero component of the divergence of Einstein's equations[5],

$$\left.\frac{\delta S[\phi, g_{\mu\nu}]}{\delta\phi(x)}\right|_{\phi=t} = \frac{\delta S_{\text{u.g.}}}{\delta t} = 2g^{0\nu}\nabla_\mu \frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}} = 0\,, \tag{2.49}$$

where the last equality holds when Einstein's equations $\frac{\delta S_{\text{u.g.}}}{\delta g_{\mu\nu}} = 0$ are inforced. Hence, this yields the fifth scalar equation, which is not independent from the others.

These five equations are for the scalar variables of the metric, namely ζ, δN, ψ and χ. To describe scalar perturbations and their physics, the Newtonian gauge is more adapted than the unitary gauge. In order to go from one to the other, a time diffeomorphism is performed

$$t \to t + \pi(t, \vec{x})\;, \tag{2.50}$$

where π describes the fluctuations of the scalar field

$$\phi = t + \pi\,. \tag{2.51}$$

In Newtonian gauge the scalar part of the metric is parametrized as

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j\;. \tag{2.52}$$

One can relate the metric perturbations in unitary gauge defined in eq. (2.34) to he metric perturbations Φ and Ψ, as well as the scalar fluctuation π by[6]

$$\delta N = \Phi - \dot{\pi}\;, \qquad \zeta = -\Psi + H\pi\;, \qquad \psi = a^{-2}\pi\,, \quad \chi = 0\;. \tag{2.53}$$

Then, the five equations can be put in the following form (in Fourier space):

[5] Since we assumed the presence of a Jordan frame, where matter is minimally coupled, its stress energy tensor is conserved independently.

[6] More precisely, to remove also the variable χ one needs a spatial diffeomorphism $x^i \to x^i + \partial_i\beta$,.

- The Hamiltonian constraint ((00) component of Einstein's equation) is

$$
\begin{aligned}
&6(1+\alpha_B)H\dot\Psi + (6-\alpha_K+12\alpha_B)H^2\Phi + 2(1+\alpha_H)\frac{k^2}{a^2}\Psi + (\alpha_K - 6\alpha_B)\,H^2\dot\pi \\
&+6\left[(1+\alpha_B)\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + \frac{1}{3}\frac{k^2}{a^2}(\alpha_H-\alpha_B)\right]H\pi = -\frac{\delta\rho_{\rm m}}{M^2}\ ,
\end{aligned}
\tag{2.54}
$$

- The momentum constraint ((0i) components of Einstein's equation) reads

$$
2\dot\Psi + 2(1+\alpha_B)H\Phi - 2H\alpha_B\dot\pi + \left(2\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{M^2}\right)\pi = -\frac{(\rho_{\rm m}+p_{\rm m})v_{\rm m}}{M^2}\ . \tag{2.55}
$$

- The traceless part of the ij components of Einstein's equation gives

$$
(1+\alpha_H)\Phi - (1+\alpha_T)\Psi + (\alpha_M-\alpha_T)H\pi - \alpha_H\dot\pi = -\frac{\sigma_{\rm m}}{M^2}\ , \tag{2.56}
$$

- The trace of the same components gives, using the equation above,

$$
\begin{aligned}
&2\ddot\Psi + 2(3+\alpha_M)H\dot\Psi + 2(1+\alpha_B)H\dot\Phi \\
&+2\left[\dot H - \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + (\alpha_B H)^{\cdot} + (3+\alpha_M)(1+\alpha_B)H^2\right]\Phi \\
&-2H\alpha_B\,\ddot\pi + 2\left[\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} - (\alpha_B H)^{\cdot} - (3+\alpha_M)\alpha_B H^2\right]\dot\pi \\
&+2\left[(3+\alpha_M)H\dot H + \frac{\dot p_{\rm m}}{2M^2} + \ddot H\right]\pi = \frac{1}{M^2}\left(\delta p_{\rm m} - \frac{2}{3}\frac{k^2}{a^2}\sigma_{\rm m}\right)\ .
\end{aligned}
\tag{2.57}
$$

- Finally, the evolution equation for π reads

$$
\begin{aligned}
&H^2\alpha_K\ddot\pi + \left\{\left[H^2(3+\alpha_M)+\dot H\right]\alpha_K + (H\alpha_K)^{\cdot}\right\}H\dot\pi \\
&+6\left\{\left(\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right)\dot H + \dot H\alpha_B\left[H^2(3+\alpha_M)+\dot H\right] + H(\dot H\alpha_B)^{\cdot}\right\}\pi - 2\frac{k^2}{a^2}\dot H\pi \\
&-2\frac{k^2}{a^2}\left\{\frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + H^2\left[1+\alpha_B(1+\alpha_M)+\alpha_T-(1+\alpha_H)(1+\alpha_M)\right] + (H(\alpha_B-\alpha_H))^{\cdot}\right\}\pi \\
&+6H\alpha_B\ddot\Psi + H^2(6\alpha_B-\alpha_K)\dot\Phi + 6\left[\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + H^2\alpha_B(3+\alpha_M) + (\alpha_B H)^{\cdot}\right]\dot\Psi \\
&+\left[6\left(\dot H + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right) + H^2(6\alpha_B-\alpha_K)(3+\alpha_M) + 2(9\alpha_B-\alpha_K)\dot H + H(6\dot\alpha_B-\dot\alpha_K)\right]H\Phi \\
&+2\frac{k^2}{a^2}\left\{\alpha_H\dot\Psi + \left[H(\alpha_M+\alpha_H(1+\alpha_M)-\alpha_T)-\dot\alpha_H\right]\Psi + (\alpha_H-\alpha_B)H\Phi\right\} = 0\ .
\end{aligned}
\tag{2.58}
$$

These equations are much more involved than in the two other sectors and as such are not readily useful. Nevertheless, one has to remember that there is only one propagating degree of freedom, which means that 4 of these equations are just constraints. Therefore, the five equations can be combined into a single equation for a single variable, e.g.

$$\boxed{\begin{aligned}&\ddot{\Psi}+\frac{\beta_1\beta_2+\beta_3\alpha_B^2\,\tilde{k}^2}{\beta_1+\alpha_B^2\tilde{k}^2}H\dot{\Psi}+\frac{\beta_1\beta_4+\beta_1\beta_5\,\tilde{k}^2+c_s^2\alpha_B^2\tilde{k}^4}{\beta_1+\alpha_B^2\,\tilde{k}^2}H^2\Psi=\\&-\frac{1}{2M^2}\left[\frac{\beta_1\beta_6+\beta_7\alpha_B^2\,\tilde{k}^2}{\beta_1+\alpha_B^2\,\tilde{k}^2}\delta\rho_{\rm m}+\frac{\beta_1\beta_8+\beta_9\alpha_B^2\,\tilde{k}^2}{\beta_1+\alpha_B^2\,\tilde{k}^2}H(\rho_{\rm m}+p_{\rm m})v_{\rm m}-\frac{\alpha_K}{\alpha}\delta p_{\rm m}\right]\end{aligned}}\,, \tag{2.59}$$

where $\tilde{k}\equiv k/(aH)$, α is defined in eq. (2.36) and for simplicity, I have assumed that the anistropic stress of matter is zero. The β_i are functions of the coefficients α_j, whose–rather cumbersome–expressions are given in Appendix C of Article G in the case $\alpha_H=0$. Although this equation is enough to describe the dynamics of the scalar sector, it is useful to have the relation between the two metric potentials Φ and Ψ to connect with observations (in particular lensing). This relation takes the form

$$\begin{aligned}&\alpha_B^2\tilde{k}^2\left[\Phi-\Psi\left(1+\alpha_T+\frac{\alpha_T-\alpha_M}{\alpha_B}\right)\right]+\beta_1\left[\Phi-\Psi(1+\alpha_T)\left(1+\alpha\frac{\alpha_T-\alpha_M}{2\beta_1}\right)\right]=\\&\quad\frac{\alpha_T-\alpha_M}{2H^2M^2}\left\{\alpha_B\left[\delta\rho_{\rm m}-3H(\rho_{\rm m}+p_{\rm m})v_{\rm m}\right]+HM^2\alpha\,\dot{\Psi}+H\frac{\alpha_K}{2}\left(\rho_{\rm m}+p_{\rm m}\right)v_{\rm m}\right\}\,.\end{aligned} \tag{2.60}$$

To complete the system of equations, one needs to provide the evolution equations for the matter sector. Since it is assumed to be minimally coupled, these equations come from the conservation of the stress energy tensor. At linear order in the perturbations, treating one species of matter only for simplicity, they read

$$\dot{\delta}_{\rm m}-3H(w_{\rm m}\delta_{\rm m}-\delta p_{\rm m})-(1+w_{\rm m})\left(\frac{k^2}{a^2}v_{\rm m}+3\dot{\Psi}\right)=0\,, \tag{2.61}$$

$$\dot{v}_{\rm m}-\left[3Hw_{\rm m}-\frac{\dot{w}_{\rm m}}{1+w_{\rm m}}\right]v_{\rm m}+\frac{\delta p_{\rm m}}{1+w_{\rm m}}+\Phi=0\,, \tag{2.62}$$

with the definitions

$$w_{\rm m}\equiv\frac{p_m}{\rho_{\rm m}}\,,\quad \delta_{\rm m}\equiv\frac{\delta\rho_{\rm m}}{\rho_{\rm m}}\,, \tag{2.63}$$

where $w_{\rm m}$ is the usual equation of state parameter and $\delta_{\rm m}$ the density contrast. Note that in general, when the fluid is not at rest, the relation between the pressure perturbation and the density contrast involves more than just the speed of sound (see for example [28]) which is why I kept explicitly $\delta p_{\rm m}$ in these equations.

2.5.3.2 Interpretation

The system of equations (2.59)–(2.62) is complete (provided $\delta p_{\rm m}$ and $w_{\rm m}$ are specified) and can in principle be solved to get the evolution of the matter perturbations and gravitational potentials. To do so without approximations would require a numerical implementation. However, the physics can be discussed analytically in specific cases, that give an idea of the effects expected. In particular, I will focus on the role played by kinetic braiding. Indeed, one can see appearing in eq. (2.59) a new scale when $\alpha_B \neq 0$:

$$k_B = \frac{aH\beta_1^{1/2}}{\alpha_B} \,, \tag{2.64}$$

which has been called braiding scale [19]. We shall explore two examples that show it is associated with noticeable modifications of gravity.

- $\alpha_B = 0$:

  It can be seen as the extreme limit where $k_B \to \infty$, meaning that all modes are outside of the braiding length, $k \ll k_B$. In this case most of the scale dependences go away. We are left with the simpler expression

$$\ddot{\Psi} + (4 + 2\alpha_M + 3\Upsilon)\, H\dot{\Psi} + \left(\beta_4 H^2 + c_s^2 \frac{k^2}{a^2}\right)\Psi = \\ -\frac{1}{2M^2}\left\{c_s^2\left[\delta\rho_\mathrm{m} - 3H(\rho_\mathrm{m} + p_\mathrm{m})v_\mathrm{m}\right] + (\alpha_M - \alpha_T + 3\Upsilon)H(\rho_\mathrm{m} + p_\mathrm{m})v_\mathrm{m} - \delta p_\mathrm{m}\right\} \,, \tag{2.65}$$

  where Υ is defined in Appendix C of Article G. Although both α_M and α_T can be nonzero here, the form of this equation is very similar to that obtained in the standard k-essence case [25]. One recovers in the quasistatic limit (i.e. by neglecting time derivatives and taking $k \gg aH/c_s$)

$$-\frac{k^2}{a^2}\Psi = \frac{1}{2M^2}\delta\rho_\mathrm{m} \,, \qquad \Phi = (1 + \alpha_T)\left[1 + \alpha_K \frac{\alpha_T - \alpha_M}{2\beta_1}\right]\Psi \,. \tag{2.66}$$

  This means that no scale dependence is introduced in the effective Newton constant defined as

$$-\frac{k^2}{a^2}\Phi \equiv 4\pi G_\mathrm{eff}\, \delta\rho_\mathrm{m} \,. \tag{2.67}$$

  As we will see, this no longer necessarily holds when $\alpha_B \neq 0$.

- $\alpha_B^2 \gg \alpha_K$:

  This case corresponds to having most of the kinetic energy of the scalar field coming from kinetic braiding. Indeed, one can see in this case that the kinetic energy (the term in $\dot{\zeta}^2$ in eq. (2.35)) is dominated by the contribution of α_B.

  For simplicity we consider only the case $\alpha_T = 0$. Moreover, to avoid gradient instabilities the following relation is required (see eq. (2.39))

$$\alpha_B \lesssim \mathcal{O}(\alpha_M) \,. \tag{2.68}$$

  However, no restrictions are imposed on α_M, whose value can affect the braiding scale. Indeed, when $\alpha_B^2 \gg \alpha_K$, this is given by

$$\frac{k_B^2}{a^2} \simeq 3(H^2\alpha_M - \dot{H}) \,, \tag{2.69}$$

which can be inside the Hubble horizon. In this case, considering modes with $k \gg k_B$, eq. (2.59) simplifies to

$$\ddot{\Psi} + (3+\alpha_M)H\dot{\Psi} + \left(\frac{k_B^2\beta_5}{a^2} + c_s^2\frac{k^2}{a^2}\right)\Psi \simeq -\frac{1}{2M^2}\left(\frac{k_B^2\beta_6}{k^2} + c_s^2 + \frac{1}{3} - \frac{\alpha_M}{3\alpha_B}\right)\delta\rho_{\rm m}\,, \tag{2.70}$$

where we have neglected relativistic terms on the right hand side of (2.59). If the ratio β_5/c_s^2 is larger than one, the scale dependence cannot be neglected even in the case $k \gg k_B$. Therefore, a non vanishing α_B, or the fact that $k_B < \infty$, brings a transition scale in the effective Newton constant[7], which is a strong signal that gravity is modified.

Another interpretation would be that dark energy clusters: one can write Einstein equations as

$$G^{\mu\nu} = \frac{T_m^{\mu\nu} + T_D^{\mu\nu}}{M^2}\,, \tag{2.71}$$

which defines effective fluid variables for dark energy/modified gravity. Thus, for subhorizon scales, the Poisson equation has the form

$$-\frac{k^2}{a^2}\Phi = \frac{1}{M^2}\left(\delta\rho_{\rm m} + \delta\rho_D\right)\,. \tag{2.72}$$

For a cosmological constant, there are no perturbation in the dark energy fluid, $\delta\rho_D = 0$, and the standard behavior is recovered. However, as soon as dark energy clusters, i.e. $\delta\rho_D \sim \mathcal{O}(\delta\rho_{\rm m})$, the relation between the gravitational potential and matter is no longer as simple, leading to a different (and potentially scale dependent) effective Newton constant.

The equations (2.59) and (2.60) can be seen as the generalization to arbitrary scales of the usual parametrization in term of $G_{\rm eff}$ (defined in eq. (2.67)) and the slip parameter

$$\gamma \equiv \frac{\Psi}{\Phi}\,, \tag{2.73}$$

that are employed in the quasistatic limit. However, if this limit is clearly defined in GR where it means focusing on subhorizon scales $k \gg aH$, its definition in the presence of an extra scalar field is more ambiguous. Indeed, in general, new scales (see [29] for a general discussion concerning new scales in modified gravity) and time dependences appear and its not always clear how this limit would translate, although in general it is expected to hold well inside the sound horizon of the scalar perturbations, $kc_s \gg aH$.

To alleviate this uncertainty, one can look at what is called the extreme quasistatic limit [19] corresponding to wavenumber k much bigger than any scale in the problem, i.e. taking $k \to \infty$ in eqs. (2.59)–(2.60). This yields the following expressions

$$8\pi G_{\rm eff} = \frac{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2}{\alpha\, c_s^2} M^{-2}\,, \tag{2.74}$$

$$\gamma = \frac{\alpha\, c_s^2 + 2\alpha_B\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]}{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2}\,, \tag{2.75}$$

[7] Although the standard relation defining $G_{\rm eff}$ involves Φ and not Ψ, it easy to convince oneself that the relation between them set by eq. (2.60) does not remove this transition.

where I have expressed both quantities directly in terms of the functions α_a (recall that $\alpha = \alpha_K + 6\alpha_B^2$ and α_H is here set to zero). These two quantities are observable since the first affects directly the growth of structures and therefore affects the power spectrum of the large scale structure. The second is related to the gravitational potential felt by photon, $\Phi + \Psi$, and thus can be probed in weak lensing experiments (see for example [30]).

In this Section, I have shown that by looking at the evolution of cosmological perturbations, one can relate the parametrization of the action in eq. (2.19) to observable quantities. The simplest cases from the theoretical side are the vector and tensor sectors. They only depend on the time variation of the Planck mass, α_M, and on the deviation from unity of the tensor sound speed, α_T. However, these sectors are precisely the fields of observations where the signals are the weakest.

The more experimentally accessible scalar sector corresponds to the most complicated domain, where all five functions α_i play a role. Although their effects are understood from a theoretical point of view (see Table 2.1), they appear in a non trivial way when going to observable quantities such as the growth of structures or weak lensing. This can be seen analytically in the quasistatic limit with the modifications of the way matter sources the gravitational potential (through $G_{\rm eff}$) or the way the two potentials are related to each other (through γ). This is why, to break the degeneracies that remain, one may need to go beyond the quasistatic limit, starting for example from eq. (2.59).

One idea would be to solve perturbatively eqs. (2.59)–(2.62) around $k \to \infty$ without necessarily making assumptions on the time derivatives. This would be a way to see the range of validity of the quasistatic approximation (see also [31]). We have actually started looking into this, but taking care of the time dependence is rather subtle and requires more work.

2.6 Conclusions

In this chapter, I presented a method called the Effective Field Theory for Dark Energy, that allows to explore the vast landscape beyond the standard model of cosmology, ΛCDM. It is based on the parametrization of an action, describing scalar-tensor theories in a very broad sense. I used the preferred time foliation that the scalar field offers, along with its 3+1 geometry, to construct a very generic Lagrangian that describes linear perturbations with second-order dynamics. This Lagrangian depends only on five functions of time, provided the expansion of the Universe and its matter content are known.

This has many advantages, both theoretically and observationally. The stability conditions that one needs to impose for a theory to be sensible can be easily read from this action. Moreover, this reduces to a single channel of analysis the comparison to experiments. The straigthforward links that we developped between wide classes of models and the parameters make it particularly convenient to use, since constraints on the five parameters easily translate to constraints on models.

However, this point of view is somewhat limiting the potential of this approach. The action (2.19) explores domains beyond the models currently known, potentially leading to new models, as we shall see in the next chapter. Indeed, it is solely based on the fact that in general, the background solution of an additional field in a cosmological setting explictly breaks time reparametrization invariance. This opens the possibility of new terms in the action beside the standard Ricci scalar. It really is deviations from ΛCDM +GR that are captured by this formalism.

Because of its minimal number of parameters, the EFT of DE has started to be used by the community. It first started with people developing codes, in particular [32], that is based on the popular CMB code CAMB [33] and others doing forecasts for galaxy surveys [34]. Now, the parametrization, conveniently optimized by [19], is being used in the analysis of the Planck collaboration [35]. Hopefully, future surveys such as EUCLID [2] and LSST [3] will also use it, and the constraints on the α_a will improve.

From a theoretical point of view, there is still work to be done. As I mentioned above, there is a yet untamed wealth of information contained in eq. (2.59), which includes for example relativistic effects that become important when looking at increasingly large surveys. It would be interesting to see how much of this information can be extracted using numerical solutions, or analytical method generalizing the quasistatic limit.

Another point I have been working on recently consists of extending this formalism to the case of where the Weak Equivalence Principle (WEP) is violated, i.e. species couple to different metrics. This has been studied for ΛCDM under the name of interacting dark energy (see for example [36–39]). The idea is to investigate the interplay between these two properties, namely modifications of gravity and violation of the WEP. In particular, one can generalize the stability conditions (2.38), as well as the evolution equations (2.59)–(2.62) to include the different couplings of the matter fields, to look at the effect on the power spectrum and weak leasing.

Chapter 3

Beyond Horndeski

Although using parametrizations such as the EFT of DE (for other examples, see [40, 41]) proves useful when testing our understanding of cosmology, finding a more complete description through a specific model provides advantages. For example, it allows to go beyond the linear approximation, which is necessary when looking at smaller scales, where it breaks down. A very important step in this endeavour was the work of Horndeski [4] and its rediscovery [22, 42]. What are now known as Horndeski theories, or generalized galileons, are the most general Lorentz invariant scalar-tensor theories leading to second-order equations of motion, both for the scalar and for the tensors. This property guarantees that they are well behaved and free of ghosts. The fondness for these theories comes from the standard lore that theories ruled by EOM with more than two derivatives should be automatically discarded because they suffer from instabilities, according to Ostrogradski's theorem. However, this reasoning is too hasty. Indeed, in order for this statement to be correct, the theory needs to be non degenerate, in a sense that I will make clear later.

In this chapter, I will describe scalar-tensor theories that are not contained in Horndeski's. As a consequence, their EOM contain terms with three derivatives, but I will show that the theories are still "healthy", meaning devoid of Ostrogradski's instability. First, I will spend some time on what Horndeski theories are, before moving to these new theories, that we dubbed G^3 for "Generalized Generalized Galileons". Finally, I will use the formalism of Chapter 2 to explore the novel phenomenology that appears when going beyond Horndeksi.

3.1 Horndeski theories

As I have said before, the easiest way to modify ΛCDM is to introduce a scalar field. The goal is therefore to write a Lagrangian for this scalar field

$$L(\phi,\, \phi_\alpha \equiv \nabla_\alpha \phi,\, \phi_{\beta\gamma} \equiv \nabla_\beta \nabla_\gamma \phi,\, g_{\mu\nu},\, \ldots)\,. \tag{3.1}$$

Usually, when writing such Lagrangians, only first derivatives of the scalar field are involved. However, one can be more general and include terms such as $\Box\phi \equiv g^{\mu\nu}\phi_{\mu\nu}$. They are slightly more delicate, as they can lead to extra, unstable DOF. A sufficient condition to avoid this is to require that the EOM derived from the Lagrangian are at most second-order in derivative. Before turning to the case of a general metric $g_{\mu\nu}$ it is instructive to focus on the Minkowski limit, where the only dynamical DOF is the scalar. The key ingredient are the so-called galileons Lagrangians of [43]:

$$
\begin{aligned}
L_2^{\text{gal},1} &= X \;, && (3.2)\\
L_3^{\text{gal},1} &= X\Box\phi - \phi_\mu\phi^{\mu\nu}\phi_\nu \;, && (3.3)\\
L_4^{\text{gal},1} &= X\big[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big] - 2(\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi - \phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}) \;, && (3.4)\\
L_5^{\text{gal},1} &= X\big[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu{}_\rho\big] && (3.5)\\
&\quad - 3\big[(\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu - 2\Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\lambda}\phi_\lambda + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\lambda}\phi_\lambda\big] \;,
\end{aligned}
$$

which are can be generalized to

$$
\begin{aligned}
L_2^{\text{Mink}} &= A_2(\phi, X) \;, && (3.6)\\
L_3^{\text{Mink}} &= A_3(\phi, X)\Box\phi \;, && (3.7)\\
L_4^{\text{Mink}} &= A_4(\phi, X)\big[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big] \;, && (3.8)\\
L_5^{\text{Mink}} &= A_5(\phi, X)\big[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu{}_\rho\big] \;, && (3.9)
\end{aligned}
$$

where here $\phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$ since this is in flat space. For the choice of functions $A_a \propto X$ one recover the previous expressions up to total derivatives.

The action $S = \int d^4x \sum_a L_a^{\text{Mink}}$ constitutes the most general action for a scalar in flat space that leads to second-order EOM. What is essential in order to avoid higher derivatives is the antisymmetric structure that appears, in particular in the quartic (3.8) and quintic (3.9) galileons. Note that the same sort of structure appears in ghost-free massive gravity [44, 45] when focusing on the scalar mode (taking the so-called decoupling limit).

If we now want to write a covariant version of the most general action leading to second-order EOM in curved spacetime, the allowed Lagrangians can be decomposed into four classes

$$
\begin{aligned}
L_2^H[G_2] &\equiv G_2(\phi, X) \;, && (3.10)\\
L_3^H[G_3] &\equiv G_3(\phi, X)\,\Box\phi \;, && (3.11)\\
L_4^H[G_4] &\equiv G_4(\phi, X)\,{}^{(4)}\!R - 2G_{4X}(\phi, X)\big[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\big] \;, && (3.12)\\
L_5^H[G_5] &\equiv G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\big[(\Box\phi)^3 - 3\,\Box\phi\,\phi^{\mu\nu}\phi_{\mu\nu} + 2\,\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho{}_\mu\big] \;. && (3.13)
\end{aligned}
$$

The first type (3.10) corresponds to quintessence and k-essence, while the second (3.11) corresponds to the kinetic gravity braiding Lagrangian (2.26).

The third Lagrangian (3.12) contains the Einstein Hilbert action (2.12), for $G_4 = M_{\text{Pl}}^2/2$. When $G_{4X} \neq 0$ the second piece has the structure inherited from the quartic galileon (3.8). However, when the metric is dynamical and the partial derivatives are replaced by covariant ones, a non minimal coupling term, $G_4(\phi, X)^{(4)}R$, is needed in order to keep the EOM second-order. Finally, the last type, eq. (3.13), known as the quintic generalized galileon, is the extension of (3.12) to more fields ϕ. The list stops there because any Lagrangian with more fields satisfying Horndeski's conditions would be a total derivative.

In the following section, I am going to argue that one can in fact write a more general action that is still stable, even though it possesses terms with more than two derivatives in the EOM.

3.2 General considerations on higher order derivatives

The desire for second-order EOM stems from Ostrogradski's theorem, which can be stated as following: imagine the position $q(t)$ of a particle is described by a Lagrangian

$$L(q, \dot{q}, \ddot{q}) \,. \tag{3.14}$$

Note that, usually, the Lagrangian does not depend on the second derivative of the position. In this peculiar case, one can define the conjugate momenta to these variable as

$$P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} \,, \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \,. \tag{3.15}$$

Ostrogradski's theorem states (see for example [46]) that if the system is non-degenerate, i.e. if one can express the variable $\dot{q}$ and $\ddot{q}$ as functions of P_1 and P_2, the system will suffer from ghost instabilities as discussed in Section 2.4. In this simple case, the non degeneracy conditions translates simply to the invertibility of the 2x2 matrix

$$\frac{\partial L}{\partial q^{(i)}}\frac{\partial L}{\partial q^{(j)}} \,, \tag{3.16}$$

where $q^{(j)}$ denotes the j-th derivative of q w.r.t. time. It is easy to convince oneself that when this is the case, the EOM contains terms with more than two time derivatives.

Indeed, even though Ostrogradski's proof is formulated at the level of the action, its consequences can be directly seen in the EOM. Let's take the case of single DOF, $\psi(t)$, whose EOM contains three time derivatives[1]. This means that, in order to evolve ψ from an initial state, one needs three conditions: the usual "position" $\psi(t_0)$ and "velocity" $\dot{\psi}(t_0)$ but also the "acceleration" $\ddot{\psi}(t_0)$. This goes against the idea that a DOF is given by the couple position-momentum. It signifies the presence of an extra DOF, which, according to Ostrogradski, is a ghost (in the sense of eq. (2.32) with $\xi < 0$).

At the root of the proof is a notion of non degeneracy. This is not apparent in the case of one field since as soon as there is a higher derivative in the EOM, one must specify more initial conditions. However, when considering an action for more than one field, the non degeneracy conditions are not always this trivial: one can have the coefficient in front of higher derivative non zero but still have a degenerate system. A simple example

[1] Note that the case of three derivatives is somewhat particular, since only one additional initial condition in needed, instead of the two associated with a full DOF. Moreover, it is not possible to construct a Lagrangian for a single field ψ that gives odd number of time derivatives. However, it can happen when more than one field are present and constitutes thus a case worth mentioning.

is the following set of equations

$$\begin{aligned} &\dddot{\psi} + \dddot{\phi} + H_1\ddot{\psi} + H_2^3\phi = 0\,, \\ &\ddot{\psi} + \ddot{\phi} - H_3\dot{\phi} = 0\,, \end{aligned} \tag{3.17}$$

where the H_i are arbitrary constants. Naively, one could think this would require three initial conditions for ψ and ϕ, for a total of six, and the apparition of a third DOF. However, by plugging the second equation in the first, one can see the system is degenerate, since it is equivalent to

$$\begin{aligned} &H_3\ddot{\phi} + H_1\ddot{\psi} + H_2^3\phi = 0\,, \\ &\ddot{\psi} + \ddot{\phi} - H_3\dot{\phi} = 0\,, \end{aligned} \tag{3.18}$$

which is a standard second-order system describing two DOF.

The case of Lorentz invariant scalar-tensor theories is even more involved. Indeed, because of the gauge freedom, the system is degenerate: we saw for example in Section 2.4 that the lapse N and the shift N_i yielded constraint equations. This explains the fact that even GR, which *a priori* has ten DOF (the ten components of the metric), propagates only two.

In the case of eqs. (3.4) and (3.5) (as well as the other quartic (3.12) and quintic (3.13) Lagrangians), the degeneracy is increased by the specific antisymmetric structure of the Lagrangians: in particular, one can see that because of this structure, $\frac{\partial^2 L_a}{\partial \ddot{\phi}^2} = 0$. Of course, as I said above, the degeneracy condition is really on the full matrix (3.16), but intuitively this is a sign that the theory is more degenerate.

It is exactly this degeneracy that would render the proof of Ostrogradski inapplicable in the Lagrangians of Horndeski, even before studying the EOM. Therefore, this realization gives hope that one can construct theories that are more general than Horndeski without introducing ghost DOF by considering Lagrangians that are deignerate enough. It should be noted that this is not a miracle recipe that would get rid of every ghost. The larger the number of derivatives, the more degenerate the theory needs to be, making it harder and harder to conceive one.

3.3 Generalized Generalized Galileons G^3

Before introducing G^3 theories, let me make a general remark here. When taking the flat space limit of any scalar-tensor theory, the possibility of non trivial degeneracy disappears since only the scalar remains. There, the number of possibilities is limited to the Lagrangians (3.6)–(3.9). This is why a necessary condition for any scalar-tensor theory to be ghost free is to reduce to these Lagrangians when $g_{\mu\nu} \to \eta_{\mu\nu}$[2].

With this idea in mind, in [GLPV2, GLPV3] we studied the following Lagrangians

$$L_4 = L_4^H[B_4(\phi, X)] + F_4(\phi, X) L_4^{\text{gal},1}\,, \tag{3.19}$$

$$L_5 = L_5^H[B_5(\phi, X)] + F_5(\phi, X) L_5^{\text{gal},1}\,, \tag{3.20}$$

[2]When this is not the case, the theory might be ghost free around specific background, but the property might not be Lorentz invariant.

where $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$ the Lagrangians from eqs. (3.4) and (3.5) with the replacement

$$\eta_{\mu\nu} \to g_{\mu\nu}\,, \quad \partial_\mu \to \nabla_\mu\,. \tag{3.21}$$

Under this form it is easy to see that the Horndeski case corresponds to $F_4 = F_5 = 0$. However, when these functions are not zero, the EOM contain terms with three derivatives. More precisely, the metric equations contain three derivatives of the scalar and the scalar field equation contains three derivative of the metric. This means that when going to flat space, the scalar field recovers its second-order EOM, which is expected since it flat space these Lagrangians reduce to eq. (3.8)–(3.9) (up to total derivatives, see Article F).

This is not how we first discovered these Lagrangians. The first hint we had was when looking at the EFT of DE and realizing that, at linear order, one could be more general than Horndeski theories: we had an additional parameter α_H in eq. (2.19) that accounted for a deviation from Horndeski, while keeping the right number of DOF. We then built a non linear theory that respected this property. Therefore, when we first wrote it, it was in the context of the EFT of DE and as such it was in unitary gauge. Using eqs. (2.4) and (2.13), the Lagrangians (3.19) and (3.20) can be recast into:

$$\boxed{\begin{aligned} &L_4 \equiv A_4(\phi,X)\left(K^2 - K_{\mu\nu}K^{\mu\nu}\right) + B_4(\phi,X)R\,, \\ &L_5 \equiv A_5(\phi,X)\left(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\nu\rho}K^{\mu}{}_{\rho}\right) + B_5(\phi,X)K^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R\right)\,,\end{aligned}} \tag{3.22}$$

where

$$\begin{aligned} A_4 &\equiv -B_4 + 2XB_{4X} - X^2F_4\,, \\ A_5 &\equiv -\frac{XB_{5X}}{3} + (-X)^{5/2}F_5\,. \end{aligned} \tag{3.23}$$

This yields the expression of the Horndeski Lagrangians in terms of 3+1 quantities in the case $F_4 = F_5 = 0$, which was first derived in [GLPV1]. On top of making the connection with Chapter 2 easier, these expressions are going to allow us to prove that the theory has no extra DOF. In order to do so, we will specialize to the case where the scalar field is spacelike, so that we can go to unitary gauge.

One could rightfully argue that proving the soundness of the theory under this assumption does not guarantee it will hold on a different background. This is indeed true. However, it is a necessary condition and under this assumption one can actually say something quantitative about the number of DOF. Moreover, the fact that it also reduces to galileons in Minkowski is a strong signal that the theory is safe around any background. I will give a purely Lorentz invariant proof in a following section, which relies on knowing *a priori* a transformation of the metric that maps subsets of G^3 onto Horndeski.

3.4 Hamiltonian analysis

In unitary gauge, the proof is very general and is based on a Hamiltonian analysis. We are back to using the metric whose line element is

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) . \tag{3.24}$$

Moreover, in this gauge, the extrinsic curvatures in eq. (3.22) take their usual 3+1 expression

$$K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - D_i N_j - D_j N_i\right] , \tag{3.25}$$

and again, the other components are not needed (see eq. (2.9)).

To prove that there are no extra DOF, I will use a Hamiltonian analysis, that will allow me to count the number of DOF. The case is actually quite similar to the standard counting of DOF in GR, starting from eq. (2.12).

The first step is to compute the conjugate momenta of the "position" variables (N, N_k, h_{ij}) in order to write the Hamiltonian as a function of the twenty canonical variables (N, π_N), (N_k, π_k), (h_{ij}, π_{ij}). Since the lapse and the shift do not appear with time derivatives, their conjugate momenta vanish

$$\pi_N = 0 , \quad \pi_i = 0 , \tag{3.26}$$

and their EOM will yield constraints.

The Hamiltonian is defined as

$$H \equiv \int d^3\vec{x}\left[\pi^{ij}\dot{h}_{ij} - \mathcal{L}\right] . \tag{3.27}$$

What is left to do is to invert the relation between $\dot{h}_{ij}$ and π_{ij}. This can be done explicitly in the case of L_4. However, in the case of L_5 the relation between these two quantities is not linear: expressing $\dot{h}_{ij}$ as a function of π_{ij} requires taking the square root of a matrix. Therefore, even though the inversion is locally well defined, one cannot get an explicit expression (see Article F for a discussion on the matter).

After this inversion in the case of L_4 (3.19), the Hamiltonian can be put in the form

$$H = \int d^3\vec{x}\left[N\mathcal{H}_0(N) + N^i\mathcal{H}_i\right] , \tag{3.28}$$

with

$$\mathcal{H}_i \equiv -2D_j\pi^j{}_i , \tag{3.29}$$

$$\begin{aligned} \mathcal{H}_0 &\equiv -\frac{1}{\sqrt{h}A_4}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \sqrt{h}\,B_4\,{}^{(4)}\!R , \\ A_4 &= -B_4 - NB_{4N} - F_4 . \end{aligned} \tag{3.30}$$

The last equality stems from eq. (3.23) in unitary gauge, where $X = -1/N^2$ (choosing $\phi_0(t) = t$). The Hamiltonian of GR has exactly the same form, with $B_4 = -A_4 = 1/(16\pi G)$, implying that $\mathcal{H}_0$ is independent of N, which is not the case in general.

To count the DOF in a constrained system such as the one described by eq. (3.28), one has to sort the constraints according to their class in Dirac's terminology. A constraint can either be first-class, which implies that its Poisson bracket with all the other constraints vanish, or second class otherwise. Although these definitions are quite technical for the unfamiliar reader, let me distill their relevant properties. First class constraints are particular constraints that are in general associated with gauge freedom. This is why, on top of eliminating one variable, the freedom associated with the gauge removes an additional variable. The statement is thus that a first class constraint removes a full DOF (which corresponds to a couple of canonical variables). Second class constraints however do not stem from gauge freedom and as such remove only half a DOF (see for example [47] for a discussion on constrained Hamiltonian and number of DOF).

In the case of GR, all the constraints

$$\pi_N = 0 \,, \quad \frac{\partial H}{\partial N} = \mathcal{H}_0 = 0 \,, \quad \pi_i = 0 \,, \quad \frac{\partial H}{\partial N_i} = \mathcal{H}_i = 0 \,, \tag{3.31}$$

are first class (this is actually guaranteed by the fact that N and N_i only appear linearly in the action). This can be understood because, even though a specific foliation is chosen to decompose the Ricci scalar in 3+1 quantities, this foliation is completely arbitrary. Therefore, the time gauge freedom is still maintained in the action and N is the variable that enforces it. The same can be said about the spatial gauge freedom and N_i.

The counting can then be done as following: there are two constraints for (N, π_N) and six for (N_i, π_i), each removing a full DOF. This leaves two DOF out of the naive ten, which are the two polarizations of gravity waves.

In the case of G^3, the only difference is that the constraints associated with (N, π_N)

$$\pi_N = 0 \,, \quad \frac{\partial H}{\partial N} = \mathcal{H}_0(N) + N\mathcal{H}_0'(N) = 0 \,, \tag{3.32}$$

are in general second class and remove only half of DOF each. Technically, this is because in general $\mathcal{H}_0$ depends on N. More intuitively, this is just the expression that the time diffeomorphism invariance is broken by the choice of the unitary gauge, which represents a specific choice of time given by the scalar field.

However, as we discussed in Section 2.1, the action is still invariant under spatial diffs, so the six constraints for (N_i, π_i) remain first class. Actually, it is not exactly $\mathcal{H}_i$ that is first class, but rather $\mathcal{H}_i + \pi_N \partial_i N$ (see for example [48]), which is actually the total momentum constraint that would appear for GR plus a scalar field.

The counting therefore yields three DOF, which are the expected tensor (two) and scalar (one) modes. Contrarily to what could have been thought naively, no extra DOF appears in the theory. Notice that imposing the Horndeski conditions (3.23) does not yield anything special in this formulation.

Nevertheless, the simplicity of the unitary gauge action hides the fact that the Lagrangians (3.19) and (3.20) are quite peculiar. Two things should be kept in mind.

- The counting of DOF could have yielded four. For example, if one were to simply detune the functions G_4 and G_{4X} in eq. (3.19), when going to unitary gauge the action would contain terms in $\dot{N}K$. Indeed it can be checked using eq. (2.4) that

$$(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu} \supset -2\nabla_\mu X K n^\mu \propto \dot{N}K\,, \tag{3.33}$$

since $X \propto 1/N^2$ in unitary gauge. In this case, $\pi_N \neq 0$ and one can invert the momenta to write $\dot{N}$ and $\dot{h}_{ij}$ in terms of π_N and π_{ij}[3]. Then the equations associated to N become dynamical: they are no longer constraints. The only constraints that remain are those for N_i, which remove six of the ten initial DOF: an extra mode appears, which is a ghost according to Ostrogradski's theorem.

- The proof in unitary gauge could be extended to any Lagrangians that depends on arbitrary combinations of the extrinsic curvature, see for example [50]. However, as soon as they do not appear in the specific forms of eqs. (3.19) and (3.20), the theories do not reduce to galileons in flat space. As I mentioned above, this means that they potentially develop ghost like DOF when the unitary gauge is not defined, i.e. when the scalar field is not spacelike. As such, they might be Lorentz violating theories, which is what happens for Hořava-Lifshitz gravity [51, 52].

3.5 Field redefinitions

A well known situation where Ostrogradski's theorem does not apply even though higher derivatives are present is when there exists an invertible mapping between a theory and one that is healthy (see e.g. [53]). The term invertible is here taken in its formal mathematical definition: the mapping must be a bijection between the two set of variables.

In particular, in the case of a single variable ψ mapped to $\tilde{\psi}$, the transformation cannot involve derivatives of the field: the "inversion" is always defined up to integration constants, implying the mapping is not injective. This means that one cannot remove the extra DOF associated to a term in $\psi^{(n>2)}$ by defining a new variable $\tilde{\psi} \equiv \psi^{(n-2)}$. The extra DOF are just hidden in the solution of the equation for $\tilde{\psi}$ in terms of ψ. This is yet another way of saying that when there is only one variable, there is no room to play with degeneracies: Ostrogradski's theorem always applies.

However, as soon as more variables are at play, the situation changes. For example, a transformation of the form

$$\begin{aligned} &\psi_1 \to \psi_2\,, \\ &\phi_1 \to \phi_2 + \ddot{\psi}_2\,, \end{aligned} \tag{3.34}$$

is invertible, since there are no differential equations to solve to express the new variables in terms of the old ones. Thus, the Lagrangian

$$L_2 = -\frac{\dot{\psi}_2^2}{2} - \frac{\dot{\phi}_2^2}{2} - \dot{\phi}_2\psi_2^{(3)} - \frac{\left(\psi_2^{(3)}\right)^2}{2}\,. \tag{3.35}$$

[3]The last statement is essential. Indeed, there exist situations where $\pi_N \neq 0$ but the Lagrangian is too degenerate to allow the inversion of the momenta, so that there is actually no extra DOF. See Article F and [49] for examples of such a case.

has the same number of DOF as the standard free field Lagrangian

$$L_1 = -\frac{\dot\psi_1^2}{2} - \frac{\dot\phi_1^2}{2}\,, \tag{3.36}$$

since they are related by eq. (3.34).

One can also see this in a way similar to eq. (3.17) since the EOM derived from the Lagrangian (3.35) are

$$\ddot\psi_2 + \phi_2^{(4)} + \psi_2^{(6)} = 0\,, \tag{3.37}$$
$$\ddot\phi_2 + \psi_2^{(4)} = 0\,, \tag{3.38}$$

which are equivalent to

$$\dddot\psi_2 = 0\,, \quad \dddot\phi_2 = 0\,. \tag{3.39}$$

It turns out that in the case of G^3, we found a way to write this mapping, when restricting to the case of either (3.19) or (3.20), but not when both are considered at the same time. The key is to use disformal transformations [54], such as

$$\begin{aligned} \bar g_{\mu\nu} &= \Omega(\phi,X)^2 g_{\mu\nu} + \Gamma(\phi,X)\phi_\mu\phi_\nu\,,\\ \bar\phi &= \phi\,. \end{aligned} \tag{3.40}$$

For most choices of Ω and Γ this transformation is invertible in the sense I defined above, since no differential equation needs to be solved to express the original quantities in terms of the tilde ones. It was shown [55] that when the functions Ω and Γ do not depend on $X = \phi_\mu\phi^\mu$, this transformation leaves the structure of Horndeski theories invariant. By this I mean that when performing such a transformation on the whole theory $L = \sum_{a=2}^5 L_a$, one gets a Lagrangian $\tilde L = \sum_{a=2}^5 \tilde L_a$, where L_a and $\tilde L_a$ are of the forms (3.10)–(3.13), but with different functions G_b.

Once the functions Ω and Γ are allowed to depend on X, the Horndeski form is no longer preserved. In particular, when focusing on the case where only Γ depends on X, we showed in [GLPV3] that the transformation creates a bridge between Horndeski theories and G^3. More precisely, when considering

$$g_{\mu\nu} \to \bar g_{\mu\nu} = \Omega^2(\phi)\, g_{\mu\nu} + \Gamma(\phi,X)\,\phi_\mu\,\phi_\nu\,, \tag{3.41}$$

the geometrical quantities above change as[4]

$$\bar R \to \Omega^{-2} R\,, \tag{3.42}$$
$$\bar K_{\mu\nu} \to \frac{K_{\mu\nu}}{\sqrt{\Omega^2 + \Gamma X}}\,. \tag{3.43}$$

Since the extrinsic and intrinsic curvatures transform differently, the function of (ϕ, X) in front of the two different parts of the Horndeski Lagrangians, e.g.

$$L_4 \equiv (B_4 - 2XB_{4X})\left(K^2 - K_{\mu\nu}K^{\mu\nu}\right) + B_4(\phi,X)R\,, \tag{3.44}$$

[4]It is easier to see the effect of the transformation (3.41) on these quantities than directly on the scalar field and its derivatives. However this can be done, see for example [49].

will be modified differently. The freedom in $\Gamma(\phi, X)$ allows to detune these functions, leading to the case of eq. (3.19) with arbitrary A_4 and B_4. Conversely, if one starts from

$$L = A_4(\phi, \bar{X})(\bar{K}^2 - \bar{K}_{\mu\nu}\bar{K}^{\mu\nu}) + B_4(\phi, \bar{X})\bar{R}\,, \tag{3.45}$$

and performs a disformal transformation with Γ solution of

$$\Gamma_X = \frac{A_4 + B_4 - 2XB_{4X}}{X^2 A_4}\,, \tag{3.46}$$

the resulting Lagrangian belongs to the Horndeski class

$$\bar{L} = (\bar{B}_4 - 2\bar{X}B_{4\bar{X}})(\bar{K}^2 - \bar{K}_{\mu\nu}\bar{K}^{\mu\nu}) + \bar{B}_4(\phi, \bar{X})\bar{R}\,. \tag{3.47}$$

The same reasoning can be made for the case of L_5 alone. However, in general it is not possible to choose Γ to put both Lagrangians in the Horndeski form. Indeed, there is only a single free function of (ϕ, X), which is not enough to eliminate the two functions A_4 and A_5 from eqs. (3.19) and (3.20) simultaneously.

What about introducing a new function of (ϕ, X) by giving a X dependence to Ω? By considering such a transformation, one not only goes out of Horndeski theories, but out of G^3 as well.

If one were to use a structure involving second derivatives of ϕ in eq. (3.41), those would have to be covariant and introduce first derivatives of the metric through the Christoffel symbols. In principle, this means that the transformation for the metric becomes differential and would not conserve the number of DOF. We have not been able to find a field redefinition that brings the full action $L_4 + L_5$ to Horndeski. The very existence of such transformation is not guaranteed. This is where the strength of the Hamilonian analysis is manifest: it is a standalone procedure and does not rely on exterior knowledge.

One could argue that since the theory can be mapped to Horndeski, the two theories are equivalent. However this is not the case, in particular in the context of late time acceleration. Indeed, the Universe is not just described by gravity plus a scalar field; the matter sector has to be accounted for. When changing the metric in a way similar to eq. (3.41), the matter sector also changes: a coupling to the scalar field is introduced. This implies in particular that the stress energy tensor of matter is no longer conserved (this is similar to Brans-Dicke theory [24]) and has consequences already at the linear level.

3.6 Linear analysis and coupling to matter

In order to study the stability conditions and to see how the presence of matter affects the theory, I now turn to the linear perturbations. For that, I will rely on the formalism of Chapter 2. Indeed, the Lagrangians (3.22) are particularly adapted since they already are in terms of geometrical quantities. The main difference with Horndeski will come from the presence of α_H in eq. (2.19). I will show in this section how this brings a non standard derivative coupling between matter and the scalar field, which affects the propagation of matter perturbations. This also means that, contrarily to the standard idea of the Jeans phenomenon, gravity's effect will not be negligible at very small scales.

3.6.1 Stability and ghosts

What I have proven with the Hamiltonian analysis in section 3.4 is the absence of extra DOF. One might still be worried that some of those DOF are ghosts. In order to conduct an explicit analysis similar to the one of section 2.4 in the presence of matter, I will describe the latter as a scalar field $\sigma(t, \vec{x}) \equiv \sigma_0(t) + \delta\sigma(t, \vec{x})$ with a non standard kinetic term, that is

$$S_m = \int d^4x\sqrt{-g}\; P(Y, \sigma)\,, \qquad Y \equiv g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\,. \tag{3.48}$$

This is enough to describe a perfect fluid, characterized by the stress energy tensor

$$T_{\mu\nu} = (\rho_{\rm m} + p_{\rm m})u_\mu u_\nu - p_{\rm m} g_{\mu\nu}\,, \tag{3.49}$$

$$p_{\rm m} \equiv P\,, \quad \rho_{\rm m} \equiv 2YP_Y - P\,, \quad u_\nu \equiv \frac{\partial_\nu\sigma}{\sqrt{-X}}\,. \tag{3.50}$$

This choice allows to have a non trivial sound speed, given by $c_m^2 \equiv P_Y/(P_Y - 2\dot{\sigma}_0^2 P_{YY})$ [56].

What will interest us for stability is the kinetic mixing between the variable ζ in eq. (2.34) and the gauge-invariant variable $Q_\sigma \equiv \delta\sigma - (\dot{\sigma}_0/H)\zeta$. The presence of $\alpha_H \neq 0$ or matter does not modify the quadratic action for tensors, so the conditions will be the same as in Section 2.4. Once the constraints are solved, the kinetic par tof the quadratic Lagrangian (that is, the one where each field is derived once) is given in Fourier space by the matrix

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} \tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\omega^2 + \tilde{\mathcal{L}}_{\partial\zeta\partial\zeta}k^2 & A\big[\alpha_B\omega^2 - c_m^2(\alpha_B - \alpha_H)k^2\big] \\ A\big[\alpha_B\omega^2 - c_m^2(\alpha_B - \alpha_H)k^2\big] & -2P_Y c_m^{-2}(\omega^2 - c_m^2 k^2) \end{pmatrix}\,, \tag{3.51}$$

with

$$A = -\frac{2\dot{\sigma}_0 P_Y}{Hc_m^2(1+\alpha_B)}\,. \tag{3.52}$$

$\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}$ and $\tilde{\mathcal{L}}_{\partial\zeta\partial\zeta}$ are functions of the paramaters α_a, whose expressions are not very useful here, but can be found in section 5.2 of Article F. In order to get the no ghost conditions, the positivity of the eigenvalues of the time kinetic matrix (i.e. the above matrix with $k = 0$) is required. This leads to the conditions

$$\alpha_K + 6\alpha_B^2 > 0\,, \quad P_Y c_m^{-2} = P_Y + 2YP_{YY} < 0\,. \tag{3.53}$$

The first one is the same as in Section 2.4, while the second is the standard condition for k-essence. Once again, we can see here that the case of Horndeski is in no way special regarding ghosts, since α_H does not appear in the conditions. Nevertheless the sound speeds are modified, which changes the conditions to avoid gradient instabilities of eqs. (2.38). To see this, one first needs the dispersion relations. They can be obtained by requiring that the kinetic matrix is singular, implying that its determinant vanishes

$$(\omega^2 - c_m^2k^2)(\omega^2 - \tilde{c}_s^2k^2) = (c_s^2 - \tilde{c}_s^2)\left(\frac{\alpha_H}{1+\alpha_H}\right)^2\omega^2k^2\,, \tag{3.54}$$

with

$$\tilde{c}_s^2 \equiv c_s^2 - \frac{\rho_m + p_m}{H^2 M^2} \frac{(1+\alpha_H)^2}{\alpha_K + 6\alpha_B^2} \,. \tag{3.55}$$

This equation has two solutions, $\omega^2 = c_\pm^2$. To avoid gradient instabilities, we require that $c_\pm^2 > 0$.

One can see that when restricted to Horndeski, $\alpha_H = 0$, $\omega^2 = c_m^2 k^2$ is a solution of this equation and matter perturbations propagate at their usual sound speed. This is in itself not completely trivial, since the presence of α_B induces a kinetic braiding, which brings off-diagonal terms in (3.51).

When $\alpha_H \neq 0$, this mixing has a stronger effect: the presence of the scalar field ϕ modifies the sound speed of matter. Thinking back to the standard Newtonian picture of the pressure perturbation, $\delta p = c_m^2 \delta\rho$, this means that the scalar field act as additional pressure contribution. This will be clearer in Newtonian gauge, where the scalar field is explicit.

3.6.2 Newtonian gauge and Einstein frame

As I have said before, the Newtonian gauge is more appropriate to discuss the EOM, particularly in the Newtonian (small scales) limit. Therefore, I reintroduce again the scalar field thanks to the transformation

$$t \to t + \pi(t, \vec{x}) \,, \tag{3.56}$$

and parametrize the scalar part of the metric as

$$ds^2 = -(1+2\Phi)dt^2 + a(t)^2(1-2\Psi)\delta_{ij}dx^i dx^j \,. \tag{3.57}$$

In the action in terms of π, Φ and Ψ, even the kinetic part alone is very involved. This makes the analysis of the propagating DOF rather complicated. However, very much alike the case of Brans-Dicke theory, one can do a field transformation at the level of the metric potentials that puts the gravitational part of the action in a simpler form. By extension of the Brans-Dicke case, where such transformation leaves only the Einstein Hilbert term for gravity, I will call this new frame the Einstein frame. The Einstein metric is related to the original Jordan metric (i.e. the metric to which matter is minimally coupled) through

$$\begin{aligned} \Phi_E &\equiv \frac{1+\alpha_H}{1+\alpha_T}\Phi + \left(\frac{1+\alpha_M}{1+\alpha_T} - \frac{1+\alpha_B}{1+\alpha_H}\right)H\pi - \frac{\alpha_H}{1+\alpha_T}\dot{\pi} \,, \\ \Psi_E &\equiv \Psi + \frac{\alpha_H - \alpha_B}{1+\alpha_H}H\pi \,. \end{aligned} \tag{3.58}$$

In terms of these variables the kinetic part of the EFT action reads

$$\begin{aligned} S = \int d^4x a^3 M^2 \Bigg\{ &\frac{\alpha}{2}\frac{H^2}{(1+\alpha_H)^2}\left(\dot{\pi}^2 - \tilde{c}_s^2\frac{(\nabla\pi)^2}{a^2}\right) \\ &- 3\dot{\Psi}_E^2 + \frac{1+\alpha_T}{a^2}\left[(\nabla\Psi_E)^2 - 2\nabla\Phi_E\nabla\Psi_E\right] \Bigg\} \,. \end{aligned} \tag{3.59}$$

The new metric variables are not derivatively coupled to the scalar field, making transparent the kinetic structure of the theory. Notice that when $\alpha_H \neq 0$, Φ_E contains a derivative of π. This comes from the fact that in terms on the original variables Φ and Ψ, the quadratic action contains a term in $\nabla\Psi\nabla\dot{\pi}$, which is exactly the term leading to higher derivatives in the EOM. In this new frame however, the equations are explicitly second-order.

This is reminiscent of Section 3.5, where field redefinitions were used to map the theory with higher-order derivatives to one with only second-order EOM. And indeed, we proved in Appendix C of Article F that the term in $\dot{\pi}$ in eq. (3.58) arises exactly because of the X dependence of Γ in eq. (3.41).

This transformation also has an effect on the matter sector. In the Jordan frame, the matter action contains an interacting part

$$L_{\rm int} \equiv \frac{1}{2}\delta g_{\mu\nu}\delta T^{\mu\nu} = -(\Phi\delta\rho_m + 3\Psi\delta p_m) \,, \tag{3.60}$$

which is the standard one for minimally coupled matter. However, when working with the Einstein frame metric, a coupling between the scalar field and the matter perturbations appears explicitly. If $\alpha_H = 0$, the coupling is of the form

$$L_{\rm int} \supset C\,\pi\delta\rho_m \,, \tag{3.61}$$

where C is a function of time, not important for the present discussion. The stress energy of matter is not conserved and we have the schematic set of equations

$$\ddot{\delta\rho}_m - c_m^2\frac{\nabla^2\delta\rho_m}{a^2} + C_m\frac{\nabla^2\pi}{a^2} \approx 0 \,, \tag{3.62}$$

$$\ddot{\pi} - \tilde{c}_s^2\frac{\nabla^2\pi}{a^2} - C_\phi\delta\rho_m \approx 0 \,, \tag{3.63}$$

where the symbol $\approx$ stands for an equality in the limit $k \gg aH/c_m$. This means that, qualitatively, $\nabla^2\pi \sim \delta\rho_m$ (very akin to the Poisson equation of GR), which translates into a non derived term in eq. (3.62). This is negligible compared to the other terms at small scales, just as for the Jeans phenomenon in GR.

If now $\alpha_H \neq 0$, eq. (3.58) implies the presence of a coupling

$$L_{\rm int} \supset -\frac{\alpha_H}{1+\alpha_H}\dot{\pi}\delta\rho_m \,. \tag{3.64}$$

The equation for matter contains new derivative terms which are relevant at small scales $k \gg aH/c_m$ since the system scalar plus matter then obeys

$$\boxed{\begin{aligned} \ddot{\delta\rho}_m - c_m^2\frac{\nabla^2\delta\rho_m}{a^2} - (\rho_m+p_m)\frac{\alpha_H}{1+\alpha_H}\frac{\nabla^2\dot{\pi}}{a^2} \approx 0 \,, \\ \ddot{\pi} - \tilde{c}_s^2\frac{\nabla^2\pi}{a^2} - \frac{\alpha_H(1+\alpha_H)}{\alpha H^2 M^2}\dot{\delta\rho}_m \approx 0 \,. \end{aligned}} \tag{3.65}$$

The dispersion relations that one gets from this set of equations are exactly the same as in eq. (3.54). One can see from the second line that now $\nabla^2\pi \sim \dot{\delta\rho}_m$, which, when plugged back into the matter equation, adds a contribution to $\ddot{\delta\rho}_m$. This cannot be ignored when going to smaller and smaller scales, contrarily to the Horndeski case. Here is yet another proof that Horndeski and G^3 are not equivalent even though connections do exist between the two.

Let me end this section with a small comment on the notion of frames that I used here. The field redefinitions (3.58) are simply a convenient way of seeing the mixing of sound speeds. In a sense, it is a sort of diagonalization of the kinetic matrix. There is not more information in the Einstein frame than in the Jordan frame and if one were to get down to observable quantities, the results would be the same.

3.7 Conclusions

In this chapter, I introduced theories beyond Horndeski, that we dubbed G^3. These theories can be seen as alternative covariantazition of the flat space galileons [43]. As such, they are guaranteed to be ghost free in Minkowski space. When going to curved space, the EOM get in general terms with three derivatives, which could be worrisome for stability. However, a careful Hamiltonian analysis shows that these theories are stable, since they exhibit only the three DOF contained from the beginning.

Links with Horndeski theories can be made for subclasses of G^3 via field redefinitions of the disformal nature (see eq. (3.41)), but they cannot be used to map the full G^3 onto Horndeski. From a cosmological perspective, a new behavior is uncovered when matter is added to the picture, with novel features such as a mixing of sound speeds. Moreover, it was argued in [57] that, even with screening mechanisms, the time variation of the Planck mass coming from the non minimal coupling to the Ricci in eq. (3.12) cannot be hidden. Since one can choose B_4 constant in eq. (3.19) and still get a quartic galileon structure from the second piece, this might be a solution to alleviate this problem.

The community has started to turn its attention to these models. For example, the non gaussian features that arise from these new theories have been explored in [58] and the screening mechanism was studied in [59]. The latter is different from the one in Horndeski theories which might have an effect on the formation of stars, as shown in [60] and [SYMGD].

On a more general note, the considerations developed here shed light on the unwarranted theoretical prejudice on higher derivatives. The assumptions in Ostrogradski's proof are precise and they do not exclude completely their presence in the EOM. These sort of theories need to be thoroughly analyzed before being discarded.

Chapter 4

Predictions for primordial tensor modes

Even though the detection turned out not to be of primordial origin [61], the BICEP2 results had the merit of putting the study of tensor modes in the spotlight by showing that the sensitivity for B-modes is reaching the levels of what is expected from theories. So far, most of the attention has been devoted to scalar perturbations, since those are the ones that give rise to the temperature anisotropies in the CMB. Although more easily connected to observations, the scalar sector is much more complex. The predictions for the power spectrum depends on many parameters, such as the speed of sound for the scalar, or the shape of the potential. This means that it is difficult to use temperature measurements to put robust constraints on models of inflation. The situation is even worse, since the almost scale invariant spectrum that Planck observed can be produced without having inflation [62].

Tensor modes on the other hand are much simpler from a theoretical point of view since their power spectrum depends only on the energy scale of inflation. Using the Effective Field Theory of Inflation (EFTI) framework [12], I am going to show in this chapter that these predictions are very robust, contrarily to the scalar case.

4.1 Tensor sound speed and quadratic action

The EFTI, from which the EFT of DE in Chapter 2 is inspired, describes inflationary perturbations in unitary gauge. This specific time slicing breaks the explicit invariance under time diffs and velocities are no longer forced to be unity, as we have seen in Section 2.4. In particular, when the scalar sound speed is non trivial, the (dimensionless) power spectrum of the curvature perturbations ζ has an expression that depends both on $\epsilon \equiv -\dot{H}/H^2$ and c_s. Since this expression is estimated at horizon crossing $c_s k = aH$, any time dependence can be related to a scale dependence. Therefore, in general, the scalar spectral tilt is

$$n_s - 1 = -2\epsilon - \eta - \frac{\alpha_s}{H}\,, \quad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,, \quad \alpha_s \equiv \frac{\dot{c}_s}{c_s}\,. \tag{4.1}$$

It was argued for example in [62] that one could get nearly scale invariance, $n_s - 1 \ll 1$ without having slow roll inflation, $\epsilon \ll 1$, by a proper choice of sound speed. Thus, measurements of the scalar tilt cannot distinguish between inflation and other scenarios.

For gravity waves, the situation is somewhat different. It is true that the tensor sound speed can be modified. When considering the EFTI action

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[{}^{(4)}R - 2\big(\dot{H} + 3H^2\big) + 2\dot{H}g^{00} - \big(1 - c_T^{-2}(t)\big)\big(\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2\big)\right], \tag{4.2}$$

and parametrizing the tensor perturbations γ as

$$h_{ij} = a^2 e^{2\zeta}(e^{\gamma})_{ij}\,, \qquad \gamma_{ii} = 0 = \partial_i\gamma_{ij}\,, \tag{4.3}$$

the quadratic action for tensors reads, using eq. (2.9),

$$S_{\gamma\gamma} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^3 c_T^{-2}\left[\dot{\gamma}_{ij}^2 - c_T^2\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]. \tag{4.4}$$

The only other way to modify the tensor sound speed would be with a term in ${}^{(3)}R$ (that contains spatial derivatives of the metric), but this is equivalent to the case of (4.2) since the two are related by the Gauss-Codazzi relation eq. (2.13). One can compute the tensor power spectrum associated with eq. (4.4). For this, we do a change of variable $\mathrm{d}y = c_T\mathrm{d}t/a$

$$S_{\gamma\gamma} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^3x\mathrm{d}y\, q^2\left[(\gamma_{ij}')^2 - (\partial_k\gamma_{ij})^2\right], \quad q \equiv a c_T^{-1/2}\,, \tag{4.5}$$

where a prime denote the derivative with respect to y. One can decompose the helicity modes in Fourier space as

$$\gamma_{ij} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_s \gamma_{\vec{k}}^s\epsilon_{ij}^s e^{-i\vec{k}\cdot\vec{x}}\,, \tag{4.6}$$

with the polarization tensors ϵ_{ij}^s normalized as $\epsilon_{ij}^s\epsilon_{ij}^{s'} = 4\delta_{ss'}$ where s, s' denote the helicity states. Defining a new variable $v_{\vec{k}}^s \equiv q\gamma_{\vec{k}}^s$ we get the standard equation

$$v_{\vec{k}}'' + \left(k^2 - \frac{q''}{q}\right)v_{\vec{k}} = 0\,. \tag{4.7}$$

As usual (see e.g. [63]), if and only if one has $q \sim y^{-1}$ does one get a scale invariant power spectrum, given in the small k limit by

$$\langle\gamma_{\vec{k}}^s\gamma_{\vec{k}'}^{s'}\rangle = (2\pi)^3\delta(\vec{k} + \vec{k}')\frac{1}{2k^3}\frac{1}{M_{\rm Pl}^2q^2y^2}\delta_{ss'} = (2\pi)^3\delta(\vec{k} + \vec{k}')\frac{1}{2k^3}\frac{(H - \alpha_t/2)^2}{M_{\rm Pl}^2c_T}\delta_{ss'}\,, \tag{4.8}$$

where $\alpha_t \equiv \dot{c}_T/c_T$ and I used the scale invariance condition $(qy)' = 0$ to express y in terms of H, α_t and c_T.

A priori one might be worried that the situation is the same as for the scalar. However, what we showed in [CGNV] is that, through a disformal transformation plus a conformal one and a redefinition of time, one can always write the action in a form that is standard for the tensor modes, namely

$$S = \int \mathrm{d}\tilde{t}\mathrm{d}^3x\sqrt{-\tilde{g}}\frac{M_{\rm Pl}^2}{2}\Big\{{}^{(4)}\tilde{R} - 2(\dot{\tilde{H}} + 3\tilde{H}^2) + 2\dot{\tilde{H}}\tilde{g}^{00}$$
$$+ \left[2(1-c_T^2)\dot{\tilde{H}} - \frac{3}{2}\alpha_s^2 - c_T^2\left(\dot{\alpha}_s + \tilde{H}\alpha_s + \frac{1}{2}\alpha_s^2\right)\right] \times \left(1 - \sqrt{-\tilde{g}^{00}}\right)^2$$
$$+ 2\alpha_s\,\delta\tilde{K}\left(1 - \sqrt{-\tilde{g}^{00}}\right)\Big\}, \tag{4.9}$$

where tildes are to distinguish the quantities from those in the original frame. In this action, only the Ricci scalar ${}^{(4)}\tilde{R}$ contributes to the quadratic action for γ, which is the same as in GR. The rest only modifies the scalar sector. Therefore, in this frame the tensor power spectrum is the standard one

$$\boxed{\langle\gamma^s_{\vec{k}}\gamma^{s'}_{\vec{k}'}\rangle = (2\pi)^3\delta(\vec{k}+\vec{k}')\tfrac{1}{2k^3}\tfrac{\tilde{H}^2}{M_{\rm Pl}^2}\delta_{ss'}}\,. \tag{4.10}$$

There is of course no contradiction with the result in the original frame, since when going through all the transformations, one can see that

$$\tilde{H} = c_T^{-1/2}(H - \alpha/2)\,. \tag{4.11}$$

The tilde frame has the advantage of having a constant Planck mass (i.e. the normalization of the quadratic action), making the connection to the present Planck mass (and therefore present observations) clearer. Contrarily to the case of Chapter 3, there is no matter during inflation that would couple differently after disformal and conformal transformations.

From eq. (4.10), one can compute the tensor tilt n_T, which has its usual form in the tilde frame

$$n_T = 2\frac{\dot{\tilde{H}}}{\tilde{H}^2}\,, \tag{4.12}$$

but has a more complicated relation to the H given by eq. (4.11).

In particular, one can choose the variation of the tensor sound speed such that the tilt is blue, $n_T > 0$, without violating the Null Energy Condition (NEC) for a FLRW universe, which is $\dot{H} < 0$. Violating this condition usually leads to instabilities [64][1]. Therefore, in the original frame, one can have $n_T > 0$ without instabilities.

In the tilde frame, having $n_T > 0$ really implies $\dot{\tilde{H}} > 0$, i.e. violation of the NEC. However, the system is still devoid of instabilities, because the terms in the last two lines of eq. (4.9), in particular the one in $\delta K\delta g^{00}$, are going to contribute to the kinetic energy of the scalar field. Indeed, this term gives a non zero α_B in the language of the EFT of DE, which means the no-ghost condition is modified by its presence (see eq. (2.38) and also [65])

Thus, one can without loss of generality assume that the tensor quadratic action comes only from the usual 4D Ricci scalar. We went further in the comparison between the two frames and proved also that the non-Gaussianity was the same in both. This means

[1] The idea is that if $\dot{H} > 0$, the kinetic term for the scalar field in eq. (4.2), which is $-g^{00}$ has the wrong sign (in the sense of Section 2.4).

that it cannot be enhanced by a non trivial speed of sound, which is the case for scalars [63] and was claimed for tensors in the literature.

4.2 Other operators

In the previous section, I explained that the quadratic action for tensors can always be cast in the standard form, i.e.

$$S_{\gamma\gamma} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^3\left[\dot\gamma_{ij}^2 - \frac{(\partial_k\gamma_{ij})^2}{a^2}\right] . \tag{4.13}$$

This statement holds as long as one does not consider higher derivatives terms[2]. In an effective field theory approach, which is assumed to be the low energy limit of a more complex theory, one expects generally higher derivatives terms to be suppressed. Therefore, they can be treated as small corrections to the power spectrum (4.10). Only two terms are possible (they need to respect the spatial diffs invariance, just like for the EFT of DE), both of them violating parity

$$\varepsilon^{ijk}\partial_i\dot\gamma_{jl}\dot\gamma_{lk} \ , \qquad \varepsilon^{ijk}\partial_i\partial_m\gamma_{jl}\partial_m\gamma_{lk} \ . \tag{4.14}$$

The contribution of an arbitrary combination of the two to the quadratic action is

$$-\frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x\frac{1}{H\eta}\left[\frac{\alpha}{\Lambda}\varepsilon^{ijk}\partial_i\gamma'_{jl}\gamma'_{lk} + \frac{\beta}{\Lambda}\varepsilon^{ijk}\partial_i\partial_m\gamma_{jl}\partial_m\gamma_{lk}\right] , \tag{4.15}$$

where a prime denotes here the derivative with respect to the conformal time $\eta \equiv \int dt/a$, α and β are dimensionless coefficients and Λ is the scale that suppresses these higher dimension operators. In order to get an idea of the corrections that this brings to the power spectrum, I will take the simplest case, where α and β are constant. In addition, I will assume that they are indeed corrections, namely that the scale Λ is much higher than the energy scale of the problem, H. Then, to compute the power spectrum, we can treat (4.15) as an interaction term and use the in-in formalism [66]. In the late-time limit, $\eta \to 0$, the result does not depend on α and the power spectrum is modified to

$$\langle\gamma^{\pm}_{\vec k}\gamma^{\pm}_{\vec k'}\rangle = (2\pi)^3\delta(\vec k + \vec k')\frac{H^2}{2M_{\rm Pl}^2k^3}\left(1 \pm \beta\frac{\pi}{2}\frac{H}{\Lambda}\right) . \tag{4.16}$$

Such a parity violating power spectrum would yield non zero TB and EB power spectra in the CMB polarization. The authors of [67] quote the detectability of parity violations of order one in the power spectrum with future experiments, which is probably far from what is expected here.

Finally, another way to modify the standard predictions for tensor modes is to change the non-Gaussianity by introducing cubic terms in the EFTI Lagrangian. Since we cannot construct operators with explicit underived γ that respect the 3-D symmetry, the lowest order in derivatives is two. The only two operators that one can then construct are $\delta K_{ij}\delta K^{ij}$ and ${}^{(3)}R$. We have seen in the previous section that they can always be reabsorbed in the Ricci scalar by suitable transformations. The only other way is to

[2]With two derivatives, only the terms in eq. (4.13) can appear, the other possibilities being total derivatives.

pay the price of an additional derivative and consider operators such as $\delta K_{ij}\delta K^{ik}\delta K_k{}^j$. However, they should be suppressed with respect to lower derivatives terms and only bring small corrections to the correlator $\langle\gamma\gamma\gamma\rangle$. The same sort of reasoning can be made for the correlator $\langle\gamma\zeta\zeta\rangle$ which, at lowest order in derivatives, can only come from the term g^{00}. On the other hand, it is hard to say anything definite for $\langle\gamma\gamma\zeta\rangle$ which can be enhanced by operators such as $\delta K_{ij}\delta K^{ij}\delta N$.

4.3 Conclusions

By use of field redefinitions, I have shown that one can always put the quadratic action for tensor in the standard form. In particular, the sound speed of tensor can always be set to unity. Physically, this makes sense, since in the absence of matter like in inflation, the benchmark for velocities is the one of gravitons. It means that the power spectrum for gravity waves is always given by the simple form of eq. (4.10). This is heavy with consequences. First, it means that the amplitude of the power spectrum directly gives the energy scale of inflation. There is no degeneracy with the shape of the potential or the sound speed as for scalars. Second, this implies that measuring a scale invariant power spectrum can only mean that H is almost constant, i.e. that there was a period of inflation.

Chapter 5

Consistency relations of the large scale structure

The CMB is great source of observational knowledge in cosmology and it is has been extensively used, in particular by Planck [1]. To obtain even more information on cosmology, the next step is to rely on the large scale structure, via galaxy surveys for example. This has two main advantages. First, contrarily to the CMB which is a 2-D surface, galaxy surveys span the full 3-D space, which greatly increases their statistical power. Second, they probe the late time universe, where the effects of dark energy are expected to be the strongest, which means more constraining power. However, even within ΛCDM+GR, it is still hard to make accurate late-time predictions at small scales. Indeed, if for the CMB the physics is well described by the linear regime, at late-time the structure has grown into the non linear regime, which means a breakdown of the usual perturbative tools. Moreover, if the dark matter distribution can be predicted through N-body simulations for example, this cannot be said for galaxies. The problem is that galaxies are what we observe when doing those experiments, so that one needs models to relate their distribution to that of dark matter. This limits the theoretical control we have on predicting the galaxies' distribution.

Fortunately, there exist testable relations that do not rely on a specific description of the small scale physics. One such example are consistency relations of the large scale structure [68–70] (see also [66] for inflationary consistency relations). They allow to make a bridge between $(n + 1)$-point and n-point correlation functions in the limit where one of the fields, called the long mode, varies much less than the others. Their strength resides in the fact that very little information on the physics of the short modes is needed, which can in principle be in the non linear regime. Moreover, these relations are very robust since they are based only on two assumptions: the Gaussianity of initial conditions and the validity of the Equivalence Principle (EP). The later is particularly interesting for the late-time universe, as some models for dark energy involve a fifth force that may break the EP.

In the first part [71] of a series of three papers, these relations were derived for the large scale structure including relativistic corrections. This is necessary if one wants to follow the evolution of the modes from inflation to now. In the second part [CGSV1], that I will present in this chapter, we focused on the non relativistic case to include a resummation of infrared effects, as well as a generalization to redshift space, where observations are made. Then, I will discuss the last part [CGSV2] where we proposed to use these relations to test the Equivalence Principle.

5.1 Deriving consistency relations

When the EP is satisfied, i.e. objects respond identically to gravity, only second derivatives of the gravitational field are important. Let me work in conformal time and decompose a gravitational field Φ_L as

$$\Phi_L(\eta, \vec{x}) = \Phi_L|_0 + \partial_i \Phi_L|_0 \, x^i + \frac{1}{2} \, \partial_i \partial_j \Phi_L|_0 \, x^i x^j + \ldots \tag{5.1}$$

The first two terms of the r.h.s. can be removed by an appropriate change of coordinates which corresponds to going to the accelerated frame in the elevator argument of Einstein.

The last term, however, is physical, since it is related to tidal forces. In the non relativistic limit, a constant gravitational field has no effect, so that I will focus on the constant gradient term $\partial_i \Phi_L|_0$. In Fourier space, this can be though of as $\Phi_L(\vec{q})$ with $\vec{q} \to 0$. To remove this constant gradient, the following change of coordinates is performed

$$\tilde{\vec{x}} = \vec{x} + \delta\vec{x}(\eta) \, , \qquad \delta\vec{x}(\eta) \equiv - \int \vec{v}_L(\tilde{\eta}) \, \mathrm{d}\tilde{\eta} \, , \tag{5.2}$$

while time is left untouched. The velocity $\vec{v}_L$ satisfies the Euler equation in the presence of the homogeneous force, whose solution is

$$\vec{v}_L(\eta) = -\frac{1}{a(\eta)} \int a(\tilde{\eta}) \vec{\nabla} \Phi_L(\tilde{\eta}) \, \mathrm{d}\tilde{\eta} \, . \tag{5.3}$$

If we denote by $\delta^{(g)}(\vec{x}, \eta)$ the overdensity in the galaxy distribution[1], the EP guarantees then that

$$\langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) | \Phi_L(\vec{y}) \rangle \approx \langle \delta^{(g)}(\tilde{\vec{x}}_1, \eta_1) \cdots \delta^{(g)}(\tilde{\vec{x}}_n, \eta_n) \rangle_0 \, . \tag{5.4}$$

The notation on the l.h.s. means that the correlation function is evaluated in the presence of a constant gradient of Φ_L, while the subscript 0 on the r.h.s. signifies that it is evaluated with $\Phi_L = 0$. Note that in order for this relation to hold, the fact that short and long mode are not correlated is essential. This is where the assumption of Gaussianity plays a role.

The next step is to express the displacement $\delta\vec{x}(\eta)$ on the r.h.s. as a function of an overdensity. For this, one combines eqs. (5.2) and (5.3) with the continuity equation $\delta' + \nabla \cdot \vec{v} = 0$ to obtain in, Fourier space

$$\delta\vec{x}(\vec{q}, \eta) = -i \frac{\vec{q}}{q^2} \delta(\vec{q}, \eta) \equiv -i \frac{\vec{q}}{q^2} D(\eta) \delta_0(\vec{q}) \, , \tag{5.5}$$

where in the second equality we have defined $D(\eta)$, the linear growth factor of density fluctuations. $\delta_0(\vec{q})$ is a Gaussian random field with power spectrum $P_0(p)$ which represents the initial condition of the density fluctuations of the long mode [72]. Finally,

[1] Note that this is for concreteness; the argument would hold for any type of overdensity.

one multiplies each side by δ_L, takes the average over the long mode and uses standard results for Gaussian integrals. In Fourier space, this leads to

$$\boxed{\langle \delta_{\vec{q}}(\eta)\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' \approx -P(q,\eta)\sum_a \frac{D(\eta_a)}{D(\eta)}\frac{\vec{k}_a\cdot\vec{q}}{q^2}\langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' \ .} \tag{5.6}$$

The primes denote that the delta function of momentum conservation have been removed. Moreover the $\approx$ signifies that this equality is valid in the limit $q \to 0$. Note that, to derive this relation, $\delta_{\vec{q}}$ is assumed to be small and obey linear theory, but no assumption is made on the size of the displacement (5.5), allowing for

$$\frac{|\delta\vec{x}|}{|\vec{x}|} \sim \frac{k}{q}\delta_{\vec{q}} \sim 1 \ . \tag{5.7}$$

This result is very robust: nowhere in the derivation does one need to specify anything on the short modes except Gaussianity and EP. In particular, the divergence in $\frac{k_a}{q}$ in the r.h.s. disappears at equal time $\eta_a = \eta$ because $\sum_a \vec{k}_a = \vec{q}$. This can be understood more physically as the following: when looking at correlations between a long and several short modes, what we are really doing is measuring how much objects have fallen in a constant gravitational gradient. The correlation at equal time corresponds exactly to the case where we have waited for the same amount of time for each objects. Since they all feel the same field, the displacement they made is the same and translational invariance guarantees that the effect on the correlation is zero[2]. Only in the unequal time case does one get a divergent contribution.

However, this case seems less reachable from an observational point of view. Indeed, for the equal time correlators, we are basically comparing the positions at the moment we see the objects with the positions we know they had in the beginning, since in cosmology we know the initial conditions. For unequal time, we would have to observe the positions at a given time and then wait different amounts of time for different objects. But since they fall at velocities much smaller than the speed of light, they would not remain on the lightcone and therefore become unobservable as schematically shown in Fig. (5.1). Nevertheless, this is a good test for N-body simulations, where one is not restricted to measurements on the lightcone.

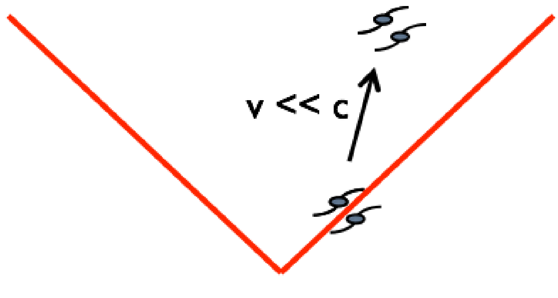

FIGURE 5.1: Galaxies getting out of the light cone for unequal time correlators.

The relation (5.6) can be generalized to the case of several "soft legs", meaning correlation functions with more than one long mode. The starting point is still eq. (5.4). If

[2]The consistency relation does not give exactly zero, because a long mode is not exactly a constant gradient, but only an approximation.

now one multiplies by m soft modes and repeat the procedure described above one gets

$$\begin{aligned}\langle\delta_{\vec{q}_1}(\tau_1)\cdots\delta_{\vec{q}_m}(\tau_m)\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' &\approx (-1)^m P(q_1,\tau_1)\cdots P(q_m,\tau_m)\\ \times\sum_{a_1}\frac{D(\eta_{a_1})}{D(\tau_1)}\frac{\vec{k}_{a_1}\cdot\vec{q}_1}{q_1^2}\cdots\sum_{a_m}\frac{D(\eta_{a_m})}{D(\tau_m)}\frac{\vec{k}_{a_m}\cdot\vec{q}_m}{q_m^2}&\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle'\,.\end{aligned} \tag{5.8}$$

This results is valid in the limit $q_a \ll k_b$ for all (a,b).

Let me reiterate that the equalities that I have just shown do not rely on any assumptions except the EP and the Gaussianity of the initial conditions. The long modes are however supposed to obey linear perturbation theory, which is the case provided they are sufficiently long ($q_a \lesssim 0.1h\text{Mpc}^{-1}$ at redshift $z=0$).

5.2 Going to redshift space

The relation derived in the previous section were set in real space, or the Fourier space associated with it. However, observations for galaxies are made in redshift space. In the plane parallel approximation, the mapping between the two spaces is given by

$$\vec{s} = \vec{x} + \frac{v_z}{\mathcal{H}}\hat{z}\,, \quad \mathcal{H} \equiv \frac{\mathrm{d}\ln a}{\mathrm{d}\eta}\,, \tag{5.9}$$

where $\hat{z}$ is the direction of the line of sight, $v_z \equiv \vec{v}\cdot\hat{z}$ and $\vec{v}$ is the peculiar velocity. Therefore, one could worry that the consistency relations do not translate nicely in redshift space, since one has to deal with peculiar velocities. As I will show, this is not the case. The ingredient needed is to see how velocities are affected by the presence of a constant gradient. This is straightforward, since we already used the Euler equation to get that

$$\vec{v} \to \vec{v} - \vec{v}_L(\eta)\,, \quad \vec{v}_L(\eta) = -(\delta\vec{x})'\,, \tag{5.10}$$

with v_L given by eq. (5.3) and $\delta\vec{x}$ by eq. (5.2). Thus, one can see how the redshift coordinates change when removing a constant gradient of the gravitational field,

$$\vec{s} \to \vec{s} + \delta\vec{x} + \frac{(\delta x_z)'}{\mathcal{H}}\hat{z}\,. \tag{5.11}$$

Using the form of the time dependence of $\delta\vec{x}$ in eq. (5.5), this can be cast into

$$\vec{s} \to \vec{s} + \delta\vec{x} + f\delta x_z\hat{z}\,, \quad f \equiv \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\,. \tag{5.12}$$

Then, to see how the density changes in redshift space, let me write its expression as

$$\rho(\vec{s}) = ma^{-3}\int \mathrm{d}^3p\mathcal{F}\left(\vec{s} - \frac{v_z}{\mathcal{H}}\hat{z}, \vec{p}\right)\,. \tag{5.13}$$

where $\mathcal{F}(\vec{x},\vec{p})$ is the real space distribution function. Therefore, the statistical properties of $\rho(\vec{s})$ are inherited from real space. In the presence of the long mode

$$\begin{aligned}\rho_s(\vec{s})_{\Phi_L} &= \frac{m}{a^3}\int \mathrm{d}^3p\,\mathcal{F}\left(\vec{s}-\frac{v_z}{\mathcal{H}}\hat{z}+\delta\vec{x},\vec{p}+am\delta\vec{v}\right)\\ &= \frac{m}{a^3}\int \mathrm{d}^3p'\,\mathcal{F}\left(\vec{s}-\frac{v_z-\delta v_z}{\mathcal{H}}\hat{z}+\delta\vec{x},\vec{p}'\right) = \rho_s(\vec{s}+\delta\vec{s}),\end{aligned} \tag{5.14}$$

and this relation does not depend on a fluid description, implying it is valid even at small scales where shell-crossings occur. Using this, we can write

$$\langle \delta^{(g,s)}(\vec{s}_1,\eta_1)\cdots\delta^{(g,s)}(\vec{s}_n,\eta_n)|\Phi_L\rangle \approx \langle \delta^{(g,s)}(\vec{\tilde{s}}_1,\eta_1)\cdots\delta^{(g,s)}(\vec{\tilde{s}}_n,\eta_n)\rangle\,, \tag{5.15}$$

which is the redshift space equivalent of eq. (5.4), since I used the redshift space density contrast $\delta^{(g,s)}$. The last step is very similar to the real space case, except for one thing. To express everything in term of the redshift space quantities, one needs to relate δ_0 contained in $\delta\vec{x}$ (see eq. (5.5)) to $\delta^{(g,s)}$. Since this is for the long mode, one can use linear perturbation theory to get [72]

$$\delta^{(g,s)}(\vec{q},\eta) = \left(b_1+f\mu_{\vec{q}}^2\right)D(\eta)\delta_0(q)\,,\quad \mu_{\vec{q}}\equiv\vec{q}\cdot\hat{z}/q\,, \tag{5.16}$$

where b_1 is the linear galaxy bias. Combining all of this, one obtains the consistency relation in redshift space

$$\boxed{\begin{aligned}\langle\delta^{(g,s)}_{\vec{q}}(\eta)\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle \approx -\frac{P_{g,s}(q,\eta)}{b_1+f\mu_{\vec{q}}^2}\sum_a\frac{D(\eta_a)}{D(\eta)}\frac{k_a}{q}\left[\hat{q}\cdot\hat{k}_a+f(\eta_a)\mu_{\vec{q}}\mu_{\vec{k}_a}\right]\\ \times\langle\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle\,,\end{aligned}} \tag{5.17}$$

with $\hat{p}\equiv\vec{p}/p$. Notice that, just as in the real space case, the divergence in the consistency relation vanishes at equal times. This adds to the robustness of the results: any deviation, even in redshift space, would be a sign of violation of the EP and/or non-Gaussianity in the initial conditions [69]. In the next section, I will focus on the constraints one can put on EP violations using these relations [CGSV2].

5.3 Violation of the Equivalence Principle

When the Equivalence Principle is not satisfied, one cannot remove the effect of a constant gravitational field with a common change of coordinates. Indeed, in principle, different objects feel differently the effect of a long mode, which is nothing more than saying that objects fall at different rates in the same potential and in general, one expects the bispectrum to be of the form

$$\lim_{q\to 0}\langle\delta_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle' = \left(\epsilon\,\frac{\vec{k}\cdot\vec{q}}{q^2}+\mathcal{O}\left[(q/k)^0\right]\right)P(q,\eta)P_{AB}(k,\eta)\,, \tag{5.18}$$

where ϵ is (model dependent) parameter that characterizes the violation of the EP and $\vec{k}\equiv(\vec{k}_1-\vec{k}_2)/2$.

The situation is actually much more complicated than when the EP is satisfied, as can be see in Fig. 5.2. On the right panel, the EP is violated, and object do no fall by the same amount (represented by the red and black arrows) in a constant gravitational field, contrarily to the left panel. Therefore, the distance between them changes in time and the force that each object has on the other (gravitational and/or electromagnetic if they are charged, represented by the blue arrow) changes as well. In general, this greatly complicates the dynamics and no definite answer can be found for the form of ϵ in eq. (5.18).

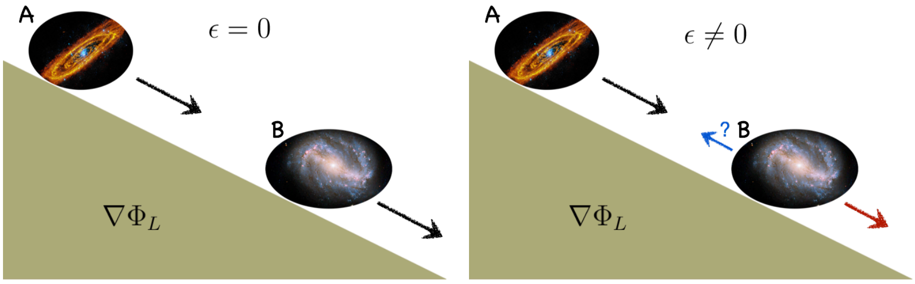

FIGURE 5.2: Two types of objects in a constant $\nabla\Phi_L$. ϵ characterizes deviations from EP, so that on the left, it is valid, while it is violated on the right.

This is why in [CGSV2], we chose a specific model, to serve as a benchmark for EP violations.

5.3.1 A toy model

The idea is to consider the case of two species A and B, in the presence of an extra scalar field φ that couples only to species B, for example through a conformal coupling [36, 73, 74]. The setup is then

$$\delta'_X + \vec{\nabla} \cdot [(1+\delta_X)\vec{v}_X] = 0 \;, \quad X = A, B \;, \tag{5.19}$$

for the continuity equations (the time evolution of φ is neglected). The Euler equation for B contains the fifth force, whose coupling is parameterized by α,

$$\vec{v}'_A + \mathcal{H}\vec{v}_A + (\vec{v}_A \cdot \vec{\nabla})\, \vec{v}_A = -\vec{\nabla}\Phi \;, \tag{5.20}$$

$$\vec{v}'_B + \mathcal{H}\vec{v}_B + (\vec{v}_B \cdot \vec{\nabla})\, \vec{v}_B = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\varphi \;. \tag{5.21}$$

Assuming that the stress energy tensor of the scalar field is negligible, Φ is related to matter densities through the standard Poisson equation

$$\nabla^2\Phi = 4\pi G\, \rho_{\rm m}\, \delta \equiv 4\pi G\, \rho_{\rm m}\, (w_A\delta_A + w_B\delta_B) \;, \tag{5.22}$$

where $\rho_{\rm m}$ is the total matter density and $w_X \equiv \rho_X/\rho_{\rm m}$. The final ingredient is a relation between φ and the matter density. In the quasistatic limit, the equation for the scalar field reduces to [74]

$$\nabla^2\varphi = \alpha \cdot 8\pi G \rho_{\rm m} w_B \delta_B \;. \tag{5.23}$$

Then one applies the standard perturbation theory tools (see e.g. [75]) to compute the bispectrum, assuming Gaussian initial conditions. For the mixed bispectrum, the result is

$$\boxed{\lim_{q\to 0}\langle \delta_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle' \simeq \tfrac{7}{5}w_B\,\alpha^2\,\tfrac{\vec{k}\cdot\vec{q}}{q^2}P(q,\eta)P^{(AB)}(k,\eta)}\,, \tag{5.24}$$

with δ defined in eq. (5.22) and

$$\langle \delta_{\vec{k}}(\eta)\delta_{\vec{k}'}(\eta)\rangle = (2\pi)^3\delta_D(\vec{k}+\vec{k}')P(k)\,, \qquad \langle \delta^{(X)}_{\vec{k}}(\eta)\delta^{(Y)}_{\vec{k}'}(\eta)\rangle = (2\pi)^3\delta_D(\vec{k}+\vec{k}')P^{(XY)}(k)\,. \tag{5.25}$$

This corresponds to having $\epsilon = \frac{7}{5}w_B\,\alpha^2$ in eq. (5.18). Now that we have an explicit form for the bispectrum $B^{(AB)}(k_1,k_2,k_3)$, defined by

$$\langle \delta_{\vec{k}_1}(\eta)\delta^{(A)}_{\vec{k}_2}(\eta)\delta^{(B)}_{\vec{k}_3}(\eta)\rangle = (2\pi)^3\delta_D(\vec{k}_1+\vec{k}_2+\vec{k}_3)B^{(AB)}(k_1,k_2,k_3)\,. \tag{5.26}$$

I will show in the next part how one can use future galaxy surveys to constrain α.

5.3.2 Estimate of the signal to noise

To see how well this effect can be measured, I will present an estimate of the signal to noise. Physically, this quantity measure how far the new bispectrum is from the standard prediction, in unit of the expected variance. Technically, this translates into the formula

$$\left(\frac{S}{N}\right)^2 = \sum_T \frac{\left[B^{(AB)}_{\alpha^2}(k_1,k_2,k_3) - B^{(AB)}_{\alpha^2=0}(k_1,k_2,k_3)\right]^2}{\Delta[B^{(AB)}]^2(k_1,k_2,k_3)}\,, \tag{5.27}$$

the sum T being on configurations for $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$ that respect $\vec{k}_1+\vec{k}_2+\vec{k}_3=0$. Furthermore, each k_i is between $k_{\rm min}$ (given by the size of the survey $k_{\rm min} = 2\pi/V^{1/3}$) and $k_{\rm max}$ that signals when linear theory breaks down. I will take $k_{\rm max} = \pi/(2R)$ where R is chosen in such a way that linear density fluctuations of the matter field in a ball of radius R have a root mean squared σ_R equal to 0.5.

The variance of the bispectrum is given by (see for example [76])

$$\Delta\left[B^{(AB)}\right]^2(k_1,k_2,k_3) = k_f^3\frac{s_{123}}{V_{123}}P(k_1)P^{(A)}(k_2)P^{(B)}(k_3)\,, \tag{5.28}$$

where I ignored the shot noise contribution to see what an ideal survey could probe. Moreover, the bispectra are computed using perturbation theory in the full case (not only in the squeezed limit $q \ll k$) to gain access to more modes. To get the limit on the detectability of EP violations in our model, one requires that the signal to noise (5.27) is of order one. The constraints are shown in Fig. 5.3.

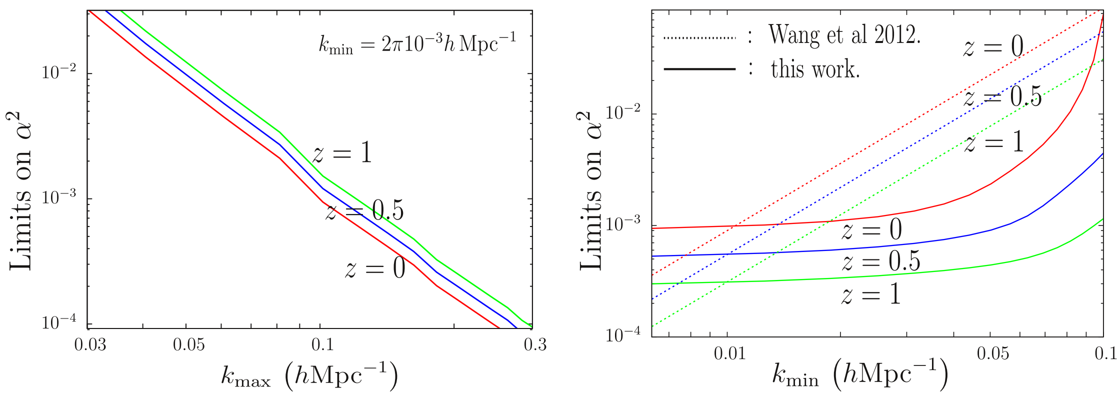

FIGURE 5.3: Limits on α^2 for a survey with volume $V = 1(\mathrm{Gpc}/h)^3$ at three different redshifts, $z = 0$, $z = 0.5$ and $z = 1$. Left: Expected bound on α^2 as a function of $k_{\rm max}$. We have chosen $k_{\rm min} = 2\pi/V^{1/3}$ so that the violation of the EP extends to the whole survey. Right: Expected bound on α^2 as a function of $k_{\rm min}$. $k_{\rm max}$ is given by 0.10, 0.14, 0.19 $h\,\mathrm{Mpc}^{-1}$ for $z = 0, 0.5, 1$ respectively. The dotted lines represent $\alpha^2 \lesssim 10^{-6}(m/H)^2$, i.e. the bound on α^2 from screening the Milky Way [77].

On the left panel, the constraints are compared with that for chameleon models derived in Ref. [77] from requiring that the Milky Way must be screened. This yields

$$\alpha^2 \lesssim 10^{-6}(m/H)^2 \; . \tag{5.29}$$

On the left panel one sees a improvement of the bound when increasing the redshift. This comes from the fact that, when going back in time, structures are less formed and the linear regime extends to larger k. This is why the choice of $k_{\rm max}$ increases with z.

Let me comment now on the applicability of such results.

First, it should be kept in mind that this is only a toy model, to get an estimate. The form of ϵ in (5.18) is model dependent and in general is different from the value obtained in eq. (5.24). The robustness is really that ϵ vanishes when there are no violation of EP. However, the simple model gives an order of magnitude of what can be expected.

The next to leading order $\mathcal{O}\left[(q/k)^0\right]$ is also very model dependent. If one wants to use as much modes as possible and not restrict to $q/k \ll 1$, this form has to be specified. For example, a scale dependent bias gives in general contributions and one should marginalize over it, which would deteriorate the constraints. Nevertheless, the peculiar scale and angular dependences of the signal we want to probe give hope that this effect should not be large.

There are two main scenarios for A and B where our model with a fifth force could apply.

- Species A are baryons and B dark matter. While the absence of fifth force is well tested on Earth for baryons [78], the dark matter sector is less constrained, even though Planck already constrains $\alpha^2 \lesssim 10^{-4}$ [79].

  However, this situation is not ideal from an observational point of view: it is hard to separate galaxies into baryons and dark matter, since they all have fairly similar baryon to dark matter ratio.

- The second scenario would be when A represents screened objects and B unscreened.This is also challenging. Indeed, for chameleon theories, the requirement that the Milky way is screened implies [77]

$$\alpha^2 \lesssim 10^{-6}(m/H)^2 \,, \tag{5.30}$$

  where m is the Compton mass of the chameleon. In this case $k_{\rm min}$ can be identified with m, the inverse of the Compton wavelength of the chameleon. Fig. 5.3 shows that the condition (5.30) is already pretty restrictive, though for $m \gtrsim 0.01\,h\,\mathrm{Mpc}^{-1}$, our constraints are better.

  Another difficulty is that for galaxies to be unscreened, they need to have a smaller gravitational potential than the Milky Way, while it is typically the opposite in galaxy surveys.

5.4 Conclusions

On cosmological scales, there are few tests as robust and simple as consistency relations. Using the Equivalence Principle as well as the Gaussianity of the initial conditions, one can derive relations between the $(n+1)$-point and n-point correlation functions, when one of the mode is much longer than the others. This long mode is the only one that needs to be dealt with explicitly (using linear perturbation theory), while no additional information on the short modes is necessary. This means one does not have to worry about baryons, bias, shell-crossing, etc, when using these relations, which makes them very robust. In this chapter, I proved that this robustness extends further. Indeed, they hold regardless of the size of the displacement caused by the long mode. Moreover, they translate very easily in redshift space, where observations are made. Therefore, by looking for potential violations of these relations, one can put constraints on non-Gaussianity[3] [76] that will in the future surpass those from Planck [82].

For the late universe, they allow to test deviations from the Equivalence Principle, which is a central property of ΛCDM+GR. Once the accelerated expansion is assumed to come from a scalar field, this opens the gate to new couplings that may not obey this principle.

By means of a simple toy model, I have given the bounds on EP violations one can expect from testing consistency relations in large scale structure. Although the bounds are not competitive with local tests, I want to emphasize that this is a unique test that probes the EP on cosmological scales. It is precisely at this scales that the laws of gravity need to be modified to account for the acceleration and we do not have yet a definite idea on how to do it. Thus, the model independence of this test makes it essential to understand better our Universe.

[3] Another promising mean of probing non-Gaussianity in the large scale structure is through scale-dependent bias [80, 81].

Chapter 6

Conclusion

6.1 Summary

In this thesis, I have condensed what I thought were the most interesting results that I obtained during my Ph.D. I have voluntarily left out a large part of the technical details and focused on the physical origin of these results, as well as their impact.

In Chapter 2, I explained how we developed a very general parametrization for linear perturbations. It is largely model independent since it is based mainly on symmetry considerations inherited from the FLRW structure of our Universe. In this apronach, the deviations from ΛCDM are given in terms of a minimal set of five functions of time. These functions can be related to virtually every model of modified cosmology, but the real strength of this approach is that it is not necessary to do so. Theoretical, as well as observational [35] constraints can be put directly on these parameters, shaping our understanding of linear cosmology without having to rely on a specific model.

By extending the stability conditions found in this chapter to the non linear case, we devised a set of Lagrangians that go beyond what was believed to be the most general stable scalar-tensor theories. I gave an overview of these new theories, called G^3, and their genesis in Chapter 3, along with a very general procedure that allows to identify well posed theories. The main goal behind this work was to convince the community not to discard every theory with higher order derivatives in their equations of motion. In the case of G^3, I have presented the unusual mixing that occurs between matter and scalar perturbations already at the linear level, using the EFT of DE.

Turning now specifically to tensor modes, in Chapter 4 I have shown that their standard predictions from single field inflation are very robust. In principle, the action for tensors can be non standard because of the presence of an extra scalar. However, using field redefinitions, I have shown that one can always return to the usual case at the linear level. Even at the next order in perturbations, the choice of modifications is rather limited. This means that the power spectrum is always given solely in term of the Hubble parameter H, which represents the energy scale of inflation and that non-Gaussianity cannot be enhanced. As a corollary, this proves that a scale invariant power spectrum for gravity waves constitutes very strong evidence for inflation.

Chapter 5 was dedicated to an approach somewhat different from the others. It was not focused on scalar-tensor theories *per se*, but rather on very robust tests called consistency relations. They are relations between correlation functions of the density contrast $\delta_{\vec{k}}$ in the limit where one (or several) of the momenta becomes much smaller than the others. To use them does not require any knowledge of the short scale behavior, where non linearities and baryonic physics play an important role. They are very useful to probe the non-Gaussianity in the initial conditions and/or the Equivalence Principle, that is not always respected in alternatives to ΛCDM.

6.2 Outlook

While writing the thesis, I have also been working on extending the formalism of the EFT of DE to the case where dark matter is non minimally coupled to the scalar field, whereas baryons are only coupled to gravity (a sort of generalization of Section 5.3.1). This scenario has been well studied in the literature, beginning with conformal couplings of the form ϕT^μ_μ [36] and more recently disformal ones $\phi_\mu\phi_\nu T^{\mu\nu}$ in [39] for example. What is usually done in these studies is to assume that gravity is described by GR, on top of which one adds a quintessence scalar field. The idea we had was to consider a general conformal plus disformal coupling for dark matter (that is, $\tilde{g}_{\mu\nu}T^{\mu\nu}$, with $\tilde{g}_{\mu\nu}$ given by eq. (3.41)) combined with modifications of gravity as in Chapter 2. In particular, this brings two additional functions of time to the analysis, which are going to change the stability conditions of Section 2.4 and the phenomenology discussed in Section 2.5.3.2.

Another possible direction of research is to investigate the equation for Ψ, eq. (2.59), which is the combination of Einstein's equations into a single one. It would certainly be interesting to solve it numerically. Even analytically, this should allow oneself to probe the modifications of gravity in a regime where the quasistatic approximation starts to break down. Bellini and Sawicki have started to look into this recently [31], where they show that in general the quasistatic regime breaks down at the sound horizon, $kc_s \sim aH$. I think there are still much information that can be extracted from this equation, in particular concerning relativistic effects.

The exploration of theories beyond Horndeski is just at its beginning. The goal would be to find a necessary and sufficient condition for Lorentz invariant scalar-tensor theories to be stable, that can be checked straightforwardly from the action. Requiring second-order EOM does not fulfill all these requirements since I showed in Chapter 3 that it is not necessary. The Hamiltonian analysis in unitary gauge of Section 3.4 also falls short, since it is not a Lorentz invariant proof. Doing it from the covariant Lagrangian, i.e. without choosing a specific gauge, is bound to be an extremely cumbersome computation, which cannot be classified as straightforward. The existence of field redefinitions that map the theory to a stable one, on top of not being necessary, cannot qualify as a straightforward check either, since one has in general to guess a specific transformation for each theory.

Concerning more particularly our proposal, G^3, people have started to look at its non linear behavior [59]. It was shown that contrarily to the Horndeski case, where non linearities allow to fully recover GR on small scales (through Vainshtein screening), in G^3 the gravitational potentials differs from that of GR inside sources, such as stars. This was latter used in [60] to study the evolution of stars for a specific G^3 Lagrangian. In [SYMGD], we derived the very general modification due to this non standard behavior in the Lane-Emden equation [83], that governs the profile of stars for a polytropic fluid. In particular, we found a generic bound on the parameter α_H of Chapter 2 for the existence of physical solutions to this equation.

Let me end with some considerations on consistency relations. The theoretical community has really shown a frank enthusiasm regarding these relations, as seen by the number of authors that have recently published on the subject [84–87]. It has been proposed to test the origin of magnetic fields [88], and as a mean to compute higher order corrections to the linear power spectrum [89]. What I think would be interesting is to check these relations in N-body simulations, where one is not limited to equal times correlators, but can actually look at the r.h.s. of eq. (5.6).

List of Publications

[GLPV1] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Essential Building Blocks of Dark Energy," *JCAP* **1308** (2013) 025, 1304.4840.

[GLV] J. Gleyzes, D. Langlois, and F. Vernizzi, "A unifying description of dark energy," *Int.J.Mod.Phys.* **D23** (2014) 3010, 1411.3712.

[GLPV2] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Healthy theories beyond Horndeski," *Phys.Rev.Lett.* **114** (2015), no. 21 211101, 1404.6495.

[GLPV3] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Exploring gravitational theories beyond Horndeski," *JCAP* **1502** (2015), no. 02 018, 1408.1952.

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[CGSV2] P. Creminelli, J. Gleyzes, L. Hui, M. Simonović, and F. Vernizzi, "Single-Field Consistency Relations of Large Scale Structure. Part III: Test of the Equivalence Principle," *JCAP* **1406** (2014) 009, 1312.6074.

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Article A

Essential Building Blocks of Dark Energy

Essential Building Blocks of Dark Energy

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Abstract

We propose a minimal description of single field dark energy/modified gravity within the effective field theory formalism for cosmological perturbations, which encompasses most existing models. We start from a generic Lagrangian given as an arbitrary function of the lapse and of the extrinsic and intrinsic curvature tensors of the time hypersurfaces in unitary gauge, i.e. choosing as time slicing the uniform scalar field hypersurfaces. Focusing on linear perturbations, we identify seven Lagrangian operators that lead to equations of motion containing at most two (space or time) derivatives, the background evolution being determined by the time dependent coefficients of only three of these operators. We then establish a dictionary that translates any existing or future model whose Lagrangian can be written in the above form into our parametrized framework. As an illustration, we study Horndeski's—or generalized Galileon—theories and show that they can be described, up to linear order, by only six of the seven operators mentioned above. This implies, remarkably, that the dynamics of linear perturbations can be more general than that of Horndeski while remaining second order. Finally, in order to make the link with observations, we provide the entire set of linear perturbation equations in Newtonian gauge, the effective Newton constant in the quasi-static approximation and the ratio of the two gravitational potentials, in terms of the time-dependent coefficients of our Lagrangian.

Contents

1 Introduction

Dark energy has now become a generic name that includes a huge number of models trying to account for the present cosmic acceleration [1, 2]. Given their proliferation, the confrontation of such models with present and future cosmological data would be greatly facilitated by an effective approach that can mediate between observational data and theory. Ideally, such a phenomenological approach would provide an effective parameterisation that minimizes the number of free functions and deals directly with the relevant low-energy degrees of freedom, which in our context are the cosmological perturbations (together with the background evolution). A precise dictionary rephrasing the various models into this common language would then simplify the confrontation with the data and point out possible degeneracies between different theories. Within its unifying picture, this effective approach should have the extra virtue of stimulating theorists to study previously unexplored regions of the parameter space which could lead to interesting new models or, conversely, to better understanding why certain regions might be forbidden.

A few steps in this direction have been undertaken recently. The so-called effective field theory (EFT) of cosmological perturbations is a powerful tool that allows to deal directly with the relevant low-energy degrees of freedom of the problem at hand. Such an approach was proposed and intensively used for inflation [3, 4], in particular to characterize high-energy corrections to slow-roll models and to predict high-order correlation functions (see e.g. [5, 6]). The EFT of inflation has now become a standard way of parametrising primordial non-Gaussianity and was used, for instance, in the interpretation of the most recent WMAP [7] and Planck [8] data. This approach has also been applied to dark energy, first in the minimally-coupled case [9] where it was proven a useful tool to study the stability in full generality and, for models with vanishing sound speed, the clustering of dark energy down to very nonlinear scales [10].

More recently, the EFT formalism has been extended to dark energy with non-minimal couplings [11, 12], providing a unifying theoretical framework for practically all single-field dark energy and modified gravity models.[1] This approach relies on two basic steps [11]: *a)* assume the weak equivalence principle and therefore the existence of a metric $g_{\mu\nu}$ universally coupled to all matter fields (it is straightforward to relax this assumption, but at the price of complicating the formalism); *b)* write the *unitary gauge* action, i.e. the most general gravitational action for such a metric compatible with the (unbroken) spatial diffeomorphism invariance on hypersurfaces of constant dark energy field.

In [11] it was argued that the EFT of dark energy has all the virtues advocated at the beginning of this section. The goal of this article is to provide a systematic procedure to translate an arbitrary dark energy model into the EFT language, as well as to establish a firm minimal setting of Lagrangian operators within this framework. In particular, here we focus our attention on the operators of the unitary gauge action that lead to at most two derivatives in the equations of motion for linear perturbations.[2] This minimal set of operators encompasses most of the theoretical models of dark energy and/or modified gravity discussed in the current literature.

The key ingredient of our derivation is a $3+1$ decomposition *à la* ADM, where time slicings coincide with the uniform scalar field hypersurfaces. With this time choice, the dynamics of the underlying degree of freedom is embodied in the dynamics of the 3-dimensional metric. In Sec. 2 we consider a generic Lagrangian given as an arbitrary function of the lapse $N \equiv 1/\sqrt{-g^{00}}$ and of the 3-dimensional metric $h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$, where n^μ is the unit vector perpendicular to constant time hypersurfaces, more specifically of its extrinsic and intrinsic curvature tensors, respectively $K_\nu{}^\mu \equiv h_\nu{}^\rho \nabla_\rho n^\mu$ and ${}^{(3)}\!R^\mu{}_\nu$,

$$S = \int d^4x \sqrt{-g}\, L(N, K^\mu{}_\mu, K_{\mu\nu}K^{\mu\nu}, {}^{(3)}\!R, {}^{(3)}\!R_{\mu\nu}{}^{(3)}\!R^{\mu\nu}, \ldots; t) \,. \tag{1}$$

In our construction we include combinations of these 3-dimensional objects *without* taking their derivatives. This automatically prevents the appearance of higher (more than two) *time* derivatives in the equations of motion. However, it is not enough to also remove higher *spatial* derivatives. By expanding this Lagrangian up to quadratic order in the cosmological perturbations and making use of an ADM analysis in unitary gauge (see for instance [3, 19, 20]) we obtain specific conditions that ensure the absence of higher spatial derivatives in Sec. 2.2.

Moreover, we also show how the parameters in front of the standard EFT operators of [11] can be expressed in terms of the time-dependent coefficients of the expansion of (1). Since the action of most of the existing theoretical models can be written as eq. (1), this can be used to derive a dictionary between theoretical models and our EFT language. As an illustration, in Sec. 3 we explicitly derive this dictionary for the most general scalar field theory leading to at most second order equations of motion, i.e. the Horndeski theory [21] (see also [22]), recently rediscovered in the context of the so-called Galileon field [23, 24] under the name of "generalized Galileons" [25, 26].

Let us summarize here the main results of Secs. 2 and 3:

[1] An alternative formulation of a background independent effective approach to dark energy and modified gravity was given in [13]. A covariant EFT of cosmological acceleration was developed in [14] for inflation and generalized to the case of dark energy in [15, 16]—see discussion in [11] for a comparison between the latter approach and the one advocated here. For a different unifying framework to cosmological perturbations for dark energy and modified gravity see, for instance, [17].

[2] This is sufficient to ensure that we only have a single propagating degree of freedom. Note, however, that higher time derivatives do not lead to higher degrees of freedom if they can be treated perturbatively, i.e. evaluating them using the lower order equations of motion [18].

- The most general EFT action, up to quadratic order, for single-field dark energy, in the Jordan frame, leading to at most second-order equations of motion for *linear* perturbations can be written as

$$
\begin{aligned}
S = \int d^4x \sqrt{-g} \, \bigg[& \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \, \delta K \delta g^{00} \\
& - m_4^2(t) \left(\delta K^2 - \delta K^\mu_{\ \nu} \, \delta K^\nu_{\ \mu} \right) + \frac{\tilde m_4^2(t)}{2} \, {}^{(3)}\!R \, \delta g^{00} \bigg] \,,
\end{aligned}
\tag{2}
$$

where $\delta g^{00} \equiv g^{00}+1$, $\delta K_{\mu\nu} \equiv K_{\mu\nu} - H h_{\mu\nu}$, $K \equiv K^\mu_{\ \mu}$ and we have assumed a flat Universe so that ${}^{(3)}\!R^\mu_{\ \nu}$ vanishes on the background[3]. This action describes the propagation of one scalar degree of freedom with dispersion relation $\omega^2 = c_s^2 k^2$, where c_s is the sound speed of fluctuations given by eq. (35) with the relations (47). Stability (absence of ghosts) is ensured by the positivity of the time kinetic term given in (47). The particular combination appearing in the operator proportional to m_4^2 is such that it does not lead to higher-order spatial derivatives. One can check that also the combination

$$
{}^{(3)}\!R^\mu_{\ \nu} \, \delta K^\nu_{\ \mu} - \frac{1}{2} {}^{(3)}\!R \, \delta K
\tag{3}
$$

does not generate higher derivatives. However, this operator is not explicitly included in eq. (2) because it can be reexpressed in terms of the others (see App. A).

- In the particular case where $m_4^2 = \tilde m_4^2$, the above action is equivalent to the *linearized* Horndeski's theory/generalized Galileons and the explicit dictionary between generalized Galileons and this action is given in App. C. This implies that the dynamics of linear scalar perturbations of action (2) is more general than that of Horndeski, while remaining second order in time and space derivatives.

- Expanding the Lagrangian (1) up to quadratic order we also find three operators that lead to higher order space—but not time—derivatives. These are

$$
S_{\rm h.s.d.} = \int d^4x \sqrt{-g} \left[- \bar m_4^2(t) \, \delta K^2 + \frac{\bar m_5(t)}{2} \, {}^{(3)}\!R \, \delta K + \frac{\bar\lambda(t)}{2} {}^{(3)}\!R^2 \right] \,.
\tag{4}
$$

When one of these operators is present in the action the dispersion relation of the propagating mode receives corrections at large momenta, $\omega^2 = c_s^2 k^2 + k^4/M^2$, where M is a mass scale. These corrections may become important in the limit of vanishing sound speed, such as in the model of the Ghost Condensate [27] or for deformations of this particular limit [3, 9].

Once a Lagrangian describing matter has been included, the action (2) can be used as a benchmark for the study of physical signatures of dark energy/modified gravity in the linear regime. In this context, cosmological perturbations are usually discussed in Newtonian gauge, which is the one that we employ in Sec. 4. In order to do that, in Sec. 4.1 we restore the covariance via the Stueckelberg trick [3, 4, 27] and we vary this action with respect to all the scalar dynamical degrees of freedom. This allows to derive Einstein's equations and the evolution equation for the fluctuations of the scalar field responsible of the acceleration, recovering and generalizing previous results [9, 11]. Interestingly, in the Newtonian gauge Einstein's equations and the scalar equation of motion contain higher order derivatives when $m_4^2 \neq \tilde m_4^2$, even if the dynamical equation for the true degree of freedom is only second order. Finally, in Sec. 4.2 we use these equations to derive the effective Newton constant and the ratio between the two gravitational potentials.

[3] The case of non-vanishing spatial curvature is commented on in footnote 7.

The phenomenology of the operators appearing in action (2) was also studied in [12]. In this reference it was indeed mentioned that these operators are sufficient to describe linear perturbations of Horndeski's theories. However, no proof of this statement was given nor the particular combination in which such operators appear shown.

2 General Lagrangian in unitary gauge

In the presence of a scalar field ϕ with a non-vanishing timelike gradient, the so-called unitary gauge corresponds to a choice of time slicing where the constant time hypersurfaces are uniform ϕ hypersurfaces. The use of unitary gauge accomplishes two main objectives. First, as explained at length in Refs. [3, 4, 27], it makes it straightforward to write a generic Lagrangian for cosmological perturbations. Since the dynamics of the scalar field has been "eaten" by the metric, the most generic Lagrangian is simply that for the *metric perturbations* around a FLRW solution, compatible with the unbroken symmetry of 3-dimensional diffeomorphisms.

Second, the 3+1 splitting in unitary gauge easily allows to keep the number of time derivatives under control, while considering higher and higher space derivatives. Therefore, the unitary gauge is helpful to systematically explore the space of higher spatial derivative theories by considering geometric invariants on the $\phi =$ *constant* hypersurfaces. In practice, we will use the metric in the ADM form

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) , \tag{5}$$

where the 3-dimensional metric h_{ij} is used to lower and raise latin indices $i, j, \dots = 1, 2, 3$. Since 3-dimensional diffeomorphism invariance is preserved in unitary gauge, it is natural to write operators (with up to two spatial derivatives per field) in terms of the extrinsic and intrinsic curvatures $K_{\mu\nu}$ and ${}^{(3)}\!R_{\mu\nu}$ and their possible contractions. The Lagrangian is also an explicit function of the lapse function N in general.

Therefore, in the following, we consider a general action of the form

$$S = \int d^4x \sqrt{-g}\, L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}; t) , \tag{6}$$

where the Lagrangian L is an arbitrary function of N and of the following four scalar quantities constructed by contracting the extrinsic and intrinsic curvature tensors:

$$K \equiv K^\mu{}_\mu , \qquad \mathcal{R} \equiv {}^{(3)}\!R \equiv {}^{(3)}\!R^\mu{}_\mu , \qquad \mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu}, \quad \mathcal{Z} \equiv {}^{(3)}\!R_{\mu\nu}{}^{(3)}\!R^{\mu\nu} . \tag{7}$$

Although one should also allow, in principle, for a dependence on $\mathcal{Y} \equiv {}^{(3)}\!R_{\mu\nu}K^{\mu\nu}$, we have preferred not to include it explicitly in the main body, for simplicity. As shown in App. A, this extra dependence leads to a quadratic Lagrangian of the same form as that found later in this section, with slightly modified coefficients. Indeed, since $\mathcal{Y}$ is equivalent to $H\mathcal{R}$ at linear order, the quadratic terms in the expansion of the Lagrangian induced by its dependence on $\mathcal{Y}$ are analogous to those induced by its dependence on $\mathcal{R}$. As for the linear term, one can use the equality

$$\lambda(t){}^{(3)}\!R_{\mu\nu}K^{\mu\nu} \;=\; \frac{\lambda(t)}{2}{}^{(3)}\!R\, K \;+\; \frac{\dot\lambda(t)}{2N}\,{}^{(3)}\!R \;+ \text{boundary terms} , \tag{8}$$

which is also shown in App. A.

Moreover, one could also consider scalars that are combinations of three or more tensors, like $K^\lambda{}_\mu K^\mu{}_\nu K^\nu{}_\lambda$, but it is easy to show that also in this case they can be re-expressed in terms of the above combinations, plus corrections which are at least cubic in the perturbations. We will show this explicitly for the extended Galileon in the next section. Finally, one could take quadratic combinations of the Riemann tensor such as ${}^{(3)}\!R_{\mu\nu\rho\sigma}{}^{(3)}\!R^{\mu\nu\rho\sigma}$. However, in three dimensions the

Riemann tensor can be expressed in terms of the Ricci scalar and tensor.[4] Thus, at quadratic order in the perturbations, the action above seems to exhaust all the possibilities compatible with our requirements.

In order to explicitly write the expansion of the action (6) up to second order in the perturbations, it is useful to define the tensors

$$\delta K \equiv K - 3H, \qquad \delta K_{\mu\nu} \equiv K_{\mu\nu} - H h_{\mu\nu} \,, \tag{10}$$

which vanish on the background, and to use the decompositions

$$\mathcal{S} = 3H^2 + \delta\mathcal{S}, \qquad \delta\mathcal{S} \equiv 2H\delta K + \delta K^\mu{}_\nu \delta K^\nu{}_\mu \,. \tag{11}$$

The quantities $\mathcal{R}$ and $\mathcal{Z}$ vanish on the background and are therefore already perturbative ($\mathcal{Z}$ is even a second order quantity).

The expansion of the Lagrangian up to second order in the perturbations yields, after discarding irrelevant boundary terms, the expression

$$\begin{aligned} L(N,K,\mathcal{S},\mathcal{R},\mathcal{Z}) = {} & \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N)\,\delta N + L_{\mathcal{R}}\,\delta\mathcal{R} \\ & + \frac{\mathcal{A}}{2}\,\delta K^2 + L_{\mathcal{S}}\,\delta K^\mu{}_\nu \delta K^\nu{}_\mu + \left(\frac{1}{2}L_{NN} - \dot{\mathcal{F}}\right)\delta N^2 \\ & + \frac{1}{2}L_{\mathcal{R}\mathcal{R}}\,\delta\mathcal{R}^2 + \mathcal{B}\,\delta K\delta N + \mathcal{C}\,\delta K\delta\mathcal{R} + L_{N\mathcal{R}}\,\delta N\delta\mathcal{R} + L_{\mathcal{Z}}\delta\mathcal{Z} + \mathcal{O}(3)\,, \end{aligned} \tag{12}$$

where we have introduced the following notations for some combinations of the partial derivatives of the Lagrangian (denoting $L_N \equiv \partial L/\partial N$, etc.), to make this expression more compact:

$$\begin{aligned} \mathcal{F} &\equiv 2HL_{\mathcal{S}} + L_K \,, \\ \mathcal{A} &\equiv 4H^2 L_{\mathcal{S}\mathcal{S}} + 4HL_{\mathcal{S}K} + L_{KK} \,, \\ \mathcal{B} &\equiv 2HL_{\mathcal{S}N} + L_{KN} \,, \\ \mathcal{C} &\equiv 2HL_{\mathcal{S}\mathcal{R}} + L_{K\mathcal{R}} \,. \end{aligned} \tag{13}$$

The first term, $\bar{L}$, is the homogeneous Lagrangian and all partial derivatives of L that appear in the above expression are evaluated on the homogeneous background, i.e. for $\bar{N} = 1$, $\bar{\mathcal{S}} = 3H^2$, $\bar{K} = 3H$, $\bar{\mathcal{R}} = 0$ and $\bar{\mathcal{Z}} = 0$. Note that, in order to obtain the expression (12), we have rewritten the term linear in δK as

$$\mathcal{F}\delta K = \mathcal{F}(K - 3H) \,, \tag{14}$$

and integrated it by parts using $K = \nabla_\mu n^\mu$,

$$\int d^4x\sqrt{-g}\,\mathcal{F}K = -\int d^4x\sqrt{-g}\,n^\mu\nabla_\mu\mathcal{F} = -\int d^4x\sqrt{-g}\frac{\dot{\mathcal{F}}}{N} \,, \tag{15}$$

where n^μ is the unit vector orthogonal to constant time hypersurfaces and, in unitary gauge, has time component $n^0 = 1/N$.

[4] This can be done using the relation

$$ {}^{(3)}R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\mu\rho}h_{\nu\sigma} - {}^{(3)}R_{\nu\rho}h_{\mu\sigma} - {}^{(3)}R_{\mu\sigma}g_{\nu\rho} + {}^{(3)}R_{\nu\sigma}h_{\mu\rho} - \frac{1}{2}{}^{(3)}R(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho}) \,. \tag{9}$$

2.1 Background equations

For the background we assume a flat homogeneous FLRW metric, written in the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,. \tag{16}$$

In this case $K = 3H/N$ and $\mathcal{S} = 3H^2/N^2$, where $H \equiv \dot{a}/a$ is the Hubble rate. Note that it is crucial to explicitly keep the lapse function N, because the first Friedmann equation is the constraint arising from the invariance under time reparametrization. Linear variation of the homogeneous action S_0 with respect to the lapse N and the scale factor a yields

$$\delta S_0 = \int dt d^3x \left[a^3 \left(\bar{L} + L_N - 3H\mathcal{F}\right)\delta N + 3a^2\left(\bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}}\right)\delta a\right] \,, \tag{17}$$

where we have used $\sqrt{-g} = a^3 N$. Then the Friedmann equations are directly given by[5]

$$3H\mathcal{F} - \bar{L} - L_N = 0, \tag{18}$$

which depends on first time derivatives at most, and

$$\dot{\mathcal{F}} + 3H\mathcal{F} - \bar{L} = 0\,, \tag{19}$$

which determines the dynamics of the scale factor.

As expected, by using the above equations one can check that the first order of the total Lagrangian $\mathcal{L} \equiv \sqrt{-g}\,L$ vanishes. Indeed, using $\sqrt{-g} = \sqrt{h}\,N$, where h is the determinant of the 3-dimensional metric h_{ij} in the ADM decomposition, one easily finds

$$\mathcal{L}_1 = \left(\bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}}\right)\delta\sqrt{h} + a^3(L_N + \bar{L} - 3H\mathcal{F})\delta N + a^3 L_{\mathcal{R}}\,\delta\mathcal{R}\,, \tag{20}$$

where the last term is a total derivative and can be ignored.

2.2 Perturbations in the ADM formalism

In this sub-section we perform the analysis of the perturbations in unitary gauge and by using the ADM form of the metric, eq. (5). For the action at second order, we will only need to take into account the perturbations of $\sqrt{-g}$ at first order, $\delta\sqrt{-g} = \delta\sqrt{h} + a^3\delta N$, because the second order one multiplies the LHS of eq. (19). Thus, the quadratic Lagrangian for perturbations is given by

$$\begin{aligned}\mathcal{L}_2 = {}& \delta\sqrt{h}\left[(\dot{\mathcal{F}} + L_N)\delta N + L_{\mathcal{R}}\,\delta\mathcal{R}\right] \\ & + a^3\Bigg[\left(L_N + \frac{1}{2}L_{NN}\right)\delta N^2 + L_{\mathcal{R}}\delta_2\mathcal{R} + \frac{1}{2}\mathcal{A}\,\delta K^2 + \mathcal{B}\,\delta K\delta N + \mathcal{C}\,\delta K\delta\mathcal{R} \\ & + L_{\mathcal{S}}\,\delta K^\mu_\nu\,\delta K^\nu_\mu + L_{\mathcal{Z}}\,\delta\mathcal{R}^\mu_\nu\,\delta\mathcal{R}^\nu_\mu + \frac{1}{2}L_{\mathcal{R}\mathcal{R}}\,\delta\mathcal{R}^2 + (L_{\mathcal{R}} + L_{N\mathcal{R}})\,\delta N\delta\mathcal{R}\Bigg]\,,\end{aligned} \tag{21}$$

where $\delta_2\mathcal{R}$ denotes the expansion of $\mathcal{R}$ at second order in the perturbations.

In the ADM decomposition (5) the only relevant components of the extrinsic curvature tensor are given by

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)\,, \tag{22}$$

where ∇_i stands for the covariant derivative associated with the 3-dimensional metric h_{ij}. For explicit calculations in unitary gauge we choose to describe scalar perturbations of the spatial metric in terms of ζ [19],

$$h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}\,. \tag{23}$$

[5] We have not included explicitly the matter in the Friedmann equations, but it is straightforward to do so.

(We consider the tensor modes separately in App. B.) Thus, the perturbations of the quantities used above are given by

$$\delta\sqrt{h} = 3a^3\zeta\,, \qquad \delta K^i{}_j = \left(\dot\zeta - H\delta N\right)\delta^i_j - \frac{1}{a^2}\delta^{ik}\partial_{(k}N_{j)}\,, \tag{24}$$

and

$$\delta\mathcal{R}_{ij} = -\delta_{ij}\partial^2\zeta - \partial_i\partial_j\zeta\,, \qquad \delta_2\mathcal{R} = -\frac{2}{a^2}\left[(\partial\zeta)^2 - 4\zeta\partial^2\zeta\right]\,. \tag{25}$$

Note that in this section ∂ stands for a spatial derivative and $\partial^2 \equiv \partial_i\partial^i$. By using the above expressions, the variation of $\mathcal{L}_2$ with respect to δN yields the Hamiltonian constraint, which reads

$$\begin{aligned}&\left[L_{NN} + 2L_N + 3H\left(3H\mathcal{A} + 2HL_{\mathcal{S}} - 2\mathcal{B}\right)\right]\delta N + 3\left(\mathcal{B} - 3H\mathcal{A} - 2HL_{\mathcal{S}}\right)\dot\zeta + 3(L_N + \dot{\mathcal{F}})\zeta \\ &- \left(\mathcal{B} - 3H\mathcal{A} - 2HL_{\mathcal{S}}\right)\frac{\partial^2\psi}{a^2} - 4\left(L_{\mathcal{R}} + L_{N\mathcal{R}} - 3H\mathcal{C}\right)\frac{\partial^2\zeta}{a^2} = 0\,.\end{aligned} \tag{26}$$

By varying $\mathcal{L}_2$ with respect to the shift

$$N_i \equiv \partial_i\psi\,, \tag{27}$$

one obtains the momentum constraint, which implies

$$-\left(\mathcal{B} - 3H\mathcal{A} - 2HL_{\mathcal{S}}\right)\delta N + \left(\mathcal{A} + 2L_{\mathcal{S}}\right)\frac{\partial^2\psi}{a^2} = \left(3\mathcal{A} + 2L_{\mathcal{S}}\right)\dot\zeta - 4\mathcal{C}\frac{\partial^2\zeta}{a^2}\,. \tag{28}$$

By combining the two constraints, one can express both δN and $\partial^2\psi$ as functions of ζ and its derivatives and then substitute in the action to write it only in terms of ζ and its derivatives. In general, a term proportional to $(\partial^2\zeta)^2$ will remain. Here, in order to single out the lowest derivatives operators first, we want to find conditions under which this term disappears. If one considers the second order action before the substitution of the constraints, one finds the following terms

$$\frac{1}{a}\left[\frac{1}{2}\left(\mathcal{A} + 2L_{\mathcal{S}}\right)(\partial^2\psi)^2 + 4\mathcal{C}\,\partial^2\psi\,\partial^2\zeta + 2\left(4L_{\mathcal{R}\mathcal{R}} + 3L_{\mathcal{Z}}\right)(\partial^2\zeta)^2\right]\,. \tag{29}$$

Taking into account the momentum constraint (28), one immediately sees that imposing the three conditions[6]

$$\mathcal{A} + 2L_{\mathcal{S}} = 0\,, \qquad \mathcal{C} = 0\,, \qquad 4L_{\mathcal{R}\mathcal{R}} + 3L_{\mathcal{Z}} = 0\,, \tag{30}$$

implies the elimination of the term proportional to $(\partial^2\zeta)^2$ in the final action and the absence of higher derivatives in the equation of motion for ζ. As we will see in the next section, all generalized Galileon models satisfy the three conditions (30).

When (30) are satisfied, the momentum constraint reduces to

$$\delta N = \mathcal{D}\dot\zeta\,, \qquad \mathcal{D} \equiv \frac{4L_{\mathcal{S}}}{\mathcal{B} + 4HL_{\mathcal{S}}}\,. \tag{31}$$

By direct substitution into $\mathcal{L}_2$ and after an integration by parts to get rid of the term $\dot\zeta\partial^2\zeta$ (note that the $\zeta\dot\zeta$ term vanishes because of the background equations of motion), we finally get the following Lagrangian for ζ:

$$\mathcal{L}_2 = \frac{a^3}{2}\left[\mathcal{L}_{\dot\zeta\dot\zeta}\,\dot\zeta^2 + \mathcal{L}_{\partial_i\zeta\partial_i\zeta}\frac{(\partial_i\zeta)^2}{a^2}\right]\,, \tag{32}$$

[6] Note that these conditions are only sufficient. A more general analysis can be performed by explicitly requiring that the coefficient of $(\partial^2\zeta)^2$ vanishes once the two constraints have been solved. However, this leads to a very complicated equation involving many of the coefficients of the quadratic Lagrangian and it is not clear whether one can find physically relevant solutions that evade (30).

with

$$\mathcal{L}_{\dot\zeta\dot\zeta} \equiv 2\left(\frac{1}{2}L_{NN} + L_N - 3H\mathcal{B} - 6H^2 L_{\mathcal{S}}\right)\mathcal{D}^2 + 12L_{\mathcal{S}} \,,$$
$$\mathcal{L}_{\partial_i\zeta\partial_i\zeta} \equiv 4\left[L_{\mathcal{R}} - \frac{1}{a}\frac{d}{dt}(a\mathcal{M})\right] \,, \qquad \mathcal{M} \equiv \mathcal{D}(L_{\mathcal{R}} + L_{N\mathcal{R}})\,. \tag{33}$$

Classical and quantum stability (absence of ghosts) requires that the time kinetic energy is positive (see, e.g. [3,9]),

$$\mathcal{L}_{\dot\zeta\dot\zeta} > 0\,. \tag{34}$$

The sound speed (squared) of fluctuations can be simply computed by taking the ratio

$$c_s^2 = -\frac{\mathcal{L}_{\partial_i\zeta\partial_i\zeta}}{\mathcal{L}_{\dot\zeta\dot\zeta}}\,. \tag{35}$$

2.3 The EFT language

We are now going to express the conditions on the absence of higher derivatives in terms of the coefficients of the action of the EFT formalism of Refs. [11,12]. The action up to quadratic order in the perturbations can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_*^2}{2}fR - \Lambda - cg^{00} + \frac{M_2^4}{2}(\delta g^{00})^2 - \frac{\bar m_1^3}{2}\delta K\delta g^{00}\right.$$
$$\left. - \frac{\bar M_2^2}{2}\delta K^2 - \frac{\bar M_3^2}{2}\delta K^\mu{}_\nu\,\delta K_\mu{}^\nu + \frac{\mu_1^2}{2}{}^{(3)}\!R\delta g^{00} + \frac{\bar m_5}{2}{}^{(3)}\!R\delta K + \frac{\lambda_1}{2}{}^{(3)}\!R^2 + \frac{\lambda_2}{2}{}^{(3)}\!R^\mu{}_\nu\,{}^{(3)}\!R_\mu{}^\nu\right], \tag{36}$$

where R in the first term inside the bracket is the four-dimensional Ricci scalar. Note that, in order to make the comparison with the previous subsection simpler, we have found more convenient to use the 3-dimensional Ricci scalar and tensor in the quadratic terms, instead of the four-dimensional ones used in Ref. [11], since the link with the ADM decomposition is then transparent.

Let us first discuss how the background equations (18) and (19) translate in this language. In action (36) we have used the time-time component of the inverse metric g^{00} and its perturbation in the expansion of quadratic and higher order operators, as it is customary in the EFT formalism. However, in the previous subsections it was more convenient to work directly with the lapse function N, related to g^{00} by

$$g^{00} = -\frac{1}{N^2}\,. \tag{37}$$

Only the first three terms in brackets in eq. (36) contribute to $\bar L$, L_N and $\mathcal{F}$, and thus to the background equations of motion. Using eq. (37) and employing the decomposition of the four-dimensional curvature scalar,

$$R = {}^{(3)}\!R + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\nu(n^\nu\nabla_\mu n^\mu - n^\mu\nabla_\mu n^\nu)\,, \tag{38}$$

after an integration by parts in the action we can rewrite these terms as

$$L_0 = \frac{M_*^2}{2}\left(f\mathcal{R} + f\mathcal{S} - fK^2 - 2\dot f\frac{K}{N}\right) - \Lambda + \frac{c}{N^2}\,, \tag{39}$$

(we remind the reader that $\mathcal{R} \equiv {}^{(3)}\!R$). By expanding at linear order in δN, integrating by parts the terms linear in K, we can match the background and linear terms of this action with the first line of eq. (12), which yields

$$\bar L - \dot{\mathcal{F}} - 3H\mathcal{F} = M_*^2(3fH^2 + 2f\dot H + 2\dot f H + \ddot f) + c - \Lambda\,,$$
$$\dot{\mathcal{F}} + L_N = M_*^2(\dot f H - 2f\dot H - \ddot f) - 2c\,. \tag{40}$$

From these two relations and using the background equations of motion (18) and (19) one finds that c and Λ are given by

$$c + \Lambda = 3M_*^2 \left(fH^2 + \dot{f}H \right) , \tag{41}$$

$$\Lambda - c = M_*^2 \left(2f\dot{H} + 3fH^2 + 2\dot{f}H + \ddot{f} \right) . \tag{42}$$

This coincides with what was found in Ref. [11] in the absence of matter.

To discuss linear perturbations we only need the second-order expansion of the action (36). By rewriting the first three terms as in eq. (39), expressing g^{00} in terms of N and using the definitions (10) and (11), one immediately sees that the EFT action is of the form (6). One can thus use the second-order expansion of the Lagrangian (21) with the following dictionary:

$$\begin{aligned}
L_{\mathcal{R}} &= \frac{1}{2} M_*^2 f \, , \\
\frac{1}{2} L_{NN} + L_N &= c + 2M_2^4 \, , \\
\mathcal{A} &= -M_*^2 f - \bar{M}_2^2 \, , \\
\mathcal{B} &= \dot{f} M_*^2 - \bar{m}_1^3 \, , \\
\mathcal{C} &= \frac{\bar{m}_5}{2} \, , \\
L_{\mathcal{S}} &= \frac{1}{2} \left(M_*^2 f - \bar{M}_3^2 \right) \, , \\
L_{\mathcal{Z}} &= \frac{\lambda_2}{2} \, , \\
L_{N\mathcal{R}} &= \mu_1^2 \, , \\
L_{\mathcal{R}\mathcal{R}} &= \lambda_1 \, ,
\end{aligned} \tag{43}$$

which is completed with eq. (40).

With these relations, the conditions for the absence of higher derivatives, eq. (30), can be written in the EFT of dark energy language. They read:

$$\bar{M}_2^2 + \bar{M}_3^2 = 0 \, , \qquad \bar{m}_5 = 0 \, , \qquad 4\lambda_1 + \frac{3}{2}\lambda_2 = 0 \, . \tag{44}$$

These conditions are straightforward to verify. Using eqs. (24) and (27), δK^2 contains a higher derivative term, $(\partial^2 \psi)^2$, while $\delta K^\mu{}_\nu \delta K_\mu{}^\nu$ contains $(\partial_i \partial_j \psi)^2$. However, when the first condition in eq. (44) is satisfied the combination of higher derivative terms in eq. (36) gives an irrelevant boundary term. The second condition implies that the operator ${}^{(3)}\!R \, \delta K$, which contains $\partial^2 \psi \partial^2 \zeta$, does not appear. Finally, ${}^{(3)}\!R^2 = 16(\partial^2 \zeta)^2/a^4$ and ${}^{(3)}\!R_{ij} {}^{(3)}\!R^{ij} = [5(\partial^2 \zeta)^2 + (\partial_i \partial_j \zeta)^2]/a^4$: one can check that when the third condition is satisfied the sum of the two operators in eq. (36) vanishes up to a total derivative.

In summary, the most general EFT Lagrangian which does not generate higher derivatives in the linear equations for the perturbations is[7]

$$\begin{aligned}
L = \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \, \delta K \delta g^{00} \\
- \, m_4^2(t) \left(\delta K^2 - \delta K^\mu{}_\nu \, \delta K^\nu{}_\mu \right) + \frac{\tilde{m}_4^2(t)}{2} \, {}^{(3)}\!R \, \delta g^{00} \, ,
\end{aligned} \tag{45}$$

[7] Let us comment here on the case of a non-vanishing spatial curvature, to which our formalism can be extended straightforwardly with the following caveats. Obviously, $\delta \, {}^{(3)}\!R$ should be used instead of ${}^{(3)}\!R$ in the quadratic operators, but apart from this the Lagrangian (45) and its properties are unchanged. The background equations change (see e.g. eqs. 16 and 17 or Ref. [11]), as well as the dictionary (43), because some first order quantity will now contribute already at zeroth order. The explicit expressions (25) of ${}^{(3)}\!R$ change by a term linear in ζ but with no derivatives, which, therefore, will not produce higher derivatives in the ADM analysis of Sec 2.2.

where

$$m_3^3 \equiv \bar{m}_1^3 \,, \qquad m_4^2 \equiv \frac{1}{4}(\bar{M}_2^2 - \bar{M}_3^2) \,, \qquad \tilde{m}_4^2 \equiv \mu_1^2 \,, \tag{46}$$

as in eq. (2). Terms containing ${}^{(3)}\!R^2$ and ${}^{(3)}\!R^{\mu}_{\nu}\,{}^{(3)}\!R^{\nu}_{\mu}$ do not appear because they only contain higher spatial derivatives. By employing the dictionary (43) in eq. (33), the quadratic action for ζ is given by eq. (32) where

$$\begin{aligned}
\mathcal{L}_{\dot\zeta\dot\zeta} &= 2\left(c + 2M_2^4 - 3H^2M_*^2 f - 3HM_*^2\dot{f} + 3Hm_3^3 - 6H^2m_4^2\right)\mathcal{D}^2 + 6(M_*^2 f + 2m_4^2)\,, \\
\mathcal{L}_{\partial_i\zeta\partial_i\zeta} &= 2\left[M_*^2 f - \frac{2}{a}\frac{d}{dt}(a\mathcal{M})\right]\,,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\mathcal{D} &= \frac{2(M_*^2 f + 2m_4^2)}{2H(M_*^2 f + 2m_4^2) + M_*^2\dot{f} - m_3^3}\,, \\
\mathcal{M} &= \frac{\mathcal{D}}{2}(M_*^2 f + 2\tilde{m}_4^2)\,.
\end{aligned} \tag{48}$$

The stability of a given model is then determined by the condition $\mathcal{L}_{\dot\zeta\dot\zeta} > 0$ and the speed of sound can be straightforwardly computed from eq. (35) by using the relations above. One can check that these results agree with those found in [11] in the limit $m_4^2 = \tilde{m}_4^2 = 0$.

Finally, we can also write down the independent operators that generate higher spatial derivatives. These are

$$L_{\text{h.s.d.}} = -\,\bar{m}_4^2(t)\,\delta K^2 \;+\; \frac{\bar{m}_5(t)}{2}\,{}^{(3)}\!R\,\delta K + \frac{\bar\lambda(t)}{2}{}^{(3)}\!R^2\,. \tag{49}$$

We now turn to study a well known example of scalar-tensor theories of gravity which does not generate equations of motion with higher derivatives and, when restricting to linear perturbations, is contained in the Lagrangian (45).

3 Generalised Galileons

In four dimensions, the most general scalar tensor theory having field equations of second order in derivatives is a combination of the following generalized Galileon Lagrangians [21, 25, 26],

$$L_2 = G_2(\phi, X)\,, \tag{50}$$

$$L_3 = G_3(\phi, X)\Box\phi\,, \tag{51}$$

$$L_4 = G_4(\phi, X)R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{;\mu\nu}\phi_{;\mu\nu})\,, \tag{52}$$

$$L_5 = G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\,\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma})\,. \tag{53}$$

For notational convenience, in this section we mostly indicate covariant differentiation with a semi-colon symbol, i.e. $_{;\mu}$. Moreover, we have defined $X \equiv \phi^{;\mu}\phi_{;\mu}$ (note that X is sometimes defined differently, i.e., with a factor of $-1/2$).

In order to translate the above Lagrangians into our EFT language we will proceed in two steps. First, we will rewrite each of these Lagrangians in terms of 3-dimensional geometrical objects ($K^{\nu}{}_{\mu}$, ${}^{(3)}\!R^{\nu}{}_{\mu}$, etc.) so that their unitary gauge expression becomes easily readable. The $3+1$ decomposition that we are after loses manifest general covariance but shows straightforwardly the lack of higher *time* derivatives already at the level of the action. The second step will be to compute the corresponding coefficients of the operators (45) by simply inverting the dictionary that we derived in the previous section—eqs (40) and (43):

$$\begin{aligned}
c &= -\frac{1}{2}\left(\dot{\mathcal{F}} + L_N\right) + H\dot{L}_{\mathcal{R}} - 2L_{\mathcal{R}}\dot{H} - \ddot{L}_{\mathcal{R}} \ , \\
\Lambda &= -\bar{L} - \frac{1}{2}L_N + \frac{1}{2}\dot{\mathcal{F}} + 3H\mathcal{F} + 2\dot{H}L_{\mathcal{R}} + 6H^2 L_{\mathcal{R}} + 5H\dot{L}_{\mathcal{R}} + \ddot{L}_{\mathcal{R}} \ , \\
f &= 2L_{\mathcal{R}}M_*^{-2} \ , \\
M_2^4 &= \frac{1}{4}(L_{NN} + 3L_N + \dot{\mathcal{F}}) - \frac{1}{2}(H\dot{L}_{\mathcal{R}} - 2\dot{H}L_{\mathcal{R}} - \ddot{L}_{\mathcal{R}}) \ , \\
m_3^3 &= 2\dot{L}_{\mathcal{R}} - 2HL_{\mathcal{S}N} - L_{KN} = 2\dot{L}_{\mathcal{R}} - \mathcal{B} \ , \\
m_4^2 &= \frac{1}{2}\left(L_{\mathcal{S}} - 2L_{\mathcal{R}} - 2H^2 L_{\mathcal{S}\mathcal{S}} - 2HL_{\mathcal{S}K} - \frac{1}{2}L_{KK}\right) = \frac{1}{2}\left(L_{\mathcal{S}} - 2L_{\mathcal{R}}\right) - \frac{1}{4}\mathcal{A} \ , \\
\tilde{m}_4^2 &= L_{N\mathcal{R}} \ ,
\end{aligned} \tag{54}$$

where we have directly adopted the notation (46) which holds in absence of higher derivatives—more general relations are easily found when eq. (44) is not satisfied.

The main result of this section is that the dynamics of linear perturbations for all generalized Galileons is described by (45), with the further restriction $m_4^2 = \tilde{m}_4^2$. This is in agreement with the result [25,26] that also higher *space* derivatives are absent from the equations of motion. This section is rather technical; the reader uninterested in the details of the calculations can skip the following subsections and go directly to Sec. 3.5 where we summarize our main results.

3.1 Geometric preliminaries

In order to express in unitary gauge terms of increasing complexity, it is useful to review the geometric formalism adapted to the $3+1$ decomposition and separate the quantities into "orthogonal" and "parallel" to the hypersurface $\phi = const.$ First, we define the future directed unitary vector orthogonal to the hypersurface. Up to a factor γ, this is proportional to the gradient of ϕ ,

$$n_\mu = -\gamma\,\phi_{;\mu}, \qquad \gamma = \frac{1}{\sqrt{-X}} \ . \tag{55}$$

The metric induced on the $\phi = const.$ hypersurface is $h_{\mu\nu} = n_\mu n_\nu + g_{\mu\nu}$. Orthogonal to n_μ are also various quantities that "live" on the hypersurface, in the sense that they vanish when contracted with n_μ: the extrinsic curvature and the "acceleration" vector

$$K_{\mu\nu} = h_\mu^\sigma\, n_{\nu;\sigma}, \qquad \dot{n}_\mu = n^\nu\, n_{\mu;\nu} \ . \tag{56}$$

The last two equations can be inverted by decomposing the derivative of n_μ into parallel and parallel/orthogonal components,

$$n_{\nu;\mu} = K_{\mu\nu} - n_\mu \dot{n}_\nu \ . \tag{57}$$

By means of the quantities just defined, we can decompose the second derivative of the scalar field as

$$\phi_{;\mu\nu} = -\gamma^{-1}(K_{\mu\nu} - n_\mu\dot{n}_\nu - n_\nu\dot{n}_\mu) + \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}n_\mu n_\nu \ . \tag{58}$$

Again, this decomposition into parallel and orthogonal quantities is useful when calculating complicated products such $\phi_{;\mu\nu}\,\phi^{;\nu\sigma}\,\phi^{;\mu}{}_{;\sigma}$ that appear in L_5, see eq. (53).

Other relevant equations are the Gauss-Codazzi equations, relating the Ricci tensor and scalar intrinsic to the hypersurface, ${}^{(3)}\!R_{\mu\nu}$ and ${}^{(3)}\!R$, to the four-dimensional ones [28,29],

$${}^{(3)}\!R_{\mu\nu} = (R_{\mu\nu})_\parallel + (n^\sigma n^\rho R_{\mu\sigma\nu\rho})_\parallel - KK_{\mu\nu} + K_{\mu\sigma}K^\sigma{}_\nu, \tag{59}$$

$${}^{(3)}\!R = R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2(Kn^\mu - \dot{n}^\mu)_{;\mu} \ , \tag{60}$$

where the symbol $\parallel$ means projection on the hypersurface of all tensor indices, e.g. $(V_\mu)_\parallel \equiv h_\mu{}^\nu V_\nu$.

3.2 L_3

Since L_2 is trivial, following [11] we start from L_3 and see how to rewrite it in the EFT of dark energy formalism. First, it is convenient to define an auxiliary function $F_3(\phi, X)$ such that

$$G_3 \equiv F_3 + 2XF_{3X} \ . \tag{61}$$

Thus, L_3 in eq. (51) can be written as

$$L_3 = F_3 \Box\phi + 2XF_{3X}\Box\phi \ . \tag{62}$$

We integrate by parts the first term on the right-hand side and we rewrite the second term using $\Box\phi = -\gamma^{-1}K + \frac{1}{2}\phi^{;\mu}X_{;\mu}/X$, which is obtained by tracing eq. (58). This yields

$$L_3 = -(F_{3X}X_{;\mu} + F_{3\phi}\,\phi_{;\mu})\phi^{;\mu} - 2X\gamma^{-1}F_{3X}K + F_{3X}X_{;\mu}\phi^{;\mu} \ . \tag{63}$$

After noticing that the first term inside the parenthesis cancels with the last one we finally obtain an expression for L_3 which is of the form of eq. (12),

$$L_3 = 2(-X)^{3/2}F_{3X}K - XF_{3\phi} \ , \tag{64}$$

where we have used $\gamma = 1/\sqrt{-X}$. In unitary gauge $\phi(t, \vec{x}) = \phi_0(t)$, which implies, for instance,

$$F_{3X}(\phi, X) \to F_{3X}(\phi_0(t), -\dot{\phi}_0^2(t)/N^2) \ . \tag{65}$$

Using eq. (54), it is now straightforward to derive the corresponding EFT parameters in terms of the Lagrangian parameters evaluated on the background. They are explicitly given in App. C and coincide with those given in [11]. They only depend on four Lagrangian parameters, $G_{3\phi}$, G_{3X}, $G_{3X\phi}$ and G_{3XX}, so that the dependence on the auxiliary function F_3 disappears. As expected from eq. (64), f, m_4^2 and $\tilde{m}_4^2$ all vanish: in order to describe L_3 we only need c, Λ, M_2^4 and m_3^3.

3.3 L_4

We now proceed with L_4, defined in eq. (52). Using eq. (58) and its trace we can rewrite this as

$$\begin{aligned} L_4 &= G_4 R - 2G_{4X}\left[\left(\gamma^{-1}K + \frac{\gamma^2}{2}\phi^{;\mu}X_{;\mu}\right)^2 - \gamma^{-2}(K_{\mu\nu}K^{\mu\nu} - 2\dot{n}_\mu \dot{n}^\mu) - \frac{\gamma^4}{4}(\phi^{;\mu}X_{;\mu})^2\right] \\ &= G_4 R + 2XG_{4X}(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2G_{4X}X_{;\mu}(Kn^\mu - \dot{n}^\mu) \ , \end{aligned} \tag{66}$$

where in the second line we have used that $\gamma^{-2} = -X$. Moreover, for the last term we have replaced $\gamma^{-1}\phi^{;\mu}$ by $-n^\mu$ and used $\dot{n}_\mu = \frac{\gamma^2}{2}h_\mu{}^\nu X_{;\nu}$. In this last term we can employ that $G_{4X}X_{;\mu} = \partial_\mu G_4 - G_{4\phi}\phi_\mu = \partial_\mu G_4 + \gamma^{-1}G_{4\phi}n_\mu$. After an integration by parts this yields, using $n_\mu \dot{n}^\mu = 0$,

$$2G_{4X}X_{;\mu}(Kn^\mu - \dot{n}^\mu) = -2G_4(Kn^\mu - \dot{n}^\mu)_{;\mu} - 2\gamma^{-1}G_{4\phi}K \ . \tag{67}$$

The first term on the right-hand side of this expression can be rewritten by using the Gauss-Codazzi equation (59). Plugging all this into the second line of eq. (66) we finally obtain L_4 in 3+1 decomposition,

$$L_4 = G_4{}^{(3)}\!R + (2XG_{4X} - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2\sqrt{-X}G_{4\phi}K \, . \tag{68}$$

It is now lengthy but straightforward to apply our usual dictionary (54) to derive the corresponding EFT parameters. Their explicit expression can be found in App. C. They depend on the six Lagrangian parameters G, G_{4X}, $G_{4X\phi}$, G_{4XX}, $G_{4XX\phi}$ and G_{4XXX}. We need all the seven parameters of the EFT action (45) to describe L_4 but the last two are equal, $m_4^2 = \tilde{m}_4^2$.

3.4 L_5

This Galileon Lagrangian is more involved than the others. Let us start working on the first term on the right-hand side of eq. (53), $G_5\, G_{\mu\nu}\phi^{;\mu\nu}$. Integrating it by parts gives

$$G_5\, G_{\mu\nu}\phi^{;\mu\nu} = -G_{5X}X^{;\nu}\phi^{;\mu}G_{\mu\nu} - G_{5\phi}\gamma^{-2}G_{\mu\nu}n^\mu n^\nu \; . \tag{69}$$

At this stage, as we did for L_3, it is convenient to define an auxiliary function $F_5(\phi, X)$, such that

$$G_{5X} \equiv F_{5X} + \frac{F_5}{2X} \; , \tag{70}$$

and use this definition to integrate by parts the term proportional to G_{5X} in (69). In particular, using that

$$G_{5X}X_{;\rho} = \gamma\nabla_\rho(\gamma^{-1}F_5) + F_{5\phi}\gamma^{-1}n_\rho \; , \tag{71}$$

we obtain

$$G_5\, G_{\mu\nu}\phi^{;\mu\nu} = F_5\phi^{;\mu\nu}G_{\mu\nu} + \gamma^{-2}(F_{5\phi} - G_{5\phi})G_{\mu\nu}n^\mu n^\nu - \frac{\gamma}{2}F_5X^{;\mu}n^\nu G_{\mu\nu} \; . \tag{72}$$

Let us now work on the second term on the right-hand side of eq. (53). Using eq. (58) we can rewrite this as

$$\frac{1}{3}G_{5X}(\Box\phi^3 - 3\,\Box\phi\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{\ ;\sigma}) = -G_{5X}\frac{\gamma^{-3}}{3}\mathcal{K} + G_{5X}\mathcal{J} \; , \tag{73}$$

where

$$\mathcal{K} \equiv K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\sigma}K^\nu_{\ \sigma} \; , \tag{74}$$

$$\mathcal{J} \equiv -\frac{1}{2}\phi^{;\rho}X_{;\rho}(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2\gamma^{-3}(K\dot{n}_\mu\dot{n}^\mu - K_{\mu\nu}\dot{n}^\mu\dot{n}^\nu) \, . \tag{75}$$

The term proportional to $\mathcal{J}$ on the right-hand side of eq. (73) can be integrated by parts using the same trick as above, which yields

$$G_{5X}\mathcal{J} = -F_5\gamma^{-1}\left(\frac{1}{2}\mathcal{K} + K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} - Kn^\sigma n^\rho R_{\sigma\rho} + \dot{n}^\sigma n^\rho R_{\sigma\rho}\right) - \frac{\gamma^{-2}}{2}F_{5\phi}(K^2 - K_{\mu\nu}K^{\mu\nu}) \; . \tag{76}$$

For the last part of the calculation we need the (one time-)contracted Gauss-Codazzi relation, eq. (59), which gives

$$K^{\mu\nu}G_{\mu\nu} = K^{\mu\nu\,(3)}R_{\mu\nu} - K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} + KK^2_{\mu\nu} - K^3_{\mu\nu} - \frac{1}{2}RK \, . \tag{77}$$

Replacing $\phi^{;\mu\nu}$ with eq. (58) in eq. (72) and using this relation, the terms proportional to F_5 in eqs. (72) and (76) combine and simplify to

$$-\gamma^{-1}F_5\left({}^{(3)}G_{\mu\nu}K^{\mu\nu} - \frac{1}{6}\mathcal{K}\right) \; . \tag{78}$$

Using this and putting together all the terms of L_5 from eqs. (72), (73) and (76) we finally obtain

$$\begin{aligned} L_5 = \; & -\sqrt{-X}F_5\left(K^{\mu\nu\,(3)}R_{\mu\nu} - \frac{1}{2}K^{(3)}R\right) - \frac{1}{3}(-X)^{3/2}G_{5X}\mathcal{K} \\ & + \frac{1}{2}X(G_{5\phi} - F_{5\phi})^{(3)}R + \frac{1}{2}XG_{5\phi}(K^2 - K_{\mu\nu}K^{\mu\nu}) \; , \end{aligned} \tag{79}$$

where in the last line we have used

$$2G_{\mu\nu}n^\mu n^\nu = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} \; . \tag{80}$$

Operator	f	Λ	c	M_2^4	m_3^3	$m_4^2 = \tilde{m}_4^2$
L_2	0	✓	✓	✓	0	0
L_3	0	✓	✓	✓	✓	0
L_4	✓	✓	✓	✓	✓	✓
L_5	✓	✓	✓	✓	✓	✓

Table 1: A list of the different contributions of the generalized Galileon Lagrangians (50)–(53) to the operators of (45).

Note that the last line of (79) has the same form as the first two terms of L_4 given in eq. (68): by using eq. (70) it can be written as

$$G_4{}^{(3)}R + (2XG_{4X} - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) \,, \tag{81}$$

with $G_4 \equiv \frac{1}{2}X(G_{5\phi} - F_{5\phi})$.

In order to compute the coefficients of the various EFT operators we use the dictionary (54). To treat the term $K^{\mu\nu(3)}R_{\mu\nu}$ we employ the prescription described by eq. (126) in App. A. Moreover, it is useful to notice that, up to quadratic order, the combination $\mathcal{K}$ of the extrinsic curvature tensor can be replaced by an expression that depends only on S and K:

$$\mathcal{K} = 6H^3 - 6H^2K + 3HK^2 - 3HS + \mathcal{O}(3) \,. \tag{82}$$

The EFT operator coefficients are explicitly given in App. C. One finds that they depend on the six Lagrangian parameters $G_{5\phi}$, G_{5X}, $G_{5X\phi}$, G_{5XX}, $G_{5XX\phi}$ and G_{5XXX}—the dependence on F_5 explicitly cancels out—and, as in the case of L_4, $m_4^2 = \tilde{m}_4^2$. Thus, at linear order in the perturbations—quadratic in the action—L_5 does not bring any new operator with respect to L_4. The difference between L_4 and L_5 appears at the cubic order in the action.

3.5 Summary

We have established a dictionary between the generalized Galileon theory, eqs. (50)–(53), and the EFT of dark energy parameters entering the action (2). Such a dictionary is explicitly given in App. C. As summarised in Table 1, the EFT operators and their associated time-dependent parameters that are needed to describe the generalized Galileons are only six: c, Λ and f, the three usual parameters already present at the background level, and M_2^4, m_3^3, $m_4^2 = \tilde{m}_4^2$, progressively appearing in L_2, L_3, L_4 and L_5, contributing only to the perturbations. As already stressed, at quadratic order in the perturbations, L_4 contains the same number of independent operators as L_5—in particular, only the combination $m_4^2 = \tilde{m}_4^2$ appears in the action. The case $m_4^2 \neq \tilde{m}_4^2$ encompasses the generalized Galileons: it does not contain higher derivatives and yet does not belong to the generalized Galileon class.[8] When $m_4^2 \neq \tilde{m}_4^2$, higher derivatives are expected to appear *beyond* linear order. However, the effect of these higher derivatives can be ignored as long as perturbations remain small and linear theory is a good approximation.

4 Observables

Observables describing large scale structures are computed in the framework of linear cosmological perturbation theory. In this section we first derive the perturbation equations describing the dynamics of dark energy and modified gravity. We include a matter sector describing cosmological species such

[8]Note that our formalism easily applies to nonlinear extensions of Hordenskis theories, such as described in Ref. [30]. In this particular case, one finds that $m_4^2 \neq \tilde{m}_4^2$ but the quadratic action contains higher order spatial derivatives.

as cold dark matter, baryons, photons and neutrinos—by adding the matter Lagrangian $\mathcal{L}_m(g_{\mu\nu}, \psi_m)$ to eq. (2), so that the final Jordan frame action in unitary gauge reads

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Bigg[& \frac{M_*^2}{2} f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K \delta g^{00} \\
& - m_4^2(t)\left(\delta K^2 - \delta K_{\mu\nu}\delta K^{\mu\nu}\right) + \frac{\tilde{m}_4^2(t)}{2}{}^{(3)}\!R\,\delta g^{00} \\
& - \bar{m}_4^2(t)\delta K^2 + \frac{\bar{m}_5(t)}{2}{}^{(3)}\!R\,\delta K + \frac{\bar{\lambda}(t)}{2}{}^{(3)}\!R^2 + \mathcal{L}_m(g_{\mu\nu}, \psi_m)\Bigg] \, .
\end{aligned}
\tag{83}
$$

We then discuss the modifications of gravity expected in linear theory. We will use Newtonian gauge, which is often used in cosmology, especially to describe cosmological perturbations for modified gravity. Extension to other gauges or to so-called "gauge invariant" formalisms is straightforward.

4.1 Perturbation equations

We will first restore the general covariance of the action above and write it in a generic coordinate system. In order to do that we need to reintroduce the scalar fluctuation π via the Stueckelberg trick [3, 4, 27]. Under the time coordinate change $t \to t + \pi(t, \vec{x})$, the four-Ricci scalar R remains invariant, while functions of time such as f and the 3-dimensional quantities change as[9] [10]

$$f \to f + \dot{f}\pi + \frac{1}{2}\ddot{f}\pi^2 \, , \tag{84}$$

$$g^{00} \to g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi \, , \tag{85}$$

$$\delta K_{ij} \to \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i\partial_j\pi \, , \tag{86}$$

$$\delta K \to \delta K - 3\dot{H}\pi - \frac{1}{a^2}\partial^2\pi \, , \tag{87}$$

$${}^{(3)}\!R_{ij} \to {}^{(3)}\!R_{ij} + H(\partial_i\partial_j\pi + \delta_{ij}\partial^2\pi) \, , \tag{88}$$

$${}^{(3)}\!R \to {}^{(3)}\!R + \frac{4}{a^2}H\partial^2\pi \, . \tag{89}$$

In the new coordinates we consider a linearly perturbed FLRW metric with only scalar fluctuations,

$$ds^2 = -(1+2\Phi)dt^2 + 2\partial_i\alpha\, dtdx^i + a^2(t)\left[(1-2\Psi)\delta_{ij} + 2\chi_{ij}\right]dx^i dx^j \, , \tag{90}$$

where χ_{ij} is traceless and given in terms of the scalar perturbation β, $\chi_{ij} \equiv (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2)\beta$. The extrinsic curvature and the 3-dimensional Ricci tensor of the *new* equal-time hypersurfaces thus read

$$
\begin{aligned}
K_{ij} &= e^{-\Phi}(H - \dot{\Psi})h_{ij} + \dot{\chi}_{ij} - \partial_i\partial_j\alpha \, , \\
{}^{(3)}\!R_{ij} &= \partial_i\partial_j\Psi + \delta_{ij}\partial^2\Psi + 2\partial_k\partial_{(i}\chi_{j)}^{\ k} - \partial^2\chi_{ij} \, .
\end{aligned}
\tag{91}
$$

We also decompose the matter stress-energy tensor at linear order as

$$T^0_{\ 0} \equiv -(\rho_m + \delta\rho_m) \, , \tag{92}$$

$$T^0_{\ i} \equiv (\rho_m + p_m)\partial_i v = -a^2 T^i_{\ 0} \, , \tag{93}$$

$$T^i_{\ j} \equiv (p_m + \delta p_m)\delta^i_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\sigma \, , \tag{94}$$

[9] With an abuse of notation, here we denote the extrinsic curvature on hypersurfaces of constant time with K_{ij} even when we are *not* in unitary gauge. The reader must be aware that K_{ij} is not the same geometrical object *before* and *after* the Stueckelberg trick. The same also holds for ${}^{(3)}\!R_{ij}$. In particular, after the Stueckelberg trick K_{ij} and ${}^{(3)}\!R_{ij}$ are respectively given by eq. (91).

[10] The operator $\tilde{m}_4^2$ is also considered in Ref. [12]. However, in v1 of this reference, the variation (89) of ${}^{(3)}\!R$ under the Stueckelberg trick has been overlooked and the error propagates into the Einstein equations and the various observables. With the authors of [12] there is now agreement on this issue [31].

where ρ_m and p_m are respectively the background energy density and pressure and $\delta\rho_m$ and δp_m their perturbations, v is the 3-velocity potential and σ the scalar component of the anisotropic stress. The background equations derived from the action (83) are [11]

$$c+\Lambda = 3M_*^2(fH^2+\dot f H) - \rho_m \;, \tag{95}$$
$$\Lambda - c = M_*^2(2f\dot H + 3fH^2 + 2\dot f H + \ddot f) + p_m \;. \tag{96}$$

Using these expressions and the transformations (84)-(89) allows us to rewrite (83) as an action for the scalar fluctuations Φ, α, Ψ, β and π. We can vary eq. (83) expanded at second order and then fix the Newtonian gauge by setting $\alpha = 0 = \beta$ in the equations derived. This yields five equations,

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Phi}\bigg|_{\alpha=0=\beta} \equiv A_\Phi\Phi + A_{\dot\Psi}\dot\Psi + A_\pi\pi + A_{\dot\pi}\dot\pi + \frac{k^2}{a^2}(A^{(2)}_\Psi\Psi + A^{(2)}_\pi\pi) + \delta T^0_{\ 0}\;, \tag{97}$$

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\alpha}\bigg|_{\alpha=0=\beta} \equiv k^2\left[B_\Phi\Phi + B_{\dot\Psi}\dot\Psi + B_\pi\pi + B_{\dot\pi}\dot\pi + \frac{k^2}{a^2}(B^{(2)}_\Psi\Psi + B^{(2)}_\pi\pi)\right] - ik^i\delta T^0_{\ i}\;, \tag{98}$$

$$\begin{aligned}0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Psi}\bigg|_{\alpha=0=\beta} &\equiv C_\Phi\Phi + C_{\dot\Phi}\dot\Phi + C_{\dot\Psi}\dot\Psi + C_{\ddot\Psi}\ddot\Psi + C_\pi\pi + C_{\dot\pi}\dot\pi + C_{\ddot\pi}\ddot\pi \\ &+ \frac{k^2}{a^2}(C^{(2)}_\Phi\Phi + C^{(2)}_\Psi\Psi + C^{(2)}_\pi\pi + C^{(2)}_{\dot\pi}\dot\pi) + \frac{k^4}{a^4}(C^{(4)}_\Psi\Psi + C^{(4)}_\pi\pi) - \delta T^k_{\ k}\;,\end{aligned} \tag{99}$$

$$\begin{aligned}0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\beta}\bigg|_{\alpha=0=\beta} &\equiv \left(k_ik^j - \frac{1}{3}\delta_i^jk^2\right)\bigg[k^ik_j\Big(D^{(2)}_\Phi\Phi + D^{(2)}_\Psi\Psi + D^{(2)}_{\dot\Psi}\dot\Psi + D^{(2)}_\pi\pi + D^{(2)}_{\dot\pi}\dot\pi \\ &+ \frac{k^2}{a^2}(D^{(4)}_\Psi\Psi + D^{(4)}_\pi\pi)\Big) - \delta T^i_{\ j}\bigg]\;,\end{aligned} \tag{100}$$

$$\begin{aligned}0 = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\pi}\bigg|_{\alpha=0=\beta} &\equiv E_\Phi\Phi + E_{\dot\Phi}\dot\Phi + E_\Psi\Psi + E_{\dot\Psi}\dot\Psi + E_{\ddot\Psi}\ddot\Psi + E_\pi\pi + E_{\dot\pi}\dot\pi + E_{\ddot\pi}\ddot\pi \\ &+ \frac{k^2}{a^2}(E^{(2)}_\Phi\Phi + E^{(2)}_\Psi\Psi + E^{(2)}_{\dot\Psi}\dot\Psi + E^{(2)}_\pi\pi) + \frac{k^4}{a^4}(E^{(4)}_\Psi\Psi + E^{(4)}_\pi\pi)\;.\end{aligned} \tag{101}$$

The coefficients A_a, B_a, C_a, D_a and E_a of these equations are detailed in App. D.

The explicit expressions of the above coefficients given in the appendix contain also higher-derivative terms—those proportional to $\bar m_4^2$, $\bar m_5$ and $\bar\lambda$. To compare with the usual Einstein equations, here we rewrite these equations by replacing the components of the stress-energy tensor T^μ_ν with their expressions given in eqs. (92)–(94). For simplicity, we set $\bar m_4^2 = \bar m_5 = \bar\lambda = 0$. We obtain:

- **00-component ($\delta S/\delta\Phi = 0$):**

$$\begin{aligned}&M_*^2\bigg[-2f\left(\frac{k^2}{a^2}\Psi + 3H\dot\Psi + 3H^2\Phi\right) + \dot f\left(\frac{k^2}{a^2}\pi + 3H^2\pi - 3H(\Phi-\dot\pi) - 3(\dot\Psi + H\Phi)\right) + 3H\ddot f\pi\bigg] \\ &-(\dot c + \dot\Lambda)\pi + (2c + 4M_2^4 + 3Hm_3^3)(\Phi - \dot\pi) + (m_3^3 - 4Hm_4^2)\left[-\frac{k^2}{a^2}\pi + 3(H\Phi + \pi\dot H + \dot\Psi)\right] \\ &-4\frac{k^2}{a^2}\tilde m_4^2(\Psi + H\pi) = \delta\rho_m\;.\end{aligned} \tag{102}$$

- **0i-component ($\delta S/\delta\alpha = 0$):**

$$\begin{aligned}&M_*^2\left[(H\dot f - \ddot f)\pi + \dot f(\Phi - \dot\pi) + 2f(H\Phi + \dot\Psi)\right] - 2c\pi - m_3^3(\Phi - \dot\pi) + 4m_4^2(H\Phi + \dot\Psi + \dot H\pi) \\ &= -(p_m + \rho_m)\,v\;.\end{aligned} \tag{103}$$

- **ij-trace component ($\delta S/\delta\Psi = 0$):**

$$
\begin{aligned}
& M_*^2 \Big\{ 2f \left[-\frac{1}{3}\frac{k^2}{a^2}(\Phi - \Psi) + (3H^2 + 2\dot{H})\Phi + H(\dot\Phi + 3\dot\Psi) + \ddot\Psi \right] \\
& + \dot f \left[-\frac{2}{3}\frac{k^2}{a^2}\pi + 2H\Phi + 2H(\Phi - \dot\pi) - (3H^2 + 2\dot H)\pi + 2\dot\Psi + \dot\Phi - \ddot\pi \right] \\
& + \ddot f \left[-2H\pi + 2(\Phi - \dot\pi) \right] - f^{(3)}\pi \Big\} + (\dot\Lambda - \dot c)\pi + 2c(\Phi - \dot\pi) \\
& - \frac{4}{3}\frac{k^2}{a^2} \left[\tilde m_4^2(\Phi - \dot\pi) + \left(H m_4^2 + (m_4^2)^{\cdot} \right)\pi + m_4^2\dot\pi \right] \\
& + 4(\dot H m_4^2)^{\cdot}\pi + 4m_4^2\dot H\dot\pi - \left[(m_3^3)^{\cdot} + 3Hm_3^3 \right](\Phi - \dot\pi) - m_3^3(\dot\Phi - \ddot\pi) \\
& + 4\left[H(m_4^2)^{\cdot} + 3H^2 m_4^2 + \dot H m_4^2 \right]\Phi + 4(m_4^2)^{\cdot}\dot\Psi + 4m_4^2 H(3\dot H\pi + \dot\Phi + 3\dot\Psi) + 4m_4^2\ddot\Psi = \delta p_m \ .
\end{aligned}
\tag{104}
$$

- **ij-traceless component ($\delta S/\delta\beta = 0$):**

$$
M_*^2 \left[f(\Phi - \Psi) + \dot f\pi \right] + 2\left[m_4^2\dot\pi + m_4^2 H\pi + (m_4^2)^{\cdot}\pi \right] + 2\tilde m_4^2(\Phi - \dot\pi) = \sigma \ . \tag{105}
$$

By combining eqs. (97) and (98) we obtain the relativistic generalization of the Poisson equation,

$$
F_\Phi\Phi + F_{\dot\Psi}\dot\Psi + F_\pi\pi + F_{\dot\pi}\dot\pi + \frac{k^2}{a^2}(F_\Psi^{(2)}\Psi + F_\pi^{(2)}\pi) = \delta\rho_m - 3H(\rho_m + p_m)v \equiv \rho_m\Delta_m \ , \tag{106}
$$

which can be also written as:

- **Generalized Poisson equation:**

$$
\begin{aligned}
& -\frac{k^2}{a^2}\left[(2fM_*^2 + 4\tilde m_4^2)\Psi - (\dot f M_*^2 - m_3^3 + 4Hm_4^2 - 4H\tilde m_4^2)\pi \right] + (6M_*^2H^2\dot f - 6Hc - \dot c - \dot\Lambda + 3m_3^3\dot H)\pi \\
& - (2c + 4M_2^4)\dot\pi - (3M_*^2H\dot f - 2c - 4M_2^4)\Phi - 3M_*^2\dot f\dot\Psi + 3m_3^3(\dot\Psi + H\Phi) = \rho_m\Delta_m \ .
\end{aligned}
\tag{107}
$$

Note that when $m_4^2 \neq \tilde m_4^2$ some of the equations contain terms with higher derivates: for instance, the terms with $k^2\dot\pi$ in eq. (104), fourth line, and those with $\dot\pi$ in eq. (105). However, the scalar propagating degree of freedom satisfies a second order equation. Indeed, one can use eq. (105) to remove the higher derivative terms from eq. (104) and derive a purely second order equation for Ψ. This is even clearer in unitary gauge, where higher derivative are explicitly absent—see analysis of Sec. (2.2).

4.2 Modification of gravity

In order to derive the effective Newton constant, $G_{\rm eff}$, we consider the quasi static approximation, i.e. we neglect the time derivatives in the equations of motion and we neglect the anisotropic stress, $\sigma = 0$ in eq. (100). This is a good approximation for scales much smaller than the sound horizon scale, i.e. for $k \gg aH/c_s$. For models with small or vanishing sound speed (see e.g. [10]) or on scales longer than the sound horizon, a consistent treatment which takes into account the time derivatives should be undertaken.

In the quasi-static limit, $G_{\rm eff}$ is defined by

$$
-\frac{k^2}{a^2}\Phi \equiv 4\pi G_{\rm eff}(t,k)\rho_m\Delta_m \ . \tag{108}
$$

Following [32, 33], in order to write the Poisson equation in this form we can use eqs. (100), (101) and (106). For $c_s \sim \mathcal{O}(1)$, we can neglect D_Φ, D_Ψ, D_π, E_Φ, E_Ψ, F_Φ and F_π from these equations and the effective Newton constant is thus given by

$$4\pi G_{\rm eff} = -[\mathcal{M}^{-1}]_{13} \,, \qquad \mathcal{M} \;\equiv\; \begin{pmatrix} D_\Phi^{(2)} & D_\Psi^{(2)} + D_\Psi^{(4)}(k/a)^2 & D_\pi^{(2)} + D_\pi^{(4)}(k/a)^2 \\ E_\Phi^{(2)} & E_\Psi^{(2)} + E_\Psi^{(4)}(k/a)^2 & E_\pi(k/a)^{-2} + E_\pi^{(2)} + E_\pi^{(4)}(k/a)^2 \\ 0 & F_\Psi^{(2)} & F_\pi^{(2)} \end{pmatrix} \,. \tag{109}$$

We can write it in a slightly more compact form as

$$4\pi G_{\rm eff}(k) = \frac{a_{-2}(k/a)^{-2} + a_0 + a_2(k/a)^2 + a_4(k/a)^4}{b_{-2}(k/a)^{-2} + b_0 + b_2(k/a)^2} \,, \tag{110}$$

where

$$\begin{aligned} a_{-2} &= D_\Psi^{(2)} E_\pi \,, \\ a_0 &= D_\Psi^{(2)} E_\pi^{(2)} - D_\pi^{(2)} E_\Psi^{(2)} + D_\Psi^{(4)} E_\pi \,, \\ a_2 &= D_\Psi^{(2)} E_\pi^{(4)} - D_\pi^{(4)} E_\Psi^{(2)} - D_\pi^{(2)} E_\Psi^{(4)} + D_\Psi^{(4)} E_\pi^{(2)} \,, \\ a_4 &= -D_\pi^{(4)} E_\Psi^{(4)} + D_\Psi^{(4)} E_\pi^{(4)} \,, \\ b_{-2} &= D_\Phi^{(2)} E_\pi F_\Psi^{(2)} \,, \\ b_0 &= D_\Psi^{(2)} E_\Phi^{(2)} F_\pi^{(2)} - D_\Phi^{(2)} E_\Psi^{(2)} F_\pi^{(2)} - D_\pi^{(2)} E_\Phi^{(2)} F_\Psi^{(2)} + D_\Phi^{(2)} E_\pi^{(2)} F_\Psi^{(2)} \,, \\ b_2 &= -D_\Phi^{(2)} E_\Psi^{(4)} F_\pi^{(2)} - D_\pi^{(4)} E_\Phi^{(2)} F_\Psi^{(2)} + D_\Phi^{(2)} E_\pi^{(4)} F_\Psi^{(2)} + D_\Psi^{(4)} E_\Phi^{(2)} F_\pi^{(2)} \,. \end{aligned} \tag{111}$$

Another quantity often used to parameterize deviations from General Relativity is the ratio between the gravitational potentials $\gamma \equiv \Psi/\Phi$, which is given by

$$\gamma = \frac{[\mathrm{com}(\mathcal{M})]_{32}}{[\mathrm{com}(\mathcal{M})]_{31}} \,, \tag{112}$$

where $\mathrm{com}(\mathcal{M})$ denotes the comatrix of $\mathcal{M}$. This reads

$$\gamma = \frac{c_{-2}(k/a)^{-2} + c_0 + c_2(k/a)^2}{a_{-2}(k/a)^{-2} + a_0 + a_2(k/a)^2 + a_4(k/a)^4} \,, \tag{113}$$

with

$$c_{-2} = -D_\Phi^{(2)} E_\pi \,, \tag{114}$$

$$c_0 = D_\pi^{(2)} E_\Phi^{(2)} - D_\Phi^{(2)} E_\pi^{(2)} \,, \tag{115}$$

$$c_2 = D_\pi^{(4)} E_\Phi^{(2)} - D_\Phi^{(2)} E_\pi^{(4)} \,. \tag{116}$$

The expressions for $G_{\rm eff}$ and γ, eqs. (110) and (113), generalize those given for instance in [32] in absence of higher derivative operators, in which case $a_2 = a_4 = b_2 = c_2 = 0$. When also $a_{-2} = b_{-2} = c_{-2} = 0$ we recover the results of [11]. Finally, we note that the numerator of $G_{\rm eff}$ equals the denominator of γ, which confirms the results of Ref. [33][11].

[11]It simply follows from $[\mathcal{M}^{-1}]_{13} = (\det \mathcal{M})^{-1}[\mathrm{com}(\mathcal{M})]_{31}$.

5 Conclusion

In this paper we lay down the basic building blocks for a systematic phenomenological study of dark energy and its cosmological perturbations. Following [11], our basic assumptions are that *a)* dark energy/modified gravity brings in at most one scalar propagating degree of freedom and that *b)* the weak equivalence principle is satisfied—there exists a metric tensor universally coupled to matter. We use the effective field theory formalism developed for inflation in [3,4], that is based on an expansion in number of perturbations rather than in number of fields. Indeed, expanding the action in number of fields [15, 16] becomes unpractical every time that the background field configuration undergoes a large excursion. On the opposite, the main advantage of the present (non-covariant) approach is that an expansion in number of perturbations can always be consistently truncated at the desired order of approximation, in virtue of the empirical fact that perturbations are small on the largest scales. Moreover, our formalism is "ready to go", in the sense that there is no need of solving for the background equations first. Apart from the three operators f, c and Λ responsible for the background evolution [11], every new operator is at least quadratic in the perturbations: it does not affect the background and its dynamical effects can be studied independently.

In particular, we consider only operators that are at most quadratic in the number of perturbations—those needed for the linearized equations of motion—and we single out a set of seven operators that bring up to two derivatives in the equations of motion. To achieve this result, in Sec. 2 we provide a systematic treatment of any Lagrangian that can be written in ADM form as a general function of extrinsic and intrinsic 3-dimensional curvature tensors and of the lapse function. This is already enough to avoid higher time derivatives in the equations of motion. Then, in Sec. 2.3 we identify specific combinations of the EFT operators that are required to avoid higher-order spatial derivatives. Some operators can be re-expressed into other ones, thus simplifying the EFT Lagrangians up to quadratic order.

The entire Horndeski, or "generalized Galileon", theory can be written in this formalism (Sec. 3): a relevant amount of work has gone into re-expressing all the generalized Galileon Lagrangians in their ADM form and obtaining their EFT formulation. At linear order, Horndeski theories can be described by a total of six operators: only three quadratic operators in addition to those—f, c and Λ—accounting for the background (see eq. (2) with $m_4^2 = \tilde{m}_4^2$). This seems a substantial simplification if compared to the full covariant treatment and well represents the power of the non-covariant EFT approach. The two Galileon Lagrangians L_4 and L_5, despite their scaring looks (52)-(53), are affordable at linear order in the perturbations with the addition of just one operator with respect to those needed for L_3.

At linear order, Horndeski theory is *not* the most general scalar-tensor theory with second-order dynamics. Indeed, for $m_4^2 \neq \tilde{m}_4^2$ there exists another operator beyond Horndeski that in unitary gauge gives equations of motion limited to second order in time and space derivatives. In some gauges (for instance in Newtonian gauge, see Sec. 4.1), this operator generates higher derivatives in the equations of motion but one can show that the dynamics of the propagating degree of freedom remains second order. [At linear order, there exists another operator beyond the Horndeski theory (for $m_4^2 \neq \tilde{m}_4^2$) that still gives equations of motion limited to second order in time and space derivatives.] Finally, we analyze also some higher *spatial* derivative operators, those in eq. (4).

The time dependent coefficients of our seven plus three operators described by actions (2) and (4) remain to be constrained or measured by observations. Indeed, in Sec. 4.1 we provide the set of linear perturbation equations in Newtonian gauge by varying these actions with respect to scalar metric and field fluctuations in a generic gauge. As an illustration, using these equations we compute the effective Newton constant in the quasi-static approximation and the ratio between the two gravitational potentials (Sec. 4.2). The computation of these "observables" should be considered as a first step towards a more general and systematic study of the impact of dark energy on cosmological perturbations in order to fully exploit future observational data.

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A Lagrangian dependence on ${}^{(3)}R_{\mu\nu}K^{\mu\nu}$

In this appendix we show how to treat a dependence on

$$\mathcal{Y} \equiv {}^{(3)}R_{\mu\nu}K^{\mu\nu} \tag{117}$$

in the unitary gauge Lagrangian.

Let us first show the relation

$$\int d^4x\sqrt{-g}\,\lambda(t){}^{(3)}R_{\mu\nu}K^{\mu\nu} = \int d^4x\sqrt{-g}\left[\frac{\lambda(t)}{2}{}^{(3)}R\,K + \frac{\dot\lambda(t)}{2N}\,{}^{(3)}R\right], \tag{118}$$

or, equivalently,

$$\int d^4x\sqrt{-g}\left[\lambda(t){}^{(3)}R_{\mu\nu}K^{\mu\nu} - \frac{\lambda(t)}{2}{}^{(3)}R\,K - \frac{\dot\lambda(t)}{2N}{}^{(3)}R\right] = 0\,, \tag{119}$$

up to some irrelevant boundary terms. Since $K = \nabla_\mu n^\mu$, the last two terms in the above integral can be simplified via an integration by parts, so that the expression reduces to

$$\int d^4x\sqrt{-g}\,\lambda(t)\left({}^{(3)}R_{\mu\nu}K^{\mu\nu} + \frac{n^\mu}{2}\nabla_\mu{}^{(3)}R\right). \tag{120}$$

It remains to show that this can be written as the integral of a total derivative.

Using the explicit expressions for the extrinsic curvature in the ADM decomposition, eq. (22), and $n^\mu = -Ng^{0\mu}$, the above expression can be rewritten as

$$\int d^4x\sqrt{h}\lambda(t)\left[\frac{1}{2}\left(h^{ik}h^{jl}\dot h_{kl}{}^{(3)}R_{ij} + {}^{(3)}\dot R\right) - \nabla^i N^j{}^{(3)}R_{ij} - \frac{1}{2}N^i\nabla_i{}^{(3)}R\right], \tag{121}$$

where ∇_i is the covariant derivative with respect to the three-metric h_{ij}. The second term can be integrated by parts and then vanishes when combined with the last term, as a consequence of the Bianchi identity $\nabla^{i(3)}G_{ij} = 0$. Finally, the term in parenthesis can be rewritten as

$$h^{ik}h^{jl}\dot h_{kl}\;{}^{(3)}R_{ij} + {}^{(3)}\dot R = h^{ik}h^{jl}\dot h_{kl}\;{}^{(3)}R_{ij} + \dot h^{ij(3)}R_{ij} + h^{ij(3)}\dot R_{ij} = h^{ij}\;{}^{(3)}\dot R_{ij}\,. \tag{122}$$

and it is known that the last expression can be reexpressed as the divergence of a three-vector, i.e. $h^{ij}\;{}^{(3)}\dot R_{ij} = \nabla_i J^i$ (the very same property is used to derive Einstein's equations from the Einstein-Hilbert action[12]). We have thus proved Eq. (118).

Let us now assume that the Lagrangian L introduced in Eq. (6) also contains an explicit dependence on $\mathcal{Y}$. By noting that $\mathcal{Y}$ is already a perturbative quantity, i.e. vanishes in the background, and can be decomposed as

$$\mathcal{Y} = H\mathcal{R} + {}^{(3)}R_{\mu\nu}\delta K^{\mu\nu}\,, \tag{123}$$

where the first term on the right hand side is a first (and higher) order quantity while the second term is only second order, one immediately finds that the expansion of the Lagrangian, up to quadratic

[12] See for instance Eq. (7.5.14) of Ref. [29].

order, will yield the following extra terms with respect to the expression (12) obtained in the main body:

$$L(N,\mathcal{S},K,\mathcal{R},\mathcal{Y},\mathcal{Z}) \supset L_{\mathcal{Y}}\mathcal{Y} + \left(L_{N\mathcal{Y}}\delta N + L_{K\mathcal{Y}}\delta K + L_{\mathcal{S}\mathcal{Y}}\delta\mathcal{S} + L_{\mathcal{R}\mathcal{Y}}\delta\mathcal{R}\right) H\delta\mathcal{R} + \frac{1}{2}L_{\mathcal{Y}\mathcal{Y}}H^2\delta\mathcal{R}^2 \,. \quad (124)$$

The first term can be expressed in terms of $\mathcal{R}$ and K by using eq. (118) with $\lambda = L_{\mathcal{Y}}$. Expanding up to second order then yields

$$L_{\mathcal{Y}}\,\mathcal{Y} = \frac{1}{2}\left(\dot{L}_{\mathcal{Y}} + 3HL_{\mathcal{Y}}\right)\delta\mathcal{R} + \frac{1}{2}\left(L_{\mathcal{Y}}\delta K - \dot{L}_{\mathcal{Y}}\delta N\right)\delta\mathcal{R} + \mathcal{O}(3) + \text{boundary terms}\,, \quad (125)$$

so that the expansion of the full Lagrangian now reads

$$\begin{aligned} L(N,\mathcal{S},K,\mathcal{R},\mathcal{Y},\mathcal{Z}) = {} & \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + L_N\,\delta N + \frac{1}{2}\left(2L_{\mathcal{R}} + \dot{L}_{\mathcal{Y}} + 3HL_{\mathcal{Y}}\right)\delta\mathcal{R} \\ & + L_{\mathcal{S}}\,\delta K^\mu_\nu \delta K^\nu_\mu + \left(2H^2L_{\mathcal{S}\mathcal{S}} + 2HL_{\mathcal{S}K} + \frac{1}{2}L_{KK}\right)\delta K^2 + \frac{1}{2}L_{NN}\delta N^2 \\ & + \frac{1}{2}\left(L_{\mathcal{R}\mathcal{R}} + H^2L_{\mathcal{Y}\mathcal{Y}} + 2HL_{\mathcal{Y}\mathcal{R}}\right)\delta\mathcal{R}^2 + \left(2HL_{\mathcal{S}N} + L_{KN}\right)\delta K\delta N \\ & + \left(2HL_{\mathcal{S}\mathcal{R}} + L_{K\mathcal{R}} + HL_{K\mathcal{Y}} + 2H^2L_{\mathcal{S}\mathcal{Y}} + \frac{1}{2}L_{\mathcal{Y}}\right)\delta K\delta\mathcal{R} \\ & + \left(L_{N\mathcal{R}} + HL_{N\mathcal{Y}} - \frac{1}{2}\dot{L}_{\mathcal{Y}}\right)\delta N\delta\mathcal{R} + \mathcal{O}(3)\,. \end{aligned} \quad (126)$$

In summary, an explicit dependence of the action on $\mathcal{Y}$ can easily be included in our treatment, via the following substitutions in Eq. (12),

$$\begin{aligned} L_{\mathcal{R}} &\to L_{\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{Y}} + \frac{3}{2}HL_{\mathcal{Y}}\,, \\ L_{\mathcal{R}\mathcal{R}} &\to L_{\mathcal{R}\mathcal{R}} + H^2L_{\mathcal{Y}\mathcal{Y}} + 2HL_{\mathcal{Y}\mathcal{R}}\,, \\ L_{N\mathcal{R}} &\to L_{N\mathcal{R}} + HL_{N\mathcal{Y}} - \frac{1}{2}\dot{L}_{\mathcal{Y}}\,, \\ \mathcal{C} &\to \mathcal{C} + HL_{K\mathcal{Y}} + 2H^2L_{\mathcal{S}\mathcal{Y}} + \frac{1}{2}L_{\mathcal{Y}}\,. \end{aligned} \quad (127)$$

B Tensor modes

In this appendix we study the propagation of tensor modes in the action (2). We consider the spatial metric [19]

$$h_{ij} = a^2(t)e^{2\zeta}\hat{h}_{ij}\,, \qquad \det\hat{h} = 1\,, \qquad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj}\,, \quad (128)$$

with γ_{ij} traceless and divergence-free, $\gamma_{ii} = 0 = \partial_i\gamma_{ij}$. Since tensor modes decouple from scalars, we can simply replace this metric into the action (2) by setting scalar perturbations to zero, which yields

$$S^{(2)}_\gamma = \int d^4x\, a^3\frac{M_*^2 f}{8}\left[\left(1 + \frac{2m_4^2}{M_*^2 f}\right)\dot{\gamma}_{ij}^2 - \frac{1}{a^2}(\partial_k\gamma_{ij})^2\right]\,, \quad (129)$$

where we used that, up to integration by parts,

$$^{(3)}R = -\frac{1}{4a^2}(\partial_i\gamma_{kj})^2\,, \qquad K = 3H\,, \quad (130)$$

$$\delta K_{ij}^2 = \frac{1}{4}\dot{\gamma}_{ij}^2\,, \qquad K_{ij}K^{ij} - K^2 = -6H^2 + \frac{1}{4}\dot{\gamma}_{ij}^2\,, \quad (131)$$

and the Gauss-Codazzi relation (38). Thus, for $m_4^2 \neq 0$ the speed of sound of gravity waves is different from the speed of light,

$$c_T^2 = \left(1 + \frac{2m_4^2}{M_*^2 f}\right)^{-1} , \tag{132}$$

which confirms [34, 35] in the case of generalised Galileon theories.

C EFT parameters for generalized Galileons

Here we explicitly give the EFT of dark energy parameters in terms of the Lagrangian (12), for the generalized Galileon Lagrangians eqs. (51)–(53). All quantities in the expressions below are calculated on the background.

- L_3:

$$f = 0 \,, \tag{133}$$
$$\Lambda = \dot\phi^2(\ddot\phi + 3H\dot\phi)G_{3X} \,, \tag{134}$$
$$c = \dot\phi^2(-\ddot\phi + 3H\dot\phi)G_{3X} + \dot\phi^2 G_{3\phi} \,, \tag{135}$$
$$M_2^4 = \frac{\dot\phi^2}{2}(\ddot\phi + 3H\dot\phi)G_{3X} - 3H\dot\phi^5 G_{3,XX} - \frac{\dot\phi^4}{2}G_{3,X\phi} \,, \tag{136}$$
$$m_3^3 = 2\dot\phi^3 G_{3X} \,, \quad m_4^2 = \tilde m_4^2 = 0 \,. \tag{137}$$

- L_4:

$$M_*^2 f = 2G_4 \,, \tag{138}$$
$$\Lambda = \frac{1}{2}\dot{\tilde{\mathcal{F}}} + 3H\dot X G_{4X} - 18H^2 G_{4X}\dot\phi^2 + 6HG_{4X\phi}\dot\phi^3 + 12H^2 G_{4XX}\dot\phi^4 \,, \tag{139}$$
$$c = -\frac{1}{2}\dot{\tilde{\mathcal{F}}} + 3H\dot X G_{4X} - 6H^2 G_{4X}\dot\phi^2 + 6HG_{4X\phi}\dot\phi^3 + 12H^2 G_{4XX}\dot\phi^4 \,, \tag{140}$$
$$M_2^4 = \frac{1}{4}\dot{\tilde{\mathcal{F}}} - \frac{3}{2}H\dot X G_{4X} + 6HG_{4X\phi}\dot\phi^3 + 18H^2 G_{4XX}\dot\phi^4 - 6HG_{4XX\phi}\dot\phi^5 - 12H^2 G_{4XXX}\dot\phi^6 \,, \tag{141}$$
$$m_3^3 = 2\dot X G_{4X} - 8HG_{4X}\dot\phi^2 + 4G_{4X\phi}\dot\phi^3 + 16HG_{4XX}\dot\phi^4 \,, \tag{142}$$
$$m_4^2 = \tilde m_4^2 = 2G_{4X}\dot\phi^2 \,, \tag{143}$$

with

$$\tilde{\mathcal{F}} \equiv 2M_*^2 Hf + M_*^2\dot f + \mathcal{F} = 2\dot X G_{4X} - 8HG_{4X}\dot\phi^2 \,. \tag{144}$$

- L_5:

$$M_*^2 f = -G_{5\phi}\dot\phi^2 + 2G_{5X}\dot\phi^2\ddot\phi \,, \tag{145}$$
$$c = -\frac{1}{2}\dot{\tilde{\mathcal{F}}} + \frac{3}{2}M_*^2 H\dot f - 3H^2 G_{5\phi}\dot\phi^2 - 3H^3 G_{5X}\dot\phi^3 + 3H^2 G_{5X\phi}\dot\phi^4 + 2H^3 G_{5XX}\dot\phi^5 \,, \tag{146}$$
$$\Lambda = \frac{1}{2}\dot{\tilde{\mathcal{F}}} + 3M_*^2 H^2 f + \frac{3}{2}M_*^2 H\dot f - 6H^2 G_{5\phi}\dot\phi^2 - 7H^3 G_{5X}\dot\phi^3 + 3H^2 G_{5X\phi}\dot\phi^4 + 2H^3 G_{5XX}\dot\phi^5 \,, \tag{147}$$
$$M_2^4 = \frac{1}{4}\dot{\tilde{\mathcal{F}}} - \frac{3}{4}M_*^2 H\dot f - \frac{3}{2}H^3 G_{5X}\dot\phi^3 + 6H^2 G_{5X\phi}\dot\phi^4 + 6H^3 G_{5XX}\dot\phi^5 - 3H^2 G_{5XX\phi}\dot\phi^6 - 2H^3 G_{5XXX}\dot\phi^7 \,, \tag{148}$$
$$m_3^3 = M_*^2\dot f - 4HG_{5\phi}\dot\phi^2 - 6H^2 G_{5X}\dot\phi^3 + 4HG_{5X\phi}\dot\phi^4 + 4H^2 G_{5XX}\dot\phi^5 \,, \tag{149}$$
$$m_4^2 = \tilde m_4^2 = G_{5\phi}\dot\phi^2 + HG_{5X}\dot\phi^3 - G_{5X}\dot\phi^2\ddot\phi \,, \tag{150}$$

with

$$\tilde{\mathcal{F}} \equiv 2M_*^2 Hf + M_*^2 \dot{f} + \mathcal{F} = 2M_*^2 fH + M_*^2 \dot{f} - 2HG_{5\phi}\dot{\phi}^2 - 2H^2 G_{5X}\dot{\phi}^3 \, . \tag{151}$$

D Coefficients of the perturbation equations

In this appendix we define the coefficients appearing in eqs. (97)–(101). For convenience we have used M_4^2 defined as

$$M_4^2 \equiv 2m_4^2 + 3\bar{m}_4^2 \, . \tag{152}$$

Moreover, from the background equations (95) and (96) and using the background conservation equation for matter, $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, one obtains

$$\dot{c} + \dot{\Lambda} = -6Hc + 6H^2 M_*^2 \dot{f} + 3M_*^2 \dot{f}\dot{H} \, . \tag{153}$$

We will make use of this relation to simplify some of the terms and eliminate the dependence with respect to $\dot{\Lambda}$ and $\ddot{\Lambda}$.

- **Variation with respect to Φ:**

$$A_\Phi = 2c + 4M_2^4 - 6H\left[fHM_*^2 + M_*^2\dot{f} - m_3^3 + HM_4^2\right] , \tag{154}$$

$$A_{\dot{\Psi}} = -3\left[2H\left(fM_*^2 + M_4^2\right) + M_*^2\dot{f} - m_3^3\right] , \tag{155}$$

$$\begin{aligned} A_\pi &= 3H^2 M_*^2 \dot{f} + 3m_3^3 \dot{H} - \dot{c} - \dot{\Lambda} - 6M_4^2 H\dot{H} + 3M_*^2 \ddot{f} \\ &= 6Hc - 3(H^2 + \dot{H})M_*^2 \dot{f} + 3M_*^2 \ddot{f} + 3m_3^3 \dot{H} - 6M_4^2 H\dot{H} \, , \end{aligned} \tag{156}$$

$$A_{\dot{\pi}} = -2c - 4M_2^4 - 3H(m_3^3 - M_*^2 \dot{f}) \, , \tag{157}$$

$$A_\Psi^{(2)} = -2fM_*^2 + 6H\bar{m}_5 - 4\tilde{m}_4^2 \, , \tag{158}$$

$$A_\pi^{(2)} = M_*^2 \dot{f} - m_3^3 + 2HM_4^2 - 4H\tilde{m}_4^2 + 6H^2 \bar{m}_5 \, . \tag{159}$$

- **Variation with respect to α:**

$$B_\Phi = -m_3^3 + 2H\left(fM_*^2 + M_4^2\right) + M_*^2 \dot{f} \, , \tag{160}$$

$$B_{\dot{\Psi}} = 2\left(fM_*^2 + M_4^2\right) , \tag{161}$$

$$B_\pi = -2c + 2M_4^2 \dot{H} + M_*^2 (H\dot{f} - \ddot{f}) \, , \tag{162}$$

$$B_{\dot{\pi}} = m_3^3 - M_*^2 \dot{f} \, , \tag{163}$$

$$B_\Psi^{(2)} = -2\bar{m}_5 \, , \tag{164}$$

$$B_\pi^{(2)} = -2(\bar{m}_4^2 + H\bar{m}_5) \, . \tag{165}$$

- **Variation with respect to Ψ:**

$$
\begin{aligned}
C_\Phi &= 3\Big[2c + 2(3H^2+\dot H)M_4^2 \\
&\quad +2fM_*^2(3H^2+2\dot H) - (m_3^3)^{\cdot} + H\left(-3m_3^3 + 4M_*^2\dot f + 2(M_4^2)^{\cdot}\right) + 2M_*^2\ddot f\Big] \ , && (166)\\
C_{\dot\Phi} &= -3m_3^3 + 6H\left(fM_*^2 + M_4^2\right) + 3M_*^2\dot f \ , && (167)\\
C_{\dot\Psi} &= 6\left(3fHM_*^2 + 3HM_4^2 + M_*^2\dot f + (M_4^2)^{\cdot}\right) \ , && (168)\\
C_{\ddot\Psi} &= 6\left(fM_*^2 + M_4^2\right) \ , && (169)\\
C_\pi &= -3\left[\dot c - \dot\Lambda - 2\dot H(M_4^2)^{\cdot} - 2(\ddot H + 3H\dot H)M_4^2 + M_*^2(2\dot f\dot H + f^{(3)} + 3H^2\dot f + 2H\ddot f)\right] \\
&= -3\left[2\dot c + 6Hc - 2\dot H(M_4^2)^{\cdot} - 2(\ddot H + 3H\dot H)M_4^2 + M_*^2\left(f^{(3)} - (3H^2+\dot H)\dot f + 2H\ddot f\right)\right] \ , && (170)\\
C_{\dot\pi} &= 3\left(-2c + 3Hm_3^3 - 2HM_*^2\dot f + 2M_4^2\dot H + (m_3^3)^{\cdot} - 2M_*^2\ddot f\right) \ , && (171)\\
C_{\ddot\pi} &= 3(m_3^3 - M_*^2\dot f) \ , && (172)\\
C_\Phi^{(2)} &= -\left(2fM_*^2 - 6H\bar m_5 + 4\tilde m_4^2\right) \ , && (173)\\
C_\Psi^{(2)} &= 2fM_*^2 - 6H\bar m_5 - 6\dot{\bar m}_5 \ , && (174)\\
C_\pi^{(2)} &= -2\left(M_*^2\dot f + (M_4^2)^{\cdot} + HM_4^2 + 3H^2\bar m_5 + 3H\dot{\bar m}_5\right) \ , && (175)\\
C_{\dot\pi}^{(2)} &= -2(M_4^2 - 2\tilde m_4^2 + 3H\bar m_5) \ , && (176)\\
C_\Psi^{(4)} &= 16\bar\lambda \ , && (177)\\
C_\pi^{(4)} &= -2\bar m_5 + 16H\bar\lambda \ . && (178)
\end{aligned}
$$

- **Variation with respect to β:**

$$
\begin{aligned}
D_\Phi^{(2)} &= M_*^2 f + 2\tilde m_4^2 - 3\bar m_5 H \ , && (179)\\
D_\Psi^{(2)} &= -M_*^2 f \ , && (180)\\
D_{\dot\Psi}^{(2)} &= -3\bar m_5 \ , && (181)\\
D_\pi^{(2)} &= M_*^2\dot f + 2m_4^2 H + 2(m_4^2)^{\cdot} - 3\dot H\bar m_5 \ , && (182)\\
D_{\dot\pi}^{(2)} &= 2m_4^2 - 2\tilde m_4^2 \ , && (183)\\
D_\Psi^{(4)} &= -8\bar\lambda \ , && (184)\\
D_\pi^{(4)} &= \bar m_5 - 8H\bar\lambda \ . && (185)
\end{aligned}
$$

• **Variation with respect to π:**

$$
\begin{aligned}
E_\Phi &= 6cH + \dot c + H^2(9m_3^3 - 6M_*^2\dot f) + 3(2m_3^3 - M_*^2\dot f)\dot H - \dot\Lambda + 3H\left[4M_2^4 - 2M_4^2\dot H + (m_3^3)^\cdot\right] + 4(M_2^4)^\cdot \\
&= 12cH + 2\dot c + 3m_3^3(3H^2 + 2\dot H) - 6M_*^2\dot f(2\dot H + H^2) + 3H\left[4M_2^4 - 2M_4^2\dot H + (m_3^3)^\cdot\right] + 4(M_2^4)^\cdot \ ,
\end{aligned} \tag{186}
$$

$$E_{\dot\Phi} = 2c + 4M_2^4 + 3H(m_3^3 - M_*^2\dot f) \ , \tag{187}$$

$$E_\Psi = 3\left[6cH + \dot c + \dot\Lambda - 3M_*^2\dot f(2H^2 + \dot H)\right] = 0 \ , \tag{188}$$

$$E_{\dot\Psi} = 3\left[2c + 3Hm_3^3 - 4HM_*^2\dot f - 2M_4^2\dot H + (m_3^3)^\cdot\right] \ , \tag{189}$$

$$E_{\ddot\Psi} = 3(m_3^3 - M_*^2\dot f) \ , \tag{190}$$

$$
\begin{aligned}
E_\pi &= -\left[6M_4^2\dot H^2 - 3(m_3^3)^\cdot\dot H + 6H\dot c - 9H\dot H m_3^3 + \ddot c - 3M_*^2\dot H\ddot f - 6H^2M_*^2\ddot f - 3m_3^3\ddot H + \ddot\Lambda\right] \\
&= -\left[6M_4^2\dot H^2 - 3(m_3^3)^\cdot\dot H + 6\dot H c + 3M_*^2(\ddot H + 4H\dot H)\dot f - 9H\dot H m_3^3 - 3m_3^3\ddot H\right] \ ,
\end{aligned} \tag{191}
$$

$$E_{\dot\pi} = -2\left[3H\left(c + 2M_2^4\right) + \dot c + 2(M_2^4)^\cdot\right] \ , \tag{192}$$

$$E_{\ddot\pi} = -2\left(c + 2M_2^4\right) \ , \tag{193}$$

$$E_\Phi^{(2)} = -\left[m_3^3 + H\left(-2M_4^2 - 6H\bar m_5 + 4\tilde m_4^2\right) - M_*^2\dot f\right] \ , \tag{194}$$

$$E_\Psi^{(2)} = -2\left[2H\tilde m_4^2 + M_*^2\dot f - 3\bar m_5\dot H + 2(\tilde m_4^2)^\cdot\right] \ , \tag{195}$$

$$E_{\dot\Psi}^{(2)} = 2M_4^2 + 6H\bar m_5 - 4\tilde m_4^2 \ , \tag{196}$$

$$E_\pi^{(2)} = -\left[2c - 4M_4^2\dot H + 4\tilde m_4^2\dot H + (m_3^3)^\cdot + 4H^2\tilde m_4^2 + Hm_3^3 - 12\bar m_5 H\dot H + 4H(\tilde m_4^2)^\cdot\right] \ , \tag{197}$$

$$E_\Psi^{(4)} = -2\bar m_5 + 16H\bar\lambda \ , \tag{198}$$

$$E_\pi^{(4)} = -2(\bar m_4^2 + 2H\bar m_5) + 16H^2\bar\lambda \ . \tag{199}$$

• **Generalized Poisson equation:**

$$F_\Phi = -3M_*^2H\dot f + 2c + 4M_2^4 + 3Hm_3^3 \ , \tag{200}$$

$$F_{\dot\Psi} = -3M_*^2\dot f + 3m_3^3 \ , \tag{201}$$

$$F_\pi = 6M_*^2H^2\dot f - 6Hc - \dot c - \dot\Lambda + 3m_3^3\dot H \ = -3\dot H(M_*^2\dot f - m_3^3) \ , \tag{202}$$

$$F_{\dot\pi} = -(2c + 4M_2^4) \ , \tag{203}$$

$$F_\Psi^{(2)} = -(2fM_*^2 + 4\tilde m_4^2) \ , \tag{204}$$

$$F_\pi^{(2)} = \dot f M_*^2 - m_3^3 + 4Hm_4^2 - 4H\tilde m_4^2 \ . \tag{205}$$

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Article B

Single-Field Consistency Relations of Large Scale Structure Part II: Resummation and Redshift Space

Single-Field Consistency Relations of Large Scale Structure
Part II: Resummation and Redshift Space

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Abstract

We generalize the recently derived single-field consistency relations of Large Scale Structure in two directions. First, we treat the effect of the long modes (with momentum q) on the short ones (with momentum k) non-perturbatively, by writing resummed consistency relations which do not require $k/q \cdot \delta_q \ll 1$. These relations do not make any assumptions on the short-scales physics and are extended to include (an arbitrary number of) multiple long modes, internal lines with soft momenta and soft loops. We do several checks of these relations in perturbation theory and we verify that the effect of soft modes always cancels out in equal-time correlators. Second, we write the relations directly in redshift space, without assuming the single-stream approximation: not only the long mode affects the short scales as a homogeneous gravitational field, but it also displaces them by its velocity along the line-of-sight. Redshift space consistency relations still vanish when short modes are taken at equal time: an observation of a signal in the squeezed limit would point towards multifield inflation or a violation of the equivalence principle.

1 Introduction

Our detailed knowledge of the Universe is mostly based on the study of correlation functions of perturbations around a homogeneous background. A considerable effort has been devoted over the years to the calculation of these correlators during inflation, for the CMB temperature fluctuations and for the present distribution of dark and luminous matter. It is by now well understood that calculations dramatically simplify in the parametric limit in which one (or more) of the momenta (that we call q in this paper) becomes much smaller than the others (denoted by k). Recently [1, 2, 3], these arguments have been applied to the matter (or Λ)-dominated phase to show that the leading term as $q \to 0$ of any correlation function with $(n+1)$ legs can be written in terms of an n-point function: the so-called consistency relations. Although the arguments work in a fully relativistic treatment [3], which is mandatory if we want to follow the evolution of the modes back in time and connect with inflation, in this paper we focus on the non-relativistic limit, which is valid deep inside the horizon.

The physical argument behind the consistency relations in the non-relativistic limit is that at leading order in q a long mode gives rise to a homogeneous gravitational field $\vec{\nabla}\Phi$. The effect of this mode on the short-scale physics can be derived exactly using the equivalence principle and erasing the long mode with a suitable change of coordinates. This logic makes virtually no assumption about the physics at short scales, including the complications due to baryons. However, the cancellation of the long mode by a change of coordinates can be performed only assuming that gravity is all there is: no extra degrees of freedom during inflation (i.e. single-field inflation) and no extra forces (violation of the equivalence principle) at present. Therefore the consistency relations can be seen as a test of these two assumptions.

In this paper, which is the natural continuation of [3], we follow our study of the subject in two directions. First, we want to extend the consistency relations non-linearly in the long mode (Section 2). The displacement due to a homogeneous gravitational field scales with time as $\Delta\vec{x} \sim \vec{\nabla}\Phi_L\, t^2$, so that the effect on the short modes of momentum k goes as

$$\vec{k} \cdot \Delta\vec{x} \sim k\, q\, \Phi_L\, t^2 \sim \frac{k}{q}\delta_L \,, \tag{1}$$

where δ_L is the long-mode density contrast[1]. Notice that this is parametrically larger than δ_L, the natural expansion parameter of perturbation theory, and this is why one is able to capture the leading $q \to 0$ behaviour. Obviously, the fact that we can erase a homogeneous gravitational field by going to a free falling frame is an exact statement, that does not require the gravitational field to be small. This implies that we do not need to expand in $k/q \cdot \delta_L$ that can be large, while we keep δ_L small to allow for a perturbative treatment of the long mode. In Section 2 we are going to give a resummed version of the consistency relations which is exact in $k/q \cdot \delta_L$. This allows to discuss the case of multiple soft modes and check the relations with the perturbation theory result. With the same logic, we will study the effect of internal soft modes and loops of soft modes.

The second topic of the paper (Section 3) is to derive consistency relations directly in redshift space, since this is where the distribution of matter is measured. We will do so without assuming anything about the short modes, in particular the single-stream approximation that breaks down in virialized objects. The redshift consistency relations contain an extra piece because the long mode,

[1] This is the leading effect in the non-relativistic limit: relativistic corrections are further suppressed by powers of $k/aH \ll 1$ [3], which are negligible well inside the horizon.

besides inducing a homogeneous gravitational field in real space, also affects the position of the short modes in redshift space along the line-of-sight. The redshift space consistency relations state that the correlation functions vanish at leading order for $q \to 0$ when the short modes are taken at the same time, as it happens in real space. Given that it is practically impossible, as we will discuss, to study correlation functions of short modes at different times, it is hard to believe that these relations will be verified with real data. However, if a signal is detected at equal times, the consistency relations are not satisfied and this would indicate that at least one of the assumptions does not hold. This would represent a detection of either multi-field inflation or violation of the equivalence principle (or both!).

As explained in [3], one of the conditions for the validity of the consistency relations is that the long mode has always been out of the sound horizon since inflation. Indeed, a well-understood example where the consistency relations are not obeyed is the case of baryons and cold dark matter particles after decoupling. Before recombination, while dark matter follows geodesics, baryons are tightly coupled to photons through Thomson scattering and display acoustic oscillations. Later on, baryons recombine and decouple from photons. Thus, as their sound speed drops they start following geodesics, but with a larger velocity than that of dark matter on comoving scales below the sound horizon at recombination. As discussed in [4], the long-wavelength relative velocity between baryons and CDM reduces the formation of early structures on small scales, through a genuinely nonlinear effect.

The fact that baryons have a different initial large-scale velocity compared to dark matter implies, if the long mode is shorter than the comoving sound horizon at recombination, that the change of coordinates that erases the effect of the long mode is not the same for the two species. Thus the effect of the long mode does not cancel out in the equal-time correlators involving different species [5, 6]. In particular, the amplitude of the short-scale equal-time n-point functions becomes correlated with the long-wavelength isodensity mode, so that the $(n+1)$-point functions in the squeezed limit do not vanish at equal time. This effect, however, becomes rapidly negligible at low redshifts because the relative comoving velocity between baryons and dark matter decays as the scale factor, $|\vec{v}_{\rm b} - \vec{v}_{\rm CDM}| \propto 1/a$.[2] Hence, while a deviation can be sizable at high redshifts, it can be neglected in galaxy surveys and the consistency relations apply also when the long mode is shorter than the comoving sound horizon at recombination. We conclude that the vanishing of the correlation functions at leading order in $q \to 0$ is very robust.

2 Resumming the long mode

Let us consider a flat unperturbed FRW universe and add to it a homogenous gradient of the Newtonian potential Φ_L.[3] Provided all species feel gravity the same way—namely, assuming the equivalence principle—we can get rid of the effect of $\vec{\nabla}\Phi_L$ by going into a frame which is free falling in the constant gravitational field. The coordinate change to the free-falling frame is (we are using conformal time

[2]The violation of the consistency relations decays as $(D_{\rm iso}/D)^2 \propto (a^2 HfD)^{-2} \sim (1+z)^{3/2}$ where $D_{\rm iso} \propto |\vec{v}_{\rm b} - \vec{v}_{\rm CDM}|/(aHf)$ is the growth function of the long-wavelength isodensity mode, D is the growth function of the long-wavelength adiabatic growing mode, f is the growth rate and H is the Hubble rate (see [5] for details); in the last approximate equality we have used matter dominance. Thus, the effect is already sub-percent at $z \sim 40$.

[3]Since we are interested in the non-relativistic limit, we do not consider a constant value of Φ_L, which is immaterial in this limit.

$d\eta \equiv dt/a(t)$)

$$\vec{x} \to \vec{x} + \delta\vec{x}(\eta) \ , \qquad \delta\vec{x}(\eta) \equiv -\int \vec{v}_L(\tilde{\eta})\, \mathrm{d}\tilde{\eta} \ , \tag{2}$$

while time is left untouched. The velocity $\vec{v}_L$ satisfies the Euler equation in the presence of the homogenous force, whose solution is

$$\vec{v}_L(\eta) = -\frac{1}{a(\eta)} \int a(\tilde{\eta}) \vec{\nabla}\Phi_L(\tilde{\eta})\, \mathrm{d}\tilde{\eta} \ . \tag{3}$$

To derive the consistency relations we start from real space. Here, for definiteness, we denote by $\delta^{(g)}$ the density contrast of the galaxy distribution. However, the relations that we will derive are more general and hold for any species—halos, baryons, etc., irrespectively of their bias with respect to the underlying dark matter field. Following the argument above, any n-point correlation function of short wavelength modes of $\delta^{(g)}$ in the presence of a slowly varying $\Phi_L(\vec{y})$ is equivalent to the same correlation function in displaced spatial coordinates, $\vec{\tilde{x}} \equiv \vec{x} + \delta\vec{x}(\vec{y},\eta)$, where the displacement field $\delta\vec{x}(\vec{y},\eta)$ is given by eq. (2) and $\vec{y}$ is an arbitrary point—e.g., the midpoint between $\vec{x}_1, \dots, \vec{x}_n$—whose choice is irrelevant at order q/k. This statement can be formulated with the following relation,

$$\begin{aligned}\langle \delta^{(g)}(\vec{x}_1,\eta_1)\cdots\delta^{(g)}(\vec{x}_n,\eta_n)|\Phi_L(\vec{y})\rangle &\approx \langle \delta^{(g)}(\vec{\tilde{x}}_1,\eta_1)\cdots\delta^{(g)}(\vec{\tilde{x}}_n,\eta_n)\rangle_0 \\ &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle_0\, e^{i\sum_a \vec{k}_a\cdot(\vec{x}_a+\delta\vec{x}(\vec{y},\eta_a))} \ ,\end{aligned} \tag{4}$$

where in the last line we have simply taken the Fourier transform of the right-hand side of the first line. Here and in the following, by the subscript 0 after an expectation value we mean that the average is taken setting $\Phi_L = 0$ (and not averaging over it); while by $\approx$ we mean an equality that holds in the limit in which there is a separation of scales between long and short modes. In momentum space this holds when the momenta of the soft modes is sent to zero. In other words, corrections to the right-hand side of $\approx$ are suppressed by $\mathcal{O}(q/k)$.

From eq. (2) and using the continuity equation $\delta' + \vec{\nabla}\cdot\vec{v} = 0$, we can rewrite each Fourier mode of the displacement field as

$$\delta\vec{x}(\vec{p},\eta) = -i\frac{\vec{p}}{p^2}\delta(\vec{p},\eta) \equiv -i\frac{\vec{p}}{p^2}D(\eta)\delta_0(\vec{p}) \ , \tag{5}$$

where in the second equality we have defined $D(\eta)$, the growth factor of density fluctuations of the *long* mode and $\delta_0(\vec{p})$, a Gaussian random field with power spectrum $P_0(p)$ which represents the initial condition of the density fluctuations of the long mode [7]. Notice that the first equality of eq. (4) is based on the crucial assumption that the long mode is statistically uncorrelated with the short ones. This only works in single-field models of inflation, which we assume throughout. Notice also that eq. (5), when going beyond the linear theory, will only receive corrections of order δ, that we can neglect for our purposes since we are only interested in corrections which are enhanced by $1/p$.

At this stage, we can compute an $(n+1)$-point correlation function in the squeezed limit by multiplying the left-hand side of eq. (4) by δ_L and averaging over the long mode. Since the only dependence on Φ_L in eq. (4) is in the exponential of $i\sum_a \vec{k}_a\cdot\delta\vec{x}(\vec{y},\eta_a)$, we obtain

$$\begin{aligned}\langle \delta_L(\vec{x},\eta)\langle\delta^{(g)}(\vec{x}_1,\eta_1)\cdots\delta^{(g)}(\vec{x}_n,\eta_n)|\Phi_L\rangle\rangle_{\Phi_L} \approx &\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3k_n}{(2\pi)^3}\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle_0\, e^{i\sum_a \vec{k}_a\cdot\vec{x}_a} \\ &\times \int \frac{\mathrm{d}^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \langle \delta_{\vec{q}}(\eta) e^{i\sum_a \vec{k}_a\cdot\delta\vec{x}(\vec{y},\eta_a)}\rangle_{\Phi_L} \ .\end{aligned} \tag{6}$$

It is then convenient to rewrite this exponential as

$$\exp\Big[i\sum_a \vec{k}_a\cdot\delta\vec{x}(\vec{y},\eta_a)\Big] = \exp\Big[\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} J(\vec{p})\delta_0(\vec{p})\Big]\,, \tag{7}$$

where

$$J(\vec{p}) \equiv \sum_a D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}\, e^{i\vec{p}\cdot\vec{y}}\,. \tag{8}$$

The integral is restricted to soft momenta, smaller than a UV cut-off Λ, which must be much smaller than the hard modes of momenta k_a. Averaging the right-hand side of eq. (7) over the long wavelength Gaussian random initial condition $\delta_0(\vec{p})$ yields[4]

$$\Big\langle \exp\Big[\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} J(\vec{p})\delta_0(\vec{p})\Big]\Big\rangle_{\Phi_L} = \exp\left[\frac{1}{2}\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} J(\vec{p})J(-\vec{p})P_0(p)\right]\,. \tag{9}$$

We can use this relation to compute the expectation value of δ_L with the exponential,

$$\begin{aligned}\Big\langle \delta_{\vec{q}}(\eta)\exp\Big(i\sum_a \vec{k}_a\cdot\delta\vec{x}(\vec{y},\eta_a)\Big)\Big\rangle_{\Phi_L} &= (2\pi)^3 D(\eta)\frac{\delta}{\delta J(\vec{q})}\Big\langle \exp\Big[\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} J(\vec{p})\delta_0(\vec{p})\Big]\Big\rangle_{\Phi_L} \\ &= P(q,\eta)\frac{J(-\vec{q})}{D(\eta)}\exp\left[\frac{1}{2}\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} J(\vec{p})J(-\vec{p})P_0(p)\right]\,,\end{aligned} \tag{10}$$

where we have defined the power spectrum at time η: $P(q,\eta)\equiv D^2(\eta)P_0(q)$. Finally, rewriting eq. (6) in Fourier space using the above relation and the definition of J, eq. (8), we obtain the resummed consistency relations in the squeezed limit,

$$\begin{aligned}\langle \delta_{\vec{q}}(\eta)\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' \approx &-P(q,\eta)\sum_a \frac{D(\eta_a)}{D(\eta)}\frac{\vec{k}_a\cdot\vec{q}}{q^2}\langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle'_0 \\ &\times \exp\left[-\frac{1}{2}\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(\sum_a D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}\Big)^2 P_0(p)\right]\,,\end{aligned} \tag{11}$$

where, here and in the following, primes on correlation functions indicate that the momentum conserving delta functions have been removed. However, what one observes in practice is not the expectation value $\langle\ldots\rangle_0$ with the long modes set artificially to zero: one wants to rewrite the right-hand side of eq. (11) in terms of an average over the long modes. Using eq. (9) one gets:

$$\langle\langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)|\Phi_L\rangle\rangle_{\Phi_L} \approx \exp\left[-\frac{1}{2}\int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(\sum_a D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}\Big)^2 P_0(p)\right]\langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle_0\,. \tag{12}$$

Once written in terms of the observable quantity the consistency relation comes back to the simple form:

$$\langle \delta_{\vec{q}}(\eta)\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' \approx -P(q,\eta)\sum_a \frac{D(\eta_a)}{D(\eta)}\frac{\vec{k}_a\cdot\vec{q}}{q^2}\langle \delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle'\,. \tag{13}$$

[4]This result will receive corrections due to primordial non-Gaussianities. Indeed, even in single-field models of inflation, the statistics of modes with comparable wavelength can deviate from Gaussianity. We neglect these corrections in the following.

This equation has the same form as the consistency relations obtained in Refs. [1, 2, 3], but now it *does not* rely on a linear expansion in the displacement field,

$$\frac{|\delta\vec{x}|}{|\vec{x}|} \sim \frac{k}{q}\delta_L \ll 1 \ . \tag{14}$$

Indeed, to derive eq. (11) we have assumed that the long mode is in the linear regime, i.e. $\delta_L \ll 1$, but no assumption has been made on $(k/q)\delta_L$, which can be as large as one wishes. For equal-time correlators the right-hand side vanishes at leading order in q because $\sum_a \vec{k}_a = \vec{q}$, in the same way as in the linearized version [1, 2, 3]. The resummation of long wavelengths in terms of a global translation of spatial coordinates—whose effect vanishes in equal-time correlation functions—was also performed in [5, 6] by using the so-called *eikonal* approximation of the equations of motion of standard perturbation theory[5]

It is important to stress that here we made practically no assumptions on the short modes. We did not assume that they are in the linear regime or that the single-stream approximation holds. The relation also takes into account all complications due to baryon physics and it does not assume a description in terms of a Vlasov-Poisson system. We did not assume any model of bias between the short-scale $\delta^{(g)}$ and the underlying dark matter distribution δ. We did not assume that the number of galaxies is conserved at short-scales, so the relation is valid including the formation and merging history. We thus believe that our derivation, rooted only on the equivalence principle, is more robust than the one of [1, 2] based on the explicit equations for dark matter and for the galaxy fluid. Notice however that, while we are completely general about the short-modes physics, the long mode is treated in perturbation theory including its bias. Of course what enters in the consistency relations is only the velocity field of the long mode eq. (3), related to Φ_L by the Euler equation. In converting this quantity in the density of some kind of objects, one has to rely on the conservation equation and this introduces the issue of bias and of its time-dependence. However, one can measure the large-scale potential in many ways, minimizing the systematic and cosmic-variance uncertainty [8].

As shown below, one can straightforwardly extend this procedure and derive consistency relations involving an arbitrary number of soft legs in the correlation functions or use it to study the effect of soft loops and internal lines.

2.1 Several soft legs

The generalisation of the consistency relations above to multiple soft legs (for an analogous discussion in inflation see [9]) relies on taking successive functional derivatives with respect to $J(\vec{q}_i)$ of eq. (9). As an example, we can explicitly compute the consistency relations with two soft modes. In this case the

[5] It is not surprising that the consistency relation eq. (13) remains the same even non-linearly in $(k/q)\delta_L$ working directly in terms of the expectation values $\langle \ldots \rangle$ averaged over the long modes. Indeed, neglecting primordial non-Gaussianities, the effect of the mode with momentum $\vec{q}$ is the same as a change of coordinates, even when the short-scale correlation functions are averaged over all long modes. Since, as we discussed, also eq. (5) does not require an expansion in $(k/q)\delta_L$, eq. (13) follows.

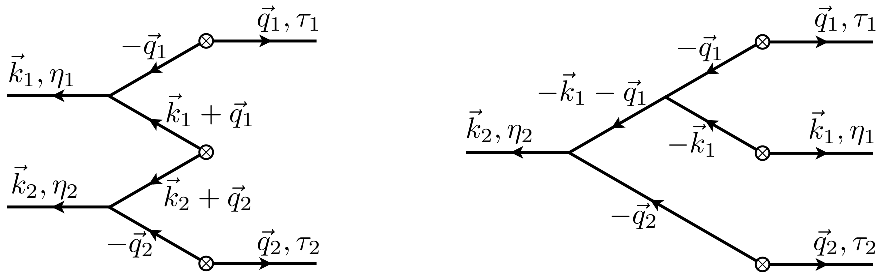

Figure 1: Two diagrams that contribute to the tree-level trispectrum. Left: T_{1122}. Right: T_{1113}.

$(n+2)$-point function reads

$$\langle \delta_L(\vec{y}_1,\tau_1)\delta_L(\vec{y}_2,\tau_2)\delta^{(g)}(\vec{x}_1,\eta_1)\cdots\delta^{(g)}(\vec{x}_n,\eta_n)\rangle \approx \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3k_n}{(2\pi)^3}\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle_0\, e^{i\sum_a \vec{k}_a\cdot\vec{x}_a}$$
$$\times\int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}e^{i(\vec{q}_1\cdot\vec{y}_1+\vec{q}_2\cdot\vec{y}_2)}\left\langle\delta_{\vec{q}_1}(\tau_1)\delta_{\vec{q}_2}(\tau_2)e^{\int^\Lambda\frac{\mathrm{d}^3p}{(2\pi)^3}J(\vec{p})\delta_0(\vec{p})}\right\rangle. \tag{15}$$

To compute the average over the long modes in the last line, it is enough to take two functional derivatives of eq. (9),

$$\begin{aligned}\left\langle\delta_{\vec{q}_1}(\tau_1)\delta_{\vec{q}_2}(\tau_2)e^{\int^\Lambda\frac{\mathrm{d}^3p}{(2\pi)^3}J(\vec{p})\delta_0(\vec{p})}\right\rangle &= (2\pi)^6 D(\tau_1)D(\tau_2)\frac{\delta}{\delta J(\vec{q}_1)}\frac{\delta}{\delta J(\vec{q}_2)}\left\langle e^{\int^\Lambda\frac{\mathrm{d}^3p}{(2\pi)^3}J(\vec{p})\delta_0(\vec{p})}\right\rangle\\ &= \frac{J(-\vec{q}_1)}{D(\tau_1)}\frac{J(-\vec{q}_2)}{D(\tau_2)}P(q_1,\tau_1)P(q_2,\tau_2)e^{\frac{1}{2}\int^\Lambda\frac{\mathrm{d}^3p}{(2\pi)^3}J(\vec{p})J(-\vec{p})P_0(p)},\end{aligned} \tag{16}$$

where we have assumed $\vec{q}_1+\vec{q}_2\neq 0$ to get rid of unconnected contributions. In Fourier space, this yields

$$\begin{aligned}\langle\delta_{\vec{q}_1}(\tau_1)\delta_{\vec{q}_2}(\tau_2)\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle' &\approx P(q_1,\tau_1)P(q_2,\tau_2)\\ &\times\sum_a\frac{D(\eta_a)}{D(\tau_1)}\frac{\vec{k}_a\cdot\vec{q}_1}{q_1^2}\sum_b\frac{D(\eta_b)}{D(\tau_2)}\frac{\vec{k}_b\cdot\vec{q}_2}{q_2^2}\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle',\end{aligned} \tag{17}$$

where again we have used eq. (12) to write the result in terms of correlation functions averaged over the long modes.

As a simple example, let us consider eq. (17) in the case where $n=2$ and $\delta^{(g)}$ describes dark matter perturbations, i.e. $\delta^{(g)}\equiv\delta$. In this case, at lowest order in $\frac{k}{q}\delta(\vec{q},\eta)$—i.e. setting the exponential in the third line to unity—the above relation reduces to

$$\langle\delta_{\vec{q}_1}(\tau_1)\delta_{\vec{q}_2}(\tau_2)\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'\approx\frac{(D(\eta_1)-D(\eta_2))^2}{D(\tau_1)D(\tau_2)}\frac{\vec{q}_1\cdot\vec{k}_1}{q_1^2}\frac{\vec{q}_2\cdot\vec{k}_1}{q_2^2}P(q_1,\tau_1)P(q_2,\tau_2)\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'. \tag{18}$$

We can check that this expression correctly reproduces the tree-level trispectrum computed in perturbation theory in the double-squeezed limit. This can be easily computed by summing the two types of diagrams displayed in Fig. 1. The diagram on the left-hand side represents the case where the density perturbations of the short modes are both taken at second order, yielding

$$
\begin{aligned}
T_{1122} =& D(\tau_1)D(\tau_2)D(\eta_1)D(\eta_2)P_0(q_1)P_0(q_2)F_2(-\vec{q}_1,\vec{k}_1+\vec{q}_1)F_2(-\vec{q}_2,\vec{k}_2+\vec{q}_2)\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' + \text{perms} \\
\approx& -8\frac{\vec{q}_1\cdot\vec{k}_1}{2q_1^2}\frac{\vec{q}_2\cdot\vec{k}_1}{2q_2^2}\frac{D(\eta_1)D(\eta_2)}{D(\tau_1)D(\tau_2)}P(q_1,\tau_1)P(q_2,\tau_2)\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' ,
\end{aligned}
\tag{19}
$$

where, on the right-hand side of the first line, $F_2(\vec{p}_1,\vec{p}_2)$ is the usual kernel of perturbation theory, which in the limit where $p_1 \ll p_2$ simply reduces to $\vec{p}_1\cdot\vec{p}_2/(2p_1^2)$ [7]. The second type of diagram, displayed on the right-hand side of Fig. 1, is obtained when one of the short density perturbations is taken at third order; it gives

$$
\begin{aligned}
T_{1113} =& D(\eta_2)^2D(\tau_1)D(\tau_2)P_0(q_1)P_0(q_2)F_3(-\vec{q}_1,-\vec{q}_2,-\vec{k}_1)\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' + \text{perms} \\
\approx& \, 4\frac{\vec{q}_1\cdot\vec{k}_1}{2q_1^2}\frac{\vec{q}_2\cdot\vec{k}_1}{2q_2^2}\frac{D(\eta_2)^2}{D(\tau_1)D(\tau_2)}P(q_1,\tau_1)P(q_2,\tau_2)\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' ,
\end{aligned}
\tag{20}
$$

where, on the right-hand side of the first line, $F_3(\vec{p}_1,\vec{p}_2,\vec{p}_3)$ is the third-order perturbation theory kernel, which in the limit where $p_1,p_2 \ll p_3$ reduces to $(\vec{p}_1\cdot\vec{p}_3)(\vec{p}_2\cdot\vec{p}_3)/(4p_1^2p_2^2)$ [7]. As expected, summing up all the contributions to the connected part of the trispectrum, i.e. $T_{1122}+T_{1131}+T_{1113}$, using eqs. (19) and (20) and $\vec{k}_2 \approx -\vec{k}_1$ one obtains eq. (18).

2.2 Soft Loops

So far we have derived consistency relations where the long modes appear explicitly as external legs. We now show that our arguments can also capture the effect on short-scale correlation functions of soft modes running in loop diagrams. We already did this in eq (12)

$$
\langle\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)|\Phi_L\rangle\rangle_{\Phi_L} \approx \exp\left[-\frac{1}{2}\int^{\Lambda}\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sum_a D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}\right)^2P_0(p)\right]\langle\delta^{(g)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{\vec{k}_n}(\eta_n)\rangle_0 \, .
\tag{21}
$$

The exponential in this expression can be expanded at a given order, corresponding to the number of soft loops dressing the n-point correlation function. Each loop carries a contribution $\propto k^2\int \mathrm{d}pP_0(p)$ to the correlation function. However, this expression makes it very explicit that at all loop order these contributions have no effect on equal-time correlators, because in this case the exponential on the right-hand side is identically unity. This confirms previous analysis on this subject [10, 11, 5, 6, 12, 13]. It is important to notice again, however, that in our derivation this cancellation is more general and robust that in those references, as it takes place independently of the equations of motion for the short modes and is completely agnostic about the short-scale physics. It simply derives from the equivalence principle.

Nevertheless, soft loops contribute to unequal-time correlators. As a check of the expression above, one can compute the contribution of soft modes to the 1-loop unequal-time matter power spectrum, $\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'$, and verify that this reproduces the standard perturbation theory result. Expanding at order $(\frac{k}{p}\delta)^2$ the exponential in eq. (12) for $n=2$, one obtains the 1-loop contribution to the power spectrum,

$$
\langle\delta^{(g)}_{\vec{k}}(\eta_1)\delta^{(g)}_{-\vec{k}}(\eta_2)\rangle'_{1-\text{soft loop}} \approx -\frac{1}{2}\left(D(\eta_1)-D(\eta_2)\right)^2\int^{\Lambda}\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\frac{\vec{p}\cdot\vec{k}}{p^2}\right)^2P_0(p)\langle\delta^{(g)}_{\vec{k}}(\eta_1)\delta^{(g)}_{-\vec{k}}(\eta_2)\rangle'_0 \, .
\tag{22}
$$

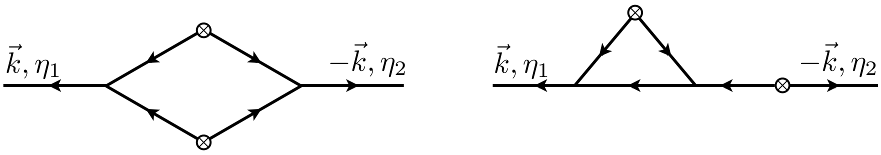

Figure 2: Two diagrams that contribute to the 1-loop power spectrum. Left: P_{22}. Right: P_{31}.

Let us now compute the analogous contribution in perturbation theory. Two types of diagrams are going to be relevant; these are shown in Fig. 2. The one on the left, usually called P_{22}, yields

$$P_{22} \approx 4D(\eta_1)D(\eta_2) \int^{\Lambda} \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\vec{p}\cdot\vec{k}}{2p^2} \right)^2 P_0(p) \langle \delta_{\vec{k}}(\eta_1)\delta_{-\vec{k}}(\eta_2)\rangle_0' \,, \tag{23}$$

while the diagram on the right, P_{31}, gives

$$P_{31} \approx -2D(\eta_1)^2 \int^{\Lambda} \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\vec{p}\cdot\vec{k}}{2p^2} \right)^2 P_0(p) \langle \delta_{\vec{k}}(\eta_1)\delta_{-\vec{k}}(\eta_2)\rangle_0' \,. \tag{24}$$

Summing up all the different contributions, $P_{22} + P_{31} + P_{13}$, one obtains eq. (22).

2.3 Soft internal lines

Another kinematical regime in which the consistency relations can be applied is the limit in which the sum of some of the external momenta becomes very small, for instance $|\vec{k}_1 + \cdots + \vec{k}_m| \ll k_1, \ldots, k_m$. In this limit, the dominant contribution to the n-point function comes from the diagram where m external legs of momenta $\vec{k}_1, \ldots, \vec{k}_m$ exchange soft modes with momentum $\vec{q} = \vec{k}_1 + \cdots + \vec{k}_m$ with $n-m$ external legs with momenta $\vec{k}_{m+1}, \ldots, \vec{k}_n$ (for an analogous case in inflation see [14, 15]). In the language of our approach, this contribution comes from averaging a product of m-point and $(n-m)$-point functions under the effect of long modes.

In this case, the n-point function in real space can be written as

$$\begin{aligned} &\langle \delta(\vec{x}_1,\eta_1)\cdots\delta(\vec{x}_m,\eta_m)\, \delta(\vec{x}_{m+1},\eta_{m+1})\cdots\delta(\vec{x}_n,\eta_n)\rangle \\ &\qquad \approx \langle \langle \delta(\vec{x}_1,\eta_1)\cdots\delta(\vec{x}_m,\eta_m)|\Phi_L\rangle \langle \delta(\vec{x}_{m+1},\eta_{m+1})\cdots\delta(\vec{x}_n,\eta_n)|\Phi_L\rangle\rangle_{\Phi_L} \,, \end{aligned} \tag{25}$$

where here and in the rest of the section we drop the superscript $^{(g)}$ on the galaxy density contrast to lighten the notation. Now we can straightforwardly apply the equations from the previous sections. As before, the long mode can be traded for the change of coordinates. Rewriting the right-hand side in Fourier space we get

$$\begin{aligned} &\langle \delta(\vec{x}_1,\eta_1)\cdots\delta(\vec{x}_m,\eta_m)\, \delta(\vec{x}_{m+1},\eta_{m+1})\cdots\delta(\vec{x}_n,\eta_n)\rangle \\ &\qquad \approx \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_m}(\eta_m)\rangle_0 \langle \delta_{\vec{k}_{m+1}}(\eta_{m+1})\cdots\delta_{\vec{k}_n}(\eta_n)\rangle_0 \, e^{i\sum_a \vec{k}_a\cdot\vec{x}_a} \\ &\qquad \times \left\langle \exp\left[i\sum_{a=1}^{m} \vec{k}_a\cdot\delta\vec{x}(\vec{y}_1,\eta_a)\right] \cdot \exp\left[i\sum_{a=m+1}^{n} \vec{k}_a\cdot\delta\vec{x}(\vec{y}_2,\eta_a)\right] \right\rangle_{\Phi_L} \,, \end{aligned} \tag{26}$$

where $\vec{y}_1$ and $\vec{y}_2$ are two different points respectively close to $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m)$ and $(\vec{x}_{m+1}, \vec{x}_{m+2}, \dots, \vec{x}_n)$. The average over the long mode can be rewritten as

$$\left\langle \exp\left[\int^{\Lambda} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\big(J_1(\vec{p}) + J_2(\vec{p})\big)\delta_0(\vec{p})\right] \right\rangle_{\Phi_L} \tag{27}$$

with

$$J_1(\vec{p}) = \sum_{a=1}^{m} D(\eta_a)\frac{\vec{k}_a \cdot \vec{p}}{p^2} e^{i\vec{p}\cdot\vec{y}_1}\ , \quad J_2(\vec{p}) = \sum_{a=m+1}^{n} D(\eta_a)\frac{\vec{k}_a \cdot \vec{p}}{p^2} e^{i\vec{p}\cdot\vec{y}_2}\ . \tag{28}$$

Taking the expectation value over the long mode using the expression for averaging the exponential of a Gaussian variable, i.e. eq. (9), eq. (26) can be written as

$$\begin{aligned} &\langle \delta(\vec{x}_1,\eta_1)\cdots\delta(\vec{x}_m,\eta_m)\ \delta(\vec{x}_{m+1},\eta_{m+1})\cdots\delta(\vec{x}_n,\eta_n)\rangle \\ &\quad\approx \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3k_n}{(2\pi)^3}\langle\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_m}(\eta_m)\rangle'\langle\delta_{\vec{k}_{m+1}}(\eta_{m+1})\cdots\delta_{\vec{k}_n}(\eta_n)\rangle'\, e^{i\sum_a \vec{k}_a\cdot\vec{x}_a} \\ &\quad\times\exp\left[-\int^{\Lambda}\frac{\mathrm{d}^3p}{(2\pi)^3}J_1(\vec{p})J_2(\vec{p})P_0(\vec{p})\right]\ . \end{aligned} \tag{29}$$

We are interested in the soft internal lines, that come from the cross term, i.e. the last line of eq. (29). Notice that $J_1(\vec{p})$ and $J_2(\vec{p})$ are evaluated at different points $\vec{y}_1$ and $\vec{y}_2$ separated by a distance $\vec{x}$.[6] It is lengthy but straightforward to take the Fourier transform of this equation, which yields

$$\begin{aligned} &\langle\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_m}(\eta_m)\delta_{\vec{k}_{m+1}}(\eta_{m+1})\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' \\ &\quad\approx\langle\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_m}(\eta_m)\rangle'\,\langle\delta_{\vec{k}_{m+1}}(\eta_{m+1})\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' \\ &\quad\times\int \mathrm{d}^3x\, e^{-i\sum_{i=1}^m \vec{k}_i\cdot\vec{x}}\exp\left[-\int^{\Lambda}\frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}\sum_{a=1}^m D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}\sum_{a=m+1}^n D(\eta_a)\frac{\vec{k}_a\cdot\vec{p}}{p^2}P_0(p)\right]. \end{aligned} \tag{30}$$

The last line encodes the effect of soft modes with total momentum $\vec{q} = \vec{k}_1 + \cdots + \vec{k}_m$ exchanged between m external legs of momenta $\vec{k}_1, \dots, \vec{k}_m$ and $n-m$ external legs with momenta $\vec{k}_{m+1}, \dots, \vec{k}_n$, in the limit $q/k_i \to 0$. Expanding the exponential at a given order in $P_0(p)$ yields the number of soft lines exchanged. The integral in d^3x ensures that the sum of the internal momenta is $\vec{q}$.

Equation (30) can be easily generalized to consider the case where more than two sums of momenta become small, i.e. when soft internal lines are exchanged between more than two hard-modes diagrams. The conclusion is always the same: soft internal lines do not contribute to equal time correlators at order $\propto k^2\int \mathrm{d}pP_0(p)$. Again, this statement is very general irrespectively of the assumption about the short scales.

As a concrete example, let us consider the case $m=2, n=4$, i.e. a 4-point function in the collapsed limit $|\vec{k}_1+\vec{k}_2| \ll k_1, k_2$, and the exchange of a single soft line. In this case, expanding the exponential at first order in $P_0(p)$, the above equation yields

$$\begin{aligned} &\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\delta_{\vec{k}_3}(\eta_3)\delta_{\vec{k}_4}(\eta_4)\rangle_c' \\ &\quad\approx -\,\langle\delta(\vec{k}_1,\eta_1)\delta(\vec{k}_2,\eta_2)\rangle'\langle\delta(\vec{k}_3,\eta_3)\cdots\delta(\vec{k}_4,\eta_4)\rangle' \\ &\quad\times\int^{\Lambda}\mathrm{d}^3p\big(D(\eta_1)-D(\eta_2)\big)\frac{\vec{k}_1\cdot\vec{p}}{p^2}\big(D(\eta_3)-D(\eta_4)\big)\frac{\vec{k}_3\cdot\vec{p}}{p^2}P_0(p)\delta_D(\vec{p}-\vec{k}_1-\vec{k}_2)\,, \end{aligned} \tag{31}$$

[6]For definiteness, we can choose $\vec{y}_1 = \frac{1}{m}\sum_{a=1}^m \vec{x}_a$ and $\vec{y}_2 = \frac{1}{n-m}\sum_{a=m+1}^n \vec{x}_a$.

where we have considered only the connected diagram and, for simplicity, we are neglecting soft loops attached to each lines. To compare with perturbation theory, we need to compute the tree-level exchange diagram. The contribution from taking $\vec{k}_1$ and $\vec{k}_3$ at second order yields

$$T_{2121} \approx -4D(\eta_1)D(\eta_3)P_0(|\vec{k}_1+\vec{k}_2|)\frac{\vec{k}_1\cdot(\vec{k}_1+\vec{k}_2)}{2|\vec{k}_1+\vec{k}_2|^2}\frac{\vec{k}_3\cdot(\vec{k}_1+\vec{k}_2)}{2|\vec{k}_1+\vec{k}_2|^2}\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'\langle\delta_{\vec{k}_3}(\eta_3)\delta_{\vec{k}_4}(\eta_4)\rangle' \ , \tag{32}$$

and summing up the other permutations lead to

$$\begin{aligned}\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\delta_{\vec{k}_3}(\eta_3)\delta_{\vec{k}_4}(\eta_4)\rangle_c' &\approx -\big(D(\eta_1)-D(\eta_2)\big)\big(D(\eta_3)-D(\eta_4)\big)P_0(|\vec{k}_1+\vec{k}_2|)\\ &\times\frac{\vec{k}_1\cdot(\vec{k}_1+\vec{k}_2)}{|\vec{k}_1+\vec{k}_2|^2}\frac{\vec{k}_3\cdot(\vec{k}_1+\vec{k}_2)}{|\vec{k}_1+\vec{k}_2|^2}\langle\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'\langle\delta_{\vec{k}_3}(\eta_3)\delta_{\vec{k}_4}(\eta_4)\rangle' \ ,\end{aligned} \tag{33}$$

which confirms eq. (31). One can easily extend this check to the case of several soft-lines.

3 Going to redshift space

The derivation of the consistency relations has been done in real space, but the galaxy distribution will of course be observed in redshift space. It is thus natural to ask if it is possible to write relations directly in terms of redshift space correlation function. Before doing so, let us stress that it will be difficult—if not impossible—to measure consistency relations at different times. To see the effect of the long mode, one would like to measure at quite different redshifts the short-scale correlation function at a spatial distance which is much smaller than Hubble. This is of course impossible since we can only observe objects on our past lightcone. This implies that, although one can check the consistency relations at different times in simulations, for real data we will have to stick to correlation functions at the same time. Given that the consistency relations vanish at equal time, their main phenomenological interest will be to look for their possible violations, which would indicate that one of the assumptions does not hold. This would represent a detection of either multi-field inflation or violation of the equivalence principle (or both!)

The mapping between real space $\vec{x}$ and redshift space $\vec{s}$ in the plane-parallel approximation is given by

$$\vec{s} = \vec{x} + \frac{v_z}{\mathcal{H}}\hat{z} \ , \tag{34}$$

where $\hat{z}$ is the direction of the line of sight, $v_z \equiv \vec{v}\cdot\hat{z}$, and $\vec{v}$ is the peculiar velocity. Also the relation between z and η receives corrections due to peculiar velocities. These corrections are small for sufficiently distant objects for which $v \ll Hx$. Notice that we do not assume that the peculiar velocity is a function of the position $\vec{x}$ since this holds only in the single-stream approximation, which breaks down for virialized objects on small scales [16, 17].

The derivation of the consistency relations follows closely what we did in real space, once we observe that also in redshift space the long mode induces a (time-dependent) translation. Indeed we have

$$\vec{x} \to \vec{x} + D\,\vec{\nabla}\Phi_{0,L} \ , \tag{35}$$

$$\vec{v} \to \vec{v} + f\mathcal{H}D\,\vec{\nabla}\Phi_{0,L} \ , \tag{36}$$

where $D(\eta)$ is the growth factor, $f(\eta) \equiv d\ln D/d\ln a$ is the growth rate and $\vec{\nabla}\Phi_{0,L}$ a homogenous gradient of the initial gravitational potential $\Phi_{0,L}$, related to δ_0 defined in eq. (5) by $\nabla^2\Phi_{0,L} = \delta_{0,L}$.

This corresponds to a redshift space translation

$$\vec{\tilde{s}} = \vec{s} + \delta\vec{s} \, , \tag{37}$$

$$\delta\vec{s} \equiv D\,(\vec{\nabla}\Phi_{0,L} + f\nabla_z\Phi_{0,L}\hat{z}) \, , \tag{38}$$

where we have applied to eq. (34) a spatial translation of the real-space coordinates and a shift of the peculiar velocity along the line of sight, respectively eqs. (35) and (36). As in real space, we can thus conclude that a redshift-space correlation function in the presence of a long mode Φ_L is the same as the correlation function in the absence of the long mode but in *translated* redshift-space coordinates:

$$\begin{aligned}\langle\delta^{(g,s)}(\vec{s}_1,\eta_1)\cdots\delta^{(g,s)}(\vec{s}_n,\eta_n)|\Phi_L\rangle &\approx \langle\delta^{(g,s)}(\vec{\tilde{s}}_1,\eta_1)\cdots\delta^{(g,s)}(\vec{\tilde{s}}_n,\eta_n)\rangle \\ &= \sum_a \delta\vec{s}_a\langle\delta^{(g,s)}(\vec{s}_1,\eta_1)\cdots\vec{\nabla}_a\delta^{(g,s)}(\vec{s}_a,\eta_a)\cdots\delta^{(g,s)}(\vec{s}_n,\eta_n)\rangle \, ,\end{aligned} \tag{39}$$

where $\delta\vec{s}_a \equiv D_a\,(\vec{\nabla}_a\Phi_{0,L} + f_a\nabla_{a,z}\Phi_{0,L}\hat{z})$. To show this notice that the density in redshift space can be written in terms of the real-space distribution function [16, 17]

$$\rho_s(\vec{s}) = ma^{-3}\int \mathrm{d}^3p\, f\left(\vec{s} - \frac{v_z}{\mathcal{H}}\hat{z}, \vec{p}\right) \, , \tag{40}$$

where m is the mass of the particles and $\vec{p}$ is the physical momentum. The statistical properties of $\rho_s(\vec{s})$ in the presence of the long mode are inherited by its expression in real space

$$\rho_s(\vec{s})_{\Phi_L} = \frac{m}{a^3}\int \mathrm{d}^3p\, f\left(\vec{s} - \frac{v_z}{\mathcal{H}}\hat{z} + \delta\vec{x}, \vec{p} + am\delta\vec{v}\right) = \frac{m}{a^3}\int \mathrm{d}^3p'\, f\left(\vec{s} - \frac{v_z - \delta v_z}{\mathcal{H}}\hat{z} + \delta\vec{x}, \vec{p}'\right) = \rho_s(\vec{s}+\delta\vec{s}), \tag{41}$$

where δx and $\delta\vec{v}$ are given by eqs. (35) and (36).

Again this statement can be directly applied to the galaxy distribution and it thus includes the bias with respect to the dark matter distribution. Notice that in the plane-parallel approximation redshift space is still translationally invariant (although it is not rotationally invariant, since the line-of-sight is a preferred direction): correlation function only depends on the distance between points. This implies that the consistency relations will be zero when the short modes are taken at equal time, since the common translation does not change distances.

In the Fourier space conjugate to redshift space, eq. (39) becomes

$$\langle\Phi_0(\vec{q})\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle \approx P_\Phi(q)\sum_a D(\eta_a)\big[\vec{q}\cdot\vec{k}_a + f(\eta_a)q_z\,k_{a,z}\big]\langle\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle \, . \tag{42}$$

By using for the long mode the linear relation between the density contrast in redshift space δ and the gravitational potential Φ, i.e.

$$\delta^{(g,s)}(\vec{q},\eta) = -(b_1 + f\mu^2_{\vec{q}})D(\eta)q^2\Phi_0(\vec{q}) \, , \tag{43}$$

where b_1 is a linear bias parameter between galaxies and dark matter and $\mu_{\vec{k}} \equiv \vec{k}\cdot\hat{z}/k$, the consistency relation above becomes

$$\begin{aligned}\langle\delta^{(g,s)}_{\vec{q}}(\eta)\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle \approx -\frac{P_{g,s}(q,\eta)}{b_1 + f\mu^2_{\vec{q}}}\sum_a \frac{D(\eta_a)}{D(\eta)}\frac{k_a}{q}\big[\hat{q}\cdot\hat{k}_a + f(\eta_a)\mu_{\vec{q}}\,\mu_{\vec{k}_a}\big] \\ \times\langle\delta^{(g,s)}_{\vec{k}_1}(\eta_1)\cdots\delta^{(g,s)}_{\vec{k}_n}(\eta_n)\rangle \, .\end{aligned} \tag{44}$$

We can check that this relation holds in perturbative calculation of redshift space distortions. The redshift space bispectrum reads [7]

$$\begin{aligned}&\langle\delta_{\vec q}^{(g,s)}(\eta)\delta_{\vec k_1}^{(g,s)}(\eta_1)\delta_{\vec k_2}^{(g,s)}(\eta_2)\rangle' =\\ &2Z_2(-\vec q,-\vec k_2;\eta_1)Z_1(\vec q;\eta)Z_1(\vec k_2;\eta_2)\langle\delta(\vec q,\eta)\delta(-\vec q,\eta_1)\rangle'\langle\delta(\vec k_1,\eta_1)\delta(\vec k_2,\eta_2)\rangle' + \text{cyclic}\ ,\end{aligned} \tag{45}$$

where

$$\begin{aligned}Z_1(\vec k;\eta) &\equiv (b_1+f\mu_{\vec k}^2)\ ,\\ Z_2(\vec k_a,\vec k_b;\eta) &\equiv b_1F_2(\vec k_a,\vec k_b)+f\mu_{\vec k}^2G_2(\vec k_a,\vec k_b)+\frac{f\mu_{\vec k}k}{2}\left[\frac{\mu_{\vec k_a}}{k_a}(b_1+f\mu_{\vec k_b}^2)+\frac{\mu_{\vec k_b}}{k_b}(b_1+f\mu_{\vec k_a}^2)\right]+\frac{b_2}{2}\ .\end{aligned} \tag{46}$$

Here b_1 and b_2 are the linear and non-linear bias parameters and F_2 and G_2 are the standard second-order perturbation kernels for density and velocity respectively [7]. In the limit $q\to 0$ we have

$$2Z_2(-\vec q,-\vec k_2;\eta_1)\approx(b_1+f_1\mu_{\vec k_2}^2)\frac{\vec q\cdot\vec k_2}{q^2}+(b_1+f_1\mu_{\vec k_2}^2)f_1\frac{k_2}{q}\mu_{\vec q}\mu_{\vec k_2}\ . \tag{47}$$

This gives

$$\begin{aligned}&\langle\delta_{\vec q}^{(g,s)}(\eta)\delta_{\vec k_1}^{(g,s)}(\eta_1)\delta_{\vec k_2}^{(g,s)}(\eta_2)\rangle'\\ &\quad\approx\frac{P_{g,s}(q,\eta)}{b_1+f\mu_{\vec q}^2}\frac{D(\eta_1)}{D(\eta)}(b_1+f_1\mu_{\vec k_2}^2)\left(\frac{\vec q\cdot\vec k_2}{q^2}+f_1\frac{k_2}{q}\mu_{\vec q}\mu_{\vec k_2}\right)Z_1(\vec k_2;\eta_2)\langle\delta(\vec k_1,\eta_1)\delta(\vec k_2,\eta_2)\rangle'\\ &\quad\approx-\frac{P_{g,s}(q,\eta)}{b_1+f\mu_{\vec q}^2}\frac{D(\eta_1)}{D(\eta)}\frac{k_1}{q}(\hat q\cdot\hat k_1+f_1\mu_{\vec q}\mu_{\vec k_1})\langle\delta(\vec k_1,\eta_1)\delta(\vec k_2,\eta_2)\rangle' + (1\leftrightarrow 2)\ .\end{aligned} \tag{48}$$

The consistency relation is satisfied.

As in real space, it is possible to derive a resummed version of eq. (44). The translation in redshift space introduces a factor

$$\exp\Big[i\sum_a\vec k_a\cdot\delta\vec s(\vec y,\eta_a)\Big]=\exp\Big[\int^\Lambda\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_a D(\eta_a)\left(\vec p\cdot\vec k_a+f(\eta_a)p_z\,k_{a,z}\right)\,e^{i\vec p\cdot\vec y}\Phi_0(\vec p)\Big] \tag{49}$$

in the correlation functions. It is then straightforward to show that, as in Sec. 2, the consistency relation in redshift space eq. (44) remains the same even when the effect of all soft modes is resummed. Moreover, using the same procedures developed in the previous section, one can easily extend the consistency relations with multiple soft legs, softs loops and soft internal lines to redshift space.

4 Conclusions

In this paper we showed that one can have a complete control of soft modes at any order in $\frac{k}{q}\cdot\delta_q$. The known cancellation of these effects for equal time correlators [10, 11, 5, 6, 12, 13] is now on more general grounds: it is physically a consequence of the equivalence principle and the lack of statistical correlation between long and short modes, which holds in single-field inflation. Therefore this cancellation is very robust and holds beyond the single-stream approximation, and including the effects of baryons on short scales. These regimes are beyond the usual arguments based on perturbation theory. Moreover, we now know exactly what is the effect of soft modes on correlators at different times. To make contact with observations one has to understand if the consistency relations can be written directly in redshift space. We showed that this is the case, without adding any assumption about the short modes: for example one does not need to assume the single-stream approximation, which breaks down on short scales.

Besides the theoretical interest of these results, the main conclusion for observations is that a detection in the squeezed limit of a $1/q$ behaviour at equal time would be a robust detection of either multi-field inflation or a violation of the equivalence principle. The next step is to evaluate how constraining measurements will be for explicit models that do not respect equivalence principle, taking into account that in the data one is obviously limited in the hierarchy between k and q. We will come back to this in a future publication [18].

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While finishing this paper reference [19] appeared. There is no disagreement with our results: in particular, we both agree that a violation of the EP implies a breaking of the consistency relations in the form of eq. (11). We thank M. Peloso and M. Pietroni for discussions. We acknowledge related work by A. Kehagias, J. Noreña, H. Perrier and A. Riotto: where comparison is possible, the results agree. It is a pleasure to thank V. Desjacques, R. Scoccimarro and M. Zaldarriaga for useful discussions, and the anonymous referee for useful comments. JG and FV acknowledge partial support by the ANR *Chaire d'excellence* CMBsecond ANR-09-CEXC-004-01.

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Article C

Single-Field Consistency Relations of Large Scale Structure Part III: Test of the Equivalence Principle

Single-Field Consistency Relations of Large Scale Structure Part III: Test of the Equivalence Principle

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Abstract

The recently derived consistency relations for Large Scale Structure do not hold if the Equivalence Principle (EP) is violated. We show it explicitly in a toy model with two fluids, one of which is coupled to a fifth force. We explore the constraints that galaxy surveys can set on EP violation looking at the squeezed limit of the 3-point function involving two populations of objects. We find that one can explore EP violations of order $10^{-3} \div 10^{-4}$ on cosmological scales. Chameleon models are already very constrained by the requirement of screening within the Solar System and only a very tiny region of the parameter space can be explored with this method. We show that no violation of the consistency relations is expected in Galileon models.

1 Introduction

Experimental tests of the Equivalence Principle (EP) are, from Galileo to modern torsion balance experiments, a prototypical example of the scientific method. The impressive modern limits on the equality of inertial and gravitational mass testify that we understand gravity very well, at least on scales much shorter than the Hubble size. On the other hand, the observed acceleration of the Universe may suggest that something new happens to gravity at very large distances. It is at first difficult to imagine how to test the EP on scales comparable to the size of the Universe, since even the most patient experimentalist cannot follow the fall of astrophysical objects for lengths and timescales comparable to Hubble. In this paper we show that this kind of test is indeed possible: we do not have to *wait* for things to fall, we just have to look at their *final position*, provided we make the correct guess about their initial conditions long back in time. It is like saying that Galileo could have simply studied the arrival time of the different rolling balls along the inclined plane, provided somebody had told him in advance the initial conditions at the top of the plane. Usually, initial conditions are part of the experimental setup and not something that can be predicted from the theory, or at least this was the situation for Galileo. Nowadays we think we know the initial conditions of our Universe, at least in a statistical, if not deterministic, sense. All the experiments are compatible with the simple picture of Gaussian initial conditions and this is what we are going to assume throughout this paper, keeping of course in mind that a deviation from this assumption would be a big discovery on its own[1]. The absence of non-Gaussianity, i.e. the statistical independence of the Fourier modes, tells us that the homogeneous gravitational field where the experiment will take place does not affect the (statistics of) initial conditions for the small objects whose fall we test. This educated assumption about initial conditions allows to test the EP on cosmological scales by simply measuring the position of different astrophysical objects at a given time.

This paper is a natural continuation of [1, 2], where we showed, following [3, 4], that for single-field inflationary models and if the EP holds, certain consistency relations for cosmological correlation functions can be derived. The violation of the consistency relations in modified gravity theories has been recently discussed in [5, 6, 7, 8] (with some differences that we are going to point out). In this paper we will concentrate on equal-time correlators, which are the most relevant observationally, and on the 3-point function which, in the non-relativistic limit, reads

$$\lim_{q\to 0}\langle \delta_{\vec q}(\eta)\delta^{(A)}_{\vec k_1}(\eta)\delta^{(B)}_{\vec k_2}(\eta)\rangle' = \left(\epsilon\,\frac{\vec k\cdot\vec q}{q^2} + \mathcal{O}\big[(q/k)^0\big]\right) P(q,\eta)P_{AB}(k,\eta)\,. \tag{1}$$

The notation requires some explanation. A prime on the correlation function on the left-hand side indicates that the momentum conserving Dirac function has been removed. $\delta^{(A)}$ and $\delta^{(B)}$ are the number densities of the two classes of objects (e.g. galaxies with different mass) we want to compare in their fall and $P_{AB}(k,\eta)$ their cross power spectrum, with $\vec k \equiv (\vec k_1 - \vec k_2)/2$. The third mode $\delta_{\vec q}$ with small momentum q corresponds to the approximately homogeneous gravitational field where objects A and B fall. It is treated in the linear regime and can be measured using any probe we like. If objects A and B fall in the same way, then ϵ vanishes. Conversely, as we will see, a deviation from the EP for the two classes of objects induces a non-zero ϵ. Equation (1) represents a violation of the consistency relation, which tells us that there should be no k/q term in such an equal-time correlator, if the EP and single-field initial condition were respected. The actual size of the violation of the consistency relation

[1] What we need to assume is the absence of non-Gaussianity in the squeezed limit.

is model-dependent. In Section 2 we are going to calculate ϵ in a simple model in which the objects A and B have a different coupling with a long range fifth force. Although modified gravity models can be significantly more complicated, this will represent our benchmark model. In Section 3 we are going to study the limits one will be able to put on the parameter ϵ in future surveys with a simple estimate of the cumulative signal-to-noise in the bispectrum. Notice that exchanging A with B is equivalent to flipping the sign of $\vec{q}$ so that the relation for $A = B$ trivially vanishes: only for two different kinds of objects the consistency relation can be violated[2].

What kind of models are expected to violate the EP on cosmological scales? One possibility is the existence of some non-universal long-range force, another is the EP violation induced on macroscopic objects by one of the screening mechanisms, which hide the deviations from GR on short scales, where stringent experimental bounds apply. We will review these possibilities in Sec. 4. It is fair to anticipate that most of these models give a negligible signal for our test, either because of other experimental constraints or because the EP violation is anyway suppressed. We think, however, that this does not diminish the interest in testing the EP on cosmological scales. Indeed one has to admit that none of the models which modify gravity on large scales addresses the cosmological constant problem, which is the main reason why we are interested in modifications of gravity in the first place. Therefore, if gravity changes on large scales in a way connected with the cosmological constant, we expect something much more dramatic and interesting than the theories studied so far. From this point of view, a test of the basic tenet of GR on cosmological scales is surely worthwhile.

One can read eq. (1), when the EP holds, i.e. $\epsilon = 0$, as the statement that there is no velocity bias between species A and B on large scales: the long mode induces exactly the same velocity for all objects. It is important to stress that this holds even considering *statistical* velocity bias. Objects do not form randomly, but in special places of the density field: therefore, even if they locally fall together with the dark matter, there can be a velocity bias in a statistical sense [9, 10]. However, the arguments of [1, 2, 3] tell us that the long mode (at leading order in q) is equivalent to a change of coordinates. Apart from this change of coordinates the long mode affects neither the dynamics nor the *statistics* of short modes. Therefore, the EP implies that the statistical velocity bias disappears on large scales: again, this statement is completely non-perturbative in the short scales and includes the effect of baryons. For the case of dark matter only, we know that the statistical velocity bias vanishes on large scales as $\sim q^2 R^2$, where R is a length scale of order the Lagrangian size of the object; this can be calculated by looking at the statistics of peaks [9, 10] and verified in numerical simulations [11]. One expects that the effect of statistical velocity bias is therefore subdominant with respect to the unknown corrections in eq. (1) since $k \lesssim R^{-1}$.([3])

[2]This point has not been made explicit in Ref. [6], where they concentrate on correlation functions for the same class of objects.

[3] Reference [5] quotes from [11] that, for $q = 0.05 h\mathrm{Mpc}^{-1}$, objects in the range $(25 \div 40) \cdot 10^{12} h^{-1} M_\odot$ have a velocity bias of 1.05 compared to dark matter. This effect is more important than the unknown corrections $\mathcal{O}[(q/k)^0]$ to the consistency relation (1) only for $k \gtrsim R^{-1}/\sqrt{0.05}$, where R is the Lagrangian size of the objects. However, it is difficult to measure the correlation function of objects on scales smaller than their Lagrangian size.

2 An example of equivalence principle violation

In this section we are going to study a toy model in which the Universe is composed of two non-relativistic fluids A and B, with the latter coupled to a scalar field mediating a fifth force. For example, the two fluids could be baryons and dark matter but, with some modifications that we will discuss below, the model can also describe two populations of astrophysical objects, say different types of galaxies. If the scalar field φ has a negligible time evolution, the continuity equations of the two fluids are the same,

$$\delta'_X + \vec{\nabla} \cdot [(1 + \delta_X)\vec{v}_X] = 0 \, , \quad X = A, B \, , \tag{2}$$

where a prime denotes the derivative with respect to the conformal time $\eta \equiv \int dt/a(t)$ (we assume a flat FRW metric, with scale factor a), $' \equiv \partial_\eta$. The Euler equation of B contains the fifth force, whose coupling is parameterized by α,

$$\vec{v}'_A + \mathcal{H}\vec{v}_A + (\vec{v}_A \cdot \vec{\nabla})\, \vec{v}_A = -\vec{\nabla}\Phi \, , \tag{3}$$

$$\vec{v}'_B + \mathcal{H}\vec{v}_B + (\vec{v}_B \cdot \vec{\nabla})\, \vec{v}_B = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\varphi \, , \tag{4}$$

where $\mathcal{H} \equiv \partial_\eta a/a$ is the comoving Hubble parameter. To close this system of equations we need Poisson's equation and the evolution equation of the scalar field. Assuming that the scalar field stress-energy tensor is negligible, only matter appears as a source in the Poisson's equation,

$$\nabla^2\Phi = 4\pi G\, \rho_{\rm m}\, \delta = 4\pi G\, \rho_{\rm m}\, (w_A\delta_A + w_B\delta_B) \, , \tag{5}$$

where $\rho_{\rm m}$ is the total matter density and $w_X \equiv \rho_X/\rho_{\rm m}$ is the density fraction of the X species. Moreover, in the non-relativistic approximation we can neglect time derivatives in comparison with spatial gradients and the equation for the scalar field reads

$$\nabla^2\varphi = \alpha \cdot 8\pi G\rho_{\rm m} w_B\delta_B \, , \tag{6}$$

where we have neglected the mass of the scalar field, assuming we are on scales much shorter than its Compton wavelength.

Let us start with the linear theory and, following [12], look for two of the four independent solutions of the system in which the density and the velocity of the species B differ from those of the species A by a (possibly time-dependent) bias factor b,

$$\delta^{(A)}_{\vec{k}}(\eta) = D(\eta)\, \delta_0(\vec{k}) \, , \tag{7}$$

$$\theta^{(A)}_{\vec{k}}(\eta) = -\mathcal{H}(\eta) f(\eta) \delta^{(A)}_{\vec{k}}(\eta) \, , \tag{8}$$

$$\delta^{(B)}_{\vec{k}}(\eta) = b(\eta) \delta^{(A)}_{\vec{k}}(\eta) \, , \tag{9}$$

$$\theta^{(B)}_{\vec{k}}(\eta) = -\mathcal{H}(\eta) f(\eta) \delta^{(B)}_{\vec{k}}(\eta) \, , \tag{10}$$

where we have defined $\theta^{(X)} \equiv \vec{\nabla} \cdot \vec{v}_X$ and $\delta_0(\vec{k})$ is a Gaussian random variable. Plugging this ansatz in eqs. (2)–(6) and using the background Friedmann equations for a flat universe, we find, at linear order,

$$f = \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\ , \tag{11}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\ln a} + f^2 + \left(2 - \frac{3}{2}\Omega_{\mathrm{m}}\right) f - \frac{3}{2}\Omega_{\mathrm{m}}(w_A + w_B b) = 0\ , \tag{12}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\ln a} = 0\ , \tag{13}$$

$$w_B b + w_A\left(1 - \frac{1}{b}\right) - w_B(1 + 2\alpha^2) = 0\ . \tag{14}$$

Using eqs. (11) and (12), the linear growth factor D satisfies a second-order equation,

$$\frac{\mathrm{d}^2 D}{\mathrm{d}\ln a^2} + \left(2 - \frac{3}{2}\Omega_{\mathrm{m}}\right)\frac{\mathrm{d}D}{\mathrm{d}\ln a} - \frac{3}{2}\Omega_{\mathrm{m}}(w_A + w_B b)D = 0\ , \tag{15}$$

whose growing and decaying solutions are D_+ and D_-. Note that eq. (13) implies that the bias b is time independent. In the absence of EP violation ($\alpha = 0$) we get $b = 1$ (using $w_A + w_B = 1$) and we recover from eq. (15) the usual evolution of the growth of matter perturbations.

Following [13, 14], we introduce $y \equiv \ln D_+$ as the time variable. Defining the field multiplet

$$\Psi_a \equiv \begin{pmatrix} \delta^{(A)} \\ -\theta^{(A)}/\mathcal{H}f_+ \\ \delta^{(B)} \\ -\theta^{(B)}/\mathcal{H}f_+ \end{pmatrix}\ , \tag{16}$$

the equations of motion of the two fluids can be then written in a very compact form as

$$\partial_y \Psi_a(\vec{k}) + \Omega_{ab}\Psi_b(\vec{k}) = \gamma_{abc}\Psi_b(\vec{k}_1)\Psi_c(\vec{k}_2)\ , \tag{17}$$

where integration over $\vec{k}_1$ and $\vec{k}_2$ is implied on the right-hand side. The entries of γ_{abc} vanish except for

$$\begin{aligned} \gamma_{121} = \gamma_{343} &= (2\pi)^3\delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2)\,\frac{\vec{k}_1\cdot(\vec{k}_1 + \vec{k}_2)}{k_1^2}\ , \\ \gamma_{222} = \gamma_{444} &= (2\pi)^3\delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2)\,\frac{\vec{k}_1\cdot\vec{k}_2(\vec{k}_1 + \vec{k}_2)^2}{2k_1^2 k_2^2}\ , \end{aligned} \tag{18}$$

the matrix Ω_{ab} reads

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}w_A & \frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}(w_A + bw_B) - 1 & -\frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}w_B & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}w_A & 0 & -\frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}\left(w_B b + w_A\left(1 - \frac{1}{b}\right)\right) & \frac{3}{2}\frac{\Omega_{\mathrm{m}}}{f_+^2}(w_A + bw_B) - 1 \end{pmatrix}\ , \tag{19}$$

and we have employed eq. (14) to replace the dependence on α^2 by a dependence on the bias b. The solution of eq. (17) can be formally written as

$$\Psi_a(y) = g_{ab}(y)\phi_b + \int_0^y \mathrm{d}y'\, g_{ab}(y - y')\gamma_{bcd}\Psi_c(y')\Psi_d(y')\ , \tag{20}$$

where ϕ_b is the initial condition, $\phi_b = \Psi_b(y=0)$, and $g_{ab}(y)$ is the linear propagator which is given by [13]

$$g_{ab}(y) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \mathrm{d}\omega \, (\omega I + \Omega)^{-1}_{ab} e^{\omega y} \, , \tag{21}$$

where ξ is a real number larger than the real parts of the poles of $(\omega I + \Omega)^{-1}$.

In the following we consider small couplings to the fifth force, $\alpha^2 \ll 1$, which by virtue of eq. (14) implies $b \simeq 1$. In this case, it is reasonable to use the approximation $f_+^2 \simeq \Omega_\mathrm{m}$, which for $b = 1$ is very good throughout the whole evolution [15]. We choose to use this approximation because it considerably simplifies the presentation but one can easily drop it and make an exact computation.

The linear evolution is characterized by four modes. Expanding for small $b-1$, apart from the "adiabatic" growing and decaying modes already introduced above, respectively going as $D_+ = e^y$ and $D_- = e^{-\frac{3}{2}[1+w_B(b-1)]y}$, one finds two "isodensity" modes, one decaying as $D_\mathrm{i} = e^{-\frac{1}{2}[1+3(1+w_A)(b-1)]y}$ and an almost constant one going as $D_\mathrm{c} = e^{3w_A(b-1)\alpha^2 y}$.[4]

We are interested in the equal-time 3-point function involving the two species. In particular, we compute

$$\langle \delta_{\vec{k}_3}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle = w_A\langle \Psi_1(k_3,\eta)\Psi_1(k_1,\eta)\Psi_3(k_2,\eta)\rangle + w_B\langle \Psi_3(k_3,\eta)\Psi_1(k_1,\eta)\Psi_3(k_2,\eta)\rangle \, , \tag{22}$$

where $\delta \equiv w_A\delta^{(A)} + w_B\delta^{(B)}$. The calculation can be straightforwardly done at tree level by perturbatively expanding the solution (20) as $\Psi_a = \Psi_a^{(1)} + \Psi_a^{(2)} + \ldots$, which up to second order in δ_0 yields

$$\begin{aligned} \Psi_a^{(1)}(y) &= g_{ab}(y)\phi_b \, , \\ \Psi_a^{(2)}(y) &= \int_0^y \mathrm{d}y' g_{ab}(y-y')\gamma_{bcd}\Psi_c^{(1)}(y')\Psi_d^{(1)}(y') \, , \end{aligned} \tag{23}$$

and by applying Wick's theorem over the Gaussian initial conditions. In the squeezed limit, the expression for (22) simplifies considerably. Assuming that the initial conditions are in the most growing mode, i.e. they are given by $\phi_a(\vec{k}) = u_a\delta_0(\vec{k})$ with $u_a = (1,1,b,b)$, at leading order in $b-1$ one finds

$$\begin{aligned} \lim_{q\to 0}\langle \delta_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle' \simeq &-(b-1)P(q,\eta)P_{AB}(k,0)\frac{\vec{k}\cdot\vec{q}}{q^2} \\ &\times \int_0^y \mathrm{d}y' e^{2y'}\left[g_{11}(y-y') + g_{12}(y-y') - g_{31}(y-y') - g_{32}(y-y')\right] \, , \end{aligned} \tag{24}$$

which shows that the long wavelength adiabatic evolution has no effect on the 3-point function[5] [16, 17]. As before, the prime on the correlation function denotes that the delta function of momentum conservation has been dropped. Retaining the most growing contribution and using $b \simeq 1 + 2w_B\alpha^2$ one finally finds

$$\lim_{q\to 0}\langle \delta_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle' \simeq \frac{7}{5}w_B\,\alpha^2\,\frac{\vec{k}\cdot\vec{q}}{q^2}P(q,\eta)P_{AB}(k,\eta) \, . \tag{25}$$

---

[4]With an abuse of language, we denote the modes $(+)$ and $(-)$ as adiabatic and (i) and (c) as isodensity even though, strictly speaking, they do not correspond to the usual notion of adiabatic and isocurvature. Indeed, $(+)$ and $(-)$ correspond to $\delta_A = \delta_B/b$ and not to $\delta_A = \delta_B$ as in the standard adiabatic case without a fifth force, while (i) and (c) yield $w_A\delta^{(A)} + bw_B\delta^{(B)} = 0$ instead of $w_A\delta^{(A)} + w_B\delta^{(B)} = 0$ which one finds in the standard isodensity case (see [16] for a discussion of adiabatic and isodensity modes in the standard case $b = 1$).

[5]For $b = 1$ one finds $g_{11}^{(+)} = g_{31}^{(+)}$, $g_{12}^{(+)} = g_{32}^{(+)}$, $g_{11}^{(-)} = g_{31}^{(-)}$ and $g_{12}^{(-)} = g_{32}^{(-)}$.

The result that we obtained remains qualitatively the same if A or B represent extended objects, for example a fluid of galaxies of a given kind [18]. For instance, one can take A to represent galaxies that are not coupled to the fifth force because they are screened (see Sec. 4), while B represents the dark matter fluid. In this case one should start from initial conditions in which there is a bias between the galaxy and the dark matter overdensities: $u_a = (b_g, 1, b, b)$. This is equivalent to exciting decaying modes, given that asymptotically the galaxy bias becomes unity (b_g is the initial galaxy bias). Consequently, the result (25) will be different. Still, it is straightforward to check that, as expected, there is no $1/q$ divergence if the EP is not violated, i.e. $\alpha = 0$. In the limit $w_A \ll 1$ (i.e. the screened galaxies contribute a subdominant component of the overall mass density) and keeping only the slowest decaying mode, one gets (in this case we take the long mode to be dark matter only)

$$\lim_{q\to 0}\langle \delta^{(B)}_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta)\rangle' \simeq \frac{7}{5}\,\alpha^2\,\frac{\vec{k}\cdot\vec{q}}{q^2}\left(1+\frac{10}{7}(b_g-1)e^{-(y-y_0)}\right)P(q,\eta)P_{AB}(k,\eta)\,. \tag{26}$$

Notice that y_0 here represents the initial value when the local galaxy bias b_g is set up.

Another complication in this case comes from the fact that objects become screened only at a certain stage of their evolution, so that the coupling of the fluid A with the scalar is time-dependent. All this modifies the numerical value on the right-hand side of eq. (25). In any case, given the model-dependence of the result, we stick to eq. (25) as our benchmark model when discussing the capabilities of experiments to constrain EP violation.

3 Detecting an equivalence principle violation

In this section we want to explore how well we can constrain the violation of the EP in our toy model using large scale structure surveys. We will use this bound to comment on the possible detection of EP violation in different modified gravity scenarios.

3.1 Signal to noise for the bispectrum

The signal to noise calculation closely follows the standard calculation for the case of primordial non-Gaussianities (see for example [19]). We will assume a survey of a given comoving volume V which defines the fundamental scale in momentum space, $k_f = 2\pi/V^{1/3}$. In this setup, the bispectrum estimator is given by

$$B(k_1,k_2,k_3) = \frac{V_f}{V_{123}}\int_{k_1}\mathrm{d}^3q_1\int_{k_2}\mathrm{d}^3q_2\int_{k_3}\mathrm{d}^3q_3\;\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3)\cdot\delta_{\vec{q}_1}\delta_{\vec{q}_2}\delta_{\vec{q}_3}\;, \tag{27}$$

where $V_f = (2\pi)^3/V$ is the volume of the fundamental cell, the integration is done over the spherical shells with bins defined by $q_i \in (k_i - \delta k/2, k_i + \delta k/2)$ and

$$V_{123} \equiv \int_{k_1}\mathrm{d}^3q_1\int_{k_2}\mathrm{d}^3q_2\int_{k_3}\mathrm{d}^3q_3\;\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3) \approx 8\pi^2\,k_1k_2k_3\,\delta k^3\;. \tag{28}$$

We will assume no significant correlation among different triangular configurations or, in other words, that the bispectrum covariance matrix is diagonal and given by a Gaussian statistics. It can be shown that in this case the variance is given by [19]

$$\Delta B^2(k_1,k_2,k_3) = k_f^3\frac{s_{123}}{V_{123}}P_{\rm tot}(k_1)P_{\rm tot}(k_2)P_{\rm tot}(k_3)\;, \tag{29}$$

where $s_{123} = 6, 2, 1$ for equilateral, isosceles and general triangles, respectively. The power spectrum $P_{\rm tot}(k)$ is given by

$$P_{\rm tot}(k) = P(k) + \frac{1}{(2\pi)^3}\frac{1}{\bar{n}} , \tag{30}$$

where the last term on the right hand side accounts for the shot noise and $\bar{n}$ is the number density of galaxies in the survey. In what follows we will neglect the shot noise contribution because we want to estimate the total amount of signal in principle available for a survey of a given volume, without restricting our analysis specifically to galaxy surveys. Moreover, for our estimates we will use only modes that are in the linear regime where the shot noise is expected to be negligible.

Given these definitions, the signal-to-noise ratio is calculated as

$$\left(\frac{S}{N}\right)^2 = \sum_T \frac{(B_{\rm new\ physics}(k_1,k_2,k_3) - B_{\rm standard}(k_1,k_2,k_3))^2}{\Delta B^2(k_1,k_2,k_3)} , \tag{31}$$

where the sum runs over all possible triangles formed by $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ given $k_{\rm min}$ and $k_{\rm max}$. Typically, the sum is written down such that the same triangles are not counted twice and the symmetry factor s_{123} takes care of special configurations. In our case, with two different species of particles, the bispectrum is not symmetric when momenta are exchanged and the previous equations have to be modified accordingly. We will impose $s_{123} = 1$ for all configurations and the sum over triangles will be

$$\sum_T \equiv \sum_{k_1=k_{\rm min}}^{k_{\rm max}} \sum_{k_2=k_{\rm min}}^{k_{\rm max}} \sum_{k_3=k^*_{\rm min}}^{k^*_{\rm max}} , \tag{32}$$

where $k^*_{\rm min} \equiv \max(k_{\rm min}, |\vec{k}_1 - \vec{k}_2|)$, $k^*_{\rm max} \equiv \min(|\vec{k}_1 + \vec{k}_2|, k_{\rm max})$ and the discrete sum is done with $|\vec{k}_{\rm max} - \vec{k}_{\rm min}|/\delta k$ steps where δk is a multiple of k_f. In the following we fix $\delta k = k_f$.

3.2 Estimate for our toy model

Now that we have defined the estimator, we apply it to the case of violation of the EP. We will not restrict ourselves to squeezed triangle configurations but we exploit all possible triangular configurations of eq. (22).

In the case at hand, the signal to noise takes the form

$$\left(\frac{S}{N}\right)^2 = \sum_T \frac{\left[B^{(AB)}_{\alpha^2}(k_1,k_2,k_3) - B^{(AB)}_{\alpha^2=0}(k_1,k_2,k_3)\right]^2}{\Delta[B^{(AB)}]^2(k_1,k_2,k_3)} , \tag{33}$$

where the bispectrum $B^{(AB)}(k_1,k_2,k_3)$ is defined by

$$\langle \delta_{\vec{k}_1}(\eta)\delta^{(A)}_{\vec{k}_2}(\eta)\delta^{(B)}_{\vec{k}_3}(\eta)\rangle = (2\pi)^3\delta_D(\vec{k}_1+\vec{k}_2+\vec{k}_3)B^{(AB)}(k_1,k_2,k_3) , \tag{34}$$

where the left hand side of this equation is computed from eq. (22) at leading order in α^2. For the computation we employ

$$\delta^{(A)}_{\vec{k}}(\eta) \equiv \frac{1}{w_A + w_B b}\delta_{\vec{k}}(\eta) , \qquad \delta^{(B)}_{\vec{k}}(\eta) \equiv \frac{1}{w_A/b + w_B}\delta_{\vec{k}}(\eta) , \tag{35}$$

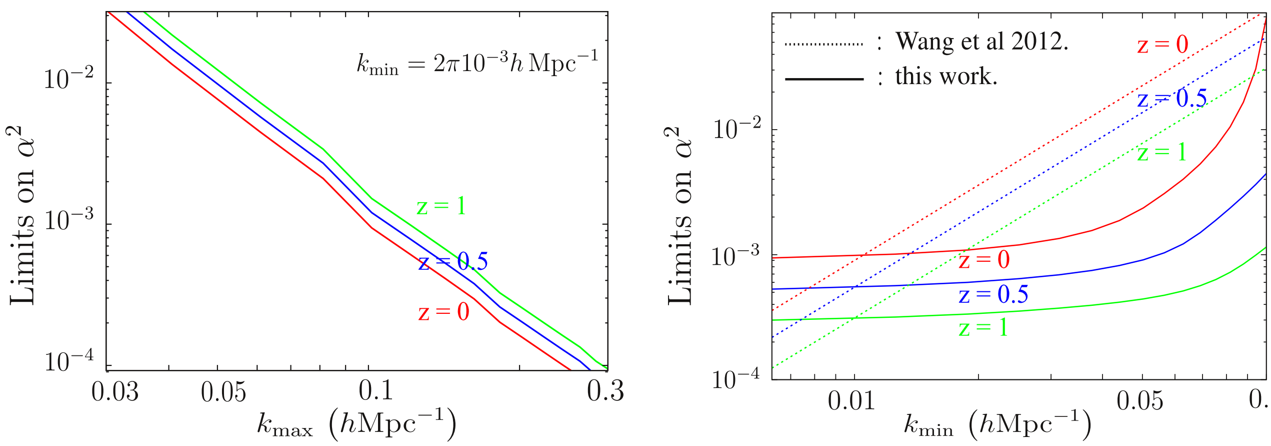

Figure 1: Expected error on α^2, $\sigma(\alpha^2)$, for a survey with volume $V = 1({\rm Gpc}/h)^3$ at three different redshifts, $z = 0$, $z = 0.5$ and $z = 1$. Left: $\sigma(\alpha^2)$ is plotted as a function of $k_{\rm max}$. We have chosen $k_{\rm min} = 2\pi/V^{1/3}$ so that the violation of the EP extends to the whole survey. Right: $\sigma(\alpha^2)$ is plotted as a function of $k_{\rm min}$. $k_{\rm max}$ is given by 0.10, 0.14, 0.19 for $z = 0$, 0.5, 1 respectively. The dotted lines represent $\alpha^2 \lesssim 10^{-6}(m/H)^2$, i.e. the bound on α^2 from screening the Milky Way [23].

where we compute the transfer function of the total matter density contrast δ using the code CAMB [20]. We then define

$$\langle \delta_{\vec k}(\eta)\delta_{\vec k'}(\eta)\rangle = (2\pi)^3\delta_D(\vec k + \vec k')P(k)\ , \qquad \langle \delta^{(X)}_{\vec k}(\eta)\delta^{(X)}_{\vec k'}(\eta)\rangle = (2\pi)^3\delta_D(\vec k + \vec k')P^{(X)}(k)\ . \tag{36}$$

Following eq. (29), for the variance of the bispectrum we use

$$\Delta\left[B^{(AB)}\right]^2(k_1,k_2,k_3) = k_f^3\frac{s_{123}}{V_{123}}P(k_1)P^{(A)}(k_2)P^{(B)}(k_3)\ . \tag{37}$$

Figure 1 shows the estimated error on α^2, $\sigma(\alpha^2)$, for three different surveys of volume $V = 1(h^{-1}{\rm Gpc})^3$ at redshift $z = 0$, $z = 0.5$ and $z = 1$, respectively. On the left panel this is shown as a function of $k_{\rm max}$ for the smallest possible $k_{\rm min}$, i.e. $k_{\rm min} = k_f = 2\pi/V^{1/3}$. The smallest measurable value of α^2 roughly scales as $k_{\rm max}^{-2.8}$, so that it crucially depends on our ability to capture the shortest scales.[6]

On the right panel, the estimated relative variance is shown as a function of $k_{\rm min}$. For each survey, we take $k_{\rm max}$ such that we are still in a quasi-linear regime where theoretical control in perturbation theory is possible. In particular, we fix $k_{\rm max} = \pi/(2R)$ where R is chosen in such a way that σ_R, the root mean squared linear density fluctuation of the matter field in a ball of radius R, is 0.5.([7]) This

[6]For $k_{\rm max} \sim 0.1h^{-1}$Mpc we roughly agree with [6] but we find a different dependence on $k_{\rm max}$. Our result can be roughly understood as follows. Recall that the experimental constraint on $f_{\rm NL}^{\rm loc}$ goes like $\Delta f_{\rm NL}^{\rm loc} \sim 5\sqrt{10^6/N}$, where N is the number of modes. This is consistent with the fact that the Planck limit is about $f_{\rm NL}^{\rm loc} \lesssim 5$, for $N \sim 10^6$ [21]. For the large scale structure we are interested in here, $N \sim (k_{\rm max}/k_{\rm min})^3 \sim 4\cdot 10^3$, and so $\Delta f_{\rm NL}^{\rm loc} \sim 80$, which is consistent with Fig. 3 of Ref. [19]. In the case of the EP violation we effectively have $f_{\rm NL}^{\rm loc} \sim \alpha^2 \times q\,k/(\Omega_m H_0^2)$. Assuming $k/q \sim 10^2/(2\pi)$, we therefore have a bound of α^2 that is about $4\cdot 10^{-3}$. This argument also tells us the scaling with $k_{\rm max}$: $\sqrt{N} \propto k_{\rm max}^{3/2}$, and the scaling of the effective $f_{\rm NL}^{\rm loc}$ adds one more power of $k_{\rm max}$, giving us a limit on α^2 that scales as $k_{\rm max}^{-2.5}$, roughly agreeing with our $k_{\rm max}^{-2.8}$ scaling.

[7]Apart from the theoretical uncertainty in understanding the nonlinear regime of density fluctuations, other

yields $k_{\rm max} = 0.10,\ 0.14,\ 0.19$ for $z = 0,\ 0.5,\ 1$ respectively. From Fig. 1 we see that the dependence on $k_{\rm min}$ is very mild when going to zero. This seems counterintuitive, because eq. (1) indicates that the bispectrum diverges as $1/q$ at small q, giving more signal. However, in that limit the power spectrum of matter fluctuations scales as q, $P(q) \propto q$, canceling the enhancement. This differs from the familiar case of local non-Gaussianity where the divergence scales as $1/q^2$, causing the known increase of precision on $f_{\rm NL}^{\rm loc}$ when going to larger surveys. The improvement of the constraints at higher redshifts, discussed also in [6], is due to the fact that $k_{\rm max}$ increases and, assuming a fixed volume, we have access to more modes. Our constraints can be compared with that for chameleon models derived in Ref. [23] from requiring that the Milky Way must be screened. This yields

$$\alpha^2 \lesssim 10^{-6} (m/H)^2 \,, \tag{38}$$

where m is the Compton mass of the chameleon. In this case $k_{\rm min}$ can be identified with m^{-1}, the Compton wavelength of the chameleon and one sees that for $m^{-1} \gtrsim 0.01$ our constraints can improve that of Ref. [23].

When looking for EP violation, a possible contaminant is the initial density or velocity bias between two different species. For instance, even in single-field inflation we know that baryons and dark matter have different initial conditions on scales below the sound horizon at recombination, because at recombination baryons are tightly coupled to photons through Thomson scattering, while dark matter particles are free falling. As discussed in [24, 16, 17], the relative velocity between baryons and dark matter excites long wavelength isodensity modes that couple to small scales reducing the formation of early structures. However, one can check that this effect decays more rapidly than the one described by eq. (25). For instance, assuming no violation of the EP but an initial density and velocity bias between the two species A and B, $u_a = (b_A, b_A, b_B, b_B)$, one obtains

$$\lim_{q\to 0} \langle \delta_{\vec q}(\eta) \delta^{(A)}_{\vec k_1}(\eta) \delta^{(B)}_{\vec k_2}(\eta) \rangle' \simeq 4\,(b_A - b_B) e^{-\frac{3}{2}(y-y_0)} \, \frac{\vec k \cdot \vec q}{q^2} P(q,\eta) P_{AB}(k,\eta) \,, \tag{39}$$

independently of w_A and w_B. Thus, the effect is still divergent as $1/q$ but rapidly decays, so that it is typically suppressed by a factor $\sim (1+z_0)^{-3/2}$ where z_0 represents the initial redshift. For the example discussed above of baryons and dark matter we can take $z_0 \simeq 1100$ and today this effect is thus suppressed by $\sim \mathcal{O}(10^{-5})$. Moreover, if we use galaxies to probe the EP it will be further suppressed by the fact that the baryon-to-dark matter ratio is rather constant in different galaxies.

When using galaxies, one should also remember that their density field is a biased tracer and that in general we expect the bias to contain nonlinearities. Thus, other contributions are expected in eq. (1), for instance of $\mathcal{O}[(k/q)^0]$ if the nonlinear bias is scale independent. To compute the signal-to-noise ratio correctly taking into account this effect, one should include these nonlinear contributions and marginalise over the bias parameters, similarly to what done in the context of non-Gaussianity, for instance in Ref. [19]. However, due to its different scale and angular dependence, we do not expect the marginalization over nonlinear bias to dramatically change our estimates.

Before concluding, it is important to stress that our estimates so far assume that we know which are the two classes of objects that violate the EP. In practice, one will have to classify objects either based

effects neglected here hinder the access to small scales. In redshift surveys, the smallest scales are affected by the radial smearing due to redshift distortion that are uncorrelated with the density fluctuations, such as the one coming from the Doppler shift due to the virialized motion of galaxies within clusters or the one due to the redshift uncertainty of spectroscopic galaxy samples. See for instance [22] for a discussion.

on some intrinsic property (mass, luminosity, dark matter content) or some environmental property (like being inside an overdense or underdense region), and astrophysical uncertainties in the selection of these objects may significantly suppress the signal. In particular, if the kind of objects we aim for is quite rare, the shot noise, which we have neglected so far, will be an important limitation. In this sense the limits discussed above are the most stringent one can get for an ideal survey of a given volume.

Despite these limitations, the absence of any signal when the EP holds is very robust. The long mode cannot give any $1/q$ effect, independently of the bias of the objects and of the selection strategy we use. Furthermore, the same statement holds also in redshift space [2, 6], which makes the connection with observations even more straightforward. In other words, all the complications that enter when one wants to use the data to infer the underlying dark matter 3-point function are not relevant here if we only want to show that the EP is violated. Of course, once a violation is detected, it would be much more challenging to better characterize the source of the violation.

4 Modifications of gravity and equivalence principle violation

A violation of the consistency relations requires a macroscopic violation of the EP: different astrophysical objects must fall at a different rate. One can envisage various possibilities depending on which is the relevant feature that determines the EP violation.

Baryon content. If dark matter and baryons have a different coupling with a light scalar, one has a violation of the EP at the fundamental level. This causes different astrophysical objects, with a different baryon/dark matter ratio to fall at a different rate in an external field. This scenario is however very constrained: Planck [25] limits this kind of couplings to be $\lesssim 10^{-4}$ smaller than gravity. This is far from what we can achieve with our method, since most astrophysical objects have a quite similar baryon content and this suppresses substantially the EP violation.

Amount of screening. The screening of extra forces to satisfy the gravity tests in the solar system induces violations of the EP [26]. We can distinguish various cases, depending on the screening mechanism.

For *chameleon* [27] or *symmetron* [28, 29, 30] screening the EP violation can be of order unity between screened and unscreened objects. However, the necessity of screening inside the solar system limits the impact of the fifth force on cosmological scales. Indeed, one can find a model-independent limit on the mass of the scalar [23, 31]

$$m^2 \gtrsim 10^6 \alpha^2 H^2 \ . \tag{40}$$

This inequality, which is valid at low redshifts, limits the effect of the scalar on short scales $k/a \lesssim m$. In Fig. 1 we compare this limit with our signal to noise forecast at different redshifts: a detection of EP violation is possible, though quite challenging. The screening here depends on the typical value of the gravitational potential GM/r of the object. Given that we know the Milky Way is screened, one should look for objects with a lower Φ to find unscreened objects. This looks challenging since in a survey one is typically sensitive to galaxies which are more luminous and therefore more massive than the Milky Way.

For *Galileon* screening [32] the issue of EP violation is rather subtle. On one hand, one can show that an object immersed in an external field which is constant over the size of the object will receive an acceleration proportional to the mass and independent of the possible Vainshtein screening of the object [26]. On the other hand, given the nonlinearity of the scalar equations, the value of the external field may not be the same before and after the object is put into place. For example, the Moon changes the solution of the Galileon around the Earth and the nonlinearity of the system is such that the acceleration the Moon experiences is different from the one of a test particle orbiting around the Earth [33, 34]. This complicated nonlinear behaviour is difficult to control in general, but we can however prove that the Galileon models do *not* lead to violations of our consistency relations. Well inside the horizon, structure formation in the presence of the Galileon π follows the equations

$$\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\pi \ , \tag{41}$$

$$F(\partial_i \partial_j \pi) = \alpha 8\pi G\rho \ , \tag{42}$$

$$\nabla^2\Phi = 4\pi G\rho \ , \tag{43}$$

where F is the equation of motion for the Galileon, which only depends on the second derivatives of π. The point is that one can run the same argument as in the absence of the Galileon: a homogeneous $\vec{\nabla}(\Phi + \alpha\pi)$ can be removed by a change of coordinates that brings us to an accelerated frame [8]. For this to happen the symmetry of Galileons is crucial, since it makes a homogeneous gradient of π drop out of eq. (42) [35]. (This does not work, for example, in the case of the chameleon.) The homogenous gradient can describe a long mode in the linear regime (simulations [36, 37, 38, 39, 40] show that the scalar force is active, i.e. not Vainshtein suppressed, on sufficiently large scales) so that, barring primordial non-Gaussianity, the effect of a long mode boils down to the change of coordinates, which does not give any effect at equal time. [9]

An intermediate case between the ones above is given by *K-mouflage* [41], where the screening depends neither on the value of the field—like in the chameleon—nor on the value of the second derivatives—like in the Galileon—but on the first derivative. This happens when we have a generic kinetic term of the form $P(X)$ with $X \equiv (\partial\phi)^2$. Although this case has not been thoroughly studied, there is no reason to expect our consistency relations to work since, in the absence of Galileon symmetry, the argument above does not go through. In this case the screening depends on the typical value of $\nabla\Phi$ of the object.

Gravitational potential. The no-hair theorem implies that black-holes do not couple with a scalar force. More generally, the mass due to self-gravity will violate the EP in the presence of a fifth force. Unfortunately, it seems impossible to observe isolated objects with a sizable component of gravitational mass. The mass of clusters only receives a contribution in the range $10^{-5} \div 10^{-4}$ from the gravitational potential and the correction is even smaller for less massive objects. Black holes, whose mass is entirely gravitational in origin, do not significantly contribute to the mass of the host galaxy.

Environment. Another possibility is to divide the objects depending not on some intrinsic feature but on their environment, for example comparing galaxies in a generic place against galaxies in voids

[8] Notice that we can remove a homogeneous field, with arbitrary time-dependence. This is not a symmetry of the full Galileon theory, but it holds deep inside the Hubble radius, when time-derivatives can be neglected in eq. (42).

[9] The reader might wonder how one can reconcile the lack of consistency relation violation, with the known equivalence principle violation (at a small level) in the case of the Galileon. The point is that the boundary condition in the Earth-Moon example is quite different from that in the cosmological example. In cosmology, we know from numerical simulations that π is in the linear regime on large scales; in the Earth-Moon example, it is a computation entirely within the Vainshtein radius of the system.

[6]. The fifth force tends to be screened in a dense environment (blanket screening), while it is active in voids. Notice that this is not a test of the Galilean EP (different objects fall at the same rate in the same external field), but it still checks whether the effect of the long mode can be reabsorbed completely by a change of coordinates. The arguments made above for the Galileon case work also here and we expect no violation of the consistency relation in this case. This effect will be present in the case of chameleon screening (with the same limitations on the Compton wavelength discussed above) and in K-mouflage.

5 Conclusions

In this paper we discussed a method to test the Equivalence Principle on cosmological scales based on the recently proposed consistency relations for Large Scale Structure. The idea is simply that a homogeneous gravitational potential can be *exactly* removed by a suitable change of coordinates [1]. This is not true if the EP is violated, in which case $\epsilon \neq 0$ in eq. (1).

The method that we propose is very robust because the absence of a $1/q$ signal when the EP holds is not affected by nonlinearities at short scales, baryon physics, the issue of bias, redshift-space distortions and the way objects are selected [1, 2, 3, 4, 6]. Moreover, the signal of EP violation in the 3-point function cannot be confused with one due to primordial non-Gaussianity. The reason is that, due to the parity of the 2-point function, the squeezed limit of the primordial 3-point function cannot have a dipolar structure of the form (1). Indeed, there are models of inflation which induce $1/q$ dependence of the 3-point function in the squeezed limit, such as Quasi-Single Field [43] or Khronon Inflation [44], but in these cases the 3-point function in the squeezed limit is a function of q only and not of its direction. In models where the 3-point function in the squeezed limit depends on the direction of $\vec{q}$, such as Solid Inflation [45], this dependence has a quadrupolar structure.

In conclusion, assuming there is no primordial non-Gaussianity, any appearance of $1/q$ divergences in the 3-point function would be a clear signal of violation of the EP. Therefore, even though most of the models that violate the EP are either very constrained or produce small effects, the proposed test is so general that it deserves to be done. One can even take an agnostic point of view and, without referring to any particular model, try to explore correlations among different types of objects in N-body simulations or directly in the data. For instance, as explained above, one aspect that has not been studied in the literature is EP violation in scalar-tensor theories with a generic kinetic term $P(X)$ [41]. It would be interesting to analyze the screening in these theories and directly observe violations of the type of eq. (1) in N-body simulations. Testing the EP in the data using our method will become particularly relevant for forthcoming large scale structure surveys, whose volumes will be large enough to put interesting limits on the violation of the EP on cosmological scales. In this case, one needs to go beyond what done in [1, 2] and carefully include relativistic effects in galaxy surveys and a treatment of redshift-space distortions beyond the plane-parallel approximation. We leave this for the future.

Notice that the same limit of the 3-point function of eq. (1), when the long mode is taken outside the Hubble scale, induces a dipolar modulation of the cross power spectrum between objects A and B. The modulation is of order[10]

$$\epsilon\, \Phi_{\rm L} \frac{\vec{k} \cdot \hat{q}}{H} \,, \tag{44}$$

[10]The $1/q$ behavior is only valid in the non-relativistic limit and it saturates at the Hubble scale, see [1].

where the direction $\hat{q}$ is fixed by the average over long modes. Although the anisotropy is suppressed by the long-mode amplitude, it grows going to short scales where it can become significant. Limits on the anisotropy of the auto-power spectrum are presently of order $\mathcal{O}(10^{-2})$ [42] and it would be interesting to see what can be done using different objects, although it is difficult that this will do better than directly looking at the 3-point function.

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Article D

Healthy theories beyond Horndeski

Healthy theories beyond Horndeski

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We introduce a new class of scalar-tensor theories of gravity that extend Horndeski, or "generalized galileon", models. Despite possessing equations of motion of higher order in derivatives, we show that the true propagating degrees of freedom obey well-behaved second-order equations and are thus free from Ostrogradski instabilities, in contrast to standard lore. Remarkably, the covariant versions of the original galileon Lagrangians—obtained by direct replacement of derivatives with covariant derivatives—belong to this class of theories. These extensions of Horndeski theories exhibit an uncommon, interesting phenomenology: The scalar degree of freedom affects the speed of sound of matter, even when the latter is minimally coupled to gravity.

The discovery of the present cosmological acceleration has spurred the exploration of gravitational theories that could account for this effect. Many extensions of general relativity (GR) are based on the inclusion of a scalar degree of freedom (DOF) in addition to the two tensor propagating modes of GR (see e.g. [1] for a review). In this context, a recent important proposal is the so-called galileon models [2], with Lagrangians that involve second-order derivatives of the scalar field and lead, nevertheless, to equations of motions of second order. Such a property guarantees the avoidance of Ostrogradski instabilities, *i.e.* of the ghost-like DOF that are usually associated with higher time derivatives (see e.g. [3]).

Initially introduced in Minkowski spacetime, galileons have then been generalized to curved spacetimes [4–6], where they turn out to be equivalent to a class of theories originally constructed by Horndeski forty years ago [7]. Today, Horndeski theories, which include quintessence, k-essence and $f(R)$ models, constitute the main theoretical framework for scalar-tensor theories, in which cosmological observations are interpreted. The purpose of this Letter is to show that this framework is not as exhaustive as generally believed, and can in fact be extended to include new Lagrangians. Indeed, having equations of motion of second order in derivatives—while indeed sufficient—is not necessary to avoid Ostrogradski instabilities, as already pointed out in e.g. [8, 9]. The theories beyond Horndeski that we propose lead to distinct observational effects and are thus fully relevant for an extensive comparison of scalar-tensor theories with observations.

The model. The theories that we consider here can be viewed as a broader generalization of the galileons to curved spacetimes . They are described by linear combinations of the Lagrangians

$$L_2^\phi \equiv G_2(\phi, X) \,, \tag{1}$$

$$L_3^\phi \equiv G_3(\phi, X) \,\Box\phi \,, \tag{2}$$

$$\begin{aligned} L_4^\phi \equiv\ & G_4(\phi, X)\, {}^{(4)}\!R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}) \\ & + F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_\sigma\, \epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'} \,, \end{aligned} \tag{3}$$

$$\begin{aligned} L_5^\phi \equiv\ & G_5(\phi, X)\, {}^{(4)}G_{\mu\nu}\phi^{\mu\nu} \\ & + \frac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^\nu{}_\sigma) \\ & + F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'} \,, \end{aligned} \tag{4}$$

which depend on a scalar field ϕ (and its derivatives $\phi_\mu \equiv \nabla_\mu\phi$, $\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$), on $X \equiv g^{\mu\nu}\phi_\mu\phi_\nu$, and on a metric $g_{\mu\nu}$ with respect to which matter is assumed to be minimally coupled; $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor and a comma denotes a partial derivative with respect to the argument. Horndeski theories correspond to a subset of the above theories, subjected to the restricting conditions

$$F_4(\phi, X) = 0 \,, \qquad F_5(\phi, X) = 0 \,, \tag{5}$$

which ensure that the equations of motion (EOM) are second order. By contrast, we allow here arbitrary functions F_4 and F_5, which means that our theories contain two additional free functions with respect to the Horndeski ones.

The new terms proportional to F_4 and F_5 are, respectively, the covariant version of the original quartic and quintic galileon Lagrangians proposed in Ref. [2]. This guarantees second-order dynamics for the scalar field in the absence of gravity. When the metric is dynamical, the EOM involve up to third-order derivatives in these extended theories, but this does not imply the presence of unwanted extra DOF, as we show below.

Arnowitt-Deser-Misner (ADM) formulation. In cosmology, where the scalar field gradient is timelike, it is convenient to perform an ADM decomposition of spacetime, with metric

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \,, \tag{6}$$

by choosing the uniform scalar field (ϕ = const) hypersurfaces as constant-time hypersurfaces. The above Lagrangians then have a very simple form in terms of the intrinsic and extrinsic 3-d curvature tensors of the spatial slices, R_{ij} and K_{ij}, as well as the lapse function N. This reformulation uses the unit vector $n^\mu \equiv -\phi^\mu/\sqrt{-X}$ normal to the uniform ϕ hypersurfaces, in terms of which the extrinsic curvature is given by $K_{\mu\nu} \equiv (g^\sigma{}_\mu + n^\sigma n_\mu)\nabla_\sigma n_\nu$. We also make use of the Gauss-Codazzi relations to relate the 4-d curvature to the 3-d one.

After cumbersome but straightforward manipulations, one finds that any combination of the L_a^ϕ leads to an ADM Lagrangian density of the form $\mathcal{L} = \sqrt{-g}\sum_a L_a$, with

$$\begin{aligned} &L_2 = A_2\,, \qquad L_3 = A_3\, K\,, \\ &L_4 = A_4\, \mathcal{K}_2 + B_4\, R\,, \\ &L_5 = A_5\, \mathcal{K}_3 + B_5\, K^{ij}\left[R_{ij} - h_{ij}R/2\right]\,, \end{aligned} \tag{7}$$

where $K \equiv h^{ij}K_{ij}$, $R \equiv h^{ij}R_{ij}$, and the quantities $\mathcal{K}_2$ and $\mathcal{K}_3$ are, respectively, quadratic and cubic combinations of $K_{ij} \equiv (\dot{h}_{ij} - D_iN_j - D_jN_i)/(2N)$ (where D_i is the covariant derivative of h_{ij}), explicitly defined as

$$\mathcal{K}_2 \equiv K^2 - K_{ij}K^{ij}\,, \tag{8}$$

$$\mathcal{K}_3 \equiv K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j{}_k\,. \tag{9}$$

The coefficients in eq. (7) are related to the original functions in eqs. (1)–(4) by

$$A_2 = G_2 - (-X)^{\frac{1}{2}}\int \tfrac{G_{3,\phi}}{2\sqrt{-X}}dX\,, \tag{10}$$

$$A_3 = -\int G_{3,X}\sqrt{-X}dX - 2\sqrt{-X}G_{4,\phi}\,, \tag{11}$$

$$A_4 = -G_4 + 2XG_{4,X} + \tfrac{X}{2}G_{5,\phi} - X^2F_4\,, \tag{12}$$

$$B_4 = G_4 + \sqrt{-X}\int \tfrac{G_{5,\phi}}{4\sqrt{-X}}dX\,, \tag{13}$$

$$A_5 = -\tfrac{(-X)^{\frac{3}{2}}}{3}G_{5,X} + (-X)^{\frac{5}{2}}F_5\,, \tag{14}$$

$$B_5 = -\int G_{5,X}\sqrt{-X}dX\,. \tag{15}$$

In this ADM formulation, these functions of ϕ and X can also be seen as functions of t and N via the relations $\phi = \phi_0(t)$ and $X = -\dot{\phi}_0^2(t)/N^2$.

By using eqs. (12)–(15), the Horndeski conditions (5) translate into

$$A_4 = -B_4 + 2XB_{4,X}\,, \qquad A_5 = -XB_{5,X}/3\,. \tag{16}$$

Hamiltonian analysis. In general, higher derivative theories are pathological because they lead, according to Ostrogradski's theorem, to *extra* DOF that behave like ghosts. Here we show, by resorting to a simple counting of the number of DOF in the Hamiltonian formalism, that the theories (7) do not contain more than three degrees of freedom. Thus, there is no room for an extra DOF in addition to the scalar DOF initially built in and the two tensor modes similar to those of GR.

The Hamiltonian is obtained from the Lagrangian via a Legendre transform,

$$H = \int d^3x\left[\pi^{ij}\dot{h}_{ij} - \mathcal{L}\right]\,, \tag{17}$$

where the π^{ij} are the conjugate momenta associated with the h_{ij}, defined by

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}}\,. \tag{18}$$

Ignoring L_5 for simplicity, one can easily invert the above relation to express $\dot{h}_{ij}$ as a function of π^{ij} and obtain the explicit Hamiltonian, which can be written in the form

$$H = \int d^3x\left[N\mathcal{H}_0 + N^i\mathcal{H}_i\right]\,, \tag{19}$$

with

$$\begin{aligned} \mathcal{H}_0 \equiv &-\sqrt{h}\Big[A_2 - \frac{3A_3^2}{8A_4} + \frac{A_3\pi}{2\sqrt{h}A_4} + B_4R \\ &+ \frac{1}{2hA_4}\left(2\pi_{ij}\pi^{ij} - \pi^2\right)\Big]\,, \end{aligned} \tag{20}$$

$$\mathcal{H}_i \equiv -2D_j\pi^j{}_i\,. \tag{21}$$

We leave aside the uninteresting case $A_4 = 0$, which does not contain propagating tensor DOF.

In GR, variation with respect to N and N^i yields, respectively, the Hamiltonian constraint $\mathcal{H}_0 = 0$ and the momentum constraints $\mathcal{H}_i = 0$. These constraints are, in Dirac's terminology, first class and eventually eliminate eight out of the initial ten degrees of freedom (see e.g. [10]). In our case, the gauge invariance under *spatial* diffeomorphims is preserved, leading to first-class constraints analogous to the momentum constraints of GR and eliminating six DOF (see [11] for details). However, variation with respect to N now gives the constraint $\tilde{\mathcal{H}}_0 \equiv \mathcal{H}_0 + N\partial\mathcal{H}_0/\partial N = 0$, which is in general second class, instead of first class. This can be understood as a consequence of the scalar field that fixes the preferred slicing and thus breaks the full spacetime diffeomorphism invariance. This entails the elimination of only one DOF (instead of two in GR). Note that this reasoning crucially depends on the absence of $\dot{N}$ from the Lagrangians (7), which is guaranteed by the specific form of the new terms proportional to F_4 and F_5 introduced in eqs. (3) and (4). The final number of physical DOF is therefore three, which correspond to the two standard tensor modes plus a scalar mode, as will be clear from the linear analysis below.

When L_5 is included, the full Hamiltonian cannot be written in closed form because one cannot invert explicitly the relation (18), even if the inversion is in general well defined locally [11]. For this reason, we have not been able to compute explicitly the constraint algebra in the full case. However, our counting depends only on the nature of the constraints. Since the full Hamiltonian is, by construction, invariant under spatial diffeomorphims, the associated constraints should remain first class and thus eliminate six DOF as before. Taking into account the other constraints, one thus expects at most three DOF and, therefore, the absence of any ghostly extra DOF. The counting is also similar if one includes matter, with the matter DOF adding to the three from the gravitational sector. Finally, note that our analysis could also be applied almost straightforwardly to general ADM Lagrangians invariant under spatial diffeomorphisms involving arbitrary combinations of the extrinsic and intrinsic curvature tensors and their spatial derivatives. However, such a wider set of possibilities is not necessarily a covariant extension of galileons as eqs. (1)–(4).

Covariant formulation. The above Hamiltonian analysis is based on our ADM reformulation of the theories and requires the gradient of the scalar field to be timelike so that uniform scalar-field hypersurfaces are spacelike. Although this is the case in cosmology, which is the main motivation to study these models, one can wonder whether our findings are still valid for more general situations.

For simplicity, let us consider theories involving up to L_4^ϕ, but not L_5^ϕ. We have found that the analysis of their equations of motion can be greatly simplified via the use of disformal transformations. Indeed, the gravitational action with the Lagrangians (1)–(3) reexpressed in terms of ϕ and of the new metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(\phi, X)\partial_\mu\phi\partial_\nu\phi\,, \tag{22}$$

with

$$\Gamma = \int \frac{F_4}{G_4 - 2XG_{4X} + X^2 F_4} dX\,, \tag{23}$$

turns out to belong to the Horndeski class. This means that the equations of motion obtained by varying the action with respect to the metric $\tilde{g}_{\mu\nu}$ are second order. By using this property and by combining the (third-order) equations of motion for ϕ and $g_{\mu\nu}$ derived from the full action (including that of matter minimally coupled to $g_{\mu\nu}$), one can explicitly replace higher-order time derivatives of ϕ by at most second order time derivatives (see details in [11], and related ideas in [9]). This shows that the equations of motion can be reduced to second order in time derivatives and do not require additional initial conditions, thus extending the conclusions of our Hamiltonian analysis to general configurations. The same method applies to theories without L_4^ϕ, although one cannot simultaneously map L_4^ϕ and L_5^ϕ to Horndeski for general combinations of these Lagrangians.

Quadratic action. The above arguments exclude the presence of extra DOF, but one still needs to check that the remaining scalar and tensor DOF are not themselves ghostlike, for which we need to calculate the quadratic action for perturbations of the propagating DOF and make sure that the kinetic terms have the right signs. We perform this calculation around a spatially flat FLRW metric and follow the general procedure developed in [12] for the specific Lagrangian L given by eq. (7). Namely, we expand at second order the action $S = \int d^4x\sqrt{-g}L$, using ζ-gauge, *i.e.* $h_{ij} = a^2(t)e^{2\zeta}(\delta_{ij} + \gamma_{ij})$, $\gamma_{ii} = 0 = \partial_i\gamma_{ij}$, and splitting the shift as $N^i = \partial_i\psi + N_V^i$, $\partial_i N_V^i = 0$. Because of the particular structure of the terms in eqs. (8) and (9), the Lagrangian (7) satisfies the criteria obtained in [12] that ensure that the linear equations of motion contain no more than two spatial derivatives. In particular, terms proportional to $(\partial^2\psi)^2$ cancel up to a total derivative. By varying the action with respect to N^i, one obtains the momentum constraints, whose solution is $N_V^i = 0$ and

$$N = 1 + \mathcal{D}\dot{\zeta}\,, \qquad \mathcal{D} \equiv \frac{4\mathcal{A}_4}{2H(2\mathcal{A}_4 + \mathcal{A}_4') - \mathcal{A}_3'}\,. \tag{24}$$

Above and in the following a dot and a prime, respectively, denote derivative with respect to t and N. Furthermore, we use the new functions

$$\begin{aligned}
\mathcal{A}_2 &\equiv A_2 + 3HA_3 + 6H^2A_4 + 6H^3A_5\,,\\
\mathcal{A}_3 &\equiv A_3 + 6HA_4 + 12H^2A_5\,,\\
\mathcal{A}_4 &\equiv A_4 + 3HA_5\,,\\
\mathcal{B}_4 &\equiv B_4 + \frac{1}{2N}\dot{B}_5|_{N=1} - (N-1)\frac{HB_5'}{2}\,.
\end{aligned} \tag{25}$$

After substitution of eq. (24) into the action all the terms containing ψ drop out, up to boundary terms [13]. After some manipulations the quadratic action becomes $S^{(2)} = \int d^4x\, a^3 L^{(2)}$ with

$$L^{(2)} = \alpha\dot{\zeta}^2 - \beta\frac{(\partial_i\zeta)^2}{a^2} + \frac{1}{4}\left[-\mathcal{A}_4\dot{\gamma}_{ij}^2 - \mathcal{B}_4\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]\,, \tag{26}$$

where the functions α and β are defined as

$$\begin{aligned}
\alpha &\equiv \left[\frac{(N^2\mathcal{A}_2')'}{2} - 3H\mathcal{A}_3' + 6H^2(N\mathcal{A}_4)'\right]\mathcal{D}^2 - 6\mathcal{A}_4\,,\\
\beta &\equiv -2\mathcal{B}_4 + \frac{2}{a}\frac{d}{dt}\left[a\mathcal{D}(N\mathcal{B}_4)'\right]\,,
\end{aligned} \tag{27}$$

evaluated on the background ($N = 1$). As expected from the previous Hamiltonian analysis, the quadratic Lagrangian (26) does not contain higher time derivatives. Moreover, for $\alpha > 0$ and $-\mathcal{A}_4 > 0$ we ensure that the propagating DOF are not ghostlike. Gradient instabilities are avoided for $c_s^2 \equiv \beta/\alpha > 0$ and $c_\gamma^2 \equiv -\mathcal{B}_4/\mathcal{A}_4 > 0$.

Coupling with matter. In cosmology, the power of gravity at large scales—and its irrelevance at short distances—is well illustrated by the Jeans phenomenon.

A matter overdensity $\delta\rho_m$ of a given Fourier mode k evolves, schematically, as

$$\left(\partial_t^2 + c_m^2 k^2 - \text{gravity}\right)\,\delta\rho_m = 0\,. \qquad (28)$$

In the above, c_m^2 is the square of the speed of sound, proportional to the pressure perturbation, $c_m^2 = \delta p_m/\delta\rho_m$. For $c_m^2 > 0$, the positive sign in front of the k^2 term guarantees an oscillating solution at sufficiently short distances, where the overdensity is supported by its own pressure gradients. The last term in parentheses stands for k-independent contributions roughly of Hubble size $\sim H^2$. Only at distances larger than $\sim c_m H^{-1}$ do these terms dominate, leading to gravitational (Jeans) instability. This well-known feature of standard cosmological perturbation theory holds true at small scales also in most modified gravity models—say, for definiteness, in all Horndeski theories as long as matter fields are minimally coupled to the metric.

The extension of Horndeski theories that we are proposing provides a counterexample to such an apparently universal behavior, even when matter is minimally coupled to the metric tensor. Let us illustrate this with a matter scalar field σ (not to be confused with the dark energy field ϕ), described by the k-essence type action,

$$S_m = \int d^4x\sqrt{-g}\,P(Y,\sigma)\,, \qquad Y \equiv g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\,, \quad (29)$$

with sound speed $c_m^2 \equiv P_{,Y}/(P_{,Y} - 2\dot\sigma_0^2 P_{,YY})$. One can then repeat the procedure discussed earlier in order to obtain the quadratic action for the scalar fluctuations expressed in terms of ζ, N, ψ and the matter field perturbation $\delta\sigma$. Making use of the momentum constraints, the final Lagrangian expressed in terms of ζ and of the gauge-invariant variable $Q_\sigma \equiv \delta\sigma - (\dot\sigma_0/H)\zeta$, reads

$$L^{(2)} = \left(\alpha - \frac{c_m^2 g_t^2}{4P_{,Y}}\right)\dot\zeta^2 - \left(\beta + \frac{P_{,Y}\dot\sigma_0^2}{H^2} - \frac{\dot\sigma_0 g_s}{H}\right)\frac{(\partial_i\zeta)^2}{a^2} - \frac{P_{,Y}}{c_m^2}\left(\dot Q_\sigma^2 - c_m^2\frac{(\partial_i Q_\sigma)^2}{a^2}\right) + g_t\dot\zeta\dot Q_\sigma + g_s\frac{\partial_i\zeta\partial_i Q_\sigma}{a^2} + \dots\,, \qquad (30)$$

where $g_s \equiv -c_m^2 g_t + 2\dot\sigma_0 P_{,Y}\Delta$, with

$$g_t \equiv \frac{2\dot\sigma_0 P_{,Y}}{c_m^2}\left(\mathcal{D} - \frac{1}{H}\right), \quad \Delta \equiv \mathcal{D}\left(1 + \frac{(N\mathcal{B}_4)'}{\mathcal{A}_4}\right) \quad (31)$$

and we have included only the terms quadratic in time or space derivatives, the other terms (in the ellipses) being irrelevant for the following discussion. The dispersion relations for the propagating DOF can be obtained by requiring that the determinant of the matrix of the kinetic and spatial gradient terms vanishes, which yields

$$\begin{aligned}&(\omega^2 - c_m^2k^2)(\omega^2 - \tilde c_s^2 k^2) = \frac{(\rho_m + p_m)}{2\alpha}\,\Delta^2\,\omega^2k^2\,,\\ &\tilde c_s^2 \equiv \left[\beta - (1/2)(\rho_m + p_m)(\mathcal{D} - \Delta)^2\right]/\alpha\,,\end{aligned} \qquad (32)$$

where we have used $2\dot\sigma_0^2 P_{,Y} = -(\rho_m + p_m)$. From this equation one derives the two dispersion relations $\omega^2 = c_\pm^2 k^2$. In Horndeski theories, $\Delta \propto \mathcal{A}_4 + (N\mathcal{B}_4)' = 0$ because of eq. (16), and we thus find that, despite the couplings in the action between the time and space derivative of ζ and Q_σ, the matter sound speed is unchanged as a consequence of the special relation $g_s = -c_m^2 g_t$. This is no longer the case in our non-Horndeski extensions, where $\Delta \neq 0$ and the two couplings are "detuned". This remarkable difference between Horndeski and non-Horndeski theories was not pointed out in the recent work Ref. [14], which also extends our previous analysis [12] to compute the quadratic action of dark energy coupled to a scalar field.

This unusual behavior can also be seen by writing the perturbed EOM derived from the manifestly covariant Lagrangian for ϕ, together with eq. (28). On sufficiently small scales, we find (see [11] for details)

$$(\partial_t^2 + \tilde c_s^2 k^2)\delta\phi - C_\phi\dot\phi\,\partial_t\delta\rho_m \approx 0\,, \qquad (33)$$

$$(\partial_t^2 + c_m^2 k^2)\delta\rho_m - C_m k^2\,\partial_t(\delta\phi/\dot\phi) \approx 0\,, \qquad (34)$$

with

$$C_m \equiv \frac{\Delta(\rho_m + p_m)}{\Delta - \mathcal{D}}\,, \qquad C_\phi \equiv -\frac{\Delta(\Delta - \mathcal{D})}{2\alpha}\,, \qquad (35)$$

which leads to the same dispersion relation as in eq. (32). This clearly shows that, in contrast to the standard Jeans lore, the gravitational scalar mode $\delta\phi$ cannot be decoupled from matter by going at sufficiently short distances. The origin of the special coupling between matter and the scalar field in eq. (33) can also be understood as follows. Taking the example of L_4 for simplicity, one can see that the variation of (3) with respect to ϕ yields a term of the form $\phi^\lambda(g^{\mu\nu} + n^\mu n^\nu)\nabla_\nu R_{\lambda\mu}$. Using Einstein's equations (this assumes to separate L_4 into a GR term and an effective additional term), one can express the Ricci tensor in terms of the matter energy-momentum tensor, which leads to the term $\dot\phi\,\partial_t\delta\rho_m$ in eq. (33).

Conclusion. We have introduced a novel class of scalar-tensor theories, which include and extend Horndeski theories. For configurations where the scalar field gradient is timelike, these theories can be formulated in a very simple form via an ADM description of spacetime based on uniform ϕ slicing. This formulation allows to absorb the scalar degree of freedom in the spatial metric, and makes it particularly transparent to show the absence of Ostrogradski instabilities. For generic configurations, one can use disformal transformations to relate subclasses of these theories to theories with manifest second-order equations of motion. However, this procedure cannot be simultaneously applied to the most general case that includes both L_4^ϕ and L_5^ϕ, which means that a complete understanding of the full covariant theory requires further investigation.

An important corollary of this work applies to the original galileons proposed in [2]: Their direct covariantization, obtained by substituting ordinary derivatives with covariant ones, belongs to the class of theories considered here. Our work suggests that such theories are already free of ghosts instabilities and do not need the gravitational "counterterms" prescribed in [4].

We have also uncovered a remarkable phenomenological property of the non-Horndeski subclass of our theories: When *minimally* coupled to ordinary matter, they exhibit a kinetic-type coupling, leading to a mixing of the dark energy and matter sound speeds. It would be interesting to study further the phenomenology of these theories.

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Article E

Resilience of the standard predictions for primordial tensor modes

Resilience of the standard predictions for primordial tensor modes

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We show that the prediction for the primordial tensor power spectrum cannot be modified at leading order in derivatives. Indeed, one can always set to unity the speed of propagation of gravitational waves during inflation by a suitable disformal transformation of the metric, while a conformal one can make the Planck mass time-independent. Therefore, the tensor amplitude unambiguously fixes the energy scale of inflation. Using the Effective Field Theory of Inflation, we check that predictions are independent of the choice of frame, as expected. The first corrections to the standard prediction come from two parity violating operators with three derivatives. Also the correlator $\langle\gamma\gamma\gamma\rangle$ is standard and only receives higher derivative corrections. These results hold also in multifield models of inflation and in alternatives to inflation and make the connection between a (quasi) scale-invariant tensor spectrum and inflation completely robust.

Introduction - We are entering an exciting period for primordial gravitational waves, since BICEP2 [1] has shown that the experimental sensitivity to B-modes is now at the level of an interesting regime for tensors, provided foreground contamination is under control. The importance of primordial tensor modes lies in their robustness: while scalar perturbations are sensitive to many details (the shape of the potential, the speed of propagation of scalar fluctuations c_s, the number of fields and their conversion to adiabatic perturbations) and can also be viably produced in non-inflationary models, tensor modes are much more model independent. In this Letter we strengthen this robustness, showing that one cannot change the tensor quadratic and cubic action at leading order in derivatives. Since the inflaton defines a preferred frame, the time and spatial kinetic term of gravitons can have in general different time-dependent coefficients. However, without loss of generality, one can always make the graviton speed equal to unity by doing a suitable disformal transformation. A conformal transformation can then remove any time dependence of the overall normalization of the action, i.e., the Planck mass, so that the dynamics of gravitons is completely standard.

Disformal vs Einstein frame - We work here with the (single-field) Effective Field Theory of Inflation [2, 3] and we will comment on generalizations later. Working in unitary gauge, where the inflaton perturbations are set to zero, the speed of gravitons can be changed by the operator $\delta K_{\mu\nu}\delta K^{\mu\nu}$, where $\delta K_{\mu\nu}$ is the perturbation of the extrinsic curvature of the spatial slices, $K_{\mu\nu}$ [3–5]. This kind of modifications arises when considering higher derivative operators for the inflaton, such as in Horndeski theories [6]. We are free to subtract δK^2, which at quadratic order contains only scalars. As shown below, the combination $\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2$ does not change the sound speed of scalar fluctuations. Thus, we consider the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\Big[R - 2\big(\dot H + 3H^2\big) + 2\dot H g^{00} - \big(1 - c_T^{-2}(t)\big)\big(\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2\big)\Big] , \qquad (1)$$

where $H \equiv \dot a/a$ is the Hubble rate and the first line describes a minimal slow-roll model [3].

We will use the usual ADM decomposition,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(N^i\mathrm{d}t + \mathrm{d}x^i)(N^j\mathrm{d}t + \mathrm{d}x^j) , \qquad (2)$$

and describe scalar and tensor perturbations as [7]

$$h_{ij} = a^2 e^{2\zeta}(e^\gamma)_{ij} , \qquad \gamma_{ii} = 0 = \partial_i\gamma_{ij} . \qquad (3)$$

In these variables the extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N}\big(\dot h_{ij} - \nabla_i N_j - \nabla_j N_i\big) . \qquad (4)$$

The coefficient in the second line of eq. (1) is chosen such that the tensor quadratic action reads

$$S_{\gamma\gamma} = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x a^3 c_T^{-2}\left[\dot\gamma_{ij}^2 - c_T^2\frac{(\partial_k\gamma_{ij})^2}{a^2}\right] . \qquad (5)$$

The second line of (1) modifies the time kinetic term of gravitons; the only other way to change tensor modes at quadratic order is to modify the spatial kinetic term with the operator ${}^{(3)}\!R$, the 3d Ricci tensor. The two choices are related by the Gauss-Codazzi identity,

$$R = {}^{(3)}\!R + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu) , \qquad (6)$$

where n^μ is the unit vector perpendicular to the surfaces of constant time.

The main point of this paper is that it is possible to set to one the speed of propagation of gravitons in action (5) by a proper redefinition of the metric. Metric transformations that change the light-cone are known as disformal transformations [8], so that we denote the metric used to write eq. (1) as the disformal metric. We first perform a disformal transformation which leaves the spatial metric in unitary gauge unchanged,[1 2]

$$g_{\mu\nu} \mapsto g_{\mu\nu} + (1 - c_T^2(t)) n_\mu n_\nu \, . \tag{7}$$

This transformation does not affect N^i and h_{ij} while $N \mapsto c_T N$. Thus $K_{ij} \mapsto K_{ij}/c_T$, while ${}^{(3)}R$ is not changed. In this way the relative coefficient between the time and the spatial kinetic term of gravitons can be set to one and combined to give the 4d Ricci scalar through (6). However, the normalization of the Einstein-Hilbert term is now non-standard and given by $\frac{1}{2} M_{\rm Pl}^2 R / c_T(t)$. This can be cast in the standard form by going to the Einstein frame with a conformal transformation of the metric,

$$g_{\mu\nu} \mapsto c_T^{-1}(t)\, g_{\mu\nu} \, . \tag{8}$$

Notice that in doing the disformal and conformal transformations the FLRW line element becomes $\mathrm{d}\tilde{s}^2 = c_T^{-1}[-c_T^2 \mathrm{d}t^2 + a^2 \mathrm{d}\vec{x}^2]$. It is thus convenient to redefine the time coordinate and the scale factor as

$$\tilde{t} \equiv \int c_T^{1/2}(t) \mathrm{d}t \, , \qquad \tilde{a}(\tilde{t}) \equiv c_T^{-1/2} a(t) \, . \tag{9}$$

Under this combined set of transformations the components of the metric in Einstein frame read $\tilde{g}^{00} = g^{00}$ ($g^{00} = -1/N^2$), $\tilde{N}^i = c_T^{1/2} N^i$ and $\tilde{h}_{ij} = c_T^{-1} h_{ij}$. Using these relations it is straightforward to compute the Einstein-frame action,

$$\begin{aligned} S = \int \mathrm{d}\tilde{t} \mathrm{d}^3 x \sqrt{-\tilde{g}} \frac{M_{\rm Pl}^2}{2} \bigg\{ & \tilde{R} - 2\big(\dot{\tilde{H}} + 3\tilde{H}^2\big) + 2\dot{\tilde{H}} \tilde{g}^{00} \\ & + \left[2\big(1 - c_T^2\big)\dot{\tilde{H}} - \frac{3}{2}\alpha^2 - c_T^2 \left(\dot{\alpha} + \tilde{H}\alpha + \frac{1}{2}\alpha^2 \right) \right] \\ & \times \left(1 - \sqrt{-\tilde{g}^{00}}\right)^2 + 2\alpha\, \delta\tilde{K} \left(1 - \sqrt{-\tilde{g}^{00}}\right) \bigg\} \, , \end{aligned} \tag{10}$$

where $\alpha \equiv \dot{c_T}/c_T$. Here and in the action above time derivatives are with respect to $\tilde{t}$. The last term in the action is obtained when using the Gauss-Codazzi identity to combine 3d quantities to form the 4d Ricci scalar, by integrating by parts the last term of (6). The first line has the expected dependence on the background evolution in Einstein frame, while the rest starts quadratic in the perturbations. In this frame, the kinetic term of gravitons is the standard one, given by the Einstein-Hilbert term. If $\alpha = 0$ we just have a polynomial in $\tilde{g}^{00} + 1$, which describes an inflationary model with a Lagrangian of the form $P(\phi, (\partial\phi)^2)$.

We stress that in doing disformal and conformal transformations one changes the way other particles are coupled to the metric; this however is immaterial, since it does not enter in the inflationary predictions.

Frame independence of predictions - Since the definition of ζ and γ_{ij} is the same in the disformal and Einstein frame, we expect all the inflationary predictions to remain unchanged, as we are now going to show. We start by discussing the scalar fluctuations. It is important to note that in the disformal frame, for significant modifications of c_T^2, the coefficient in front of $\delta K_{\mu\nu}\delta K^{\mu\nu} - \delta K^2$ in action (1) is of order $M_{\rm Pl}^2$. Thus, one cannot rely on the decoupling limit when deriving predictions from this action.

As anticipated above, the operator in the second line of eq. (1) does not contribute to scalar fluctuations up to quadratic order. Indeed, to fix N we need the solution of the momentum constraint, which is the same as in the standard $c_T = 1$ case, i.e. $N = 1 + \dot{\zeta}/H$ [7] (use for instance eq. (74) of [10]). Thus, from eq. (4) the scalar contributions to K_{ij} from N and $\dot{h}_{ij}$ cancel and we are left with those coming from N^i which, in the combination that appears in eq. (1), only give a total derivative. Thus, the scalar sound speed in the disformal frame is $c_s = 1$.

Since in the Einstein frame tensor modes propagate on the light-cone, we expect the scalar speed of propagation to be $\tilde{c}_s = 1/c_T$. For a constant c_T ($\alpha = 0$), this can be easily seen from the first term on the second line of action (10). Indeed, introducing the scalar Goldstone boson $\tilde{\pi}$ associated with the breaking of time-diff invariance by the time transformation $\tilde{t} \mapsto \tilde{t} + \tilde{\pi}(\tilde{t}, \vec{x})$, and expanding up to cubic order in the decoupling limit, the action becomes

$$\mathcal{L} = \tilde{a}^3 M_{\rm Pl}^2 |\dot{\tilde{H}}| c_T^2 \left[\dot{\tilde{\pi}}^2 - c_T^{-2} \frac{(\partial_i \tilde{\pi})^2}{\tilde{a}^2} - (1 - c_T^{-2}) \dot{\tilde{\pi}} \frac{(\partial_i \tilde{\pi})^2}{\tilde{a}^2} \right] . \tag{11}$$

One can verify that $\tilde{c}_s = 1/c_T$, as expected, also when $\alpha \neq 0$ (use e.g. eq. (69) of [11]).

Let us now check that the spectrum of gravitational waves is the same when computed in either frame. For the quadratic action (5), scale invariance is obtained for $a\, c_T^{-1/2} \int (c_T/a) dt \simeq$ const. (we do not assume c_T slowly varying, see [12]). Perturbations evolve with an effective scale factor $a\, c_T^{-1/2}$ so that the gravitational wave spec-

[1] In terms of the inflaton field ϕ, the new metric reads $g_{\mu\nu} \mapsto g_{\mu\nu} - (1 - c_T^2) \partial_\mu\phi \partial_\nu\phi / (\partial\phi)^2$.

[2] A similar transformation was also employed for instance in [9] to set an action with modified graviton sound speed in the standard Einstein-frame form.

trum becomes

$$\langle \gamma^s_{\vec{k}} \gamma^{s'}_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \frac{1}{2k^3} \frac{(H-\alpha/2)^2}{M_{\rm Pl}^2 c_T} \delta_{ss'} \; . \qquad (12)$$

(The polarization tensors ϵ^s_{ij} are normalized as $\epsilon^s_{ij}\epsilon^{s'}_{ij} = 4\delta_{ss'}$ where s, s' denote the helicity states.) Using eq. (9), the Einstein frame Hubble rate is $\tilde{H} = c_T^{-1/2}(H-\alpha/2)$, implying that eq. (12) is simply the standard spectrum for gravitational waves with unit sound speed in Einstein frame. It is straightforward to verify, using again eq. (9), that also the scalar power spectrum is the same in both frames.

Given that the relation between the two frames does not involve the spatial coordinates, also the tilt of the tensor and scalar power spectra remains the same. For tensors, this is given by the usual formula $n_T = 2\dot{\tilde{H}}/\tilde{H}^2$. In the disformal frame, it is possible to obtain a blue tilt by a time varying c_T, keeping $\dot{H} < 0$. In this case one does not violate the Null Energy Condition (NEC) and, indeed, there is no sign of instability. It is interesting to see how this translates in the Einstein frame where a blue tilt requires $\dot{\tilde{H}} > 0$. One can check that the usual gradient instability associated with the violation of the NEC is cured by the last term of action (10), as showed in [2]. For example, this operator arises in Galileon models that violate the NEC [13].

We conclude that there is no loss of generality in assuming that gravitons have a standard kinetic term. In particular, this implies that the amplitude of tensor modes is fixed by the vacuum energy of inflation and that a blue spectrum of gravitational waves, $n_T > 0$, requires $\dot{\tilde{H}} > 0$, i.e. a violation of the NEC in Einstein frame. Moreover, the observation of an approximately scale-invariant tensor spectrum would imply an approximately time-independent $\tilde{H}$. While one can make a scale-invariant scalar power spectrum playing with a variable speed of sound c_s and equation of state $\epsilon \equiv -\dot{H}/H^2$ [12], tensors are absolutely robust and sensitive only to $\tilde{H}$. It is worthwhile to stress that these conclusions do not change if we consider multifield models of inflation, or even alternatives to inflation. However, our conclusions do not apply to cases with a different symmetry structure, like solid inflation [14] (in this case one can have $n_T > 0$ with $\dot{\tilde{H}} < 0$) or gauge-flation [15], or when tensors are produced not as vacuum fluctuations [16].

Non-Gaussianity - We now show the equivalence between the two frames beyond linear order, taking c_T time-independent for simplicity. We saw that in Einstein frame the scalar has a nontrivial sound speed $\tilde{c}_s = 1/c_T$. This implies a cubic interaction $\propto (1-\tilde{c}_s^{-2})$, as in eq. (11). In the disformal frame this is not obvious, since the second line of action (1) does not contribute to the action of π in the decoupling limit. However, as mentioned above, one cannot rely on this limit, but has to solve the constraints. The linear Hamiltonian constraint fixes the scalar part of the shift. Crucially, this gets rescaled by a factor c_T^2 with respect to the standard case (use eq. (75) of [10]),

$$\psi \equiv \partial^{-2}\partial_i N^i = -c_T^2 \frac{\zeta}{a^2 H} + \chi \; , \quad \partial^2 \chi = \epsilon c_T^2 \dot{\zeta} \; . \qquad (13)$$

Using this solution, after several manipulations and integration by parts, one obtains that the leading interaction in the slow-roll limit, up to field redefinitions which die out on super-Hubble scales, is

$$\mathcal{L}_{\zeta\zeta\zeta} = a\epsilon\left(1-c_T^2\right)\frac{\dot{\zeta}}{H}(\partial_i \zeta)^2 \; , \qquad (14)$$

which yields $f_{\rm NL} \sim 1 - c_T^2 = 1 - \tilde{c}_s^{-2}$.

Let us now discuss cubic interactions involving gravitons. As already noticed in [17], the second line of eq. (1) does not contain cubic graviton vertices. Therefore, in both frames $\langle\gamma\gamma\gamma\rangle$ coincides with the minimal slow-roll result of [7]. To study interactions involving two gravitons and one scalar we need to expand the action to cubic order and plug in the linear solutions to the constraints, i.e. $N = 1 + \dot{\zeta}/H$ and eq. (13). After some manipulations and integrations by parts (see [7]) one obtains, at leading order in slow-roll,

$$\mathcal{L}_{\gamma\gamma\zeta} = \frac{M_{\rm Pl}^2}{8} a^3 c_T^{-2}\left[\epsilon\zeta\left(\dot{\gamma}_{ij}^2 + c_T^2 \frac{(\partial\gamma_{ij})^2}{a^2}\right) - 2\dot{\gamma}_{ij}\partial\gamma_{ij}\partial\chi\right]. \qquad (15)$$

In the Einstein frame the cubic interaction is standard (see eq. (3.17) of [7]) except for a factor of c_T^2 in the solution for χ due to the scalar speed of sound (see eq. (4.9) of [18]). Taking into account eq. (9) and the different wavefunctions, one can check that $\langle\gamma\gamma\zeta\rangle$ computed in the two frames coincide. This correlator goes as $\langle\gamma\gamma\zeta\rangle \sim \epsilon\langle\zeta\zeta\rangle\langle\gamma\gamma\rangle$.[3] This differs from the result of [5] obtained in the decoupling limit. Finally, it is straightforward to verify that also the prediction for $\langle\gamma\zeta\zeta\rangle$ is the same in the two frames and coincides with the minimal slow-roll model [7].

Quadratic terms with three derivatives - We have seen that it is possible, without loss of generality, to cast the graviton kinetic term in the standard form. From now on we assume to be in Einstein frame and we drop the tildes. Notice that the operators $\dot{\gamma}_{ij}^2$ and $(\partial_l \gamma_{ij})^2$

[3] The cubic $\gamma\gamma\zeta$ action is suppressed by $\epsilon\zeta$ compared to the graviton kinetic term. This holds also for the term including χ in the limit $\tilde{c}_s \ll 1$ since, in the Einstein frame,

$$\frac{\mathcal{L}_{\gamma\gamma\zeta}}{M_{\rm Pl}^2} \supset \dot{\gamma}\partial\gamma\partial\chi \sim \epsilon\,\tilde{c}_s^{-2}\dot{\gamma}\partial\gamma\partial^{-1}\dot{\zeta} \sim \epsilon\,\dot{\gamma}\partial\gamma\frac{\partial}{\tilde{H}}\zeta \sim \epsilon\,\dot{\gamma}^2\zeta \; , \qquad (16)$$

where we used $\dot{\zeta} \sim \tilde{c}_s^2\partial^2\zeta/\tilde{H}$. Indeed, given the different dispersion relation, ζ is already frozen when tensor modes exit the Hubble radius.

are the only quadratic operators with two derivatives. Indeed, one could imagine a term with one time and one space derivative, in the parity violating combination $\varepsilon^{ijk}\partial_i\gamma_{jl}\dot\gamma_{lk}$, where ε^{ijk} is the totally antisymmetric tensor. However, it is easy to see that this is a total derivative.

The first possible corrections to the tensor power spectrum come from terms with three derivatives. The combinations with an even number of spatial derivatives, $\dot\gamma_{ij}\ddot\gamma_{ij}$ and $\partial_l\gamma_{ij}\partial_l\dot\gamma_{ij}$, are total derivatives, so we are left to consider parity-violating terms with one or three spatial derivatives. There are two possible combinations,

$$\varepsilon^{ijk}\partial_i\dot\gamma_{jl}\dot\gamma_{lk}\ , \qquad \varepsilon^{ijk}\partial_i\partial_m\gamma_{jl}\partial_m\gamma_{lk}\ . \tag{17}$$

The first term comes from $4\int \mathrm{d}^4x\ \varepsilon^{0ijk}\nabla_i\delta K_{jl}\delta K_{lk}$. The second term comes from the 3d Chern-Simons term,

$$-4\int \mathrm{d}^4x\ \varepsilon^{ijk}\left(\frac{1}{2}{}^3\Gamma^p_{iq}\partial_j{}^3\Gamma^q_{kp}+\frac{1}{3}{}^3\Gamma^p_{iq}{}^3\Gamma^q_{jr}{}^3\Gamma^r_{kp}\right)\ , \tag{18}$$

where ${}^3\Gamma^i_{jk}$ are the Christoffel symbols of the 3d metric. The impact of these terms on primordial gravitational waves has been studied in the context of Horava-Lifschitz gravity in [19, 20].[4]

It is easy to study the effect of the two 3-derivative operators on the power spectrum of tensor modes. The standard quadratic action is modified by the addition of

$$-\frac{M_{\rm Pl}^2}{8}\int \mathrm{d}^4x\frac{1}{H\eta}\left[\frac{\alpha}{\Lambda}\varepsilon^{ijk}\partial_i\gamma'_{jl}\gamma'_{lk}+\frac{\beta}{\Lambda}\varepsilon^{ijk}\partial_i\partial_m\gamma_{jl}\partial_m\gamma_{lk}\right], \tag{21}$$

where a prime denotes the derivative with respect to the conformal time $\eta\equiv\int dt/a$, α and β are dimensionless coefficients and Λ is the scale that suppresses these higher dimension operators. We are going to assume an exact de Sitter background and take α and β, which could depend on time, to be approximately constant. In this limit the dilation isometry of de Sitter guarantees the spectrum to remain scale invariant also in the presence of the new operators. We are going to treat the corrections due to these terms perturbatively, i.e. assume that the energy scale of the problem, the Hubble scale H, is small compared to Λ. The action (21) violates parity and induces opposite corrections to the power spectrum of gravitons with opposite helicities. Indeed, the polarization tensors $\epsilon^{\pm}_{ij}$ of the two helicities satisfy $ik_l\,\varepsilon^{jlm}\epsilon^{\pm}_{im}=\pm k\,\epsilon^{\pm\,j}_i$. The interaction Hamiltonian $\mathcal{H}_{\rm int}$ in Fourier space is thus given by

$$\mathcal{H}_{\rm int}=\pm\frac{M_{\rm Pl}^2}{2H\Lambda}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k}{\eta}\left[\alpha\gamma^{\pm}_{\vec k}{}'\gamma^{\pm}_{-\vec k}{}'+\beta k^2\gamma^{\pm}_{\vec k}\gamma^{\pm}_{-\vec k}\right]. \tag{22}$$

For the other helicity we would have an overall minus sign. It is straightforward to study the effect of this term in the usual in-in formalism [7]. The correction to the power spectrum is given by

$$\delta\langle\gamma^{\pm}_{\vec k}\gamma^{\pm}_{\vec k'}\rangle=\mp i\int_{-\infty}^{\eta}\mathrm{d}\tilde\eta\ \langle\gamma^{\pm}_{\vec k}(\eta)\gamma^{\pm}_{\vec k'}(\eta)\mathcal{H}^{\rm int}(\tilde\eta)\rangle+\text{c.c.}\,. \tag{23}$$

In the late-time limit, $\eta\to0$, the result does not depend on α and the power spectrum is modified to

$$\langle\gamma^{\pm}_{\vec k}\gamma^{\pm}_{\vec k'}\rangle=(2\pi)^3\delta(\vec k+\vec k')\frac{H^2}{2M_{\rm Pl}^2k^3}\left(1\pm\beta\frac{\pi}{2}\frac{H}{\Lambda}\right)\ . \tag{24}$$

The same result was obtained in [22]. For a large background of tensor modes, $r\sim0.1$, one will be able to distinguish a 50% difference in the power spectra of the two helicities [23].

Enhanced graviton non-Gaussianity? - We saw above that it is not possible, at the lowest derivative level, to change the predictions for the power spectrum of tensor modes. We now check that the same happens for the cubic correlator $\langle\gamma\gamma\gamma\rangle$. With three gravitons, the minimum number of derivatives is two.[5] If they are both with respect to time, schematically $\dot\gamma\dot\gamma\gamma$, one is forced by invariance under time-dependent spatial diffs to promote $\dot\gamma$ to the extrinsic curvature. The only operator that one can write is thus $\delta K_{ij}\delta K^{ij}$: as discussed before, this operator does not contain a cubic graviton interaction. It is straightforward to realize that it is impossible to write an operator with one time and one spatial derivative: one may include the totally antisymmetric ε tensor but cannot build an invariant geometric operator. If the derivatives are both spatial, the operator has only to do with the 3d geometry. The only scalar that one can write with two derivatives is the 3d Ricci scalar: we saw above this term can always be cast in the standard form inside the 4d Ricci. We conclude that, at two derivative

[4] Parity violation in the context of inflation [21] is usually discussed in terms of the topological current

$$K^\mu=2\varepsilon^{\mu\alpha\beta\gamma}\left(\frac{1}{2}\Gamma^\sigma_{\alpha\nu}\partial_\beta\Gamma^\nu_{\gamma\sigma}+\frac{1}{3}\Gamma^\sigma_{\alpha\nu}\Gamma^\nu_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right)\ , \tag{19}$$

which satisfies

$$\partial_\mu K^\mu=\frac{1}{4}\varepsilon^{\mu\nu\alpha\beta}R^\sigma{}_{\rho\alpha\beta}R^\rho{}_{\sigma\mu\nu}\ . \tag{20}$$

It is easy to see that the operator $-2\int\mathrm{d}^4x\ \ K^0$ gives, at quadratic order in γ, the linear combination $\varepsilon^{ijk}\partial_i\dot\gamma_{jl}\dot\gamma_{lk}-\varepsilon^{ijk}\partial_i\partial_m\gamma_{jl}\partial_m\gamma_{lk}$. Notice, however, that in general the relative coefficient of the two operators in eq. (17) is not fixed by symmetry.

[5] In pure de Sitter, i.e. in the absence of a breaking of time diffs due to the inflaton, this correlator is strongly constrained by the isometry of de Sitter space, so that it can be fixed in terms of three constants, without relying on a derivative expansion [17]. In the presence of the inflaton one cannot get such a general result, but one can rely on the derivative expansion: the correlator will be dominated by operators with the lowest number of derivatives.

level, the correlator $\langle\gamma\gamma\gamma\rangle$ has always the standard form, first calculated in [7]. Higher derivative corrections start with three derivatives: parity violating operators were discussed above, while parity-conserving ones may have three time derivatives (e.g. $\delta K_{ij}\delta K_{jl}\delta K_{li}$) or one time derivative (e.g. $\delta K_{ij}\delta^{(3)}R$).

It is difficult to reach general conclusions involving mixed correlators. For example, one can induce an arbitrarily large $\langle\zeta\gamma\gamma\rangle$ with the operators $\delta N\delta K_{ij}\delta K^{ij}$ and $\delta N\delta^{(3)}R$, though this may be quite unnatural. On the other hand, the $\langle\gamma\zeta\zeta\rangle$ correlator comes, in the standard case, from the tadpole g^{00}: it is thus impossible to enhance this correlator, unless one relies on higher-derivative operators.

Conclusions - We showed that the tensor power-spectrum formula $\langle\gamma\gamma\rangle = (H/M_{\rm Pl})^2/(2k^3)$, with H and $M_{\rm Pl}$ Einstein frame quantities, is completely general and only receives (small) higher-derivative corrections. In particular, the tensor amplitude fixes the energy scale of inflation. The tilt of the power spectrum cannot be modified by a time-dependent speed of tensor modes: a blue tensor tilt requires violation of the NEC in the Einstein frame.

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Article F

Exploring gravitational theories beyond Horndeski

Exploring gravitational theories beyond Horndeski

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Abstract

We have recently proposed a new class of gravitational scalar-tensor theories free from Ostrogradski instabilities, in Ref. [1]. As they generalize Horndeski theories, or "generalized" galileons, we call them G^3. These theories possess a simple formulation when the time hypersurfaces are chosen to coincide with the uniform scalar field hypersurfaces. We confirm that they contain only three propagating degrees of freedom by presenting the details of the Hamiltonian formulation. We examine the coupling between these theories and matter. Moreover, we investigate how they transform under a disformal redefinition of the metric. Remarkably, these theories are preserved by disformal transformations that depend on the scalar field gradient, which also allow to map subfamilies of G^3 into Horndeski theories.

Contents

1 Introduction

The fact that current cosmological observations consistently point to a recent phase of accelerated expansion has boosted the exploration of alternative theories of gravity (see *e.g.* [2] for a review), that could provide a more natural explanation than simply a cosmological constant. Even if these efforts have not led to a compelling or even realistic model, these research activities have deepened our understanding of gravity by highlighting the theoretical and observational constraints that alternatives to general relativity must satisfy.

Many models of modified gravity involve the presence of at least one scalar degree of freedom in addition to the two tensor degrees of freedom of general relativity. The underlying scalar field can sometimes be hidden in the explicit formulation of the theory. A typical example is $f(R)$ theory, where the Lagrangian is written as a function of the Ricci scalar R, but which can be reformulated as a manifestly scalar-tensor theory (see *e.g.* [3]).

A minimal requirement on alternative theories is the absence of ghost-like instabilities within their domains of validity (see *e.g.* [4] on this point). According to the so-called Ostrogradski's theorem, such instabilities arise in theories characterized by a non-degenerate Lagrangian[1] with higher time derivatives (see *e.g.* [5]). The simplest example is the Lagrangian

$$L = \frac{1}{2}\ddot{q}^2, \tag{1}$$

which leads to fourth-order equations of motion. In the Hamiltonian formulation, an extra degree of freedom appears so that the corresponding phase space is four-dimensional, with a Hamiltonian that depends linearly on one of the momenta and is thus (kinetically) unbounded from below. In this case the extra degree of freedom is a ghost and the theory is not viable.

Not all theories containing higher-order time derivatives in the Lagrangian suffer from Ostrogradski instabilities. In particular, this is the case for theories that lead to second-order equations of motion, such as the much studied galileon models [6], briefly reviewed in Sec. 2.1. Although originally introduced in Minkowski, the galileon Lagrangians can be extended to general curved spacetimes by promoting the derivatives to *covariant* derivatives. However, as discussed in Sec. 2.2, maintaining second-order equations of motion with respect to spacetime derivatives requires the addition of suitable gravitational "counterterms" [7, 8]. The largest class of these Generalized Galileons [9], or G^2, turns out to be equivalent to the more ancient Horndeski's theories [10], which correspond to the most general scalar-tensor theories with second-order field equations.

Although Horndeski theories are often considered as the most general scalar-tensor theories immune from Ostrogradski's instabilities, we have recently showed that this is not the case and proposed a new class of scalar-tensor theories, reviewed in Sec. 3 (see also Appendix A for the details of the calculations), that do not suffer from such instabilities [1]. Since our theory contains generalized galileons (Horndeski) as a special limit, we dubbed it "Generalized Generalized Galileons" or G^3 for brevity. It turns out that our theories have the same decoupling limit as Horndeski theories, as briefly showed at the end of Sec. 3.

The stability properties of G^3 are most easily seen by using the ADM formalism applied to the uniform scalar field hypersurfaces (also called unitary gauge formulation). In this formulation, the scalar field does not appear explicitly as it is part of the degrees of freedom of the metric, and the action depends only on first time derivatives of the metric (the "velocities"), as generally expected from healthy theories. Indeed, the Hamiltonian analysis confirms the absence of unwanted extra degrees of freedom, and thus the absence of Ostrogradski instabilies [1]. In Sec. 4 of the present article we give more details about the derivation of the Hamiltonian and about the counting of the degrees of freedom, which depends on the number and nature (first or second class) of the constraints between canonical variables. Our analysis clearly proves that our theories contain only three degrees of freedom and do not suffer from Ostrogradski instabilities, as stated in [1].

Hints that one could go beyond Horndeski theories without encountering fatal instabilities appeared in our previous work [11], where we studied the most general quadratic Lagrangian for linear perturbations about a homogenous and isotropic spacetime that does not induce higher derivatives on the linear propagating scalar degree of freedom. Such a Lagrangian contains an additional term, which is absent in Horndeski theories. In Sec. 5.1 we review this analysis of linear perturbations and we extend it in Sec. 5.2 by including some matter field, detailing the analysis of [1]. For convenience, we describe matter by means of a scalar field with non-standard kinetic term, which can be formulated in terms of a simple Lagrangian and which is characterized by a nontrivial speed of sound. We are thus able to derive a quadratic Lagrangian that includes both metric and matter perturbations in the unitary gauge. A similar calculation was presented in [12], and generalized to several matter scalar fields in [13]. We also give an equivalent treatment for perfect fluid matter by working directly with the equations of motion written in the Newtonian gauge, in Sec. 5.3. For this analysis we find it convenient to employ the notation proposed in Ref. [14], based on the effective

[1]A Lagrangian $L(q, \dot{q}, \ddot{q})$ is said to be nondegenerate if $\partial^2 L/\partial \ddot{q}^2 \neq 0$

approach to cosmological perturbations for dark energy, introduced in [15, 16, 11, 17]. In Appendix B we review the connection between the different notations employed in these references.

Other fissures in the standard lore concerning Horndeski theories were pointed out in [18], which studied scalar-tensor theories generated by disformal relations [19]

$$\tilde{g}_{\mu\nu} = \Omega^2(X,\phi)g_{\mu\nu} + \Gamma(X,\phi)\partial_\mu\phi\partial_\nu\phi\,, \tag{2}$$

where $X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. In particular, it was shown that starting from an action consisting of the Einstein-Hilbert term for $\tilde{g}_{\mu\nu}$ and of a standard action for ϕ, one obtains equations of motion for $g_{\mu\nu}$ and ϕ that are higher order but can be combined so that the dynamics is only second order. This is another example beyond Horndeski that is not Ostrogradski unstable. Interestingly, a very similar argument has been invoked in the context of ghost-free massive gravity in [20].

It is natural to wonder whether our theories could be formulated in a similar way, i.e. derived via a disformal transformation from a theory belonging to the Horndeski class. We discuss this issue in Sec. 6 and find that our general theory cannot be derived from Horndeski via a disformal transformation. Remarkably however, the two non-Horndeski pieces contained in our Lagrangian can be *separately* derived from a Horndeski Lagrangian combined with a disformal transformation. Since the disformal transformation that we consider conserves the number of degrees of freedom, this proves that our two non-Horndeski pieces are separately equivalent to a subset of Horndeski theories. In Appendix C we explicitly check in Newtonian gauge that the disformal metric redefinition de-mixes part of the kinetic couplings (the part containing higher derivatives) between the scalar field and the metric. In this respect, the disformal transformations considered here are analogous to the field redefinition removing higher derivatives discussed in the context of massive gravity in [20]. Since the two disformal transformations are distinct for the two non-Horndeski pieces of G^3, the procedure cannot be applied to the whole Lagrangian. However, the fact that these pieces can be mapped to Horndeski provides an alternative way to show the healthy behavior of our theories. Using a disformal transformation, in Sec. 6.5 we provide an example of naively higher-derivative equations of motion which can be reduced to second order ones, generalizing the treatment of [18].

2 Galileons and Horndeski theories

2.1 Galileon theories

One of the most explored frameworks for infra-red modifications of gravity is the so-called galileon theory [6], which distills and generalizes the interesting features of the DGP scenario [21] and emerges in the decoupling limit of massive gravity [22].

Galileon theories can be seen as the effective theory of a Goldstone boson ϕ in Minkowski space, that is invariant under a generalized shift symmetry,

$$\phi(x) \;\to\; \phi(x) + b_\mu x^\mu + c, \tag{3}$$

for the five arbitrary parameters b_μ and c. Only in Minkowski can we arbitrarily choose a *constant* vector field b^μ and thus this is where galileon theories are naturally set. At lowest order in derivatives, there exists a limited number of Lagrangian terms invariant under (3), with schematic form $\mathcal{L}_n \sim (\partial\phi)^2(\partial^2\phi)^{n-2}$, where $n \leq 5$ in four dimensions. Such operators are protected by the symmetry (3) against quantum corrections [23, 24].

These theories can be most naturally formulated as [6]

$$\mathcal{L}^{\mathrm{gal},1}_{n+1} = \left(\mathcal{A}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\phi_{\mu_1}\phi_{\nu_1}\right)\phi_{\mu_2\nu_2}\ldots\phi_{\mu_n\nu_n}\,, \tag{4}$$

where $\mathcal{A}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}$ is a tensor separately antisymmetric in the indices μ's and ν's and symmetric under the exchange $\{\mu_i\} \leftrightarrow \{\nu_i\}$, *e.g.* $\mathcal{A}^{\mu_1\mu_2\nu_1\nu_2} \propto g^{\mu_1\nu_1}g^{\mu_2\nu_2} - g^{\mu_1\nu_2}g^{\mu_2\nu_1}$ (see *e.g.* the nice review [25] for technical details). In the above expression and in the rest of this section, we use the shorthand notation $\phi_\mu \equiv \nabla_\mu \phi$, $\phi_{\mu\nu} \equiv \nabla_\nu \nabla_\mu \phi$ for convenience. More explicitly, the galileon Lagrangians are written as linear combinations of the five following Lagrangians:

$$
\begin{aligned}
L_2^{\rm gal,1} &= X \,, && (5)\\
L_3^{\rm gal,1} &= X\Box\phi - \phi_\mu \phi^{\mu\nu}\phi_\nu \,, && (6)\\
L_4^{\rm gal,1} &= X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] - 2(\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi - \phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}) \,, && (7)\\
L_5^{\rm gal,1} &= X\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu{}_\rho\right] && (8)\\
&\quad - 3\left[(\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu - 2\Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\lambda}\phi_\lambda + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\lambda}\phi_\lambda\right] .
\end{aligned}
$$

In flat space there exist alternative (in fact, infinite) versions of galileon Lagrangians, equivalent up to total derivatives. A particularly compact and popular choice (called "form 3" in [25]) is

$$
\begin{aligned}
L_2^{\rm gal,3} &= X \,, && (9)\\
L_3^{\rm gal,3} &= \frac{3}{2}X\Box\phi \,, && (10)\\
L_4^{\rm gal,3} &= 2X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] , && (11)\\
L_5^{\rm gal,3} &= \frac{5}{2}X\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu{}_\rho\right] , && (12)
\end{aligned}
$$

where we have chosen the normalization factors in order to be consistent with the original expressions (5)-(8).

2.2 Coupling to gravity and Horndeski theories

By going from (5)-(8) to (9)-(12) we have exchanged the order of partial derivatives, which can be consistently done in flat space. But in general curved spaces, while doing so for L_4 and L_5 we have to pay a commutator proportional to the curvature. Indeed, by taking f as a general function of X, we find that the two main blocks of terms appearing in $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$ are related by, respectively,

$$
f\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] = -2f_X(\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi - \phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}) + f\ {}^{(4)}\!R^{\mu\nu}\phi_\mu\phi_\nu + \text{boundary terms}\,, \quad (13)
$$

and

$$
\begin{aligned}
& f\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu{}_\rho\right] = \\
&\qquad - 2f_X\left[(\Box\phi)^2\phi_\mu\phi^{\mu\nu}\phi_\nu - 2\Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\lambda}\phi_\lambda + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\lambda}\phi_\lambda\right] \qquad (14)\\
&\qquad - 2Xf\ \left({}^{(4)}\!R_{\mu\sigma\rho\nu}\phi^\mu\phi^{\rho\sigma}\phi^\nu + {}^{(4)}\!R_{\mu\nu}\phi_\sigma\phi^{\mu\sigma}\phi^\nu - {}^{(4)}\!R_{\mu\nu}\phi^\mu\phi^\nu\Box\phi\right) + \text{boundary terms}\,.
\end{aligned}
$$

This also means that the different versions of the galileon Lagrangians, which are all equivalent in flat space, correspond to genuinely different theories once minimally coupled to gravity by trading ordinary derivatives for covariant derivatives. Of course, as realized in [7], the minimally coupled versions of galileons L_4 and L_5 bring higher (third order) derivatives into the equations of motion. For example, by varying $X(\Box\phi)^2$ with respect to ϕ, one ends up with terms containing two derivatives hitting on a Christoffel symbol, i.e., three derivatives of the metric. In order to get rid of such higher derivatives, the authors of [7] added to $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$ suitable gravitational "counterterms" and thus "re-discovered" Horndeski theories [10], which can be described by an arbitrary linear combination of the Lagrangians

$$L_2^H[G_2] \equiv G_2(\phi, X) \ , \tag{15}$$

$$L_3^H[G_3] \equiv G_3(\phi, X) \, \Box\phi \ , \tag{16}$$

$$L_4^H[G_4] \equiv G_4(\phi, X) \, {}^{(4)}\!R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}) \ , \tag{17}$$

$$L_5^H[G_5] \equiv G_5(\phi, X) \, {}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma}) \ , \tag{18}$$

following the presentation given in Ref. [9].

3 Beyond Horndeski: G^3

As we have recently shown in [1], it turns out that it is possible to extend the Horndeski Lagrangians presented above without encountering ghost-like Ostrogradski instabilities. In order to introduce these theories, it is much simpler to use the so-called unitary gauge, where the uniform scalar field (ϕ = const) hypersurfaces coincide with constant-time hypersurfaces. To do so, we assume that the gradient of the scalar field, $\partial_\mu\phi$, is time-like. Using an ADM decomposition of the metric,

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ , \tag{19}$$

we write the Lagrangian density in terms of the intrinsic and extrinsic 3-d curvature tensors of the spatial slices, respectively denoted R_{ij} and K_{ij}, their traces, $R \equiv h^{ij}R_{ij}$, $K \equiv h^{ij}K_{ij}$, as well as the lapse function N. The theories presented in [1] are then given by the action

$$S = \int d^4x\sqrt{-g}(L_2 + L_3 + L_4 + L_5) \ , \tag{20}$$

with

$$\begin{aligned}
&L_2 \equiv A_2(t, N) \ , \\
&L_3 \equiv A_3(t, N)K \ , \\
&L_4 \equiv A_4(t, N)\big(K^2 - K_{ij}K^{ij}\big) + B_4(t, N)R \ , \\
&L_5 \equiv A_5(t, N)\big(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j{}_k\big) + B_5(t, N)K^{ij}\left(R_{ij} - \frac{1}{2}h_{ij}R\right) \ ,
\end{aligned} \tag{21}$$

where A_a and B_a ($a = 2, 3, 4, 5$) are generic functions of t and N. Let us remind that, in terms of ADM variables, the extrinsic curvature reads

$$K_{ij} = \frac{1}{2N}\big(\dot{h}_{ij} - D_iN_j - D_jN_i\big) \ , \tag{22}$$

where D_i is the spatial covariant derivative. The combination $K^2 - K_{ij}K^{ij}$ in the third line is the usual GR kinetic term. Indeed, when $B_4 = -A_4 = 1/(16\pi G)$, while the other coefficients vanish, the above action corresponds to the Einstein-Hilbert action up to boundary terms, as can be easily seen upon using the Gauss-Codazzi relation (see eq. (132) in App. A). In this case the action becomes fully 4-d diff invariant and there are no propagating scalar degrees of freedom.

We now rewrite the above Lagrangians in a manifestly covariant form, *i.e.* in terms of ϕ and its spacetime derivatives. The dependence on t and N of the functions A_a and B_a will turn into a dependence on ϕ and $X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, since $\phi = \phi_0(t)$ and $X = -\dot{\phi}_0^2(t)/N^2$ in our ADM formulation. We can then introduce the unit vector normal to the uniform ϕ hypersurfaces,

$$n_\mu \equiv -\frac{\partial_\mu\phi}{\sqrt{-X}} \ , \tag{23}$$

and define the extrinsic curvature as

$$K_{\mu\nu} \equiv (g^{\sigma}{}_{\mu} + n^{\sigma} n_{\mu}) \nabla_{\sigma} n_{\nu} \,. \tag{24}$$

Using this expression and $K \equiv \nabla_\mu n^\mu$, and denoting the derivation by a lower index, *e.g.* $A_{2X} \equiv \partial A_2/\partial X$, the above Lagrangians can be rewritten, after lengthy but straightforward manipulations explicitly given in App. A, as [1]

$$L_2 = L_2^H[A_2] \,, \tag{25}$$

$$L_3 = L_3^H[C_3 + 2XC_{3X}] + L_2^H[XC_{3\phi}] \,, \tag{26}$$

$$L_4 = L_4^H[B_4] + L_3^H[C_4 + 2XC_{4X}] + L_2^H[XC_{4\phi}] - \frac{B_4 + A_4 - 2XB_{4X}}{X^2} L_4^{\rm gal,1} \,, \tag{27}$$

$$L_5 = L_5^H[G_5] + L_4^H[C_5] + L_3^H[D_5 + 2XD_{5X}] + L_2^H[XD_{5\phi}] + \frac{XB_{5X} + 3A_5}{3(-X)^{5/2}} L_5^{\rm gal,1} \,, \tag{28}$$

where A_a and B_a are now functions of ϕ and X, $A_a = A_a(\phi, X)$, $B_a = B_a(\phi, X)$, and C_3, C_4, C_5, D_5 and G_5 are defined as

$$\begin{aligned} C_3 &\equiv \frac{1}{2} \int A_3 (-X)^{-3/2}\, dX \,, \\ C_4 &\equiv - \int B_{4\phi} (-X)^{-1/2}\, dX \,, \\ C_5 &\equiv -\frac{1}{4} X \int B_{5\phi} (-X)^{-3/2} dX \,, \\ D_5 &\equiv - \int C_{5\phi} (-X)^{-1/2}\, dX \,, \\ G_5 &\equiv - \int B_{5X} (-X)^{-1/2}\, dX \,. \end{aligned} \tag{29}$$

If A_4 and A_5 are related to B_4 and B_5 by

$$A_4 = -B_4 + 2XB_{4X} \,, \qquad A_5 = -XB_{5X}/3 \,, \tag{30}$$

the last terms of both eqs. (27) and (28) vanish. In this case one is left only with the Horndeski Lagrangians, which manifestly shows that eqs. (25)–(28) (and thus action (20)) contain Horndeski theories. In general, the functions A_4 and A_5 are completely free, which means that our theories contain two additional free functions with respect to the Horndeski ones.

It is straighforward to see that the minimally coupled versions of the original galileons proposed in [6], (5)–(8), are contained in eqs. (25)–(28) by the choice of functions $B_4 = 0$, $B_5 = 0$, $A_2 = X$, $A_3 = 3X/2$, $A_4 = -X^2$ and $A_5 = (-X)^{5/2}$. As a corollary, $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$ are already healthy without the need of additional gravitational counterterms. In other words, the straightforward covariantization of galileons, i.e. substituting ordinary derivatives with covariant derivatives, is a viable covariantization. It should be noted, however, that galileon symmetry remains broken by terms proportional to the curvature, regardless of the chosen covariantization procedure.

Finally, before concluding this section, let us briefly comment on the decoupling limit of eqs. (25)–(28). In Ref. [26], the decoupling limit of Horndeski theories has been studied by expanding the metric $g_{\mu\nu}$ around Minkowski and the scalar field ϕ around a constant background value. In doing so, the following scaling of the functions $G_a(\phi, X)$ introduced in eqs. (15)–(18) was assumed [27],

$$G_2 \sim \Lambda_3^3 M_{\rm Pl} \,, \quad G_3 \sim M_{\rm Pl} \,, \quad G_4 \sim M_{\rm Pl}^2 \,, \quad G_5 \sim \Lambda_3^{-3} M_{\rm Pl}^2 \,, \tag{31}$$

where Λ_3 is a mass scale which may be associated to the current accelerated expansion of the universe (in which case $\Lambda_3^3 \sim M_{\rm Pl} H_0^2$) and $M_{\rm Pl}$ is the Planck mass. The decoupling limit is defined as $M_{\rm Pl} \to \infty$

while Λ_3 remains constant. It is easy to see that taking this limit in eqs. (25)–(28) leads to the same decoupling limit found in [26] for Horndeski, but with different dimensionless parameters. This is clearly the case for eqs. (25) and (26), because they are equivalent to the Horndeski Lagrangians L_2^H and L_3^H. Equations (27) and (28) contain non-Horndeski pieces, respectively $L_4^{\rm gal,1}$ and $L_5^{\rm gal,1}$. By expanding these terms in scalar field and metric perturbations, the only contributions that do not vanish in the decoupling limit are galileons, i.e.,

$$-\frac{B_4 + A_4 - 2XB_{4X}}{X^2} L_4^{\rm gal,1} \sim \Lambda_3^{-6} L_4^{\rm gal,1}\,, \qquad \frac{XB_{5X} + 3A_5}{3(-X)^{5/2}} L_5^{\rm gal,1} \sim \Lambda_3^{-9} L_5^{\rm gal,1}\,, \tag{32}$$

where the functions $(B_4 + A_4 - 2XB_{4X})/X^2$ and $(XB_{5X}/3 + A_5)/(-X)^{5/2}$ are evaluated on the background. In conclusion, operators leading to higher-derivative equations of motion in eqs. (27) and (28) are also higher order in the decoupling limit.

4 Hamiltonian analysis

As discussed in the introduction, theories that contain higher-order time derivatives often lead to lethal Ostrogradski instabilities. The presence of higher derivatives manifests itself in the form of extra degrees of freedom that behave like ghosts (i.e. negative energy states). For instance, the dynamics of a system with a nondegenerate Lagrangian of the form $L(q, \dot{q}, \ddot{q})$ is described by a 4-dimensional phase space, corresponding to two degrees of freedom, one of which behaves like a ghost (see *e.g.* [5]).

In the ADM formulation, our Lagrangian (21) depends on the dynamical quantities h_{ij} and their "velocities" K_{ij}: in this sense, it is already evident that the Lagrangian does not contain higher-order time derivatives and that Ostrogradski instabilities should not be there. In order to confirm this intuition, we now perform the Hamiltonian analysis for the Lagrangian (21) and show that the number of degrees of freedom remains three—i.e. two tensor modes and one scalar mode, thus excluding the appearance of dangerous extra degrees of freedom. The present analysis details that of [1] and confirms its conclusions.

The phase space of our theory is described by the variables h_{ij}, N, N^i and their conjugate momenta, given respectively by

$$\begin{aligned}\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{\sqrt{h}}{2}\Big[&\Big(A_3 + 2A_4 K + 3A_5(K^2 - K_{lm}K^{lm})\Big) h^{ij} \\ &-2(A_4 + 3A_5 K)K^{ij} + 6A_5 K^i_l K^{lj} + B_5\left(R^{ij} - \frac{1}{2}Rh^{ij}\right)\Big]\,,\end{aligned} \tag{33}$$

and

$$\pi_N \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \qquad \pi_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0\,. \tag{34}$$

The absence of time derivatives of the lapse N and the shift N^i in the action implies that their conjugate momenta automatically vanish. The relations $\pi_N = 0$ and $\pi_i = 0$ can thus be seen as restrictions of the initial 20-dimensional phase space, corresponding to so-called *primary constraints*. So far, the situation is quite similar to that of pure general relativity.

The canonical Hamiltonian is then obtained via the Legendre transform of the Lagrangian,

$$H \equiv \int d^3\vec{x}\left[\pi^{ij}\dot{h}_{ij} - \mathcal{L}\right]\,. \tag{35}$$

The Hamiltonian is expressed in terms of the canonical variables, which means that, in principle, one must invert the relation in (33) to obtain $\dot{h}_{ij}$ as a function of π^{ij}. Because of the presence of primary constraints, the time evolution is governed by the extended Hamiltonian,

$$\tilde{H} = H + \int d^3\vec{x}\, \left[\lambda_N\, \pi_N + \lambda^i\, \pi_i\right] \,, \tag{36}$$

where λ_N and λ_i play the role of Lagrange multipliers. For any function F defined on the phase space, its time evolution is given by

$$\frac{d}{dt}F = \frac{\partial F}{\partial t} + \{F, \tilde{H}\}\,. \tag{37}$$

The Poisson bracket in the above formula is defined, as usual, by the expression

$$\{F, G\} \equiv \sum_A \int d^3\vec{x} \left(\frac{\delta F}{\delta \phi^A(\vec{x})}\frac{\delta G}{\delta \pi_A(\vec{x})} - \frac{\delta F}{\delta \pi_A(\vec{x})}\frac{\delta G}{\delta \phi^A(\vec{x})}\right) \,, \tag{38}$$

where we use the collective notation $\phi^A = (h_{ij}, N, N^i)$ and $\pi_A = (\pi^{ij}, \pi_N, \pi_i)$.

4.1 Lagrangians up to L_4

It is straightforward to apply the procedure outlined above to our Lagrangians up to L_4, because the expression (33) for π^{ij} is linear in K_{ij} and can be easily inverted. Including L_5 is more involved, as (33) is quadratic in K_{ij} and we briefly discuss the procedure in the next subsection.

Therefore, assuming that L_5 is absent, i.e. $A_5 = B_5 = 0$, one can immediately invert (33) to find

$$K_{ij} = -\frac{1}{A_4\sqrt{h}}\left(\pi_{ij} - \frac{1}{2}\pi h_{ij}\right) - \frac{A_3}{4A_4}h_{ij}\,. \tag{39}$$

Using (22), it is then straightforward to express $\dot{h}_{ij}$ as a function of π_{ij} and to substitute the result in (35). Using integrations by parts to get rid of the derivatives of the shift, one finds that the Hamiltonian can be written in the form

$$H = \int d^3\vec{x}\, \left[N\mathcal{H}_0(N) + N^i\mathcal{H}_i\right] \,, \tag{40}$$

with

$$\mathcal{H}_0 \equiv -\frac{1}{\sqrt{h}A_4}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{A_3}{2A_4}\pi + \sqrt{h}\left(\frac{3{A_3}^2}{8A_4} - A_2\right) - \sqrt{h}\, B_4 R \,, \tag{41}$$

$$\mathcal{H}_i \equiv -2D_j\pi^j{}_i\,. \tag{42}$$

As mentioned in the previous section, by specializing the above expressions to the case $B_4 = -A_4 = 1/(16\pi G)$ and $A_2 = A_3 = 0$ one recovers the usual general relativity Hamiltonian. In the general case, however, the A_a and B_a are functions of N, so that $\mathcal{H}_0$ now depends on N, in contrast with general relativity. This difference plays a crucial role, as we will see below.

Let us now consider the time evolution of the primary constraints. Imposing that they are conserved in time leads to the so-called *secondary constraints*. For the first constraint, $\pi_N \approx 0$, one finds

$$\dot{\pi}_N = \{\pi_N, \tilde{H}\} \approx \{\pi_N, H\} = -\frac{\partial}{\partial N}\left(N\mathcal{H}_0\right) \,, \tag{43}$$

where the symbol $\approx$ denotes equality in a "weak" sense, i.e. restricted to the constrained phase space. Thus, the above equation yields the secondary constraint,

$$\tilde{\mathcal{H}}_0 \equiv \mathcal{H}_0 + N\frac{\partial\mathcal{H}_0}{\partial N} \approx 0\,. \tag{44}$$

Note that, in general relativity, $\mathcal{H}_0$ is independent of N, thus leading to the familiar Hamiltonian constraint $\tilde{\mathcal{H}}_0 = \mathcal{H}_0 \approx 0$. Similarly, using

$$\dot{\pi}_i = \{\pi_i, \tilde{H}\} \approx \{\pi_i, H\} = -\mathcal{H}_i \,, \tag{45}$$

the conservation in time of the three primary constraints $\pi_i \approx 0$ gives the secondary constraints

$$\mathcal{H}_i \approx 0 \,. \tag{46}$$

These constraints are exactly the same as in pure general relativity, where they are associated with the invariance under spatial diffeomorphims.

Let us now compute the Poisson brackets of the constraints. We start with the constraints $\mathcal{H}_i$, for which the treatment is very similar to general relativity. It is convenient to introduce the "momentum" function

$$\mathcal{M}_f \equiv \int d^3\vec{x}\, f^i(\vec{x})\, \mathcal{H}_i(\vec{x}) \,, \tag{47}$$

where the f^i are three arbitrary functions of space. By reproducing the general relativity calculations (see *e.g.* the appendix of [28]), one finds

$$\{\mathcal{M}_f, \mathcal{M}_g\} = \mathcal{M}_h, \qquad h^i \equiv f^k D_k g^i - g^k D_k f^i \,. \tag{48}$$

It is also straightforward to check that

$$\{\mathcal{M}_f, \mathcal{T}_g\} = -\int d^3\vec{x}\, g\, D_i(\mathcal{T} f^i) = \int d^3\vec{x}\, \mathcal{T} f^i D_i g \,, \tag{49}$$

with

$$\mathcal{T}_g \equiv \int d^3\vec{x}\, g(\vec{x})\, \mathcal{T}(\vec{x}) \,, \tag{50}$$

where g is an arbitrary function of space and $\mathcal{T}$ is any combination of the Hamiltonian that depends on π^{ij} and h_{ij}, but *not* on N. So $\mathcal{T}$ can be any of the following expressions,

$$\mathcal{T}_1 = \frac{1}{\sqrt{h}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right), \qquad \mathcal{T}_2 = \pi \,, \qquad \mathcal{T}_3 = \sqrt{h} \,, \qquad \mathcal{T}_4 = \sqrt{h} R \,, \tag{51}$$

or any linear combination of these with coefficients *independent* of N. In particular, (49) implies that in general relativity, where the constraint $\mathcal{H}_0$ does not depend on N, the Poisson bracket of $\mathcal{M}_f$ with $\mathcal{H}_0$ weakly vanishes.

If the combination $\mathcal{T}$ is now multiplied by a function of N,

$$\tilde{\mathcal{T}} = \mathcal{F}(N)\, \mathcal{T} \,, \tag{52}$$

one immediately deduces from (49) that

$$\{\mathcal{M}_f, \tilde{\mathcal{T}}_g\} = -\int d^3\vec{x}\, g\, \mathcal{F}\, D_i(\mathcal{T} f^i) \,, \tag{53}$$

and $\tilde{\mathcal{T}}$ cannot appear after integration by parts. However, by introducing the slightly modified constraints[2]

$$\tilde{\mathcal{H}}_i \equiv \mathcal{H}_i + \pi_N \partial_i N, \tag{54}$$

[2]Note that its form is similar to the total momentum constraint that would arise in general relativity with a scalar field.

one obtains

$$\{\tilde{\mathcal{M}}_f, \tilde{\mathcal{T}}_g\} = \{\mathcal{M}_f, \tilde{\mathcal{T}}_g\} - \int d^3\vec{x}\, g\, \frac{\partial \mathcal{F}}{\partial N} \mathcal{T} f^i D_i N = - \int d^3\vec{x}\, g\, D_i(\tilde{\mathcal{T}} f^i) = \int d^3\vec{x}\, \tilde{\mathcal{T}} f^i\, D_i g\,, \tag{55}$$

where now $\tilde{\mathcal{T}}$ appears explicitly.

This treatment also applies to any linear combination of $\tilde{\mathcal{T}}$ terms. In particular, it applies to $\mathcal{H}_0$, since this is given by a linear combination of $\mathcal{T}_a$ with coefficients that depend on time and N, and as a consequence it applies to $\tilde{\mathcal{H}}_0$ defined in eq. (44). Thus, from the above analysis one concludes that the Poisson brackets of the constraints $\tilde{\mathcal{H}}_i$ with $\tilde{\mathcal{H}}_0$ vanish weakly, i.e.

$$\{\tilde{\mathcal{H}}_i, \tilde{\mathcal{H}}_0\} \approx 0\,. \tag{56}$$

Using eq. (48) and the fact that $\mathcal{H}_i$ does not depend on N, π_N, N^i or π_i, it is also immediate to verify that

$$\{\tilde{\mathcal{H}}_i, \tilde{\mathcal{H}}_j\} \approx 0\,, \qquad \{\tilde{\mathcal{H}}_i, \pi_N\} \approx 0, \qquad \{\tilde{\mathcal{H}}_i, \pi_j\} \approx 0\,. \tag{57}$$

Therefore, the Poisson brackets of the three constraints $\tilde{\mathcal{H}}_i$ with all the other constraints vanish weakly. The same is true for the three primary constraints $\pi_i \approx 0$. Consequently, these six constraints, associated with the 3-dimensional diffeomorphism invariance, are *first-class* constraints.

The remaining constraints, $\tilde{\mathcal{H}}_0$ and $\pi_N \approx 0$, satisfy the relations

$$\{\pi_N(x), \pi_N(y)\} = 0, \qquad \{\tilde{\mathcal{H}}_0, \pi_N\} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial N} = 2\frac{\partial \mathcal{H}_0}{\partial N} + \frac{\partial^2 \mathcal{H}_0}{\partial N^2}\,. \tag{58}$$

Provided that the derivative of $\tilde{\mathcal{H}}_0$ with respect to N does not vanish, this shows that these two constraints are of the *second-class* type, in contrast with general relativity.

It is also useful to check that no additional constraint arises from the time evolution of the secondary constraints. Indeed, since

$$\frac{d}{dt}\tilde{\mathcal{H}}_0 = \frac{\partial \tilde{\mathcal{H}}_0}{\partial t} + \{\tilde{\mathcal{H}}_0, H\} + \lambda_N \frac{\partial \tilde{\mathcal{H}}_0}{\partial N}\,, \tag{59}$$

imposing the conservation of $\tilde{\mathcal{H}}_0$ simply fixes the Lagrange multiplier λ_N without generating any new constraint, provided $\partial \mathcal{H}_0/\partial N$ does not vanish, which is assumed here. As for the momentum constraints, we simply have

$$\frac{d}{dt}\tilde{\mathcal{H}}_i = \{\tilde{\mathcal{H}}_i, H\} \approx 0\,, \tag{60}$$

because the brackets of $\tilde{\mathcal{H}}_i$ with all the elements in H vanish weakly, according to (55) and the first relation in (57).

In conclusion, we find that the dynamical system is, in general, characterized by a 20-dimensional phase space with six first-class constraints and two second-class constraints. Each first-class constraint eliminates two canonical variables and each second-class constraint eliminates one canonical variable. In total, 14 canonical variables can be eliminated, which corresponds to a 6-dimensional physical phase space, i.e. three degrees of freedom. The difference with general relativity, where all eight constraints are first-class thus leaving only two physical degrees of freedom, is due to the presence of a preferred slicing defined by the scalar field, which breaks the full spacetime diffeomorphism invariance.

Let us briefly discuss a special case where the second Poisson bracket in (58) vanishes weakly, which happens when the whole N dependence factorizes in $\mathcal{H}_0$. Let us illustrate this case by considering the Lagrangian L_4 with

$$B_4 = -\frac{1}{A_4}\,. \tag{61}$$

In this case

$$\mathcal{H}_0 = B_4 \left[\frac{1}{2\sqrt{h}}(2\pi_{ij}\pi^{ij} - \pi^2) - \sqrt{h}R\right] \tag{62}$$

and

$$\tilde{\mathcal{H}}_0 = \left(B_4 + \frac{\partial B_4}{\partial N}\right) \left[\frac{1}{2\sqrt{h}}(2\pi_{ij}\pi^{ij} - \pi^2) - \sqrt{h}R\right]. \tag{63}$$

One then notices that the system is equivalent to general relativity, up to the redefinition of a new lapse function $\tilde{N} \equiv NB_4$.

Finally, let us make a few considerations on the restriction to the unitary gauge which is at the basis of the Hamiltonian analysis of this section. An explicit Hamiltonian analysis without fixing unitary gauge seems to be a very tedious task in view of the complicated expressions of our theories in the covariant form, eqs. (25)-(28). Indeed, resorting to the unitary gauge has the huge advantage to hide the scalar degree of freedom in the metric and to enormously simplify the analysis. Thus, the full Hamiltonian treatment in an arbitrary gauge is beyond the scope of the present work. Fortunately, in Sec. 6 we present a completely different approach, which shows that the higher-order time derivatives in the equations of motion can be eliminated by using constraints that follow from these equations. This other approach is valid in any gauge and it confirms that no additional degree of freedom is necessary to describe higher-order time derivatives.

4.2 Including the Lagrangian L_5

The inclusion of L_5 makes the Hamiltonian analysis more involved, the main subtlety in this case being inverting eq. (33) in order to obtain K_{ij} as a function of π^{ij}. However, this technical difficulty does not impair the basic counting of degrees of freedom, which is the main target of our Hamiltonian analysis.

In the case when only A_5 is considered, from the last line of (21) we obtain

$$\pi^{ij} \;=\; \frac{3\sqrt{h}A_5}{2}\left[(K^2 - K_{mn}K^{mn})h^{ij} + 2(K^i{}_l K^{lj} - KK^{ij})\right]. \tag{64}$$

Inverting the above equation is technically more involved and because K_{ij} is essentially a "square root" of π_{ij} there is generally more than one branches of solutions. However, the inversion problem is well-defined locally around some non-singular chosen value of K_{ij}. It is worth mentioning how the problem can be tackled in practice with a systematic series expansion around, for instance, a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) configuration,

$$(K_0)_i{}^j = H\delta_i^j, \qquad (\pi_0)_i{}^j = 3\sqrt{h}A_5 H^2 \delta_i^j\,. \tag{65}$$

We can then fix whatever value of the conjugate momentum through the new "shifted" variable $\hat{\pi}_i{}^j$,

$$\pi_i{}^j \equiv (\pi_0)_i{}^j + \frac{3}{2}\sqrt{h}A_5\,\hat{\pi}_i{}^j, \tag{66}$$

write a formal power expansion for $K_i{}^j$,

$$K_i{}^j \;= (K_0)_i{}^j + (K_1)_i{}^j + (K_2)_i{}^j + \dots \tag{67}$$

and solve (64) order by order. By doing this, we obtain the recursive relations

$$(K_1)_i{}^j \;=\; -\frac{1}{2H}\left(\hat{\pi}_i{}^j - \frac{\hat{\pi}}{2}\delta_i^j\right), \tag{68}$$

$$(K_2)_i{}^j \;=\; \frac{1}{4H}\left[\left((K_1)^2 - (K_1)_m^n (K_1)_n{}^m\right)\delta_i^j + 4\left((K_1)_i{}^l (K_1)_l{}^j - (K_1)(K_1)_i{}^j\right)\right], \dots\,, \tag{69}$$

where $(K_a) \equiv (K_a)_i{}^i$.

A completely analogous procedure applies to other cases, such as when the full battery of terms is present, as in eq. (33). In this case, the easily invertible part (L_2-L_4) can be used as the zeroth order piece and one can make a formal Taylor expansion in A_5.

4.3 Generalizations

Although we have focused our discussion on a specific class of theories, which represent a natural extension of Horndeski theories from the ADM point of view, similar conclusions can be drawn for a much wider class of models. Essentially, the basic ingredients that lead us to exclude the presence of unwanted additional degrees of freedom can be formulated in unitary gauge as

1. unbroken spatial diffeomorphism (producing three first-class momentum constraints as in general relativity);
2. absence of time derivatives of the lapse function N (which makes the Hamiltonian constraint an *algebraic* equation for N);
3. absence of time derivatives of the extrinsic curvature K_{ij} (which prevents that the Lagrangian depends on the "accelerations", *i.e.* the second time derivatives of h_{ij}).

Such an approach has already been used in the past to study, for instance, the behavior of specific models of Horava's gravity [29]. In analogy with Horava's gravity, one could consider various combinations of the intrinsic curvature tensor and its spatial derivatives, as well as various combinations of the extrinsic curvature tensor, as recently discussed in [30]. Note, however, that these theories do not generically have the same decoupling limit as Horndeski, as it is the case for G^3 theories (see discussion at the end of Sec. 3).

5 Linear theory and coupling with matter

The Hamiltonian analysis excludes the presence of extra degrees of freedom. However, one still needs to check that the remaining scalar and tensor degrees of freedom are not themselves ghosts. In this section we compute the quadratic action for the perturbations of the propagating degrees of freedom and derive the conditions for which the kinetic terms have the right signs. We then add matter fields minimally coupled to gravity and study the phenomenology on small scales. We first perform this analysis in unitary gauge and then in Newtonian gauge.

5.1 Unitary gauge

Let us expand action (20) around a spatially flat FLRW metric following the general procedure developed in [11, 31] (see also [32]). We use the ζ-gauge and write the spatial metric as

$$h_{ij} = a^2(t) e^{2\zeta} (\delta_{ij} + \gamma_{ij}) \, , \qquad \gamma_{ii} = 0 = \partial_i \gamma_{ij} \, , \tag{70}$$

and we split the shift as

$$N^i = \partial_i \psi + N_V^i \, , \qquad \partial_i N_V^i = 0 \, . \tag{71}$$

Moreover, it is convenient to express the dependence of the second-order action on the function A_a and B_a introduced in the Lagrangians (21) in terms of the following functions evaluated on the background,[3]

$$
\begin{aligned}
M^2 &\equiv -2(A_4+3HA_5)\,,\\
\alpha_K &\equiv -\frac{2A_2'+A_2''+3H(2A_3'+A_3'')+6H^2(2A_4'+A_4'')+6H^3(2A_5'+A_5'')}{2H^2(A_4+3HA_5)}\,,\\
\alpha_B &\equiv -\frac{A_3'+4HA_4'+6H^2A_5'}{4H(A_4+3HA_5)}\,,\\
\alpha_T &\equiv -\frac{B_4+\dot B_5/2}{A_4+3HA_5}-1\,,\\
\alpha_H &\equiv -\frac{B_4+B_4'-HB_5'/2}{A_4+3HA_5}-1\,,
\end{aligned}
\tag{72}
$$

where a prime denotes a derivative with respect to N and a dot a derivative with respect to t. We discuss in Appendix B how these functions are related to the general formalism of Ref. [11].

Higher (spatial) derivative terms proportional to $(\partial^2\psi)^2$, which are contained in quadratic products of the extrinsic curvature, cancel from the action up to a total derivative because of the particular combinations in which these products appear in eq. (21). By varying the quadratic action with respect to N^i, one obtains the momentum constraints, whose solution is $N^i_V=0$ and

$$
N = 1+\frac{1}{1+\alpha_B}\frac{\dot\zeta}{H}\,. \tag{73}
$$

After substitution of this equation into the quadratic action, all the terms containing ψ drop out, up to total derivatives [31]. For this reason, we do not need the Hamiltonian constraint, obtained by varying the action with respect to N, to solve for ψ. After some manipulations the quadratic action becomes [11, 31, 14]

$$
S^{(2)} = \frac12\int d^4x\, a^3\left[\mathcal{L}_{\dot\zeta\dot\zeta}\dot\zeta^2+\mathcal{L}_{\partial\zeta\partial\zeta}\frac{(\partial_i\zeta)^2}{a^2}+\frac{M^2}{4}\dot\gamma_{ij}^2-\frac{M^2}{4}(1+\alpha_T)\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]\,, \tag{74}
$$

where

$$
\mathcal{L}_{\dot\zeta\dot\zeta} \equiv M^2\frac{\alpha_K+6\alpha_B^2}{(1+\alpha_B)^2}\,, \tag{75}
$$

$$
\mathcal{L}_{\partial\zeta\partial\zeta} \equiv 2M^2(1+\alpha_T)-\frac{2}{a}\frac{d}{dt}\left[\frac{aM^2(1+\alpha_H)}{H(1+\alpha_B)}\right]\,. \tag{76}
$$

As expected from the previous Hamiltonian analysis, the quadratic Lagrangian (74) does not contain higher-order time derivatives. As a consequence of the particular combination of extrinsic curvature in eq. (21), neither does it contain higher space derivatives.

The condition required to ensure that the propagating degrees of freedom are not ghost-like is that their time kinetic terms are positive, $\mathcal{L}_{\dot\zeta\dot\zeta}>0$ and $M^2>0$. Moreover, gradient instabilities are avoided when the speed of sound of the scalar and tensor propagating degrees of freedom,

$$
c_s^2 \equiv -\frac{\mathcal{L}_{\partial\zeta\partial\zeta}}{\mathcal{L}_{\dot\zeta\dot\zeta}}\,, \qquad c_T^2\equiv 1+\alpha_T\,, \tag{77}
$$

are also positive, $c_s^2>0$ and $c_T^2>0$.

[3] The first four functions in eq. (72) have been introduced by Bellini and Sawicki in Ref. [14], where they consider linear perturbations in Horndeski theories, with the difference $\alpha_B^{\rm here}=-\alpha_B^{\rm there}/2$, which simplifies further the equations. In particular, M^2, α_K, α_B and α_T respectively parameterize the effective Planck mass, a modification of the scalar kinetic term [34, 35], a kinetic mixing between the scalar and the metric (the so-called braiding) [36, 37, 38, 39] and a tensor speed excess. As stressed in such a reference and also shown in Appendix B, these functions are just a convenient basis of the parameters previously introduced in the context of the so-called Effective Field Theory of Dark Energy in Refs. [15, 16, 11, 17] (see [31, 33] for reviews). Here we adopt this parameterization because it simplifies the notation. We also introduce a new function, α_H, which parametrizes the deviation from Horndeski theories [11, 1].

5.2 Adding matter: $P(\sigma, Y)$

To study our theories in the presence of matter fields minimally coupled to gravity, we add to action (20) a k-essence type action describing a matter scalar field σ (not to be confused with the dark energy field ϕ),

$$S_m = \int d^4x\sqrt{-g}\, P(Y,\sigma)\,, \qquad Y \equiv g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\,, \tag{78}$$

with sound speed $c_m^2 \equiv P_Y/(P_Y - 2\dot{\sigma}_0^2 P_{YY})$.

We can then expand at second order these actions and repeat the procedure discussed earlier. To describe matter fluctuations it is convenient to use the gauge-invariant variable $Q_\sigma \equiv \delta\sigma - (\dot{\sigma}_0/H)\zeta$. After substitution of the momentum constraints, the final action expressed in terms of ζ and Q_σ reads

$$\begin{aligned} S^{(2)} = \int d^4x a^3 \Bigg[& \frac{1}{2}\left(\tilde{\mathcal{L}}_{\dot\zeta\dot\zeta}\dot\zeta^2 + \tilde{\mathcal{L}}_{\partial\zeta\partial\zeta}\frac{(\partial_i\zeta)^2}{a^2}\right) - \frac{P_Y}{c_m^2}\left(\dot{Q}_\sigma^2 - c_m^2\frac{(\partial_i Q_\sigma)^2}{a^2}\right) \\ & - \frac{2\dot\sigma_0 P_Y}{Hc_m^2(1+\alpha_B)}\left(\alpha_B\dot\zeta\dot{Q}_\sigma - c_m^2(\alpha_B - \alpha_H)\frac{\partial_i\zeta\partial_i Q_\sigma}{a^2}\right) \\ & + m_\zeta^2\zeta^2 + m_\sigma^2 Q_\sigma^2 + m_c^2\zeta Q_\sigma + \lambda\dot\zeta Q_\sigma \Bigg]\,, \end{aligned} \tag{79}$$

with the new coefficients for the kinetic and gradient terms of ζ

$$\tilde{\mathcal{L}}_{\dot\zeta\dot\zeta} = \mathcal{L}_{\dot\zeta\dot\zeta} + \frac{\rho_m + p_m}{H^2 c_m^2}\left(\frac{\alpha_B}{1+\alpha_B}\right)^2\,, \tag{80}$$

$$\tilde{\mathcal{L}}_{\partial\zeta\partial\zeta} = \mathcal{L}_{\partial\zeta\partial\zeta} - \frac{\rho_m + p_m}{H^2}\left(1 - \frac{2(1+\alpha_H)}{1+\alpha_B}\right)\,, \tag{81}$$

where we have used $2\dot\sigma_0^2 P_Y = -(\rho_m + p_m)$. The second line contains two derivative couplings between ζ and Q_σ while the third line contains non-derivative terms, which are irrelevant for the present discussion.

The kinetic matrix for (ζ, Q_σ) reads

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} \tilde{\mathcal{L}}_{\dot\zeta\dot\zeta}\omega^2 + \tilde{\mathcal{L}}_{\partial\zeta\partial\zeta}k^2 & A\big[\alpha_B\omega^2 - c_m^2(\alpha_B - \alpha_H)k^2\big] \\ A\big[\alpha_B\omega^2 - c_m^2(\alpha_B - \alpha_H)k^2\big] & -2P_Y c_m^{-2}(\omega^2 - c_m^2 k^2) \end{pmatrix}\,, \qquad A = -\frac{2\dot\sigma_0 P_Y}{Hc_m^2(1+\alpha_B)}\,. \tag{82}$$

Requiring that its determinant vanishes yields the dispersion relation

$$(\omega^2 - c_m^2k^2)(\omega^2 - \tilde{c}_s^2k^2) = (c_s^2 - \tilde{c}_s^2)\left(\frac{\alpha_H}{1+\alpha_H}\right)^2\omega^2k^2\,, \tag{83}$$

with

$$\tilde{c}_s^2 \equiv c_s^2 - \frac{\rho_m + p_m}{H^2M^2}\frac{(1+\alpha_H)^2}{\alpha_K + 6\alpha_B^2}\,. \tag{84}$$

From this equation one derives the two dispersion relations $\omega^2 = c_\pm^2 k^2$. For Horndeski theories ($\alpha_H = 0$), the matter sound speed is unchanged, despite the presence of couplings in the action between the time and space derivative of ζ and Q_σ, i.e. the non- vanishing of the non-diagonal terms in the kinetic matrix. Indeed, these couplings are precisely proportional to $\omega^2 - c_m^2k^2$ and give the standard dispersion relation for matter. However, this is no longer true with our non-Horndeski extensions, where $\alpha_H \neq 0$.

5.3 Newtonian gauge

We now study linear perturbations for our theories in the presence of a more general type of matter by considering a gauge often employed in the study of cosmological perturbations: the Newtonian gauge, where the metric reads

$$ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)d\vec{x}^2 \;, \tag{85}$$

taking into account only scalar perturbations.

Let us directly expand the action for the sum of the Lagrangians (25)–(28) up to quadratic order around the background field solution $\phi_0(t) = t$, i.e.,

$$\phi = t + \pi(t, \vec{x}) \;, \tag{86}$$

where π describes the scalar field perturbation.[4] The quadratic action for linear perturbations is given by

$$\begin{aligned} S = \int d^4x a^3 M^2 \bigg\{ & \frac{1}{2} H^2 \alpha_K \dot{\pi}^2 + \bigg[\dot{H} + \frac{1}{2M^2} \big(\rho_m + p_m + 2(M^2 H \alpha_B)^{\cdot} - 2(H M^2 \alpha_H)^{\cdot} \big) + H^2(\alpha_B - \alpha_M) \\ & + H^2(\alpha_T - \alpha_H) \bigg] \frac{(\nabla\pi)^2}{a^2} - 3\dot{\Psi}^2 + (1+\alpha_T)\frac{(\nabla\Psi)^2}{a^2} + 2H(\alpha_B - \alpha_H)\nabla\Phi\nabla\pi \\ & - 2H(\alpha_M - \alpha_T)\frac{\nabla\Psi\nabla\pi}{a^2} + 6H\alpha_B \dot{\pi}\dot{\Psi} + H^2(6\alpha_B - \alpha_K)\Phi\dot{\pi} - 2(1+\alpha_H)\frac{\nabla\Phi\nabla\Psi}{a^2} \\ & - 6H(1+\alpha_B)\dot{\Psi}\Phi + H^2 \left(\frac{1}{2}\alpha_K - 3(1+2\alpha_B) \right) \Phi^2 + 2\alpha_H \frac{\nabla\dot{\pi}\nabla\Psi}{a^2} + \dots \bigg\} \;, \end{aligned} \tag{87}$$

where we have used the background equations to rewrite the coefficient of $(\nabla\pi)^2$. We have written explicitly all the terms that are quadratic in derivatives, as well as other terms involving Φ *without* derivatives because they also contribute to the kinetic limit as we will see below. The ellipses in the last line stand for all the other terms, irrelevant for the present discussion. As expected from the Lagrangians (27) and (28), the quadratic action in the Newtonian gauge contains a higher order derivative term, $\nabla\dot{\pi}\nabla\Psi$, which is proportional to the non-Horndeski coefficient α_H. This term generates higher order (one time- and two spatial-) derivative terms in the equations of motion, as discussed in detail in Ref. [11].

It is possible to find a redefinition of the metric perturbations that de-mixes the new metric variables from the scalar field π and removes the higher derivative term from the gravitational action. In Brans-Dicke theories such de-mixed variables are usually referred to as *Einstein-frame* quantities. In our much more general framework they are explicitly given by

$$\begin{aligned} \Phi_E &\equiv \frac{1+\alpha_H}{1+\alpha_T}\Phi + \left(\frac{1+\alpha_M}{1+\alpha_T} - \frac{1+\alpha_B}{1+\alpha_H} \right) H\pi - \frac{\alpha_H}{1+\alpha_T}\dot{\pi} \;, \\ \Psi_E &\equiv \Psi + \frac{\alpha_H - \alpha_B}{1+\alpha_H} H\pi \,. \end{aligned} \tag{88}$$

Using this change of variables into the quadratic action, one ends up with

$$\begin{aligned} S = \int d^4x a^3 M^2 \bigg\{ & \frac{H^2 \mathcal{L}_{\dot{\zeta}\dot{\zeta}}}{2M^2} \left(\frac{1+\alpha_B}{1+\alpha_H} \right)^2 \left(\dot{\pi}^2 - \tilde{c}_s^2 \frac{(\nabla\pi)^2}{a^2} \right) \\ & - 3\dot{\Psi}_E^2 + \frac{1+\alpha_T}{a^2} \big[(\nabla\Psi_E)^2 - 2\nabla\Phi_E\nabla\Psi_E \big] + \cdots \bigg\} \;, \end{aligned} \tag{89}$$

[4] Assuming a monotonic $\phi_0 = \phi_0(t)$, one can always make a field redefinition of ϕ and choose the background solution $\phi_0 = t$.

whose first line corresponds to the action of a minimally coupled scalar field. In particular, the term proportional to $\dot{\pi}$ in the definition of Φ_E entails the removal of the higher derivative term $\nabla\dot{\pi}\nabla\Psi$.

Let us now consider matter. Since it is minimally coupled to the original metric, i.e.

$$L_{\rm int} \equiv \frac{1}{2}\delta g_{\mu\nu}\delta T^{\mu\nu} = -(\Phi\delta\rho_m + 3\Psi\delta p_m) \, , \tag{90}$$

it becomes coupled to π after the field redefinition (88). When $\alpha_H = 0$, matter is coupled to the gravitational sector with standard terms, $\Phi_E\delta\rho_m$ and $\Psi_E\delta p_m$, as well as to π via fifth-force terms, $\pi\delta\rho_m$ and $\pi\delta p_m$. These couplings can be neglected on scales smaller than the matter sound horizon, *i.e.* for $k \gg Ha/c_m$, where c_m is the matter sound speed. However, in the non-Horndeski case ($\alpha_H \neq 0$), the interaction Lagrangian (90) contains a new coupling proportional to the time derivative of the scalar π,

$$L_{\rm int} \supset -\frac{\alpha_H}{1+\alpha_H}\dot{\pi}\delta\rho_m \, , \tag{91}$$

which cannot be neglected on scales smaller than the sound horizon. Indeed, on these scales, the propagation equations for the density contrast $\delta\rho_m$ and field perturbation π become

$$\ddot{\delta}\rho_m - c_m^2\frac{\nabla^2\delta\rho_m}{a^2} - (\rho_m + p_m)\frac{\alpha_H}{1+\alpha_H}\frac{\nabla^2\dot{\pi}}{a^2} \approx 0 \, , \tag{92}$$

$$\ddot{\pi} - \tilde{c}_s^2\frac{\nabla^2\pi}{a^2} - \frac{1}{H^2\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\frac{\alpha_H(1+\alpha_H)}{(1+\alpha_B)^2}\dot{\delta}\rho_m \approx 0 \, , \tag{93}$$

where the symbol $\approx$ stands for an equality in the kinetic limit. One can check that the propagation equation is given also in this case by eq. (83). In contrast to the standard Jeans lore, the gravitational scalar mode π cannot be decoupled from matter by going at sufficiently short distances. The presence of the scalar field perturbations impacts the propagation of matter fluctuations, by changing their sound speed.

6 Field redefinitions

This section is devoted to exploring some mathematical properties of the class of theories that we are proposing and to confirm their soundness for subclasses of these theories. The approach discussed in this section does not rely on the ADM formulation and we do not need to assume $\nabla_\mu\phi$ being timelike, in contrast with our Hamiltonian analysis.

First, we analyse disformal transformations and focus on a specific class of disformal transformations that act as a "morphism" on our theories, in the same way in which conformal transformations preserve the basic structure of Brans-Dicke theories. Next, we show how to relate, by means of such disformal transformations, subsets of our theories—i.e. L_4 and L_5, separately studied in Secs. 6.2 and 6.3, respectively—into Horndeski ones. As these disformal transformations conserve the number of degrees of freedom, this is yet another proof that our theories do not contain ghosts, even if they contain higher derivatives. In the cases in which the mapping with Horndeski is possible, we further clarify this issue in Sec. 6.5, by showing that naively higher-derivative equations can be reduced to second-order ones. In passing, we also verify in Sec. 6.4 that the presence of matter does not spoil the soundness of the theory.

6.1 Disformal transformations

In this section we compute the transformation properties of our theories under *disformal transformations*. More precisely, we consider a field redefinition of the metric tensor made of a conformal transformation and of a further lightcone structure-changing piece [19],

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(\phi, X)\, g_{\mu\nu} + \Gamma(\phi, X)\, \partial_\mu\phi\, \partial_\nu\phi \, . \tag{94}$$

For convenience, we directly work in unitary gauge even though the same results can be reached using a covariant approach (see *e.g.* [40, 18]). As we shall see, the use of the unitary gauge considerably simplifies the calculations.

In this gauge, the dependence of Ω and Γ on ϕ and X translates into an explicit dependence on the time variable t and on the lapse function N. Moreover, we choose time to coincide with ϕ, so that $\partial_\mu \phi = \delta^0_\mu$ and eq. (94) reads, in ADM components,

$$\bar{N}^i = N^i \,, \qquad \bar{h}_{ij} = \Omega^2(t, N)\, h_{ij} \,, \qquad \bar{N}^2 = \Omega^2(t, N)\, N^2 - \Gamma\,(t, N)\,. \tag{95}$$

Thus, the volume element is transformed accordingly,

$$\sqrt{-\bar{g}} = \sqrt{-g}\,\Omega^3 \sqrt{\Omega^2 - \Gamma/N^2}\,. \tag{96}$$

In order to find how the three-dimensional Ricci scalars, R and $\bar{R}$, are related to each other, we can apply the standard formulae to the conformal transformations of the 3-d metric (95) (see *e.g.* [41]),

$$\bar{R} = \Omega^{-2}\left[R - 4D^2 \ln \Omega - 2\partial_i(\ln \Omega)\partial^i(\ln \Omega)\right]\,. \tag{97}$$

Moreover, using the definition of the extrinsic curvature, eq. (22), one finds

$$\bar{K}^j_{\ i} \;=\; \frac{N}{\bar{N}}\left[K^j_{\ i} \;-\; N g^{0\mu}\partial_\mu \ln \Omega\, \delta^j_{\ i}\right]. \tag{98}$$

As in unitary gauge Ω depends on the spatial coordinates only through N, it makes a lot of difference whether or not Ω depends on N. If it does, the transformation (95) generates derivatives of N explicitly in the action, therefore changing the structure of action (20). Thus, transformations with Ω dependent on N do not preserve the G^3 form of the Lagrangian. On the contrary, if Ω is independent of N, eq. (95) is just an overall (spatial) coordinate-independent rescaling from the 3-dimensional point of view and the structure of our theory does not change after the field redefinition.

Thus, let us consider an N independent conformal factor, $\Omega = \Omega(t)$. Explicitly, starting from the action (20) written in terms of the barred metric quantities with coefficients $\bar{A}_a$ and $\bar{B}_a$, and making the substitution (95) with $\Omega = \Omega(t)$, one ends up with an action in terms of the unbarred quantities. Remarkably, this new action shares the same structure (20), up to a reshuffling of the coefficients:

$$\begin{aligned}
A_2 &= \frac{\Omega^3 \bar{N}}{N}\left[\bar{A}_2 + 3\frac{d\ln\Omega}{d\bar{t}}\bar{A}_3 + 6\left(\frac{d\ln\Omega}{d\bar{t}}\right)^2\bar{A}_4 + 6\left(\frac{d\ln\Omega}{d\bar{t}}\right)^3\bar{A}_5\right]\,,\\
A_3 &= \Omega^3\left[\bar{A}_3 + 4\frac{d\ln\Omega}{d\bar{t}}\bar{A}_4 + 6\left(\frac{d\ln\Omega}{d\bar{t}}\right)^2\bar{A}_5\right]\,,\\
A_4 &= \frac{\Omega^3 N}{\bar{N}}\left[\bar{A}_4 + 3\frac{d\ln\Omega}{d\bar{t}}\bar{A}_5\right]\,,\\
A_5 &= \frac{\Omega^3 N^2}{\bar{N}^2}\bar{A}_5\,,\\
B_4 &= \frac{\Omega\bar{N}}{N}\left[\bar{B}_4 - \frac{1}{2}\frac{d\ln\Omega}{d\bar{t}}\bar{B}_5\right]\,,\\
B_5 &= \Omega\bar{B}_5\,,
\end{aligned} \tag{99}$$

where $d\bar{t} \equiv \bar{N}dt$. One notes that, in this disformal transformation, a Lagrangian of a given order generally contributes also to the lower-order Lagrangians. For instance, the transformation of L_4 contains also L_3 and L_2 pieces. Only when $\Omega =$ const. does this mixing not occur.

Although we have worked specifically in the unitary gauge, it is straightforward to perform the same analysis covariantly, directly with the 4-dimensional transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(\phi)\, g_{\mu\nu} + \Gamma(\phi, X)\, \partial_\mu\phi\, \partial_\nu\phi\,. \tag{100}$$

One then obtains relations between the coefficients $A_a(\phi, X)$, $B_a(\phi, X)$ and $\bar{A}_a(\phi, \bar{X})$, $\bar{B}_a(\phi, \bar{X})$, which are essentially the above relations (99) with the correspondence $N = 1/\sqrt{-X}$ and $\bar{N} = 1/\sqrt{-\bar{X}}$. The relation between X and $\bar{X}$ can be computed by contracting the inverse metric,

$$\bar{g}^{\mu\nu} = \Omega^{-2}\left(g^{\mu\nu} - \frac{\Gamma}{\Gamma X + \Omega^2}\partial^\mu\phi\partial^\nu\phi\right) , \tag{101}$$

with $\partial_\mu\phi\partial_\nu\phi$. This gives

$$\bar{X} = \frac{X}{\Gamma X + \Omega^2} , \qquad X = \frac{\Omega^2 \bar{X}}{1 - \Gamma\bar{X}} . \tag{102}$$

We also have

$$\frac{\sqrt{-g}}{\sqrt{-\bar{g}}} = \frac{\sqrt{1-\bar{X}\Gamma}}{\Omega^4} = \frac{1}{\Omega^3\sqrt{\Gamma X + \Omega^2}} , \tag{103}$$

which implies in particular that, in unitary gauge,

$$\frac{N}{\bar{N}} = \frac{\sqrt{1-\bar{X}\Gamma}}{\Omega} = \frac{1}{\sqrt{\Gamma X + \Omega^2}} , \tag{104}$$

which can be substituted in eq. (99).

6.2 Link between L_4 and Horndeski

The disformal transformations discussed in the previous subsection can be used to relate Horndeski theories with our general Lagrangians.

First, let us start from a Horndeski Lagrangian L_4^H expressed in terms of the metric $\bar{g}_{\mu\nu}$, with coefficients $\bar{A}_4(\phi, \bar{X})$ and $\bar{B}_4(\phi, \bar{X})$ satisfying the Horndeski condition (see eq. (30))

$$\bar{A}_4 = -\bar{B}_4 + 2\bar{X}\bar{B}_{4\bar{X}} . \tag{105}$$

Substituting in this Lagrangian the expression

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma_4(\phi, X)\,\partial_\mu\phi\,\partial_\nu\phi , \tag{106}$$

leads to a G^3 Lagrangian, now expressed in terms of the metric $g_{\mu\nu}$ and X, with coefficients $A_4(\phi, X)$ and $B_4(\phi, X)$. According to the results of the previous subsection, specialized to the case $\Omega = 1$, the link between the old and new coefficients is given by the relations

$$\bar{A}_4(\phi, \bar{X}) = A_4(\phi, X)\sqrt{1 + X\Gamma_4} , \qquad A_4(\phi, X) = \bar{A}_4(\phi, \bar{X})\sqrt{1 - \bar{X}\Gamma_4} \tag{107}$$

and

$$\bar{B}_4(\phi, \bar{X}) = \frac{B_4(\phi, X)}{\sqrt{1 + X\Gamma_4}} , \qquad B_4(\phi, X) = \frac{\bar{B}_4(\phi, \bar{X})}{\sqrt{1 - \bar{X}\Gamma_4}} , \tag{108}$$

with

$$\bar{X} = \frac{X}{1 + \Gamma_4 X} , \qquad X = \frac{\bar{X}}{1 - \Gamma_4\bar{X}} . \tag{109}$$

The Horndeski condition (105) on the coefficients $\bar{A}_4$ and $\bar{B}_4$ implies the following relation between Γ_4 and the new coefficients A_4 and B_4:

$$\Gamma_{4X} = \frac{A_4 + B_4 - 2XB_{4X}}{X^2 A_4} . \tag{110}$$

It is thus clear that the new Lagrangian, expressed in terms of the metric $g_{\mu\nu}$, is not of the Horndeski type unless Γ_4 is independent of X. This is consistent with the findings of Ref. [40] that the Horndeski form of the Lagrangian is preserved under a restricted version of (100), in which the disformal function Γ, like Ω, does not depend on X.

Conversely, if we start with a G^3 Lagrangian without L_5 terms, but otherwise with arbitrary functions $A_4(\phi, X)$ and $B_4(\phi, X)$, one can always rewrite it as a Horndeski Lagrangian L_4^H, provided that the transformation function Γ_4 is a solution of the differential equation (110). Note that the field redefinition (106) is *well-defined*, in the sense that it leaves invariant the number of degrees of freedom (see other examples in [20]). Indeed, one can express $g_{\mu\nu}$ in terms of $\bar{g}_{\mu\nu}$ and ϕ without introducing additional degrees of freedom. As the set of fields $(\bar{g}_{\mu\nu}, \phi)$ obeys the Horndeski equations of motion, it describes three degrees of freedom. By the field transformation (106), also $(g_{\mu\nu}, \phi)$ obeying the equations of motion derived from the G^3 Lagrangian L_4 describe the same number of degrees of freedom, i.e. three. This essentially confirms the Hamiltonian analysis of Sec. 4 which excludes the presence of more than three degrees of freedom in G^3 theories. As expected, the field redefinition (106) partly de-mixes the metric and scalar field kinetic mixing presented in Sec. (5). In Newtonian gauge, this corresponds to removing the higher-derivative coupling $2\alpha_H \nabla\dot{\pi}\nabla\Psi$ from action (87), as explicitly shown in Appendix C.

6.3 Link between L_5 and Horndeski

The same procedure described above applies to L_5 Lagrangians along similar lines. Namely, one can always relate a G^3 Lagrangian with arbitrary A_5 and B_5, but with $A_4 = B_4 = 0$, to a Horndeski Lagrangian of the type L_5^H, provided the two metrics are related by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma_5(\phi, X)\, \partial_\mu\phi\, \partial_\nu\phi \,, \tag{111}$$

with Γ_5 satisfying the condition

$$\Gamma_{5X} = \frac{3A_5 + XB_{5X}}{3X^2 A_5} \,. \tag{112}$$

Analogously to the above discussion, this follows from requiring that $\bar{A}_5$ and $\bar{B}_5$, given by (see eq. (99))

$$\bar{A}_5(\phi, \bar{X}) = A_5(\phi, X)(1 + X\Gamma_5)\,, \qquad \bar{B}_5(\phi, \bar{X}) = B_5(\phi, X)\,, \tag{113}$$

satisfy Horndeski condition (see eq. (30)),

$$\bar{A}_5 = -\bar{X}\bar{B}_{5\bar{X}}/3 \,. \tag{114}$$

However, one cannot in general re-express an arbitrary G^3 Lagrangian as a Horndeski Lagrangian via a disformal transformation, because the would-be transformation coefficient Γ cannot satisfy simultaneously the two differential equations (110) and (112).

6.4 Coupling to matter

When the G^3 Lagrangian can be re-expressed as a Horndeski Lagrangian, i.e. in either of the two cases discussed above, the coupling between matter and the gravitational sector, now described by $\bar{g}_{\mu\nu}$ and ϕ, becomes more complicated since the matter Lagrangian depends on the combination

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - \Gamma(\phi, \bar{X})\partial_\mu\phi\partial_\nu\phi, \tag{115}$$

or its inverse,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \frac{\Gamma(\phi, \bar{X})}{1 - \Gamma(\phi, \bar{X})\bar{X}}\bar{g}^{\rho\mu}\bar{g}^{\sigma\nu}\partial_\rho\phi\partial_\sigma\phi \,. \tag{116}$$

Let us illustrate this with the simple example of an ordinary matter scalar field, minimally coupled to the metric $g_{\mu\nu}$. Its action, which intially reads

$$S_{\rm mat} = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right] , \tag{117}$$

becomes, when expressed in terms of $\bar{g}_{\mu\nu}$ and ϕ,

$$S_{\rm mat} = \int d^4x\sqrt{-\bar{g}}\sqrt{1-\Gamma\bar{X}}\left[-\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{\Gamma}{2(1-\Gamma\bar{X})}\left(\bar{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\phi\right)^2 - V(\sigma)\right]. \tag{118}$$

The equation of motion for σ is obtained by varying this action with respect to σ. Since each field is at most derived once in the action, the equation of motion for σ will be second order. The same conclusion holds with the matter contribution to the equation of motion of ϕ. Therefore, the presence of a matter scalar field does not introduce higher-order derivative terms in the equations of motion.

6.5 Equations of motion

Using a disformal transformation, we provide a new example of naively higher-derivative equations of motion which can be reduced to second-order ones. We consider a subclass of G^3 theories that can be mapped into Horndeski (where they appear with the metric $\bar{g}_{\mu\nu}$) and are minimally coupled to matter with their usual metric $g_{\mu\nu}$. The associated action can thus be written in the form

$$S = \int d^4x\sqrt{-\bar{g}}\, L^H[\bar{g}_{\mu\nu},\phi] + \int d^4x\sqrt{-g}\, L_m[g_{\mu\nu}]\,, \tag{119}$$

with

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(\phi, X)\,\partial_\mu\phi\,\partial_\nu\phi\,. \tag{120}$$

Since the theory, written in terms of $g_{\mu\nu}$, is not of the Horndeski type, one expects to find higher derivatives in the equations of motion. We show below how to reduce such a system of equations to a second order system.

The variation of the action (119) yields

$$\delta S = \int d^4x\sqrt{-\bar{g}}\,\left[\mathcal{O}_H^{\mu\nu}\delta\bar{g}_{\mu\nu} + \mathcal{S}_H\,\delta\phi\right] + \frac{1}{2}\int d^4x\sqrt{-g}\,T_m^{\mu\nu}\delta g_{\mu\nu}, \tag{121}$$

with

$$\delta\bar{g}_{\mu\nu} = \delta g_{\mu\nu} + \Gamma_X\partial_\mu\phi\,\partial_\nu\phi\delta X + \Gamma_\phi\,\partial_\mu\phi\,\partial_\nu\phi\,\delta\phi + 2\Gamma\partial_{(\mu}\phi\nabla_{\nu)}\delta\phi \tag{122}$$

and

$$\delta X = -\partial^\mu\phi\,\partial^\nu\phi\delta g_{\mu\nu} + 2\partial^\mu\phi\nabla_\mu\delta\phi\,. \tag{123}$$

The operators $\mathcal{O}_H^{\mu\nu}$ and $\mathcal{S}_H$, when expressed in terms of $\bar{g}_{\mu\nu}$ and ϕ, contain only second order derivatives since they come from a Horndeski Lagrangian. Variation of the action with respect to the metric $g_{\mu\nu}$ gives the equations of motion

$$\mathcal{O}_H^{\mu\nu} - \mathcal{O}_H^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\,\Gamma_X\partial^\mu\phi\,\partial^\nu\phi + \frac{1}{2}\Xi\,T_m^{\mu\nu} = 0\,, \tag{124}$$

where

$$\Xi \equiv \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} = \frac{1}{\sqrt{1+\Gamma X}}\,, \tag{125}$$

and we used eq. (103) for the second equality. Variation with respect to ϕ gives the scalar equation of motion:

$$2\nabla_\mu\left[\mathcal{O}_H^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\,\Gamma_X\partial^\mu\phi + \mathcal{O}_H^{\mu\nu}\partial_\nu\phi\,\Gamma\right] - \mathcal{O}_H^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\,\Gamma_\phi - \mathcal{S}_H = 0\,. \tag{126}$$

Contracting (124) with $\partial_\mu\phi\partial_\nu\phi$ yields

$$\mathcal{O}_H^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = -\frac{\Xi\,T_m^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi}{2(1-X^2\Gamma_X)}\,. \tag{127}$$

Substituting back in (124) gives the equation of motion for $g_{\mu\nu}$,

$$\mathcal{O}_H^{\mu\nu} = -\frac{\Xi\, T_m^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi}{2(1-X^2\Gamma_X)}\Gamma_X\partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\Xi\, T_m^{\mu\nu}\,, \tag{128}$$

which is second order with respect to $g_{\mu\nu}$. However, it also contains third order derivatives of ϕ since $\mathcal{O}_H^{\mu\nu}$ is second order in $\bar{g}_{\mu\nu}$, which itself depends on the gradient of ϕ. By taking the trace of (128), one can find a relation expressing the third time derivative of ϕ in terms of at most second-order time derivatives. In this way, the equations of motion (128) are effectively second order in time derivatives. Finally, substituting equation (128) in the scalar equation (126), one gets

$$\nabla_\mu\left[\Xi\,\Gamma_X\frac{(1+\Gamma X)T^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}{1-X^2\Gamma_X}\partial^\mu\phi + \Xi\Gamma T^{\mu\nu}\partial_\nu\phi\right] - \frac{1}{2}\Xi\Gamma_\phi\frac{T^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}{1-X^2\Gamma_X} + \mathcal{S}_H = 0\,, \tag{129}$$

which is manifestly second order. This procedure extends that given in [18] and illustrates how equations of motion that at first view look higher order can in fact be only second order.

7 Conclusions

Since its original appearance in [6], the galileon mechanism has proved an essential tool for modified gravity. Several concrete modified gravity proposals happen to have galileons as their basic skeleton and reduce to galileons in the appropriate decoupling limit. This leads to the possibility of classifying modified gravity scenarios according to the different inequivalent ways in which the galileons can be consistently coupled to gravity, or "covariantized". For instance, massive gravity models can be seen as non-minimal covariantizations of the galileon [22], because they involve other degrees of freedom than simply the metric and the scalar field. If we insist on having the minimal number of degrees of freedom and equations of motion strictly of second order in derivatives, we end up in the realm of Horndeski—or generalized galileons, G^2 theories [10, 9].

In this paper we have studied in details the scalar-tensor theories proposed in [1], called here G^3. This class of theories, presented in Sec. 3, covariantizes the galileons in a minimal way, i.e. without introducing any other degree of freedom than the metric and a scalar field. However, they extend Horndeski in containing two more free functions. They can display equations of motion with derivatives higher than second order in some gauges, but such higher derivatives are in fact harmless, in the sense that they do not bring in unwanted extra degrees of freedom, as we have shown with a detailed Hamiltonian analysis in Sec. 4. It turns out that the direct covariantization of the original galileons proposed in [6], obtained by simply substituting ordinary derivatives with covariant ones, belongs to our class of theories. As such, original galileons are "ready to go" without the need of the gravitational counterterms prescribed in [7]. Contrarily to what was previously thought, their simple minimally coupled versions are free of ghosts instabilities.

Despite the aspects of "minimality" just discussed, the covariant form of G^3 theories is mathematically challenging, due to the high number of derivatives and complexity of the equations involved. We have highlighted a few "handles" to manage their basic properties. First of all, the unitary gauge formulation based on a $3+1$ ADM decomposition, eq. (21), is particularly compact and reveals the basic healthy structure of the dynamical system that we are considering. Indeed, the expressions (21) only contain "velocities", i.e., *first* time derivatives of the dynamical variables.

Important insights about scalar-tensor theories can also be given by field transformations. The simplest well-known example is constituted by Brans-Dicke theories, that maintain their basic form under a conformal rescaling of the metric tensor that depends only on the scalar field. On the other hand, the structure of our G^3 theories is invariant under *disformal transformations*, as we discussed in some detail in Sec. 6. In particular, disformal transformations with the conformal factor depending on the scalar field only, and the disformal one depending on both ϕ and X,

$$\tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} + \Gamma(\phi, X)\partial_\mu\phi\partial_\nu\phi\,, \tag{130}$$

are the most general class of transformations that preserve the basic G^3 structure. This is analogous to the role played by disformal transformations with $\Gamma = \Gamma(\phi)$ for Horndeski theories, which leave them invariant [40]. We have showed that by applying eq. (130) to a Horndeski theory we end up in G^3—another way of proving the soundness of the corresponding G^3 theory—but that, conversely, not all G^3 theories can be reduced to the Horndeski form by using (130).

Finally, disformal transformations also help understanding another remarkable property of G^3: even when *minimally* coupled to ordinary matter, G^3 exhibit a kinetic type coupling, leading to a mixing of the dark energy and matter sound speeds, and thus to a modified Jeans phenomenon [1]. In linear perturbations theory, in order to isolate the scalar propagating degree of freedom, one is implicitly de-mixing the scalar from the metric with a field redefinition. For Brans-Dicke theories this can be done at full non-linear level by simply going to the *Einstein-frame* metric with a conformal transformation. In our more general set of theories the mixing terms between the scalar and the metric can be higher in derivatives, in which case they are weighted by the parameter α_H, with which we measure the departure from Horndeski. However, it is still possible to perform the de-mixing, at least at the *linear* level in perturbation theory. As we show in App. C, part of the field redefinitions (88) that bring us to this generalized Einstein frame corresponds to a disformal transformation of the type discussed above. Because such transformations contain higher derivatives, it ends up mixing matter with the scalar field at a higher order in derivatives, thus affecting the speed of sound of both components. The phenomenology of G^3, which includes this type of mixing, is an interesting development of this work that we intend to pursue in the future.[5]

Note added: While finishing this paper, Ref. [43] appeared with an analysis and results similar to those of our Sec. 4

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A Covariant theory

Let us give more details on how to go from the Lagrangians in eq. (21) to their covariant versions, eqs. (25)–(28). A crucial relation needed for this calculation is

$$K_{\mu\nu} = -\frac{\phi_{\mu\nu}}{\sqrt{-X}} + n_\mu \dot{n}_\nu + n_\nu \dot{n}_\mu - \frac{1}{2(-X)} n^\lambda \nabla_\lambda X n_\mu n_\nu\,, \qquad (\dot{n}_\mu \equiv n^\nu\, \nabla_\nu n_\mu)\,, \tag{131}$$

which follows from (23) and (24). As the covariantization of L_2 is trivial we start from L_3. To rewrite K in terms of scalar field quantities we use the trace of eq. (131), $K = -\left(\Box\phi - \phi^\lambda \nabla_\lambda X/2X\right)/\sqrt{-X}$. Integrating by parts the term proportional to $\nabla_\lambda X$ we obtain eq. (26).

[5] Besides dark energy, the other playground for these theories is inflation, as recently considered in [42].

For L_4 we replace the 3-d Ricci curvature R in terms of the 4-d one, ${}^{(4)}R$, using the Gauss-Codazzi relation,

$$
{}^{(4)}R = R - K^2 + K_{\mu\nu}K^{\mu\nu} + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu) \,, \tag{132}
$$

after which L_4 becomes

$$
L_4 = B_4{}^{(4)}R + (A_4 + B_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2B_4\nabla_\mu(Kn^\mu - \dot{n}^\mu) \,. \tag{133}
$$

Then, using eq. (131) and that $\dot{n}_\mu = h_\mu^{\ \nu}\nabla_\nu X/(-2X)$, it is possible to express the quadratic combination of extrinsic curvatures as

$$
K^2 - K_{\mu\nu}K^{\mu\nu} = -\frac{(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}}{X} - \frac{\nabla_\mu X(Kn^\mu - \dot{n}^\mu)}{X} \,. \tag{134}
$$

After an integration by parts on the last term of eq. (133) we obtain

$$
\begin{aligned}
L_4 = {} & B_4\,{}^{(4)}R - \frac{B_4 + A_4}{X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] \\
& + 2\frac{B_4 + A_4 - 2XB_{4X}}{X^2}(\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi - \phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}) \\
& + (C_4 + 2XC_{4X})\Box\phi + XC_{4\phi} \,,
\end{aligned}
$$

where the last line comes from rewriting the term proportional to $B_{4\phi}$ analogously to L_3 above. This equation can be rewritten as eq. (27) by using the definition of $L_4^{\text{gal},1}$ in eq. (7) and eqs. (15)–(17).

The case of L_5 is the most cumbersome. In addition to the relations (131) and (132), we will also need the Gauss Codazzi relation

$$
R_{\mu\nu} = \left({}^{(4)}R_{\mu\nu}\right)_\parallel + \left(n^\sigma n^{\rho\,(4)}R_{\mu\sigma\nu\rho}\right)_\parallel - KK_{\mu\nu} + K_{\mu\sigma}K^\sigma_{\ \nu}, \tag{135}
$$

where a symbol $\parallel$ denotes the projection on the hypersurface of all tensor indices, *e.g.* $(V_\mu)_\parallel \equiv h_\mu^{\ \nu}V_\nu$. For simplicity, let us treat the two parts of L_5 separately. Using eq. (131) we can rewrite the term proportional to A_5 as

$$
\begin{aligned}
& A_5\big(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\rho}K^\nu_{\ \rho}\big) \\
& = - A_5(-X)^{-3/2}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_{\ \rho}\right] \\
& \quad + 3A_5(-X)^{-3/2}\left[-\frac{1}{2}\phi^\rho\nabla_\rho X(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2(-X)^{3/2}(K\dot{n}_\mu\dot{n}^\mu - K_{\mu\nu}\dot{n}^\mu\dot{n}^\nu)\right] .
\end{aligned} \tag{136}
$$

As we did for L_3, we define an auxiliary function, F_5, satisfying $\frac{F_5}{2X} + F_{5X} = A_5(-X)^{-3/2}$ and integrate by parts the last line so that up to boundary terms the above equation reads,

$$
\begin{aligned}
& A_5\big(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\rho}K^\nu_{\ \rho}\big) \\
& = - A_5(-X)^{-3/2}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_{\ \rho}\right] \\
& \quad - 3F_5\sqrt{-X}\bigg[\frac{1}{2}\big(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\rho}K^\nu_{\ \rho}\big) + K^{\mu\nu}n^\sigma n^{\rho\,(4)}R_{\mu\sigma\nu\rho} \\
& \quad - 3Kn^\sigma n^{\rho\,(4)}R_{\sigma\rho} + \dot{n}^\sigma n^{\rho\,(4)}R_{\sigma\rho}\bigg] + \frac{X}{2}F_{5\phi}(K^2 - K_{\mu\nu}K^{\mu\nu}) \,.
\end{aligned} \tag{137}
$$

Now we need to deal with the second part. Using the Gauss-Codazzi relations, eqs. (132) and (135), this can be rewritten as

$$
\begin{aligned}
B_5\,K_{\mu\nu}G^{\mu\nu} = B_5\bigg[& K_{\mu\nu}{}^{(4)}G^{\mu\nu} + K_{\mu\nu}n_\sigma n_\rho{}^{(4)}R^{\mu\sigma\nu\rho} - Kn_\sigma n_\rho{}^{(4)}R^{\sigma\rho} \\
& + \frac{1}{2}\big(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\rho}K^\nu_{\ \rho}\big)\bigg] .
\end{aligned} \tag{138}
$$

We can now replace $K_{\mu\nu}{}^{(4)}G^{\mu\nu}$ using eq. (131) and again $\dot{n}_\mu = h_\mu^{\ \nu}\nabla_\nu X/(-2X)$. Introducing a new auxiliary function defined as $G_5 \equiv -\int B_{5X}(-X)^{-1/2}\,dX$, and integrating by parts, first on the $\phi_{\mu\nu}$ term, then on the B_{5X} term that appears, we finally obtain

$$
\begin{aligned}
B_5 K_{\mu\nu} G^{\mu\nu} = {} & G_5 \phi_{\mu\nu}{}^{(4)}G^{\mu\nu} + \left(\frac{B_{5\phi}}{\sqrt{-X}} + G_{5\phi} \right) \phi_\mu \phi_\nu{}^{(4)}G^{\mu\nu} \\
& + B_5 \bigg[\frac{1}{2} \big(K^3 - 3K K_{\mu\nu} K^{\mu\nu} + 2 K_{\mu\nu} K^{\mu\rho} K^\nu_{\ \rho} \big) \\
& + K_{\mu\nu} n_\sigma n_\rho{}^{(4)}R^{\mu\sigma\nu\rho} - K n_\sigma n_\rho{}^{(4)}R^{\sigma\rho} + \dot{n}_\mu n_\nu{}^{(4)}R^{\mu\nu} \bigg] .
\end{aligned}
\tag{139}
$$

We can now combine the two parts of L_5, eqs. (137) and (139), and use the Gauss-Codazzi relation,

$$
n_\mu n_\nu{}^{(4)}G^{\mu\nu} = \frac{1}{2}\big(R + K^2 - K_{\mu\nu}K^{\mu\nu} \big) , \tag{140}
$$

to rewrite the term $\phi_\mu\phi_\nu{}^{(4)}G^{\mu\nu}$ in eq. (139). To simplify this further, we rewrite the combination of Riemann and Ricci that remains employing again eq. (137) which yields

$$
\begin{aligned}
L_5 = {} & G_5 \phi_{\mu\nu}{}^{(4)}G^{\mu\nu} - A_5(-X)^{-3/2} \big[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_{\ \rho} \big] \\
& + (3A_5 + X B_{5X}) \bigg[\frac{1}{2} \big(K^3 - 3K K_{\mu\nu} K^{\mu\nu} + 2K_{\mu\nu} K^{\mu\rho} K^\nu_{\ \rho} \big) \\
& + (-X)^{-3/2} (\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_{\ \rho}) \bigg] \\
& - \frac{X}{2} \left(G_{5\phi} + \frac{B_{5\phi}}{\sqrt{-X}} \right) R - \frac{X}{2} G_{5\phi} (K^2 - K_{\mu\nu}K^{\mu\nu}) .
\end{aligned}
\tag{141}
$$

For the last step, we rewrite the cubic combination of extrinsic curvatures using eq. (131) and rewrite the last line analogously to L_4, which finally leads to

$$
\begin{aligned}
L_5 = {} & G_5\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} - (-X)^{-3/2} A_5 \big[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_{\ \rho} \big] \\
& - \frac{X B_{5X} + 3A_5}{(-X)^{5/2}} \big[(\Box\phi)^2 \phi_\mu \phi^{\mu\nu}\phi_\nu - 2\Box\phi\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho - \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\lambda}\phi_\lambda + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\lambda}\phi_\lambda \big] \\
& + C_5\,{}^{(4)}R - 2C_{5X} \big[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu} \big] + (D_5 + 2XD_{5X})\Box\phi + XD_{5\phi} ,
\end{aligned}
$$

where, again, the last line comes from applying the method of L_4 to the last line of eq (141). To rewrite this expression as eq. (28) we use the definition of $L_5^{\rm gal,1}$, eq. (8), and eqs. (15)–(18).

B Connection to the building blocks of dark energy

The dynamics of cosmological perturbations around a FLRW background in the presence of dark energy and modifications of gravity can be systematically studied using the Effective Field Theory of Dark Energy, introduced in Refs. [37, 15, 16, 11, 17] in the case where dark energy can be described by a single scalar degree of freedom. In particular, Ref. [11] proposed a minimal description of dark energy and modified gravity encompassing all existing models in terms of quadratic Lagrangian operators leading to at most two derivatives in the equations of motion, the so-called Building Blocks of Dark Energy. In this section we would like to make the connection between these operators, the unitary gauge Lagrangians in eq. (21) and the parametrisation introduced in Ref. [14].

As in [11], let us consider a Lagrangian which is a function of N, K, R, $\mathcal{S}$ and $\mathcal{Y}$, where $\mathcal{S} \equiv K_{ij}K^{ij}$ and $\mathcal{Y} \equiv K_{ij}R^{ij}$, i.e.,

$$
L = L(N, K, \mathcal{S}, R, \mathcal{Y}) , \tag{142}
$$

such as eq. (21). To isolate linear perturbations, we focus on the quadratic action. This can be expanded at second order in the perturbations around a flat FLRW metric, $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$, using that $\sqrt{-g} = \sqrt{h}N$ and that $\sqrt{h}|_0 = a^3$ on the background. Then, integrating by parts the term linear in K and using the background equations of motion (the details of these calculations can be found in [11]) the second-order action can be rewritten as

$$\begin{aligned} S_2 \equiv & \int d^4x \delta_2\big(\sqrt{-g}L\big) \\ = & \int d^4x \frac{M^2(t)}{2}\Big\{\delta_2\Big[\sqrt{-g}\Big({}^{(4)}R - 6H^2 + 2\rho_m/M^2 - \frac{2}{N}\big(2\dot{H} + (\rho_m + p_m)/M^2\big)\Big)\Big] \\ & + 2H\alpha_M(t)\delta_2\big[\sqrt{h}(K-2H)\big] + \alpha_T(t)\,\delta_2\big(\sqrt{h}R\big) \\ & + a^3H^2\alpha_K(t)\,\delta N^2 + 4a^3H\alpha_B(t)\,\delta N\delta K + a^3\alpha_H(t)\,\delta N R\Big\}\,, \end{aligned} \tag{143}$$

where we have introduced the time-dependent quantities

$$\begin{aligned} M^2 &\equiv 2L_{\mathcal{S}}\,, \\ \alpha_M &\equiv \frac{\dot{L}_{\mathcal{S}}}{HL_{\mathcal{S}}}\,, \\ \alpha_K &\equiv \frac{2L_N + L_{NN}}{2H^2L_{\mathcal{S}}}\,, \\ \alpha_B &\equiv \frac{2HL_{\mathcal{S}N} + L_{KN}}{4HL_{\mathcal{S}}}\,, \\ \alpha_T &\equiv \frac{L_R + \dot{L}_{\mathcal{Y}}/2 + 3HL_{\mathcal{Y}}/3}{L_{\mathcal{S}}} - 1\,, \\ \alpha_H &\equiv \frac{L_R + L_{NR} + 3HL_{\mathcal{Y}}/2 + HL_{N\mathcal{Y}}}{L_{\mathcal{S}}} - 1\,, \end{aligned} \tag{144}$$

evaluated on the background. Notice that to remove the dependence of action (142) on $\mathcal{Y}$ and obtain eq. (143) we have used the relation [11]

$$\lambda(t)\mathcal{Y} = \frac{\lambda(t)}{2}RK + \frac{\dot{\lambda}(t)}{2N}R\,, \tag{145}$$

valid up to boundary terms.

For a constant M, the first line of action (143) describes second-order metric perturbations in a ΛCDM universe. The parameters in eq. (144) appear naturally as the coefficients of the second-order expansion of L beyond this standard case. This expansion makes it also clear that these are the minimal number of parameters describing the dynamics once the background expansion history, $H(t)$, and the matter content, i.e. $\rho_m(t_0)$ and its equation of state, are given.

Not surprisingly, the first 5 of these parameters are the same as those proposed in Ref. [14]. The last one is new and parameterizes a deviation from Horndeski theories. Using $L = L_2 + L_3 + L_4 + L_5$, in eq. (72) we have written these parameters in terms of the functions A_i and B_i appearing in the Lagrangians (21).

In Ref. [11] we explicitly separated the operators affecting the perturbations from those fixed by the background evolution, writing the action as

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[& \frac{M_*^2}{2}f(t)^{(4)}R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\,\delta K\delta g^{00} \\ & - m_4^2(t)\left(\delta K^2 - \delta K^\mu{}_\nu\,\delta K^\nu{}_\mu\right) + \frac{\tilde{m}_4^2(t)}{2}\,R\,\delta g^{00}\Big]\,. \end{aligned} \tag{146}$$

As explained in [15, 16, 11, 17], the functions c and Λ are fully specified by the background expansion history. We are thus left with 6 free parameters in this action. As expected, there is a simple relation between these parameters and those in eq. (144). Indeed, at second order the above action reduces to eq. (143) with the following dictionary between the two notations,

$$\begin{aligned} M^2 &= M_*^2 f + 2m_4^2 \,, \\ \alpha_M &= \frac{2\dot{M}}{MH} \,, \\ \alpha_K &= \frac{2c + 4M_2^4}{M^2H^2} \,, \\ \alpha_B &= \frac{M_*^2\dot{f} - m_3^3}{2M^2H} \,, \\ \alpha_T &= -\frac{2m_4^2}{M^2} \,, \\ \alpha_H &= \frac{2(\tilde{m}_4^2 - m_4^2)}{M^2} \,. \end{aligned} \tag{147}$$

To see this, one can use $g^{00} = -1/N^2$ and rewrite the term proportional to c up to second order as

$$- cg^{00} = -\frac{c}{N}(1 - \delta N) - c\,\delta N^2 \,. \tag{148}$$

The last term combines with the operator proportional to M_2^4. Moreover, one can rewrite the term proportional to m_4^2, up to boundary terms, as

$$\begin{aligned} m_4^2\left(\delta K^2 - \delta K^\mu{}_\nu\,\delta K^\nu{}_\mu\right) &= m_4^2\Big({}^{(4)}R - R - 6H^2 + 4HK\Big) + 2(m_4^2)^{\cdot}\,\frac{K}{N} \\ &= m_4^2\Big({}^{(4)}R - R\Big) + \left[M_*^2\dot{f} + 2(m_4^2)^{\cdot}\right]\frac{K}{N} + M_*^2\dot{f}\delta N\delta K + \frac{M_*^2\ddot{f}}{N} + 3HM_*^2\dot{f}\frac{\delta N}{N} \,, \end{aligned} \tag{149}$$

and use the background equations of motion for the last two terms.

Finally, it is also easy to make connection with the (slightly different) notation adopted in [44], where the phenomenological aspects of dark energy were studied by using the formalism developed in [15, 16, 11, 17, 31]. There, the time-dependent "Planck mass squared" $M_*^2 f(t)$ was pulled out of the action,

$$\begin{aligned} S \;=\; \int d^4x\,\sqrt{-g}\,\frac{M_*^2 f(t)}{2}\Big[{}^{(4)}R \;-\; 2\lambda(t) \;-\; 2\mathcal{C}(t)g^{00} \\ +\,\mu_2^2(t)(\delta g^{00})^2 \;-\; \mu_3(t)\,\delta K\delta g^{00} \;+\; \epsilon_4(t)\left(\delta K^\mu{}_\nu\,\delta K^\nu{}_\mu - \delta K^2\right) + \frac{\tilde{\epsilon}_4(t)}{2}R\,\delta g^{00}\Big] \;, \end{aligned} \tag{150}$$

so that the natural order of magnitude of the time-dependent coefficients (inside the square brackets above) is the Hubble parameter to the appropriate power. This is also evident by the following dictionary

$$
\begin{aligned}
M^2 &= M_*^2 f\,(1+\epsilon_4)\,,\\
\alpha_M &= \frac{\dot\epsilon_4}{H(1+\epsilon_4)} + \frac{\mu}{H}\,,\\
\alpha_K &= \frac{2\mathcal{C}+4\mu_2^2}{H^2(1+\epsilon_4)}\,,\\
\alpha_B &= \frac{\mu-\mu_3}{2H(1+\epsilon_4)}\,,\\
\alpha_T &= -\frac{\epsilon_4}{1+\epsilon_4}\,,\\
\alpha_H &= \frac{\tilde\epsilon_4-\epsilon_4}{1+\epsilon_4}\,.
\end{aligned}
\tag{151}
$$

C Disformal transformation in Newtonian gauge

In this appendix we show that (part of) the change of variables introduced in Sec. 5.3 in order to demix the metric Newtonian potentials and the scalar field can be understood in terms of a disformal transformation. In particular, we restrict to the G^3 Lagrangian L_4 in eq. (21), given in terms of the metric $g_{\mu\nu}$, and we show that after the disformal transformation (106) with $\Gamma=\Gamma_4$ satisfying eq. (110), all the couplings proportional to α_H disappear from action (87). To maintain the usual background time-time component of the barred metric, $\bar g_{00}^{(0)}=-1$, together with the field redefinition (106) we also perform a time coordinate change,

$$
\bar t = \int \sqrt{1-\Gamma_0}\,dt - \alpha\,, \qquad \alpha \equiv \frac{\Gamma_0}{\sqrt{1-\Gamma_0}}\pi\,, \tag{152}
$$

where Γ_0 is the background value of Γ. The change $t\to t-\alpha$ ensures that $\bar g_{0i}=g_{0i}$ and that we thus remain in Newtonian gauge (see eq. (85)). Using $\phi=t+\pi$, the combination of eq. (106) and the above time redefinition gives, up to linear order,

$$
\bar g_{00} = \frac{g_{00}+\Gamma(1+2\dot\pi)}{1-\Gamma_0} - 2\frac{d\alpha}{d\bar t}\,, \qquad \bar g_{0i}=g_{0i}=0\,, \qquad \bar g_{ij}=g_{ij}\,, \tag{153}
$$

where a dot always denotes the derivative with respect to t. Expanding Γ to linear order and defining $\bar g_{00}\equiv-(1+2\bar\Phi)$ and $\bar g_{ij}=a^2(\bar t)(1-2\bar\Psi)\delta_{ij}$, we obtain, for the potentials in the barred frame,

$$
\bar\Phi = \frac{(1-\Gamma_X)\Phi+\Gamma_X\dot\pi}{1-\Gamma_0} + \frac{\dot\Gamma_0\pi}{2(1-\Gamma_0)^2}\,, \qquad \bar\Psi = \Psi - \frac{\Gamma_0}{1-\Gamma_0}H\pi\,. \tag{154}
$$

Since the time has been redefined according to eq. (152), π in the barred frame reads

$$
\bar\pi = \frac{1}{\sqrt{1-\Gamma_0}}\pi\,, \tag{155}
$$

where we have used $\pi=-\delta t$ and $\bar\pi=-\delta\bar t$.

We can rewrite the time dependent quantities Γ_0, $\dot\Gamma_0$ and Γ_X in terms of the quantities α_i and $\bar\alpha_i$, using the definitions of α_i in eq. (72) together with the metric transformation (106) and eqs. (107) and (108). This yields

$$
1-\Gamma_0 = \frac{1+\alpha_T}{1+\bar\alpha_T}\,, \qquad \dot\Gamma_0 = \frac{1+\alpha_T}{1+\bar\alpha_T}(\alpha_M-\bar\alpha_M)\,, \qquad \Gamma_X = -\alpha_H\,. \tag{156}
$$

Replacing these relations in the above equations and using $\bar{H} = H/\sqrt{1-\Gamma_0}$ due to the time redefinition, we obtain

$$
\begin{aligned}
\bar{\Phi} &= \frac{1+\bar{\alpha}_T}{1+\alpha_T}\left[(1+\alpha_H)\Phi + (\alpha_M - \bar{\alpha}_M)H\pi - \alpha_H\dot{\pi}\right] \ , \\
\bar{\Psi} &= \Psi + \frac{\alpha_T - \bar{\alpha}_T}{1+\alpha_T}H\pi \ , \\
\bar{\pi} &= \frac{1+\bar{\alpha}_T}{1+\alpha_T}\frac{H}{\bar{H}}\pi \ .
\end{aligned}
\tag{157}
$$

These are the field redefinitions in Netwonian gauge between the two frames. One can use these relations, together with an expression for $\bar{\alpha}_B$ and $\bar{\alpha}_K$ as a function of the other quantities, to rewrite action (87) in the barred frame, where all the couplings proportional to α_H disappear. Here we simply check, using the relations above and

$$
\bar{\alpha}_B = -1 + \frac{1+\alpha_B}{1+\alpha_H}\frac{1+\alpha_T}{1+\bar{\alpha}_T} \ , \tag{158}
$$

that Φ_E and Ψ_E given in eq. (88) become, as expected,

$$
\begin{aligned}
\Phi_E &= \frac{1}{1+\bar{\alpha}_T}\bar{\Phi} + \left(\frac{1+\bar{\alpha}_M}{1+\bar{\alpha}_T} - 1 - \bar{\alpha}_B\right)\bar{H}\bar{\pi} \ , \\
\Psi_E &= \bar{\Psi} - \bar{\alpha}_B\bar{H}\bar{\pi} \, .
\end{aligned}
\tag{159}
$$

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Article G

A Unifying Description of Dark Energy

A unifying description of dark energy

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Abstract

We review and extend a novel approach that we recently introduced , to describe general dark energy or scalar-tensor models. Our approach relies on an Arnowitt-Deser-Misner (ADM) formulation based on the hypersurfaces where the underlying scalar field is uniform. The advantage of this approach is that it can describe in the same language and in a minimal way a vast number of existing models, such as quintessence models, $F(R)$ theories, scalar tensor theories, their Horndeski extensions and beyond. It also naturally includes Horava-Lifshitz theories. As summarized in this review, our approach provides a unified treatment of the linear cosmological perturbations about a Friedmann–Lemaître–Robertson–Walker (FLRW) universe, obtained by a systematic expansion of our general action up to quadratic order. This shows that the behaviour of these linear perturbations is generically characterized by five time-dependent functions. We derive the full equations of motion in the Newtonian gauge. In the Horndeski case, we obtainthe equation of state for dark energy perturbations in terms of these functions. Our unifying description thus provides the simplest and most systematic way to confront theoretical models with current and future cosmological observations.

Contents

1 Introduction

The discovery of the present cosmological acceleration, consistently confirmed by various cosmological probes, has spurred an intense theoretical activity to account for this observational fact. Although a cosmological constant is by far the simplest explanation for this acceleration, the huge fine-tuning that seems required, at least from a current perspective, has motivated the exploration of alternative models.

As a consequence, the dark energy landscape is now very similar to that of inflation, containing a huge number of models with various motivations and various degrees of sophistication. In fact, many of the inflationary models have been reconverted into dark energy models, and vice-versa. A majority of models of dark energy, although not all of them, involve a scalar field, in an explicit or implicit way. This scalar component can be simply added to standard gravity, like in quintessence models, or, more subtly, intertwined with gravity itself, like in scalar-tensor gravitational theories. This illustrates the two ways to modify the dynamical equations in cosmology: either by adding a new matter component or by modifying gravity itself.

In this paper we review and extend the approach introduced in [1] to describe in a unifying and minimal way most existing dark energy or modified gravity models that contain a single scalar degree of freedom. This approach was initially inspired by the so-called effective field theory (EFT) formalism, pioneered in [2, 3] for inflation and in [4] for minimally coupled dark energy, and later developed in the context of dark energy [5, 6, 7] (see also [8] for a recent review and e.g. [9, 10, 11, 12, 13] for applications of the EFT formalism[1]), but exploits more systematically the 3+1 spacetime ADM decomposition by starting from a Lagrangian written only in terms of ADM quantities. This leads to an almost automatic treatment of the equations of motion, both at the background and perturbative levels. Our ADM approach is also at the core of several recent works [17, 18] and is very useful for the theories beyond Horndeski that we proposed in [19, 20] (see also [21, 22, 23, 24, 25]).

In the present article, we give a slightly more general presentation of our formalism than that given in [1], by parametrizing the dynamical equations with (background-dependent) functions that are constructed directly from partial derivatives of the initial Lagrangian with respect to the ADM tensors, rather than from partial derivatives with respect to a few scalar combinations of the ADM tensors. This makes our formalism readily applicable to a larger class of models without further preparation work, but the results are essentially the same. The results obtained in [1] and in [7] have been reformulated in [26] by introducing dimensionless (time-dependent) functions that are combinations of those that appear in the effective formalisms previously introduced, with the advantage of clearly parametrizing deviations from General Relativity (GR). Here we will use this notation, up to a minor redefinition and an extension to theories beyond Horndeski.

The advantage of a unified treatment of dark energy is multiple. First, it provides a global view of the lanscape of theoretical models, by translating them in the same language. They thus become easy to compare, with a clear identification of approximate or exact degeneracies between the models. Moreover, a precise map also enables theorists to identify, beyond well-known regions, unchartered territories that remain to be explored. A striking illustration of this is the recent realization that theories beyond Horndeski could be free from Ostrogradski instabilities [19, 20]: these theories were initially motivated by noticing that Horndeski theories correspond to a subset of all possibilities at the level of linear perturbations [1].

Second, a unified treatment of theoretical models enormously simplifies the confrontation of these with observational constraints. Instead of constraining separately each existing model in the literature, one can simply constrain the parametrized functions of the general formalism and then infer what this implies for each model. Our treatment reduces redundancies, ensuring that the number of parametrized functions is minimal for a given set of assumptions (number of space or time derivatives, etc.). One can also identify models that are confined to "subspaces" of the general framework and devise optimized ways to rule them out by observations.

Our plan is the following. In Sec. 2, we introduce the central starting point of our formalism, a generic Lagrangian written in the ADM formulation, and show how well-known models proposed in the literature can be reformulated in this form. In Sec. 3 we rederive the main results obtained in [1], but adopting a more general presentation than that given originally. Then, in Sec. 4 we focus our attention on the evolution of cosmological perturbations and translate the results of the previous section into the more familiar Newtonian gauge. Moreover, we derived the perturbed Einstein and scalar field equations. In the case of Horndeski, we provide an expression for the equation of state of dark energy perturbations and discuss its observational implications. In Appendix A we discuss the long wavelength limit of the perturbation equations, in Appendix B we give the perturbation equations in the synchronous gauge, while in Appendix C we provide the definitions of several parameters

[1]Other general treatments of single degree of freedom dark energy, based on the equations of motion, can be found in Refs. [14, 15, 16]. The advantage of an action formulation is, of course, that one can easily identify ghost instabilities.

useful in the paper.

2 A unifying action

2.1 General action principle

In this section we review the approach introduced in [1]. Following [5], we assume the validity of the weak equivalence principle and thus the existence of a metric $g_{\mu\nu}$ universally coupled to all matter fields. The fundamental idea is then to start from a generic action that depends on the basic geometric quantities that appear in an ADM decomposition of spacetime, with *uniform scalar field* hypersurfaces as constant time hypersurfaces. The equations governing the background evolution and the linear perturbations can then be obtained in a generic way, up to a few simplifying assumptions (which can be easily relaxed) that are verified by most existing models.

2.1.1 Geometrical quantities

Our approach relies on the existence of a scalar field characterized by a time-like spacetime gradient, which is a natural assumption in a cosmological context. As a consequence, the uniform scalar field hypersurfaces correspond to space-like hypersurfaces and can be used for a 3+1 decomposition of spacetime.

One can associate various geometrical quantities to these hypersurfaces, which will be useful in order to build a generic variational principle. The most immediate geometrical quantities are the future-oriented time-like unit vector normal to the hypersurfaces n^μ, which satisfies $g_{\mu\nu}n^\mu n^\nu = -1$, and the projection tensor on the hypersurfaces,

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \,. \tag{1}$$

One can also introduce the intrinsic curvature of the hypersurfaces, described by the Ricci tensor (which contains as much information as the Riemann tensor for three-dimensional manifolds)

$$R_{\mu\nu} \,, \tag{2}$$

and the extrinsic curvature tensor

$$K^\mu_{\ \nu} \equiv h^{\mu\rho}\nabla_\rho n_\nu \,. \tag{3}$$

Other quantities can be derived by combining the above tensors, together with the covariant derivative ∇_μ and the spacetime metric $g_{\mu\nu}$. For example, one can define the "acceleration" vector field

$$a^\mu \equiv n^\lambda \nabla_\lambda n^\mu \,, \tag{4}$$

which is tangent to the hypersurfaces (since $n_\mu a^\mu = 0$).

With the geometrical quantities introduced above, the dependence on the scalar field is implicit. Since many dark energy models are given explicitly in terms of a scalar field ϕ, it is useful to write down the correspondance between the various geometrical tensors and expressions of ϕ. The relation between the unit vector n^μ and the first derivative of ϕ is simply

$$n_\mu = -\frac{1}{\sqrt{-X}}\nabla_\mu\phi \,, \qquad X \equiv g^{\rho\sigma}\,\nabla_\rho\phi\,\nabla_\sigma\phi \,. \tag{5}$$

The extrinsic curvature tensor is related to second derivatives of ϕ, according to the expression

$$K_{\mu\nu} = -\frac{1}{\sqrt{-X}}\nabla_\mu\nabla_\nu\phi + n_\mu a_\nu + n_\nu a_\mu + \frac{1}{2X}n_\mu\, n_\nu\, n^\lambda\nabla_\lambda X \,, \tag{6}$$

which can be derived by substituting (5) into (3).

Finally, since the Lagrangian for gravitational theories often involves the four-dimensional curvature, it is useful to recall the Gauss-Codazzi relation,

$$ {}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_\mu(Kn^\mu - n^\rho\nabla_\rho n^\mu) \,, \tag{7} $$

which expresses the four-dimensional curvature ${}^{(4)}R$ in terms of the extrinsic curvature tensor and of the intrinsic curvature. We will always denote the four-dimensional curvature with the superscript (4) to distinguish it from the hypersurface intrinsic curvature.

2.1.2 ADM coordinates

So far, all geometrical quantities have been introduced intrinsically, without reference to any specific coordinate system. However, since spacetime is endowed with a preferred slicing, defined by the uniform scalar field hypersurfaces, it is convenient to use coordinate systems especially adapted to this slicing, in other words so that constant time hypersurfaces coincide with the preferred hypersurfaces.

We thus express the four-dimensional metric in the ADM form

$$ ds^2 = -N^2dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \,, \tag{8} $$

where N is the lapse and N^i the shift. In matricial form, the components of the metric and of its inverse are given respectively by

$$ g_{\mu\nu} = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^j \\ h_{ij}N^i & h_{ij} \end{pmatrix} , \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & h^{ij} - N^iN^j/N^2 \end{pmatrix} . \tag{9} $$

In ADM coordinates, we obtain

$$ X = g^{00}\dot\phi^2(t) = -\frac{\dot\phi^2(t)}{N^2} \,, \tag{10} $$

since the scalar field depends only on time, by construction. The components of the normal vector are thus given by

$$ n_0 = -N \,, \qquad n_i = 0 \,. \tag{11} $$

The components of the extrinsic curvature tensor can be written as

$$ K_{ij} = \frac{1}{2N}\left(\dot h_{ij} - D_iN_j - D_jN_i\right) \,, \tag{12} $$

where a dot stands for a time derivative with respect to t, and D_i denotes the covariant derivative associated with the three-dimensional spatial metric h_{ij}. Spatial indices are lowered and raised by the spatial metric.

In the following, we will consider general gravitational actions which can be written in terms of the geometrical quantities that we have introduced, expressed in ADM coordinates,

$$ S_g = \int d^4x\sqrt{-g}\, L(N, K_{ij}, R_{ij}, h_{ij}, D_i; t) \,, \tag{13} $$

with $\sqrt{-g} = N\sqrt{h}$, where h is the determinant of h_{ij}. Note that, by construction, the above action is automatically invariant under spatial diffeomorphisms, corresponding to a change of spatial coordinates.

2.2 Examples

To make things concrete, let us illustrate our formalism by listing briefly the main scalar tensor theories that have been studied in the context of dark energy and by presenting their explicit reformulations in the general form (13).

2.2.1 General relativity

Before introducing models with a scalar component, let us start by simply rewriting the action for general relativity in the above ADM form. Starting from the Einstein-Hilbert action

$$S_{\rm GR} = \int d^4x \sqrt{-g}\, \frac{M_{\rm Pl}^2}{2}\, {}^{(4)}\!R\,, \tag{14}$$

and substituting the Gauss-Codazzi expression (7), one can get rid of the total derivative term and express the action in terms of the extrinsic and intrinsic curvature terms only. Therefore, one easily obtains a Lagrangian of the form (13) for General Relativity (GR), which reads

$$L_{\rm GR} = \frac{M_{\rm Pl}^2}{2} \left[K_{ij} K^{ij} - K^2 + R \right] . \tag{15}$$

Note that, in contrast with the following examples that intrinsically contain a scalar degree of freedom, the slicing of spacetime is arbitrary since there is no preferred family of spacelike hypersurfaces. This means that the Lagrangian (15) contains an additional symmetry, leading to full four-dimensional invariance, which is not directly manifest in the ADM form.

2.2.2 Quintessence and k-essence

The simplest way to extend gravity with a scalar component is to add to the Einstein-Hilbert action a standard action for the scalar field, which consists of a kinetic term plus a potential. This corresponds to quintessence models. The initial covariant action

$$S = S_{\rm GR} + \int d^4x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) \tag{16}$$

leads to the ADM Lagrangian

$$L = L_{\rm GR} + L_{\rm Q}\,, \qquad L_{\rm Q}(t, N) = \frac{\dot\phi^2(t)}{2N^2} - V(\phi(t))\,. \tag{17}$$

In a similar way, one can describe k-essence theories [27, 28] by expressing their Lagrangian $P(X, \phi)$ in terms of N and t:

$$L_{k\text{-essence}}(t, N) = P\Big[-\frac{\dot\phi^2(t)}{2N^2}, \phi(t) \Big] . \tag{18}$$

2.2.3 $F({}^{(4)}R)$ theories

Theories described by a Lagrangian consisting of a nonlinear function of the four-dimensional curvature scalar ${}^{(4)}R$ are equivalent to a scalar-tensor theory. Indeed, it is easy to verify that the Lagrangian

$$L_{F(R)} = F(\phi) + F_\phi(\phi)({}^{(4)}R - \phi)\,, \tag{19}$$

is equivalent to the Lagrangian $F({}^{(4)}R)$, as they lead to the same equations of motion (as long as $F_{{}^{(4)}R\,{}^{(4)}R} \neq 0$). Given this property, one can then use eq. (7) to rewrite the above Lagrangian, after integration by parts, in the ADM form

$$L_{F(R)} = F_\phi (R + K_{\mu\nu} K^{\mu\nu} - K^2) + 2 F_{\phi\phi}\, K \sqrt{-X} + F(\phi) - \phi F_\phi\,. \tag{20}$$

2.2.4 Horndeski theories

In the last few years, a lot of activity has been focussed on a large class of theories, known as Hordenski theories [29], shown to be equivalent to Generalized Galileons [30] in [31]. Although their Lagrangians contain up to second derivatives of a scalar field, these theories correspond to the most general scalar-tensor theories that directly lead to at most second order equations of motion. As such, they include all the examples introduced above. They can be written as an arbitrary linear combination of the following Lagrangians:

$$L_2^H[G_2] \equiv G_2(\phi, X) \;, \tag{21}$$

$$L_3^H[G_3] \equiv G_3(\phi, X)\,\Box\phi \;, \tag{22}$$

$$L_4^H[G_4] \equiv G_4(\phi, X)\,{}^{(4)}\!R - 2G_{4X}(\phi, X)\big[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\big] \;, \tag{23}$$

$$\begin{aligned} L_5^H[G_5] \equiv\; & G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \frac{1}{3}G_{5X}(\phi, X)\times \\ & \big[(\Box\phi)^3 - 3\,\Box\phi\,(\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi) + 2\,(\nabla_\mu\nabla_\nu\phi)(\nabla^\sigma\nabla^\nu\phi)(\nabla_\sigma\nabla^\mu\phi)\big] \;. \end{aligned} \tag{24}$$

Rewriting these Lagrangians in the ADM form turns out to be significantly more involved than in the previous examples. This calculation was undertaken in [1], where all the details are given explicitly. The final result is that the above Lagrangians (21)–(24) yield, in the ADM form, combinations of the following four Lagrangians

$$L_2^H = F_2(\phi, X) \;, \tag{25}$$

$$L_3^H = F_3(\phi, X)\, K \;, \tag{26}$$

$$L_4^H = F_4(\phi, X)\, R + (2XF_{4X} - F_4)(K^2 - K^{\mu\nu}K_{\mu\nu}) \;, \tag{27}$$

$$L_5^H = F_5(\phi, X)\, G_{\mu\nu}K^{\mu\nu} - \frac{1}{3}XF_{5X}(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\sigma}K^\nu{}_\sigma) \;. \tag{28}$$

The functions F_a appearing here are related to the G_a in eqs. (21)–(24) through (see [1] for details)

$$\begin{aligned} F_2 =& G_2 - \sqrt{-X}\int \frac{G_{3\phi}}{2\sqrt{-X}}\,dX \;, \\ F_3 =& -\int G_{3X}\sqrt{-X}\,dX - 2\sqrt{-X}G_{4\phi} \;, \\ F_4 =& G_4 + \sqrt{-X}\int \frac{G_{5\phi}}{4\sqrt{-X}}\,dX \;, \\ F_5 =& -\int G_{5X}\sqrt{-X}\,dX \;. \end{aligned} \tag{29}$$

It is then straightforward to express the above Lagrangians in ADM coordinates (8).

2.2.5 Beyond Horndeski

Requiring equations of motion to be at most second order, which leads to Horndeski theories, has long seemed to be a necessary requirement in order to avoid ghost-like instabilities, associated with higher order time derivatives, also known as Ostrogradksi instabilities. However, it has been shown in [19, 20] (see also [24] for similar analysis and conclusion and [21, 22, 25] for extensions) that an action composed of the Lagrangians

$$\begin{aligned} L_2 &\equiv A_2(t, N) \;, \\ L_3 &\equiv A_3(t, N)K \;, \\ L_4 &\equiv A_4(t, N)\big(K^2 - K_{ij}K^{ij}\big) + B_4(t, N)R \;, \\ L_5 &\equiv A_5(t, N)\big(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j{}_k\big) + B_5(t, N)K^{ij}\left(R_{ij} - \frac{1}{2}h_{ij}R\right) \;, \end{aligned} \tag{30}$$

with arbitrary functions B_4 and B_5, i.e. *without assuming* B_4 and B_5 to depend on, respectively, A_4 and A_5 (as implied by the Hordenski Lagrangians (27) and (28)),[2] does not lead to Ostrogradski instabilities, in contrast with previous expectations. This conclusion is based on a Hamiltonian analysis of the Lagrangian (30), which applies to all configurations where the spacetime gradient of the scalar field is timelike.

Interestingly, one can also map two subclasses of the general covariant Lagrangian, namely the subclass without L_4 and the subclass without L_5, to Horndeski theories via a disformal transformation of the metric (disformal transformations are discussed in section 3.4). Since L_4 and L_5 require distinct disformal transformations to be related to Horndeski theories, such transformation cannot be applied to the whole Lagrangian [20].

2.2.6 Hořava-Lifshitz theories

An interesting class of Lorentz-violating gravitational theories has been introduced by Hořava with the goal of obtaining (power counting) renormalizability [32]. These theories, dubbed Hořava-Lifshitz gravity, assume the existence of a preferred foliation of spacelike hypersurfaces. An ADM formulation of these theories is thus very natural, even if a covariant description is also possible, via the introduction of a scalar field, often called "khronon", that describes the foliation. Several variants of Hořava-Lifshitz gravity have been proposed in the literature. In particular, the so-called healthy non-projectable extension has been shown to be free of instabitilities [33, 34]. All these theories are describable by a Lagrangian of the form (13), which can be written as (see [35] for a general discussion)

$$L_{\rm HL} = \frac{M_{\rm Pl}^2}{2}\left[K_{ij}K^{ij} - \lambda K^2 + \mathcal{V}(R_{ij},\, N^{-1}\partial_i N)\right] . \qquad (31)$$

Note that the dependence on $N^{-1}\partial_i N$ has been introduced in the healthy non-projectable extension of Hořava-Lifshitz gravity. Since the Hořava-Lifshitz Lagrangian is already in an ADM form, it is very natural to include these theories in our general approach, as discussed in [21] (see also [36]).

3 Cosmology: background equations and linear perturbations

In this section, we analyse from a general perspective the cosmological dynamics, for the background and linear perturbations, simply starting from a generic Lagrangian of the form (13).

3.1 Background evolution

We first discuss the background equations by considering a spatially flat FLRW spacetime, endowed with the metric

$$ds^2 = -\bar{N}^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j . \qquad (32)$$

In this spacetime, the intrinsic curvature tensor of the constant time hypersurfaces vanishes, i.e. $R_{ij} = 0$, and the components of the extrinsic curvature tensor are given by

$$K^i_j = H\delta^i_j, \qquad H \equiv \frac{\dot{a}}{\bar{N}a}\,, \qquad (33)$$

where H is the Hubble parameter. Substituting into the Lagrangian L of (13), one thus obtains an homogeneous Lagrangian, which is a function of $\bar{N}(t)$, $a(t)$ and of time:

$$\bar{L}(a, \dot{a}, \bar{N}) \equiv L\left[K^i_j = \frac{\dot{a}}{\bar{N}a}\,\delta^i_j, R^i_j = 0, N = \bar{N}(t)\right] . \qquad (34)$$

[2]The Lagrangians (30) describe Horndeski theories if the following relations hold: $A_4 = -B_4 + 2XB_{4X}$ and $A_5 = -XB_{5X}/3$.

The variation of the homogeneous action,

$$\bar{S}_g = \int dt\, d^3x \bar{N} a^3 \bar{L}, \tag{35}$$

leads to

$$\delta \bar{S}_g = \int dt d^3x \left\{ a^3 \left(\bar{L} + \bar{N} L_N - 3H\mathcal{F} \right) \delta \bar{N} + 3a^2 \bar{N} \left(\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} \right) \delta a \right\} , \tag{36}$$

where L_N denotes the partial derivative $\partial L/\partial N|_{\rm bgd}$, evaluated on the homogeneous background. We have also introduced the coefficient $\mathcal{F}$, which is defined from the derivative of the Lagrangian with respect to the extrinsic curvature, evaluated on the background[3]

$$\left(\frac{\partial L}{\partial K_{ij}} \right)_{\rm bgd} \equiv \mathcal{F} \bar{g}^{ij} , \tag{37}$$

where $\bar{g}^{ij} = a^{-2}\delta^{ij}$ are the spatial components of the inverse background metric.

If we add some matter minimally coupled to the metric $g_{\mu\nu}$, the variation of the corresponding action with respect to the metric defines the energy-momentum tensor,

$$\delta S_{\rm m} = \frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu} . \tag{38}$$

In a FLRW spacetime, this reduces to

$$\delta \bar{S}_{\rm m} = \int d^4x \bar{N} a^3 \left(-\rho_{\rm m} \frac{\delta \bar{N}}{\bar{N}} + 3p_{\rm m} \frac{\delta a}{a} \right) . \tag{39}$$

Consequently, variation of the total homogeneous action $\bar{S} = \bar{S}_g + \bar{S}_{\rm m}$ with respect to N and a yields, respectively, the first and second Friedmann equations in a very unusual form:

$$\bar{L} + \bar{N} L_N - 3H\mathcal{F} = \rho_{\rm m} \tag{40}$$

and

$$\bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_{\rm m} . \tag{41}$$

These two equations also imply

$$\frac{\dot{\mathcal{F}}}{\bar{N}} + \bar{N} L_N = \rho_{\rm m} + p_{\rm m} . \tag{42}$$

Although written in a very unusual form, it is easy to check that one recovers the usual Friedmann equations when gravity is described by GR. Indeed, in this case,

$$\frac{\partial L_{\rm GR}}{\partial K^i_j} = M_{\rm Pl}^2 \left(K^j_i - K\delta^j_i \right) , \tag{43}$$

which, after substituting $K^i_j = H\delta^i_j$, yields,

$$\mathcal{F}_{\rm GR} = -2M_{\rm Pl}^2 H , \tag{44}$$

whereas $\bar{L}_{\rm GR} = -3M_{\rm Pl}^2 H^2$ and $L_N = 0$.

[3] The present formulation of our approach is more general than that given explicitly in [1], where we assumed that the Lagrangian L was a function of specific scalar combinations of the geometric tensors, namely of $K \equiv K^i_i$, $S \equiv K^{ij}K_{ij}$, $R \equiv R^i_i$ and $Z \equiv R_{ij}R^{ij}$. The coefficient $\mathcal{F}$ was then related to the derivatives of L with respect to K and S, i.e. $\mathcal{F} = L_K + 2HL_S$. The definition (37) enables us to include automatically a dependence on other scalar combinations, such as $K^i{}_j K^j{}_k K^k_i$ which appears in L_5.

3.2 Quadratic action

In order to describe the dynamics of linear perturbations about the FLRW background solution, we now expand the action up to quadratic order. The tensor R_{ij} vanishes in the background and is thus a perturbative quantity. It is useful to introduce the two other perturbative quantities (remembering the definition of H in eq. (33))

$$\delta N \equiv N - \bar{N}\,, \qquad \delta K_i^j \equiv K_i^j - H\delta_i^j\,. \tag{45}$$

The expansion of the Lagrangian L up to quadratic order yields

$$L(N, K_j^i, R_j^i, \dots) = \bar{L} + L_N \delta N + \frac{\partial L}{\partial K_j^i}\delta K_j^i + \frac{\partial L}{\partial R_j^i}\delta R_j^i + L^{(2)} + \dots, \tag{46}$$

with the quadratic part given by

$$\begin{aligned} L^{(2)} =& \frac{1}{2}L_{NN}\delta N^2 + \frac{1}{2}\frac{\partial^2 L}{\partial K_j^i \partial K_l^k}\delta K_j^i \delta K_l^k + \frac{1}{2}\frac{\partial^2 L}{\partial R_j^i \partial R_l^k}\delta R_j^i \delta R_l^k + \\ &+ \frac{\partial^2 L}{\partial K_j^i\, \partial R_l^k}\delta K_j^i \delta R_l^k + \frac{\partial^2 L}{\partial N \partial K_j^i}\delta N \delta K_j^i + \frac{\partial^2 L}{\partial N \partial R_j^i}\delta N \delta R_j^i + \dots\,, \end{aligned} \tag{47}$$

where all the partial derivatives are evaluated on the FLRW background (without explicit notation, as will be the case in the rest of this paper). The coefficient L_{NN} denotes the second derivative of the Lagrangian with respect to N. The dots in the two above equations correspond to other possible terms which are not indicated explicitly to avoid too lengthy equations, but can be treated exactly in the same way. This includes for instance the spatial derivatives of the curvature or of the lapse, which appear in Horava-Lifshitz gravity.

The third term on the right hand side of (46) can be simplified as follows. Rewriting it as

$$\frac{\partial L}{\partial K_j^i}\delta K_j^i = \mathcal{F}\delta K = \mathcal{F}(K - 3H)\,, \tag{48}$$

and noting that $K = \nabla_\mu n^\mu$, one can use the integration by parts

$$\int d^4x\sqrt{-g}\,\mathcal{F}K = -\int d^4x\sqrt{-g}\,n^\mu \nabla_\mu \mathcal{F} = -\int d^4x\sqrt{-g}\frac{\dot{\mathcal{F}}}{N}\,. \tag{49}$$

This implies that the Lagrangian (46) can be replaced by the equivalent Lagrangian

$$L^{\rm new} = \bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{N} + L_N\delta N + \frac{\partial L}{\partial R_j^i}\delta R_j^i + L^{(2)}\,. \tag{50}$$

Let us now consider the quadratic part (47). Because of the background geometry, the coefficient of the second term is necessarily of the form[4]

$$\frac{\partial^2 L}{\partial K_i^j\, \partial K_k^l} = \hat{\mathcal{A}}_K\, \delta_j^i\, \delta_l^k + \mathcal{A}_K\left(\delta_l^i\, \delta_j^k + \delta^{ik}\delta_{jl}\right)\,, \tag{51}$$

[4]This is equivalent to the definition below, expressed with covariant indices for the extrinsic curvature tensors, which makes the symmetry under exchange of the indices more manifest:

$$\frac{\partial^2 L}{\partial K_{ij}\,\partial K_{kl}} \equiv \hat{\mathcal{A}}_K\, \bar{g}^{ij}\, \bar{g}^{kl} + \mathcal{A}_K\left(\bar{g}^{ik}\,\bar{g}^{jl} + \bar{g}^{il}\,\bar{g}^{jk}\right)\,.$$

where we have introduced the (a priori time-dependent) coefficients $\hat{\mathcal{A}}_K$ and $\mathcal{A}_K$. Similarly, one can write

$$\frac{\partial^2 L}{\partial R^j_i\, \partial R^l_k} = \hat{\mathcal{A}}_R\, \delta^i_j\, \delta^k_l + \mathcal{A}_R \left(\delta^i_l\, \delta^k_j + \delta^{ik}\delta_{jl}\right) , \tag{52}$$

and

$$\frac{\partial^2 L}{\partial K^j_i\, \partial R^l_k} = \hat{\mathcal{C}}\, \delta^i_j\, \delta^k_l + \mathcal{C} \left(\delta^i_l\, \delta^k_j + \delta^{ik}\delta_{jl}\right) . \tag{53}$$

The mixed coefficients that appear on the second line of eq. (47) are proportional to δ^j_i and can be written as

$$\frac{\partial^2 L}{\partial N \partial K^i_j} = \mathcal{B}\, \delta^j_i\, , \qquad \frac{\partial^2 L}{\partial N \partial R^i_j} = \mathcal{B}_R\, \delta^j_i\, . \tag{54}$$

Taking into account the term $\sqrt{-g} = N\sqrt{h}$, it is straightforward to derive the quadratic part of the full Lagrangian $\mathcal{L} \equiv \sqrt{-g}\, L$, which is relevant to study linear perturbations. After some cancellations due to the background equations of motion[5], one finds

$$\begin{aligned}\mathcal{L}_2 = {} & \bar{N}\mathcal{G}\, \delta_1 R\, \delta\sqrt{h} + a^3 \left(L_N + \frac{1}{2}\bar{N} L_{NN}\right) \delta N^2 \\ & + \bar{N} a^3 \left[\mathcal{G}\delta_2 R + \frac{1}{2}\hat{\mathcal{A}}_K\, \delta K^2 + \mathcal{B}\, \delta K \delta N + \hat{\mathcal{C}}\, \delta K \delta R + \mathcal{C}\, \delta K^i_j\, \delta R^j_i \right. \\ & \left. + \mathcal{A}_K\, \delta K^i_j\, \delta K^j_i + \mathcal{A}_R\, \delta R^i_j\, \delta R^j_i + \frac{1}{2}\hat{\mathcal{A}}_R\, \delta R^2 + \left(\frac{\mathcal{G}}{\bar{N}} + \mathcal{B}_R\right) \delta N \delta R \right] + \dots\, ,\end{aligned} \tag{55}$$

where, in analogy with the definition (37) of $\mathcal{F}$, we have introduced the coefficient $\mathcal{G}$ defined by

$$\frac{\partial L}{\partial R^i_j} = \mathcal{G}\, \delta^j_i\, . \tag{56}$$

We have also denoted as $\delta_1 R$ and $\delta_2 R$, respectively, the first and second order terms of the curvature R expressed in terms of the metric perturbations.

Note that the coefficients that enter here in the quadratic Lagrangian are more general than those introduced explicitly in [1], where the Lagrangian L was considered as a function of N, K, $S = K_{ij}K^{ij}$, R and $Z \equiv R_{ij}R^{ij}$. It is however straightforward to derive the relation between the present coefficients in terms of our former notation[6]. The present definitions have the advantage to automatically include cases with more complicated combinations involving the tensors K_{ij} or R_{ij}, such as $K^j_i K^k_j K^i_k$ in the Lagrangian term L_5 that appears in Horndeski theories and beyond.

The above quadratic expression can be further simplified, as shown in [1], by reexpressing $\delta K^i_j\, \delta R^j_i$ in terms of the other terms, thanks to the identity

$$\int d^4x \sqrt{-g}\, \lambda(t) R_{ij} K^{ij} = \int d^4x \sqrt{-g} \left[\frac{\lambda(t)}{2} R\, K + \frac{\dot{\lambda}(t)}{2N}\, R\right] . \tag{57}$$

This implies the following replacement at quadratic order:

$$\bar{N} a^3 \mathcal{C}\, \delta K^i_j\, \delta R^j_i \quad \rightarrow \quad \frac{\bar{N} a^3}{2} \left[\left(\frac{\dot{\mathcal{C}}}{\bar{N}} + H\mathcal{C}\right) \left(\delta_2 R + \frac{\delta\sqrt{h}}{a^3} \delta R\right) + \mathcal{C}\, \delta R \delta K + \frac{H\mathcal{C}}{\bar{N}} \delta N \delta R\right] . \tag{58}$$

[5] If matter is present, one must also include in the quadratic Lagrangian the terms from the expansion of the matter action with respect to the metric perturbations.

[6] The correspondence is given by $\hat{\mathcal{A}}_K = 4H^2 L_{SS} + 4HL_{SK} + L_{KK}$, $\mathcal{A}_K = L_S$, $\mathcal{B} = 2HL_{SN} + L_{KN}$, $\mathcal{B}_R = L_{NR}$, $\mathcal{A}_R = L_Z$, $\mathcal{G} = L_R$, $\hat{\mathcal{A}}_R = L_{RR}$ and $\hat{\mathcal{C}} = 2HL_{SR} + L_{KR}$.

Consequently, the quadratic Lagrangian (55) is equivalent to the new one

$$
\begin{aligned}
\mathcal{L}_2^{\rm new} = & \bar{N}\mathcal{G}^*\,\delta_1 R\,\delta\sqrt{h} + a^3\left(L_N + \frac{1}{2}\bar{N}L_{NN}\right)\delta N^2 \\
& + \bar{N}a^3\left[\mathcal{G}^*\delta_2 R + \frac{1}{2}\hat{\mathcal{A}}_K\,\delta K^2 + \mathcal{B}\,\delta K\delta N + \mathcal{C}^*\,\delta K\delta R\right. \\
& \left. +\mathcal{A}_K\,\delta K^i_j\,\delta K^j_i + \mathcal{A}_R\,\delta R^i_j\,\delta R^j_i + \frac{1}{2}\hat{\mathcal{A}}_R\,\delta R^2 + \left(\frac{\mathcal{G}^*}{\bar{N}} + \mathcal{B}^*_R\right)\delta N\delta R\right] + \dots\,,
\end{aligned}
\tag{59}
$$

with the "renormalized" coefficients [7]

$$
\begin{aligned}
\mathcal{G}^* &= \mathcal{G} + \frac{\dot{\mathcal{C}}}{2\bar{N}} + H\mathcal{C}\,, \\
\mathcal{C}^* &= \hat{\mathcal{C}} + \frac{1}{2}\mathcal{C}\,, \\
\mathcal{B}^*_R &= \mathcal{B}_R - \frac{\dot{\mathcal{C}}}{2\bar{N}^2}\,.
\end{aligned}
\tag{60}
$$

3.2.1 Tensor modes

Let us first investigate the tensor modes in the general quadratic Lagrangian (59). At linear order, tensor modes correspond to the perturbations of the spatial metric

$$h_{ij} = a^2(t)\left(\delta_{ij} + \gamma_{ij}\right)\,, \tag{61}$$

with γ_{ij} traceless and divergence-free, $\gamma_{ii} = 0 = \partial_i\gamma_{ij}$. Using

$$\delta K^i_j = \frac{1}{2\bar{N}}\dot{\gamma}^i_j \tag{62}$$

and

$$\delta_2 R = \frac{1}{a^2}\left(\gamma^{ij}\partial^2\gamma_{ij} + \frac{3}{4}\partial_k\gamma_{ij}\partial^k\gamma^{ij} - \frac{1}{2}\partial_k\gamma_{ij}\partial^j\gamma^{ik}\right)\,, \tag{63}$$

one finally obtains

$$S^{(2)}_\gamma = \int d^3x dt\, a^3\left[\frac{\mathcal{A}_K}{4}\dot{\gamma}^2_{ij} - \frac{\mathcal{G}^*}{4a^2}(\partial_k\gamma_{ij})^2\right]\,, \tag{64}$$

where here and below we set $\bar{N} = 1$. We recover the standard GR result when $\mathcal{A}_K = \mathcal{G}^* = M^2_{\rm Pl}/2$. By comparison, this suggests to define the effective Planck mass squared by

$$M^2 \equiv 2\mathcal{A}_K > 0\,, \tag{65}$$

where the sign is required to avoid ghost instabilities, and write the action as

$$S^{(2)}_\gamma = \int d^3x dt\, a^3\frac{M^2}{8}\left[\dot{\gamma}^2_{ij} - \frac{c_T^2}{a^2}(\partial_k\gamma_{ij})^2\right]\,. \tag{66}$$

The square of the graviton propagation speed is given by

$$c_T^2 \equiv 1 + \alpha_T = \frac{\mathcal{G}^*}{\mathcal{A}_K}\,, \tag{67}$$

[7] For a Lagrangian L which is a function of N, K, $S = K_{ij}K^{ij}$, R, $Z \equiv R_{ij}R^{ij}$ and *also* of $Y \equiv R_{ij}K^{ij}$, the relation between the coefficients defined in this paper and the derivatives of L with respect to the above quantities is unchanged for $\hat{\mathcal{A}}_K$, $\mathcal{A}_K$, $\mathcal{B}$ and $\mathcal{A}_R$ (see footnote 6). The other coefficients, taking into account the dependence on Y, are given by $\mathcal{B}^*_R = L^*_{NR} \equiv L_{NR} + HL_{NY} - \dot{L}_Y/2$, $\mathcal{G}^* = L^*_R \equiv L_R + \dot{L}_Y/2 + 3HL_Y/2$, $\hat{\mathcal{A}}_R = L_{RR} + H^2L_{YY} + 2HL_{YR}$ and $\hat{\mathcal{C}}^* = 2HL_{SR} + L_{KR} + HL_{KY} + 2H^2L_{SY} + L_Y/2$ with $\bar{N} = 1$.

where α_T represents the deviation with respect to the GR result.

The graviton sector is thus characterized by the two coefficients $\mathcal{A}_K$ and $\mathcal{G}^*$, or equivalently by M and α_T. In practice, it is the time variation which can distinguish the effective Planck mass defined here with respect to the standard Planck mass, so it is convenient, following [26], to introduce the dimensionless parameter

$$\alpha_M \equiv \frac{1}{H}\frac{d}{dt}\ln M^2 \,. \tag{68}$$

With these definitions, the evolution equation for tensor modes is given by

$$\ddot{\gamma}_{ij} + H(3+\alpha_M)\dot{\gamma}_{ij} - (1+\alpha_T)\frac{\nabla^2}{a^2}\gamma_{ij} = \frac{2}{M^2}\left(T_{ij} - \frac{1}{3}T\delta_{ij}\right)^{TT} \,, \tag{69}$$

where $(T_{ij} - T\delta_{ij}/3)^{TT}$ is the transverse-traceless projection of the anisotropic matter stress tensor.

3.2.2 Vector modes

Let us now study the behaviour of vector modes. In unitary gauge, these are parameterized by the transverse components of the shift vector, i.e. $N^i = N^i_V$ with $\partial_i N^i_V = 0$. The second-order action for vector modes then is

$$S^{(2)}_{N_V} = \int d^3x dt\, \frac{1}{a}\frac{M^2}{8}(\partial_i N^V_j + \partial_j N^V_i)^2 \,. \tag{70}$$

Including matter, variation of the action with respect to N^V_i gives the transverse part of the momentum constraint,

$$\frac{1}{2}\nabla^2 N^V_i = \frac{a^2}{M^2}(T^0_{\ i})^T \,, \tag{71}$$

where $(T^0_{\ i})^T$ is the transverse projection of the matter energy flux. For a perfect fluid, the conservation of the matter stress-energy tensor implies that $(T^0_{\ i})^T \propto 1/a^3$; then the metric vector perturbations scale as

$$N^i_V \propto \frac{1}{aM^2} = \frac{1}{a^{1+\alpha_M}} \,, \tag{72}$$

where the last equality holds for a constant α_M. It is interesting to note that the evolution of the vector modes is modified when the gravitational effective mass M is time-dependent, i.e. when $\alpha_M \neq 0$. Thus, in principle, measuring the time evolution of the vector and tensor perturbations could determine α_M and α_T, independently of the scalar modes.

3.2.3 Scalar modes

Without loss of generality, in unitary gauge the scalar modes can be described by the metric perturbations [37]

$$N = 1 + \delta N, \quad N^i = \delta^{ij}\partial_j\psi, \quad h_{ij} = a^2(t)e^{2\zeta}\delta_{ij} \,. \tag{73}$$

Substituting

$$\delta\sqrt{h} = 3a^3\zeta \,, \qquad \delta K^i_{\ j} = \left(\dot{\zeta} - H\delta N\right)\delta^i_j - \frac{1}{a^2}\delta^{ik}\partial_k\partial_j\psi \,, \tag{74}$$

and

$$\delta_1 R_{ij} = -\delta_{ij}\partial^2\zeta - \partial_i\partial_j\zeta \,, \qquad \delta_2 R = -\frac{2}{a^2}\left[(\partial\zeta)^2 - 4\zeta\partial^2\zeta\right] \,, \tag{75}$$

into (59), one obtains a lengthy Lagrangian in terms of δN, ψ and ζ. Since the Lagrangian does not depend on the time derivatives of the lapse and of the shift, the variation of the Lagrangian with respect to δN and ψ yields two constraints, corresponding to the familiar Hamiltonian constraint and (the scalar part of) the momentum constraint.

In the following, we will assume for simplicity the conditions

$$\hat{\mathcal{A}}_K + 2\mathcal{A}_K = 0\,, \qquad \mathcal{C}^* = 0\,, \qquad 4\hat{\mathcal{A}}_R + 3\mathcal{A}_R = 0\,, \tag{76}$$

which ensure that there are at most two spatial derivatives in the quadratic Lagrangian written in terms of ζ only. This includes Horndeski theories as well as their extensions discussed in Section 2.2.5.

Provided conditions (76) are satisfied, one finds that the momentum constraint reduces to

$$\delta N = \frac{4\mathcal{A}_K}{\mathcal{B} + 4H\mathcal{A}_K}\,\dot{\zeta} = \frac{\dot{\zeta}}{H\,(1+\alpha_B)}\,, \tag{77}$$

where we have introduced the dimensionless quantity[8]

$$\alpha_B \equiv \frac{\mathcal{B}}{4H\mathcal{A}_K}\,, \tag{78}$$

which expresses the deviation from the standard expression $\delta N = \dot{\zeta}/H$. When $\alpha_B \neq 0$, part of the kinetic term of scalar fluctuations comes from the term $\delta K \delta N$ in action (59), i.e. from kinetic mixing between gravitational and scalar degrees of freedom [2, 3, 4]. This phenomenon has been called *kinetic braiding* in [38, 39].

Substituting (77), the quadratic action for ζ is then given by

$$S^{(2)} = \frac{1}{2}\int d^3x dt\, a^3 \left[\mathcal{L}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta}^2 + \mathcal{L}_{\partial\zeta\partial\zeta}\frac{(\partial_i\zeta)^2}{a^2} + \frac{M^2}{4}\dot{\gamma}_{ij}^2 - \frac{M^2}{4}(1+\alpha_T)\frac{(\partial_k\gamma_{ij})^2}{a^2}\right]\,, \tag{79}$$

with

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}} \equiv M^2\,\frac{\alpha}{(1+\alpha_B)^2}\,, \qquad \alpha \equiv \alpha_K + 6\alpha_B^2\,, \tag{80}$$

$$\mathcal{L}_{\partial\zeta\partial\zeta} \equiv 2M^2\left\{1 + \alpha_T - \frac{1+\alpha_H}{1+\alpha_B}\left(1 + \alpha_M - \frac{\dot{H}}{H^2}\right) - \frac{1}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{1+\alpha_B}\right)\right\}\,, \tag{81}$$

where we have introduced the dimensionless time-dependent functions

$$\alpha_K \equiv \frac{2L_N + L_{NN}}{2H^2\mathcal{A}_K}\,, \qquad \alpha_H \equiv \frac{\mathcal{G}^* + \mathcal{B}^*_R}{\mathcal{A}_K} - 1\,. \tag{82}$$

Note that the coefficient of the kinetic term reduces to $\mathcal{L}_{\dot{\zeta}\dot{\zeta}} = M^2\alpha_K$ when $\alpha_B = 0$. In this case, the kinetic coefficient for ζ is directly related to the coefficient of the term δN^2 in the quadratic Lagrangian (59), which represents the kinetic energy of the scalar field fluctuations. The parameter α_H is different from zero for theories that deviate from Horndeski theories [1, 19, 20]. In particular, this includes theories that can be related to Horndeski theories via disformal transformations, as shown in [20]. Indeed, starting from a Horndeski theory for a metric $\tilde{g}_{\mu\nu}$ related to $g_{\mu\nu}$ via a disformal transformation that depends on X, the Lagrangian expressed in terms of $g_{\mu\nu}$ differs from the standard Horndeski Lagrangian, which implies $\alpha_H \neq 0$.

Classical and quantum stability (absence of ghosts) requires the kinetic coefficient to be positive,

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}} > 0 \qquad \Longrightarrow \qquad \alpha = \alpha_K + 6\alpha_B^2 > 0\,. \tag{83}$$

The sound speed (squared) of fluctuations can be simply computed by taking the ratio

$$c_s^2 \equiv -\frac{\mathcal{L}_{\partial\zeta\partial\zeta}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\,. \tag{84}$$

[8] Although we use the same symbol, our variable α_B differs from that introduced in [26] by a factor -2. This simplifies the subsequent equations.

Eq. (86)	M^2	α_M	α_K	α_B	α_T	α_H
Eq. (59)	$2\mathcal{A}_K$	$\frac{1}{H}\frac{d}{dt}\ln\mathcal{A}_K$	$\frac{2L_N+L_{NN}}{2H^2\mathcal{A}_K}$	$\frac{\mathcal{B}}{4H\mathcal{A}_K}$	$\frac{\mathcal{G}^*}{\mathcal{A}_K}-1$	$\frac{\mathcal{G}^*+\mathcal{B}_R^*}{\mathcal{A}_K}-1$
Eq. (12) of [1]	$2L_S$	$\frac{1}{H}\frac{d}{dt}\ln L_S$	$\frac{2L_N+L_{NN}}{2H^2L_S}$	$\frac{2HL_{SN}+L_{KN}}{4HL_S}$	$\frac{L_R^*}{L_S}-1$	$\frac{L_R^*+L_{NR}^*}{L_S}-1$
Eq. (30) Eq. (87)	$M_*^2f+2m_4^2$	$\frac{M_*^2\dot{f}+2(m_4^2)^{\cdot}}{M^2H}$	$\frac{2c+4M_2^4}{M^2H^2}$	$\frac{M_*^2\dot{f}-m_3^3}{2M^2H}$	$-\frac{2m_4^2}{M^2}$	$\frac{2(\tilde{m}_4^2-m_4^2)}{M^2}$

Table 1: In the first row, the parameters α_i introduced in eqs. (67), (68), (78) and (82), i.e. the Lagrangian coefficients of eq. (86). These parameters are written in terms of the Lagrangian coefficients of eq. (59), defined in eqs. (51)–(54) (second row), of the coefficients introduced in [1], where the derivative of the Lagrangian L with respect to N, K, $S=K_{ij}K^{ij}$, R, $Z\equiv R_{ij}R^{ij}$ and $Y\equiv R_{ij}K^{ij}$ (third row) and, finally, of the EFT Lagrangian, action (87) (fourth row). All these quantities are understood to be evaluated on the background, with $\bar{N}=1$.

When adding matter to the dark energy Lagrangian, the kinetic and spatial gradient terms of the scalar fluctuations acquire new contributions that modify the expression for the sound speed [19, 20]. The final expression for the sound speed, when matter is present, reads

$$c_s^2=-2\frac{(1+\alpha_B)^2}{\alpha}\left\{1+\alpha_T-\frac{1+\alpha_H}{1+\alpha_B}\left(1+\alpha_M-\frac{\dot{H}}{H^2}\right)-\frac{1}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{1+\alpha_B}\right)\right\}-\frac{(1+\alpha_H)^2}{\alpha}\frac{\rho_{\rm m}+p_{\rm m}}{M^2H^2}\,. \tag{85}$$

In the simple case of k-essence field with a Lagrangian $P(\phi,X)$, where all α_i coefficients vanish except $\alpha_K=(2\bar{X}P_X+4\bar{X}^2P_{XX})/(M^2H^2)$, the above formula yields $c_s^2=-2\dot{H}/(\alpha_K H^2)-(\rho_{\rm m}+p_{\rm m})/(\alpha_K M^2H^2)$ and one recovers $c_s^2=P_X/(P_X+2\bar{X}P_{XX})$ after using the Friedmann equation $\dot{H}=-(2\bar{X}P_X+\rho_{\rm m}+p_{\rm m})/(2M^2)$.

3.3 Link with the building blocks of dark energy

In the previous subsection, we have focussed our attention on Lagrangians that satisfy the conditions (76) in order to get propagation equations with no more than two (space) derivatives. At quadratic order, the most general action of the form (59) that satisfies these conditions can be written in the form

$$\begin{aligned}S^{(2)}=\;&\int d^3x dt a^3\,\frac{M^2}{2}\bigg[\delta K_{ij}\delta K^{ij}-\delta K^2+(1+\alpha_T)\left(R\frac{\delta\sqrt{h}}{a^3}+\delta_2 R\right)\\&+\alpha_K H^2\delta N^2+4\alpha_B H\,\delta K\,\delta N+(1+\alpha_H)R\,\delta N\bigg]\,,\end{aligned} \tag{86}$$

where, for convenience, we summarize in Table 1 the definitions of the parameters α_i introduced in the previous subsection, in terms of the original coefficients defined in Sec. 3.2 (second row) and those introduced explicitly in Ref. [1] (third row).

The action leading to the quadratic Lagrangian (86) can also be written in the standard EFT form, with an explicit dependence on the four-dimensional scalar curvature, g^{00} and several quadratic operators. This action reads [1]

$$\begin{aligned}S=\int d^4x\sqrt{-g}\,\bigg[&\frac{M_*^2}{2}f(t)^{(4)}R-\Lambda(t)-c(t)g^{00}+\frac{M_2^4(t)}{2}(\delta g^{00})^2-\frac{m_3^3(t)}{2}\,\delta K\delta g^{00}\\&-m_4^2(t)\left(\delta K^2-\delta K^\mu{}_\nu\,\delta K^\nu{}_\mu\right)+\frac{\tilde{m}_4^2(t)}{2}\,R\,\delta g^{00}\bigg]\,.\end{aligned} \tag{87}$$

It leads to the background equations of motion [1]

$$c + \Lambda = 3M_*^2(fH^2 + \dot{f}H) - \rho_{\rm m} \,, \tag{88}$$

$$\Lambda - c = M_*^2(2f\dot{H} + 3fH^2 + 2\dot{f}H + \ddot{f}) + p_{\rm m} \,, \tag{89}$$

and to the quadratic action for the linear perturbations (86), where the relation between the coefficients α_i and the seven parameters appearing in (87) is given in Table 1. The two background equations of motion (88) and (89) imply that only five of the EFT parameters are independent, thus setting the minimal number of functions parametrizing deviations from General Relativity [1].

3.4 Disformal transformations and dependence on $\dot{N}$

In our discussion, we have assumed that the initial Lagrangian depends on N, but not on its time derivative $\dot{N}$. Allowing a dependence on $\dot{N}$ leads in general to an additional propagating degree of freedom. However, this is not always the case, as illustrated by considering disformal transformations of the metric, originally introduced in [40], of the form

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(\phi, X)\, g_{\mu\nu} + \Gamma(\phi, X)\, \partial_\mu\phi\, \partial_\nu\phi \,. \tag{90}$$

As shown in [41], Horneski theories are invariant under a restricted class of disformal transformations where Ω and Γ depend on ϕ only, not on X. In [20], we showed explicitly that one could use disformal transformations with an X-dependent function Γ to relate subclasses of theories beyond Horndeski to Horndeski theories. A similar result for a disformal transformation of the Einstein-Hilbert Lagrangian was previously obtained in [42].

In unitary gauge, Ω and Γ become functions of the time variable t and of the lapse function N. By choosing time to coincide with ϕ, i.e. $\partial_\mu\phi = \delta^0_\mu$, the disformal transformation (90) corresponds, in the ADM language, to the transformations [20]

$$\tilde{N}^i = N^i \,, \qquad \tilde{h}_{ij} = \Omega^2(t, N)\, h_{ij} \,, \qquad \tilde{N}^2 = \Omega^2(t, N)\, N^2 - \Gamma\,(t, N) \,. \tag{91}$$

Moreover, the relations between the old and new curvature tensors are given by

$$\tilde{R} = \Omega^{-2}\left[R - 4D^2 \ln\Omega - 2\partial_i(\ln\Omega)\partial^i(\ln\Omega)\right] \,, \tag{92}$$

and

$$\tilde{K}^j_{\ i} \;=\; \frac{N}{\tilde{N}}\left[K^j_{\ i} \;-\; N g^{0\mu}\partial_\mu \ln\Omega\, \delta^j_{\ i}\right] . \tag{93}$$

The last relation can be expanded into

$$\tilde{K}^j_{\ i} \;=\; \frac{N}{\tilde{N}}\left[K^j_{\ i} + \frac{1}{N\Omega}\left(\Omega_t + \Omega_N\big(\dot{N} - N^i\partial_i N\big)\right)\, \delta^j_{\ i}\right] . \tag{94}$$

Consequently, a Lagrangian that depends initially on tilded quantities, will finally depend on $\dot{N}$ when reexpressed in terms of untilded quantities. The quadratic Lagrangian will now depend on $\delta\dot{N}$, in addition to all the terms discussed previously. However, according to (74) and (94), one sees that $\delta\dot{N}$ will always appear associated with $\dot{\zeta}$ in the combination

$$\dot{\zeta} + \frac{\Omega_N}{\tilde{N}\Omega}\delta\dot{N} \,, \tag{95}$$

which implies that the matrix of the kinetic coefficients is degenerate. Thus, one can introduce a new degree of freedom

$$\zeta_{\rm new} = \zeta + \frac{\Omega_N}{\tilde{N}\Omega}\delta N \,, \tag{96}$$

which absorbs all time derivatives of δN. Contrarily to what could have been expected, the explicit dependence on $\dot{N}$ does not lead, in this particular case, to an extra degree of freedom.

4 Evolution of the cosmological perturbations

In this section we follow [1] and derive the evolution equations for linear scalar perturbations described by the action (86), together with some matter field minimally coupled to the metric $g_{\mu\nu}$. We first restore the general covariance of the action and write it in a generic coordinate system. In order to do so, we perform the time diffeomorphism [43, 2, 3]

$$t \to t + \pi(t, \vec{x}) \,, \tag{97}$$

where π describes the fluctuations of the scalar degree of freedom. Under this time diffeomorphism, any function of time f changes up to second order as

$$f \to f + \dot{f}\pi + \frac{1}{2}\ddot{f}\pi^2 + \mathcal{O}(\pi^3) \,, \tag{98}$$

while the metric component $g^{00} = -1/N^2$ exactly transforms as

$$g^{00} \to g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi \,. \tag{99}$$

For the other perturbed geometric quantities, we only need their change at linear order in π, i.e. [1]

$$\delta K_{ij} \to \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i\partial_j\pi + \mathcal{O}(\pi^2) \,, \tag{100}$$

$$\delta K \to \delta K - 3\dot{H}\pi - \frac{1}{a^2}\partial^2\pi + \mathcal{O}(\pi^2) \,, \tag{101}$$

$$R_{ij} \to R_{ij} + H(\partial_i\partial_j\pi + \delta_{ij}\partial^2\pi) + \mathcal{O}(\pi^2) \,, \tag{102}$$

$$R \to R + \frac{4}{a^2}H\partial^2\pi + \mathcal{O}(\pi^2) \,. \tag{103}$$

We stress that in the above expressions K_{ij} and R_{ij} respectively denote the extrinsic and intrinsic curvature on hypersurfaces of constant time, even when we are *not* in unitary gauge. Therefore they are not the same geometrical quantities *before* and *after* the change of time.

We can then expand the covariant action up to quadratic order, considering a linearly perturbed FLRW metric. Varying the action with respect to the four scalar perturbations in the metric and the scalar fluctuation π we obtain five scalar equations; see Ref. [1] for details on their derivations. We turn to a discussion of these equations restricting to Newtonian gauge.

4.1 Perturbation equations in Newtonian gauge

We assume a perturbed FLRW metric in Newtonian gauge with only scalar perturbations, i.e.,

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j \,. \tag{104}$$

The metric perturbations Φ and Ψ and the scalar fluctuation π are related to the metric perturbations in unitary gauge defined in eq. (73) by

$$\Phi = \delta N + (a^2\psi)^{\cdot} \,, \qquad \Psi = -\zeta - a^2 H\psi \,, \qquad \pi = a^2\psi \,. \tag{105}$$

Moreover, in this gauge we decompose the total matter stress-energy tensor at linear order as

$$T^0_{\ 0} \equiv -(\rho_{\rm m} + \delta\rho_{\rm m}) \,, \tag{106}$$

$$T^0_{\ i} \equiv \partial_i q_{\rm m} \equiv (\rho_{\rm m} + p_{\rm m})\partial_i v_{\rm m} = -a^2 T^i_{\ 0} \,, \tag{107}$$

$$T^i_{\ j} \equiv (p_{\rm m} + \delta p_{\rm m})\delta^i_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\sigma_{\rm m} \,, \tag{108}$$

where $\delta\rho_{\rm m}$ and $\delta p_{\rm m}$ are the energy density and pressure perturbations, $q_{\rm m}$ and $v_{\rm m}$ are respectively the 3-momentum and the 3-velocity potentials; $\sigma_{\rm m}$ is the anisotropic stress potential.

The Hamiltonian constraint ((00) component of the Einstein equation) is

$$\begin{aligned} & 6(1+\alpha_B)H\dot{\Psi} + (6-\alpha_K+12\alpha_B)H^2\Phi + 2(1+\alpha_H)\frac{k^2}{a^2}\Psi \\ & + (\alpha_K - 6\alpha_B)\, H^2\dot{\pi} + 6\left[(1+\alpha_B)\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + \frac{1}{3}\frac{k^2}{a^2}(\alpha_H-\alpha_B)\right] H\pi = -\frac{\delta\rho_{\rm m}}{M^2} \,, \end{aligned} \tag{109}$$

while the momentum constraint ((0i) components of the Einstein equation) reads

$$2\dot{\Psi} + 2(1+\alpha_B)H\Phi - 2H\alpha_B\dot{\pi} + \left(2\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{M^2}\right)\pi = -\frac{(\rho_{\rm m}+p_{\rm m})v_{\rm m}}{M^2} \,. \tag{110}$$

The traceless part of the ij components of the Einstein equation gives

$$(1+\alpha_H)\Phi - (1+\alpha_T)\Psi + (\alpha_M-\alpha_T)H\pi - \alpha_H\dot{\pi} = -\frac{\sigma_{\rm m}}{M^2} \,, \tag{111}$$

while the trace of the same components gives, using the equation above,

$$\begin{aligned} & 2\ddot{\Psi} + 2(3+\alpha_M)H\dot{\Psi} + 2(1+\alpha_B)H\dot{\Phi} \\ & + 2\left[\dot{H} - \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + (\alpha_B H)^{\cdot} + (3+\alpha_M)(1+\alpha_B)H^2\right]\Phi \\ & - 2H\alpha_B\,\ddot{\pi} + 2\left[\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} - (\alpha_B H)^{\cdot} - (3+\alpha_M)\alpha_B H^2\right]\dot{\pi} \\ & + 2\left[(3+\alpha_M)H\dot{H} + \frac{\dot{p}_{\rm m}}{2M^2} + \ddot{H}\right]\pi = \frac{1}{M^2}\left(\delta p_{\rm m} - \frac{2}{3}\frac{k^2}{a^2}\sigma_{\rm m}\right) \,. \end{aligned} \tag{112}$$

The evolution equation for π reads

$$\begin{aligned} & H^2\alpha_K\ddot{\pi} + \left\{\left[H^2(3+\alpha_M)+\dot{H}\right]\alpha_K + (H\alpha_K)^{\cdot}\right\}H\dot{\pi} \\ & + 6\left\{\left(\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right)\dot{H} + \dot{H}\alpha_B\left[H^2(3+\alpha_M)+\dot{H}\right] + H(\dot{H}\alpha_B)^{\cdot}\right\}\pi \\ & - 2\frac{k^2}{a^2}\left\{\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + H^2\left[1+\alpha_B(1+\alpha_M)+\alpha_T-(1+\alpha_H)(1+\alpha_M)\right] + (H(\alpha_B-\alpha_H))^{\cdot}\right\}\pi \\ & + 6H\alpha_B\ddot{\Psi} + H^2(6\alpha_B-\alpha_K)\dot{\Phi} + 6\left[\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + H^2\alpha_B(3+\alpha_M) + (\alpha_B H)^{\cdot}\right]\dot{\Psi} \\ & + \left[6\left(\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2}\right) + H^2(6\alpha_B-\alpha_K)(3+\alpha_M) + 2(9\alpha_B-\alpha_K)\dot{H} + H(6\dot{\alpha}_B-\dot{\alpha}_K)\right]H\Phi \\ & + 2\frac{k^2}{a^2}\left\{\alpha_H\dot{\Psi} + \left[H(\alpha_M+\alpha_H(1+\alpha_M)-\alpha_T)-\dot{\alpha}_H\right]\Psi + (\alpha_H-\alpha_B)H\Phi\right\} = 0 \,. \end{aligned} \tag{113}$$

These equations have been previously derived in [1] in terms of the effective field theory parameters. Restricting to Horndeski theories ($\alpha_H=0$), they have been also obtained in [7] and later reproduced in [26], where the notation used here was introduced. Note that as a consequence of the parameterization where M^2 is in factor of the full gravitational Lagrangian (79), matter quantities always appear divided by a M^2 factor. In Appendix A we discuss the long wavelength behaviour of these equations for adiabatic initial conditions; in Appendix B we write these equations in synchronous gauge and conformal time, which is the coordinate system usually employed in CMB codes.

4.2 Fluid description

It is sometimes convenient to describe the dark energy, both in the background and perturbative equations, as an effective fluid. In order to do so, we define the background energy density and pressure for dark energy, respectively, as

$$\rho_D \equiv 3M^2H^2 - \rho_{\rm m} \;, \qquad p_D \equiv -M^2(2\dot H + 3H^2) - p_{\rm m} \;. \tag{114}$$

These are simply derived quantities that can be computed once the evolution of the expansion history, the matter content and the effective Planck mass M are known. With these definitions, and using the conservation of the background matter stress-energy tensor,

$$\dot\rho_{\rm m} + 3H(\rho_{\rm m} + p_{\rm m}) = 0 \;, \tag{115}$$

the conservation of the background dark energy stress-energy tensor reads

$$\dot\rho_D = -3H(\rho_D + p_D) + 3\alpha_M M^2 H^3 = 3H(\rho_{\rm m} + p_{\rm m}) + 6M^2H(\dot H + \alpha_M H^2) \;. \tag{116}$$

Another useful relation that one can use to express $\dot p_D$ in terms of matter and geometry is

$$\dot p_D = -\dot p_{\rm m} - M^2\big[2\ddot H + 2H\dot H(3+\alpha_M) + 3\alpha_M H^3\big] \;, \tag{117}$$

which can be derived from the equations above.

Equations (109)–(112) can then be rewritten in the usual form,

$$\frac{k^2}{a^2}\Psi + 3H\big(\dot\Psi + H\Phi\big) = -\frac{1}{2M^2}\sum_I \delta\rho_I \;, \tag{118}$$

$$\dot\Psi + H\Phi = -\frac{1}{2M^2}\sum_I q_I \;, \tag{119}$$

$$\Psi - \Phi = \frac{1}{M^2}\sum_I \sigma_I \;, \tag{120}$$

$$\ddot\Psi + H\dot\Phi + 2\dot H\Phi + 3H\big(\dot\Psi + H\Phi\big) = \frac{1}{2M^2}\sum_I\left(\delta p_I - \frac{2}{3}\frac{k^2}{a^2}\sigma_I\right) \;, \tag{121}$$

where the sum is over the matter and the dark energy components. These equations implicitely define the quantities $\delta\rho_D$, q_D, δp_D and σ_D as the energy density perturbation, momentum, pressure perturbation and anisotropic stress of the dark energy fluid. An explicit definition is given in Newtonian gauge in Appendix A and in synchronous gauge in Appendix B.

With these definitions, one can verify that the evolution equation for π, eq. (113), is equivalent to a conservation equation of the dark energy fluid quantities,

$$\delta\dot\rho_D + 3H(\delta\rho_D + \delta p_D) - 3(\rho_D + p_D)\dot\Psi - \frac{k^2}{a^2}q_D = \alpha_M H\sum_I \delta\rho_I \;. \tag{122}$$

The Euler equation,

$$\dot q_D + 3Hq_D + (\rho_D + p_D)\Phi + \delta p_D - \frac{2}{3}\frac{k^2}{a^2}\sigma_D = \alpha_M H\sum_I q_I \;, \tag{123}$$

is identically satisfied by the definitions of q_D, δp_D and σ_D. Conservation of matter in the Jordan frame implies a continuity and Euler equations for matter with vanishing right-hand side.

To close the system, one needs to provide an equation of state for dark energy or, at least, a relation between δp_D and σ_D in terms of $\delta\rho_D$, q_D and the other matter variables. In order to do so in the simpler case where $\alpha_H = 0$, we solve eqs. (109)–(111) for Ψ, $\dot\Psi$ and $\dot\pi$ and then we plug these solutions in eqs. (118) and (119) to express π and Φ in terms of $\delta\rho_{\rm m}$, $q_{\rm m}$, $\sigma_{\rm m}$, $\delta\rho_D$ and q_D. $\dot\Phi$ is obtained from the first derivative of (111). To obtain $\ddot\Psi$ and $\ddot\pi$ we use eqs. (112) and (113). Combining all these solutions we can finally express σ_D and δp_D in terms of the other fluid variables. We obtain

$$\begin{aligned}\delta p_D = & \frac{\gamma_1\gamma_2+\gamma_3\alpha_B^2\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_1\gamma_4+\gamma_5\alpha_B^2\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}Hq_D \\ & + \gamma_7(\delta\rho_{\rm m}-3Hq_{\rm m}) + \frac{\gamma_1\gamma_6+3\gamma_7\alpha_B^2\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}Hq_{\rm m} - \frac{6\alpha_B^2}{\alpha}\delta p_{\rm m}\,, \end{aligned} \tag{124}$$

$$\begin{aligned}\sigma_D = & \frac{a^2}{2k^2}\left[\frac{\gamma_1\alpha_T+\gamma_8\alpha_B^2\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}(\delta\rho_D-3Hq_D) + \frac{\gamma_9\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}Hq_D \right. \\ & \left. + \alpha_T(\delta\rho_{\rm m}-3Hq_{\rm m}) + \frac{\gamma_{10}\tilde k^2}{\gamma_1+\alpha_B^2\tilde k^2}Hq_{\rm m}\right]\,, \end{aligned} \tag{125}$$

where we use the notation $\tilde k \equiv k/(aH)$ and we have introduced dimensionless coefficients γ_a, whose expressions are explicitly given in Appendix C. These relations for δp_D and σ_D are derived, to our knowledge, for the first time and represent the most general description of dark energy in the context of Horndeski theories. In particular, eqs. (124) and (125) extend the equations of state for perturbations derived in Refs. [45, 10].

One can check that for adiabatic initial conditions, i.e.

$$\pi \approx -(\dot\Psi + H\Phi)/\dot H\,, \qquad \delta\rho_{\rm m} \approx \dot\rho_{\rm m}\pi\,, \qquad \delta p_{\rm m} \approx \dot p_{\rm m}\pi\,, \qquad v_{\rm m} \approx -\pi\,, \tag{126}$$

where the symbol $\approx$ denotes equality in the long wavelength limit $\tilde k \ll 1$, the dark energy equation of state satisfies

$$\begin{aligned} \delta\rho_D &\approx -3H(\rho_D+p_D)\pi \qquad & \delta p_D &\approx -\dot p_D\pi\,, \\ q_D &\approx -(\rho_D+p_D)\pi\,, \qquad & \sigma_D &\approx -\alpha_T M^2\Psi - (\alpha_T-\alpha_M)M^2H\pi\,, \end{aligned} \tag{127}$$

which is what expected from the equations of motion in Sec. 4.1, see discussion in Appendix A.

Going back to arbitrary scales, let us discuss two illustrative examples.

- $\alpha_B = 0$: there is no braiding and the kinetic term of scalar fluctuations depends on α_K only, $\alpha = \alpha_K$. In this case eqs. (124) and (125) reduce to

$$\begin{aligned}\delta p_D = & \,\tilde c_s^2(\delta\rho_D-3Hq_D) - \left[\frac{\dot p_D + 3\alpha_M H^3M^2}{\rho_D+p_D} + H(\alpha_T-\alpha_M)\left(1-\frac{2M^2}{3(\rho_D+p_D)}\frac{k^2}{a^2}\right)\right]q_D \\ & + \frac{\alpha_T}{3}\delta\rho_{\rm tot} - \frac{\alpha_K}{6}(\alpha_T-\alpha_M)Hq_{\rm m}\,, \end{aligned}\tag{128}$$

$$\sigma_D = -\,\alpha_T M^2\Psi + H(\alpha_T-\alpha_M)\frac{M^2}{\rho_D+p_D}q_D\,, \tag{129}$$

  where we have used eqs. (118), (119) and $\tilde c_s^2 \equiv c_s^2 - 2(\alpha_T-\alpha_M)/\alpha_K = (\rho_D+p_D)/(\alpha_K H^2M^2)$, $\delta\rho_{\rm tot} \equiv \delta\rho_{\rm m} + \delta\rho_D$. For $\alpha_T = 0 = \alpha_M$ we recover the standard k-essence pressure perturbation [44] and no anisotropic stress. For $\alpha_T \neq 0$ or $\alpha_T - \alpha_M \neq 0$, the dark energy anisotropic stress is nonzero and simply given in terms of the total curvature Ψ and the dark energy momentum q_D. Note that the term containing k^2 in the pressure perturbation δp_D cancels from the combination $\delta p_D - (2k^2/3a^2)\sigma_D$, which appears as a source in the Euler equation and in the evolution equation for Ψ.

- $\alpha_B^2 \gg \alpha_K$: braiding dominates the time kinetic term, $\alpha \simeq 6\alpha_B^2$. However, one needs $\alpha_B \lesssim 1$ to avoid gradient instabilities [4]. In this case, from the definition of γ_1, eq. (184), we have $\gamma_1 \simeq -3\alpha_B^2 \dot{H}/H^2$ so that, if we concentrate on sub-horizon scales, $k \gg aH$, eqs. (124) and (125) reduce to

$$\begin{aligned}
\delta p_D = & \left(c_s^2 + \frac{\alpha_T}{3} + \frac{\xi}{3} - \frac{2\dot{H} + \ddot{H}/H - \xi \dot{H}}{\alpha_B} \frac{a^2}{k^2} \right) (\delta\rho_D - 3Hq_D) - (1+\xi)\, Hq_D \\
& - 3\frac{\dot{H}a^2}{k^2} \xi H q_{\rm m} - (1+\xi)\, \frac{\delta\rho_{\rm m}}{3} - \delta p_{\rm m} \, , \qquad (130)\\
\sigma_D = & \, \xi \frac{a^2}{k^2} \left[\frac{1}{2} (\delta\rho_D - 3Hq_D) + \frac{3}{2} H\alpha_B q_{\rm tot} \right] \, , \qquad (131)
\end{aligned}$$

  where $\xi \equiv (\alpha_T - \alpha_M)/\alpha_B$ and $q_{\rm tot} \equiv q_{\rm m} + q_D$. As expected in this case [46], the behavior of dark energy is very different from that of a perfect fluid. In particular, for $\tilde{k}^2 \lesssim \gamma_2/\gamma_3$ the relation between pressure and density perturbations is scale dependent. For $\xi \neq 0$, the anisotropic stress is nonzero and has a scale dependence that differs from the $\alpha_B = 0$ case discussed above.

4.3 Interface with the observations

In Sec. 4.1 we have described the full set of evolution equations including the standard matter species directly using the scalar fluctuation π. These equations can be solved in a modified Boltzmann code; for instance, they have been recently implemented in a code in [47, 48]. Alternatively, in Sec. 4.2 we have rewritten these equations in terms of dark energy fluid quantities and we have provided the full equations of state for Horndeski theories ($\alpha_H = 0$), eqs. (124) and (125). In this approach, the equations of state fully encode the description of dark energy.

To discuss more easily the relation with late time observations we can use the Einstein equations in the fluid form, eqs. (118)–(121), and rewrite these two equations as an evolution equation for the gravitational potential Ψ and a relation between Ψ and Φ. For simplicity, we restrict again our discussion to the case $\alpha_H = 0$. To do that we can first combine eqs. (118)–(120) to solve for Φ, $\delta\rho_D$ and q_D in terms of Ψ, $\dot{\Psi}$, σ_D and the matter field. Moreover, we can use eq. (121) to express δp_D as a function of the other quantities. Using these relations, it is straighforward to show that eqs. (124) and (125) are equivalent to a dynamical equation for the gravitational potential Ψ, sourced by the matter fields,[9]

$$\begin{aligned}
\ddot{\Psi} + & \frac{\beta_1\beta_2 + \beta_3\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\tilde{k}^2} H\dot{\Psi} + \frac{\beta_1\beta_4 + \beta_1\beta_5\,\tilde{k}^2 + c_s^2\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2\Psi = -\frac{1}{2M^2} \left[\frac{\beta_1\beta_6 + \beta_7\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2} \delta\rho_{\rm m} \right.\\
& \left. + \frac{\beta_1\beta_8 + \beta_9\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2} Hq_{\rm m} + \frac{\beta_1\beta_{10} + \beta_1\beta_{11}\,\tilde{k}^2 + \frac{2}{3}\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2\sigma_{\rm m} - \frac{\alpha_K}{\alpha}\delta p_{\rm m} - 2H\dot{\sigma}_{\rm m} \right] \, ,
\end{aligned} \qquad (132)$$

where the dimensionless parameters β_a are explicitly given in Appendix C, and a relation between Φ and Ψ, involving $\dot{\Psi}$ and the matter fields,

$$\begin{aligned}
& \alpha_B^2\tilde{k}^2 \left[\Phi - \Psi \left(1 + \alpha_T + \frac{2\gamma_9}{\alpha\alpha_B} \right) + \frac{\sigma_{\rm m}}{M^2} \right] + \beta_1 \left[\Phi - \Psi(1+\alpha_T)\frac{\gamma_1}{\beta_1} + \frac{\sigma_{\rm m}}{M^2} \right] = \\
& \frac{\gamma_9}{H^2M^2} \left[\frac{\alpha_B}{\alpha} (\delta\rho_{\rm m} - 3Hq_{\rm m}) + HM^2\,\dot{\Psi} + H\frac{\alpha_K}{2\alpha}\, q_{\rm m} - H^2\,\sigma_{\rm m} \right] \, .
\end{aligned} \qquad (133)$$

[9]An alternative derivation of eq. (132) is to combine eqs. (109)–(111) to solve for π, $\dot{\pi}$ and Φ in terms of Ψ, $\dot{\Psi}$ and the matter field. We can then use the time derivative of (111) to solve for $\dot{\Phi}$ and the scalar field equation (113) to solve for $\ddot{\pi}$. Using these solutions, it is possible to eliminate the scalar field fluctuations and derive (132) [26]. We can then derive eq. (133) from (111).

Combined with the evolution equations for matter, these equations form a close system. They generalize those given in [26], which we recover for $\delta p_{\rm m} = 0 = \sigma_{\rm m}$. Following [26], the parameter β_1 appears in eq. (132) to make explicit the existence of a transition scale in the dynamics, $k_B \equiv aH\beta_1^{1/2}/\alpha_B$, which has been called *braiding scale*. Here we find that for $\alpha_T \neq \alpha_M$ this scale is different from the transition scale $aH\gamma_1^{1/2}/\alpha_B$ appearing in eqs. (124) and (125). In particular, β_1 is related to γ_1 by

$$\beta_1 = \gamma_1 - \gamma_9 \;, \tag{134}$$

(see the explicit definition in terms of the α_i in eq. (198)). Note that eq. (133) displays both scales.

Let us consider again the two limits discussed before (see also [26]).

- $\alpha_B = 0$: In this case most of the scale dependences go away. We are left with the simpler expression

$$\ddot\Psi + (4 + 2\alpha_M + 3\Upsilon)\, H\dot\Psi + \left(\beta_4 H^2 + c_s^2 \frac{k^2}{a^2}\right)\Psi = \\ -\frac{1}{2M^2}\left[c_s^2(\delta\rho_{\rm m} - 3Hq_{\rm m}) + (\alpha_M - \alpha_T + 3\Upsilon)Hq_{\rm m} + \left(\beta_{10}H^2 + \frac{2k^2}{3a^2}\right)\sigma_{\rm m} - \delta p_{\rm m} + 2H\dot\sigma_{\rm m}\right] . \tag{135}$$

  Although both α_M and α_T can be nonzero here, the form of this equation is very similar to that obtained in the standard k-essence case.

- $\alpha_B^2 \gg \alpha_K$: For simplicity we consider only the case $\alpha_T = 0$. Moreover, to avoid negative gradient instabilities we require $\alpha_B \lesssim \mathcal{O}(1)$ [4]. However, no such a restriction is imposed on α_M, whose value can affect the braiding scale. Indeed, when $\alpha_B^2 \gg \alpha_K$, this is given by

$$\frac{k_B^2}{a^2} \simeq 3(H^2\alpha_M - \dot H) \;. \tag{136}$$

  Considering modes with $k \gg k_B$, eq. (132) simplifies to

$$\ddot\Psi + (3+\alpha_M)H\dot\Psi + \left(\frac{k_B^2\beta_5}{a^2} + c_s^2\frac{k^2}{a^2}\right)\Psi \simeq -\frac{1}{2M^2}\left(\frac{k_B^2\beta_6}{k^2} + c_s^2 + \frac{1}{3} - \frac{\alpha_M}{3\alpha_B}\right)\delta\rho_{\rm m} \;, \tag{137}$$

  where we have neglected relativistic terms on the right hand side of (132). The mass scale $k_B^2\beta_5/a^2$ corresponds to the so-called Compton mass. Depending on the value of β_5, this scale may be inside the horizon and induce a transition on the behaviour of the effective Newton constant, which is considered a strong signal of modified gravity.

To make the link with observations without resorting to numerical calculations, one often relies on the quasi static approximation, which corresponds to neglecting time derivatives with respect to spatial ones on scales much below the sound horizon, i.e. for $k \gg aH/c_s$. In this regime, modifications of gravity can be captured by two quantities, the effective Newton constant $G_{\rm eff}$, defined by

$$-\frac{k^2}{a^2}\Phi = 4\pi G_{\rm eff}\delta\rho_{\rm m} \;, \tag{138}$$

and the gravitational slip $\gamma \equiv \Psi/\Phi$.

Both these quantities can be computed using eqs. (132) and (133). However, as discussed in [26] this does not give the same result as neglecting time derivatives in eq. (113) and using eqs. (109) and (111) to derive $G_{\rm eff}$ and γ. The two procedures are consistent if k is much larger than the other scales, i.e. in the limit $k \to \infty$. In this case we recover (compare for instance with the results of [1] in the same limit)

$$8\pi G_{\rm eff} = \frac{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2}{\alpha\, c_s^2}\, M^{-2} \;, \tag{139}$$

$$\gamma = \frac{\alpha\, c_s^2 + 2\alpha_B\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]}{\alpha\, c_s^2(1+\alpha_T) + 2\left[\alpha_B(1+\alpha_T) + \alpha_T - \alpha_M\right]^2} \;, \tag{140}$$

where we have expressed both quantities directly in terms of the functions α_i (recall that $\alpha = \alpha_K + 6\alpha_B^2$ and α_H is here set to zero), obtained from the derivatives of the initial ADM Lagrangian.

5 Conclusions

In this article, we have presented a very general approach to parametrize theoretically motivated deviations from the ΛCDM standard model. This approach combines several advantages, both from the theoretical and observational points of view. On the one hand, it provides a unified treatment of theoretical models, using as a starting point a Lagrangian expressed in terms of ADM geometrical quantities defined for a foliation of uniform scalar field hypersurfaces. On the other hand, it expresses all the relevant information about the linear cosmological perturbations in terms of a minimal set of five time-dependent functions, which can be constrained by observations. These five functions, together with the background time evolution (and a constant parameter), are sufficient to fully characterize the background and linear perturbations, within the large class of models we have considered (corresponding to the conditions (76) to avoid a non trivial dispersion relation for the scalar mode).

The link between these two endpoints, theoretical and observational, is direct since the five functions correspond to combinations of the derivatives of the initial Lagrangian. One can thus automatically derive the observational predictions for any existing or novel model by computing these functions from the ADM Lagrangian. Conversely, one can use this approach in a model-independent way by trying to constrain the five arbitrary functions (this requires some parametrization of these free functions expressed for instance in terms of the redshift; see e.g. [11]) with observations. Of course, since the bounds on parameters are less stringent as the number of parameters increases, it would be interesting to analyse the data with a scale of increasing complexity, corresponding to the number of free functions, thus covering the range from the simplest theory, i.e. ΛCDM, where all functions are zero, to more and more general theories.

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A Superhorizon evolution

In this appendix we extend the arguments of [49, 50] and check that the Einstein equations of Sec. 4.1 satisfy the usual adiabatic solution on superhorizon scales. To this end, it is convenient to define the quantities

$$\mathcal{P} \equiv M^2(\dot\pi - \Phi) \;, \qquad \mathcal{Q} \equiv M^2(\dot\Psi + H\Phi + \dot H\pi) \;, \qquad \mathcal{R} \equiv M^2(\Psi + H\pi) \;, \tag{141}$$

(note that $Q = \dot{\mathcal{R}} - H(\mathcal{P} + \alpha_M \mathcal{R})$) and rewrite the Einstein equations (109)–(112) respectively as

$$
\begin{aligned}
&-2\frac{k^2}{a^2}\left[(1+\alpha_H)\mathcal{R} - (1+\alpha_B)M^2 H\pi\right] \\
&\quad -6H(1+\alpha_B)\mathcal{Q} - H^2(\alpha_K - 6\alpha_B)\mathcal{P} = \delta\rho_{\rm m} - \dot{\rho}_{\rm m}\pi \ , \qquad (142) \\
&\qquad\qquad -2\mathcal{Q} + 2\alpha_B H\mathcal{P} = (\rho_{\rm m} + p_{\rm m})(v_{\rm m} + \pi) \ , \qquad (143) \\
&M^2(\Psi - \Phi) + \alpha_T \mathcal{R} - \alpha_M H\pi + \alpha_H \mathcal{P} = \sigma_{\rm m} \ , \qquad (144) \\
&2\dot{\mathcal{Q}} + 6H\mathcal{Q} + 2\left(\frac{\rho_{\rm m} + p_{\rm m}}{2M^2} - 3\alpha_B H^2\right)\mathcal{P} - 2(H\alpha_B\mathcal{P})^{\cdot} = \delta p_{\rm m} - \dot{p}_{\rm m}\pi - \frac{2}{3}\frac{k^2}{a^2}\sigma_{\rm m} \ . \qquad (145)
\end{aligned}
$$

The evolution of π reads

$$
\begin{aligned}
&(H^2\alpha_K\mathcal{P})^{\cdot} + 6(H\alpha_B\mathcal{Q})^{\cdot} + 3H(\alpha_K H^2 - 2\alpha_B\dot{H})\mathcal{P} + 36\left(\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + 3H^2\alpha_B\right)\mathcal{Q} \\
&+ \frac{k^2}{a^2}\Bigg\{2\alpha_H\mathcal{Q} + 2\Big[H\alpha_H + (M^2\alpha_H)^{\cdot}M^{-2} + H\alpha_M - H\alpha_T\Big]\mathcal{R} \qquad (146) \\
&- 2\left[\dot{H} + \frac{\rho_{\rm m}+p_{\rm m}}{2M^2} + H^2\alpha_B + (M^2 H\alpha_B)^{\cdot}M^{-2}\right]M^2\pi - 2M^2 H\alpha_B\Phi\Bigg\} = 0 \ .
\end{aligned}
$$

Moreover, in terms of these quantites, the definitions of the fluid variables introduced in eqs. (118)–(121) are given by

$$
\begin{aligned}
\delta\rho_D &\equiv 2\frac{k^2}{a^2}\big(\alpha_H\mathcal{R} - \alpha_B M^2 H\pi\big) - 3H\big[(\rho_D + p_D)\pi - 2\alpha_B\mathcal{Q}\big] + H^2(\alpha_K - 6\alpha_B)\mathcal{P} \ , \qquad (147) \\
q_D &\equiv -2\alpha_B H\mathcal{P} - (\rho_D + p_D)\pi \ , \qquad (148) \\
\sigma_D &\equiv \alpha_M M^2 H\pi - \alpha_T\mathcal{R} - \alpha_H\mathcal{P} \ , \qquad (149) \\
\delta p_D &\equiv \big[\dot{p}_D + \alpha_M H M^2(2\dot{H} + 3H^2)\big]\pi - 2\alpha_M H\mathcal{Q} \\
&\quad + \left(\frac{\rho_D + p_D}{M^2} + 6\alpha_B H^2\right)\mathcal{P} + 2\big(\alpha_B H\mathcal{P}\big)^{\cdot} + \frac{2}{3}\frac{k^2}{a^2}\sigma_D \ . \qquad (150)
\end{aligned}
$$

Independently of the constituents of the Universe, the $k = 0$ mode of the field equations for scalar fluctuations in Newtonian gauge is invariant under the coordinate transformation [49]

$$
\begin{aligned}
t &\to t + \epsilon(t) \ , \qquad (151) \\
x^i &\to x^i(1 - \lambda) \ , \qquad (152)
\end{aligned}
$$

where ϵ is an arbitrary infinitesimal function of time and λ an arbitrary infinitesimal constant. In particular, using these transformations one can start from an unperturbed FLRW solution and generate a solution in Newtonian gauge with metric perturbations

$$
\Psi = H\epsilon - \lambda \ , \qquad \Phi = -\dot{\epsilon} \ , \qquad (153)
$$

and, assuming the Universe filled by several fluids and scalar fields, with matter perturbations

$$
\delta\rho_X = -\dot{\rho}_X\epsilon \ , \qquad \delta\varphi_X = -\dot{\varphi}_X\epsilon \ . \qquad (154)
$$

Note that these solutions remain valid also if individual matter components are not separately conserved [49], as in the case of dark energy components that are non-minimally coupled to gravity.

Let us check that this is solution of the above equations. For our dark energy and matter components, eq. (154) becomes

$$
\delta\rho_{\rm m} = -\dot{\rho}_{\rm m}\epsilon \ , \qquad \pi = -\epsilon \ . \qquad (155)
$$

This second equality, together with eq. (153), implies that $\mathcal{P} = 0$ and $\mathcal{Q} = 0$ from which it follows that eqs. (142), (145) and the evolution equation for π, eq. (146), are satisfied for $k = 0$.

The remaining equations, i.e. (143) and (144), are automatically satisfied because they multiply an overall factor of k and k^2, respectively, that has been dropped here. However, for the solutions (153) and (154) to be physical we must require that these equations are satisfied for finite k in the $k \to 0$ limit [49], which implies

$$v_{\rm m} = -\pi \,, \tag{156}$$

and

$$(M^2\epsilon)^{\cdot} + HM^2\epsilon - \sigma_{\rm m} = M^2(1+\alpha_T)\lambda \,, \tag{157}$$

with solution

$$\epsilon = \frac{1}{M^2 a}\int^t a\left[M^2(1+\alpha_T)\lambda + \sigma_{\rm m}\right] dt' \,. \tag{158}$$

Equations (153) and (155), with the conditions (156) and (158) correspond to the well-known super-horizon adiabatic solution. One can define the quantity ζ from the metric perturbation as

$$\zeta_{\rm tot} \equiv -\Psi + H\frac{\dot\Psi + H\Phi}{\dot H} \,, \tag{159}$$

which is known to be conserved in the $k \to 0$ limit for adiabatic perturbations [51]. Indeed, one can replace the solutions (153) in its definition and check that in this limit it coincides with the constant λ in eq. (152),

$$\zeta_{\rm tot} = \lambda = -\mathcal{R}M^{-2} \,, \qquad k \to 0 \,. \tag{160}$$

We note that on super-Hubble scales and for adiabatic initial conditions, eq. (126), eqs. (132)-(133) reduce to the conservation of the total comoving curvature perturbation, i.e., $\dot\zeta_{\rm tot} \approx 0$.

B Evolution equations in synchronous gauge

For completeness, in this section we give the perturbation equations in synchronous gauge. In this gauge, the perturbed FLRW metric including scalar perturbations reads

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \,, \tag{161}$$

with

$$h_{ij} \equiv \frac{1}{3}h\delta_{ij} + \left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)(h + 6\eta) \,. \tag{162}$$

Using gauge transformations (see for instance [52]), we can express the variables in Newtonian gauge into those in synchronous gauge, which yields

$$\begin{aligned}
\pi^{(N)} &= \pi^{(S)} + \delta t \,, \\
\Phi &= \delta\dot t \,, \\
\Psi &= \eta - H\delta t \,, \\
\delta\rho_{\rm m}^{(N)} &= \delta\rho_{\rm m}^{(S)} - 3H(\rho_{\rm m} + p_{\rm m})\delta t \,, \\
\delta p_{\rm m}^{(N)} &= \delta p_{\rm m}^{(S)} + \dot p_{\rm m}\delta t \,, \\
v_{\rm m}^{(N)} &= -\frac{\theta_{\rm m}^{(S)}}{k^2} - \delta t \,,
\end{aligned} \tag{163}$$

with

$$\delta t \equiv \frac{a^2}{k^2}\left(\dot h + 6\dot\eta\right) \,, \tag{164}$$

and where we have introduced the divergence of the velocity, $\theta \equiv \vec{\nabla} \cdot \vec{v}$, related to the velocity potential, in Fourier space, by $\theta = -k^2 v$. The anisotropic stress is gauge invariant.

We can now apply these gauge transformations to eqs. (109)–(112). We will use conformal time, $\tau \equiv \int dt/a$, which is usually employed in numerical codes. Using this time, it is convenient to rescale the scalar fluctuation π and the velocity divergence θ by the conformal factor and redefine

$$\pi \to \pi/a\,, \quad \theta \to \theta/a\,. \tag{165}$$

By denoting by a prime the derivative with respect to conformal time, the Einstein equations in synchronous gauge read ((00) component)

$$\begin{aligned} &2k^2(1+\alpha_H)\eta - \mathcal{H}(1+\alpha_B)h' - \mathcal{H}^2(6\alpha_B - \alpha_K)\pi' \\ &+ \mathcal{H}\left[2k^2(\alpha_H - \alpha_B) + \mathcal{H}^2(\alpha_K - 12\alpha_B) + 6\mathcal{H}'\alpha_B\right]\pi = -\frac{a^2}{M^2}\left[\delta\rho_{\rm m} - 3\mathcal{H}(\rho_D + p_D)\pi\right]\,, \end{aligned} \tag{166}$$

((0i) component)

$$\eta' - \mathcal{H}\alpha_B\pi' - \mathcal{H}^2\alpha_B\pi = \frac{a^2}{2M^2}\left[(\rho_{\rm m} + p_{\rm m})\theta_{\rm m}/k^2 + (\rho_D + p_D)\pi\right]\,, \tag{167}$$

((ij)-traceless)

$$h'' + 6\eta'' + \mathcal{H}(2+\alpha_M)(h' + 6\eta') - 2k^2(1+\alpha_T)\eta - 2k^2\alpha_H\pi' - 2k^2\mathcal{H}\left(\alpha_H + \alpha_T - \alpha_M\right) = -\frac{2k^2}{M^2}\sigma_{\rm m}\,, \tag{168}$$

and ((ii)-trace)

$$\begin{aligned} &h'' + \mathcal{H}(2+\alpha_M)h' - 2k^2(1+\alpha_T)\eta + 6\alpha_B\pi'' + 2\left[3\mathcal{H}^2\alpha_B(3+\alpha_M) + (3\alpha_B\mathcal{H})' - k^2\alpha_H\right]\pi' \\ &\left\{3\mathcal{H}^2[3\alpha_M + 2\alpha_B(2+\alpha_M)] + 6\alpha_B\mathcal{H}' + 6(\alpha_B\mathcal{H})' - 2k^2(\alpha_H + \alpha_T - \alpha_M)\right\}\mathcal{H}\pi \\ &= \frac{a^2}{M^2}\left[-3(\rho_D + p_D)(\pi' + \mathcal{H}\pi) - 3p_D'\pi - 3\delta p_{\rm m} - 2\frac{k^2}{a^2}\sigma_{\rm m}\right]\,, \end{aligned} \tag{169}$$

where

$$\rho_D + p_D = -\rho_{\rm m} - p_{\rm m} - 2\frac{M^2}{a^2}\left(\mathcal{H}' - \mathcal{H}^2\right)\,, \qquad \mathcal{H} \equiv \frac{a'}{a}\,. \tag{170}$$

The evolution equation for the scalar fluctuation reads

$$\begin{aligned} &-\mathcal{H}^2\alpha_K\pi'' - \left[\mathcal{H}^2\alpha_K(2+\alpha_M) + \mathcal{H}'\alpha_K + (\alpha_K\mathcal{H})'\right]\mathcal{H}\pi' \\ &+ 2k^2\left\{H^2\left[(\alpha_B - \alpha_{\mathcal{H}})\alpha_M + \alpha_T - \alpha_M\right] + \left[(\alpha_B - \alpha_{\mathcal{H}})\mathcal{H}\right]' - \frac{a^2}{2M^2}(\rho_D + p_D)\right\}\pi \\ &+ \left\{\mathcal{H}^4[6\alpha_B\alpha_M - \alpha_K(1+\alpha_M)] - 3\mathcal{H}^2\mathcal{H}'[\alpha_K - 2\alpha_B(3-\alpha_M)]\right. \\ &\left. - 6\alpha_B\mathcal{H}'^2 + \mathcal{H}^3(6\alpha_B' - \alpha_K') - 6\mathcal{H}(\alpha_B\mathcal{H}')' - \frac{3a^2}{2M^2}(\rho_D + p_D)(\mathcal{H}^2 - \mathcal{H}')\right\}\pi \\ &+ \mathcal{H}\alpha_B h'' - 2k^2\alpha_H\eta' + \left[\mathcal{H}^2\alpha_B(1+\alpha_M) + (\alpha_B\mathcal{H})' - \frac{a^2}{2M^2}(\rho_D + p_D)\right]h' \\ &- 2k^2\left\{\mathcal{H}[\alpha_M + \alpha_H(1+\alpha_M) - \alpha_T] + \alpha_{\mathcal{H}}'\right\}\eta = 0\,. \end{aligned} \tag{171}$$

Moreover, we can write these equations in terms of fluid quantities,

$$k^2\eta - \frac{1}{2}\mathcal{H}h' = -\frac{a^2}{2M^2}\sum_I \delta\rho_I \,, \tag{172}$$

$$k^2\eta' = \frac{a^2}{2M^2}\sum_I (\rho_I + p_I)\theta_I \,, \tag{173}$$

$$h'' + 6\eta'' + 2\mathcal{H}(h' + 6\eta') - 2k^2\eta = -\frac{2k^2}{M^2}\sum_I \sigma_I \,, \tag{174}$$

$$h'' + 2\mathcal{H}h' - 2k^2\eta = -\frac{3}{M^2}\sum_I \left(\delta p_I + \frac{2}{3}\frac{k^2}{a^2}\sigma_I\right) \,, \tag{175}$$

where we have defined

$$\begin{aligned}\delta\rho_D \equiv \frac{M^2}{a^2}\Big\{&2k^2\alpha_H\eta - \mathcal{H}\alpha_B h' + \mathcal{H}^2(\alpha_K - 6\alpha_B)\pi' + \Big[-2k^2(\alpha_B - \alpha_{\mathcal{H}}) + 6\mathcal{H}'\alpha_B \\ &+ \mathcal{H}^2(\alpha_K - 12\alpha_B) - 3\frac{a^2}{M^2}(\rho_D + p_D)\Big]\mathcal{H}\pi\Big\} \,, \end{aligned} \tag{176}$$

$$\theta_D \equiv \frac{k^2M^2}{a^2(\rho_D + p_D)}\left\{2\mathcal{H}\alpha_B\pi' + \left[2\mathcal{H}^2 + \frac{a^2}{M^2}(\rho_D + p_D)\right]\pi\right\} \,, \tag{177}$$

$$\sigma_D \equiv -\frac{M^2}{k^2}\left[k^2\eta\alpha_T - \frac{\mathcal{H}\alpha_M}{2}\left(h' + 6\eta'\right) + k^2\alpha_H\pi' + k^2\left(\alpha_H + \alpha_T - \alpha_M\right)\mathcal{H}\pi\right] \,, \tag{178}$$

$$\begin{aligned}\delta p_D \equiv \frac{2}{3}\frac{k^2}{a^2}\sigma_D^S + \frac{M^2}{a^2}\Big\{&-2\mathcal{H}\alpha_M\eta' + 2\mathcal{H}\alpha_B\pi'' + \Big[2[\mathcal{H}^2\alpha_B(3 + \alpha_M) + (\alpha_B\mathcal{H})'] + \frac{a^2}{M^2}(\rho_D + p_D)\Big]\pi' \\ &+ \Big[\mathcal{H}^3\left[3\alpha_M + 2\alpha_B(2 + \alpha_M)\right] + 2\mathcal{H}\mathcal{H}'\alpha_B + 2\mathcal{H}(\alpha_B\mathcal{H})' + \frac{a^2}{M^2}\left[\mathcal{H}(\rho_D + p_D) + p_D'\right]\Big]\pi\Big\} \,. \end{aligned} \tag{179}$$

The equation for π is equivalent to the continuity equation in conformal synchronous gauge, namely

$$\delta\rho_D' + 3\mathcal{H}(\delta\rho_D + \delta p_D) + (\rho_D + p_D)\left(\theta_D + \frac{h'}{2}\right) = \mathcal{H}\alpha_M\sum_I \delta\rho_I \,, \tag{180}$$

and the Euler equation

$$\theta_D' + \mathcal{H}\left[1 + \frac{p_D'}{\mathcal{H}(\rho_D + p_D)} + \alpha_M\sum_I \frac{\rho_I}{\rho_D + p_D}\right]\theta_D + \frac{1}{\rho_D + p_D}\left(\delta p_D - \frac{2}{3}\frac{k^2}{a^2}\sigma_D\right) = \mathcal{H}\alpha_M\sum_I \frac{\rho_I + p_I}{\rho_D + p_D}\theta_I \,, \tag{181}$$

is an identity just as in the Newtonian gauge case.

C Scale dependence and definitions of the parameters

We report here the two "equations of state" for the dark energy fluid, eqs. (124) and (125),

$$\begin{aligned}\delta p_D = &\frac{\gamma_1\gamma_2 + \gamma_3\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_1\gamma_4 + \gamma_5\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D \\ &+ \gamma_7(\delta\rho_{\rm m} - 3Hq_{\rm m}) + \frac{\gamma_1\gamma_6 + 3\gamma_7\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_{\rm m} - \frac{6\alpha_B^2}{\alpha}\delta p_{\rm m}\,, \end{aligned} \tag{182}$$

$$\begin{aligned}\sigma_D = &\frac{a^2}{2k^2}\left[\frac{\gamma_1\alpha_T + \gamma_8\alpha_B^2\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}(\delta\rho_D - 3Hq_D) + \frac{\gamma_9\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_D\right. \\ &\left. + \alpha_T(\delta\rho_{\rm m} - 3Hq_{\rm m}) + \frac{\gamma_{10}\tilde{k}^2}{\gamma_1 + \alpha_B^2\tilde{k}^2}Hq_{\rm m}\right]\,, \end{aligned} \tag{183}$$

and provide the definitions of the parameters γ_a, for which we have assumed $\alpha_H = 0$:

$$\gamma_1 \equiv \alpha_K\frac{\rho_D + p_D}{4H^2M^2} - 3\alpha_B^2\frac{\dot{H}}{H^2}\,, \tag{184}$$

$$\gamma_2 \equiv c_s^2 + \frac{\alpha_T}{3} - 2\frac{\alpha_B(2+\Gamma) + (1+\alpha_B)(\alpha_M - \alpha_T)}{\alpha}\,, \tag{185}$$

$$\gamma_3 \equiv c_s^2 + \frac{\gamma_8}{3}\,, \tag{186}$$

$$\begin{aligned}\gamma_4 \equiv &\frac{1}{\rho_D + p_D}\left\{ -\dot{p}_D/H + \alpha_M\left[\rho_D + p_D - 3H^2M^2\right]\right. \\ &\left. + 6\frac{\alpha_B^2}{\alpha}\Big[(3 + \alpha_M + \Gamma)(\rho_{\rm m} + p_{\rm m}) - \dot{p}_{\rm m}/H\Big]\right\}\,, \end{aligned} \tag{187}$$

$$\gamma_5 \equiv -1 - \frac{(6\alpha_B - \alpha_K)(\alpha_T - \alpha_M)}{6\alpha_B^2} + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \tag{188}$$

$$\gamma_6 \equiv -6\alpha_B^2\frac{2+\Gamma}{\alpha} + \frac{\alpha_K\alpha_M - 6\alpha_B^2}{\alpha}\,, \tag{189}$$

$$\gamma_7 \equiv \frac{\alpha_K\alpha_M - 6\alpha_B^2}{3\alpha} - \frac{(6\alpha_B - \alpha_K)(\alpha_T - \alpha_M)}{3\alpha}\,, \tag{190}$$

$$\gamma_8 \equiv \alpha_T + \frac{\alpha_T - \alpha_M}{\alpha_B}\,, \tag{191}$$

$$\gamma_9 \equiv \alpha\frac{\alpha_T - \alpha_M}{2}\,, \tag{192}$$

$$\gamma_{10} \equiv 3\alpha_B^2(\alpha_T - \alpha_M)\,, \tag{193}$$

where

$$\gamma_1\Gamma \equiv \frac{\alpha_K}{4H^2M^2}\left[(3+\alpha_M)(\rho_{\rm m} + p_{\rm m}) - \dot{p}_{\rm m}/H - \frac{\alpha_B^2(\rho_D + p_D)}{\alpha_K H}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot}\right] - \alpha\frac{\ddot{H}}{2H^3}\,, \tag{194}$$

and

$$c_s^2 = -\frac{2(1+\alpha_B)\Big[\dot{H} - (\alpha_M - \alpha_T)H^2 + H^2\alpha_B(1+\alpha_T)\Big] + 2H\dot{\alpha}_B + (\rho_{\rm m} + p_{\rm m})/M^2}{H^2\alpha}\,. \tag{195}$$

As explained in Sec. 4.3, the equations of state are equivalent to the dynamical equation for Ψ, eq. (132) and eq. (133),

$$
\ddot{\Psi} + \frac{\beta_1\beta_2 + \beta_3\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\tilde{k}^2} H\dot{\Psi} + \frac{\beta_1\beta_4 + \beta_1\beta_5\,\tilde{k}^2 + c_s^2\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2\Psi = -\frac{1}{2M^2}\Bigg[\frac{\beta_1\beta_6 + \beta_7\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2}\delta\rho_{\rm m}
$$

$$
+ \frac{\beta_1\beta_8 + \beta_9\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H q_{\rm m} + \frac{\beta_1\beta_{10} + \beta_1\beta_{11}\,\tilde{k}^2 + \frac{2}{3}\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2\sigma_{\rm m} - \frac{\alpha_K}{\alpha}\delta p_{\rm m} - 2H\dot{\sigma}_{\rm m}\Bigg] \tag{196}
$$

$$
\alpha_B^2\tilde{k}^2\left[\Phi - \Psi\left(1 + \alpha_T + \frac{2\gamma_9}{\alpha\alpha_B}\right) + \frac{\sigma_{\rm m}}{M^2}\right] + \beta_1\left[\Phi - \Psi(1+\alpha_T)\frac{\gamma_1}{\beta_1} + \frac{\sigma_{\rm m}}{M^2}\right] =
$$

$$
\frac{\gamma_9}{H^2M^2}\left[\frac{\alpha_B}{\alpha}\left(\delta\rho_{\rm m} - 3Hq_{\rm m}\right) + HM^2\,\dot{\Psi} + H\frac{\alpha_K}{2\alpha}\,q_{\rm m} - H^2\,\sigma_{\rm m}\right] . \tag{197}
$$

Here the parameters β_a, for which we have assumed again $\alpha_H = 0$, are:[10]

$$
\beta_1 \equiv \gamma_1 - \gamma_9 = -\alpha_K\frac{\rho_{\rm m} + p_{\rm m}}{4H^2M^2} - \frac{1}{2}\alpha\left(\frac{\dot{H}}{H^2} + \alpha_T - \alpha_M\right) , \tag{198}
$$

$$
\beta_2 \equiv 2(2+\alpha_M) + 3\Upsilon , \tag{199}
$$

$$
\beta_3 \equiv 3 + \alpha_M + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} , \tag{200}
$$

$$
\beta_4 \equiv (1+\alpha_T)\big[2\dot{H}/H^2 + 3(1+\Upsilon) + \alpha_M\big] + \dot{\alpha}_T/H , \tag{201}
$$

$$
\beta_5 \equiv c_s^2 - \frac{2\alpha_B(\beta_3 - \beta_2)}{\alpha} + \frac{\alpha_B^2}{\beta_1}(1+\alpha_T)(\beta_3 - \beta_2) + \frac{\alpha_B^2\beta_4}{\beta_1} , \tag{202}
$$

$$
\beta_6 \equiv \beta_7 - 2\frac{\alpha_B(\beta_3 - \beta_2)}{\alpha} , \tag{203}
$$

$$
\beta_7 \equiv c_s^2 + 2\frac{\alpha_B^2(1+\alpha_T) + \alpha_B(\alpha_T - \alpha_M)}{\alpha} , \tag{204}
$$

$$
\beta_8 \equiv \beta_9 - \frac{(\alpha_K - 6\alpha_B)(\beta_3 - \beta_2)}{\alpha} , \tag{205}
$$

$$
\beta_9 \equiv -(1 + 3c_s^2 + \alpha_T) + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} , \tag{206}
$$

$$
\beta_{10} \equiv -6(1+\Upsilon) - 4\dot{H}/H^2 , \tag{207}
$$

$$
\beta_{11} \equiv \frac{2}{3} - 2\frac{\alpha_B^2}{\beta_1}\big[(2-\alpha_M) + 2\dot{H}/H^2\big] - 2\frac{\alpha_B^4}{\beta_1 H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} , \tag{208}
$$

with

$$
\begin{aligned}
12\beta_1 H^3M^2\Upsilon \equiv\ & 2\alpha M^2\Big\{\big[\dot{H} + (\alpha_T - \alpha_M)H^2\big]^{\cdot} + (3+\alpha_M)H\big[\dot{H} + (\alpha_T - \alpha_M)H^2\big]\Big\} \\
& + \alpha_K\dot{p}_{\rm m} - (\rho_{\rm m} + p_{\rm m})H(\alpha_K - 6\alpha_B)(\alpha_T - \alpha_M) + 6(\rho_{\rm m} + p_{\rm m})\frac{\alpha_B^4}{\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} .
\end{aligned} \tag{209}
$$

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[10] These parameters do not exactly correspond to those introduced in [26]. Indeed, here the β_a and Υ have been made dimensionless by dividing by the appropriate power of H and, because of the different definition of α_B, we have divided β_1 by a factor 4 so that ${\beta_1}^{(\rm here)} = \beta_1^{(\rm there)}/(4H^2)$. Moreover, we have corrected minor typos in the definitions of Υ and β_7. We thank the authors of Ref. [26] for having checked and privately agreed on these corrections.

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