Three-Point Functions in $\mathcal{N} = 4$ Super-Yang-Mills Theory From *Integrability*

Subtle is the Lord, but malicious He is not.

—Albert Einstein

Abstract

Cette thèse est dédiée à l'étude de la fonction à trois points dans la théorie de jauge super-symétrique (SYM) N=4, dans la limite du grand nombre de couleurs, à l'aide de l'intégrabilité. La théorie de jauge N=4 SYM est invariante conforme au niveau quantique est on pense qu'elle est résoluble exactement. Par la correspondance AdS/CFT, elle est duale à la théorie des cordes de type IIB dans l'espace courbe $Ads_5 \times S^5$. Les fonctions à trois points sont des quantités qui contiennent de l'information essentielle sur la dynamique de la théorie.

Nous passons en revue les méthodes déjà existantes et outils de l'intégrabilité qui sont nécessaires pour le calcul de la fonction à trois points. Nous présentons le calcul de la fonction à trois points dans le secteur $su(3)$, de rang supérieur à un, nous avons utilisé une représentation sous forme de déterminant, qui nous permets de prendre la limite semi-classique. En exploitant la relation entre des chaines de spin à langue portée et la chaine de Heisenberg inhomogène, nous avons développé une nouvelle pur calculer la fonction à trois points dans le secteur $su(2)$ à l'ordre d'une boucle qui nous permets d'obtenir le résultat dans une forme très compacte. Dans la limite de Frolov-Tseytlin ce résultat est en accord avec celui qu'on obtient au couplage fort.

Nous avons exploré des nouvelles formulations de la fonction à trois points. En nous inspirant de la formulation de la théorie des champs des cordes dans la jauge du cône de lumière nous avons construit un vertex de spin, qui est la version de couplage faible du vertex des cordes, pour tous les secteurs à l'ordre des arbres. Cette approche peut être reliée au programme des facteurs de forme pour les théories de champs bi-dimensionnelles intégrables, dont nous rappelons ici les bases. Nous étudions la dépendance dans la taille du système pour une classe spéciale de fonction à trois points qui correspond aux facteurs de forme diagonaux.

Abstract

This thesis is devoted to the study of three-point functions of $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory in the planar limit by using integrability. $\mathcal{N} = 4$ SYM theory is conformal invariant at quantum level and is believed to be completely solvable. By the AdS/CFT correspondence, it is dual to the type IIB superstring theory on the curved background $\mathrm{AdS}_5 \times S^5$. The three-point functions are important quantities which contain essential dynamic information of the theory.

The necessary tools in integrability and the existing methods of computing three-point functions are reviewed. We compute the three-point functions in the higher rank $su(3)$ sector and obtain a determinant representation for one special configuration, which allows us to take the semi-classical limit. By exploring the relation between long-range interacting spin chain and inhomogeneous XXX spin chain, we develop a new approach to compute three-point functions in the $su(2)$ sector at one-loop and obtain a compact result. In the Frolov-Tseytlin limit, this result matches the result at strong coupling.

We also explore new formulations of the three-point functions. In one formulation inspired by the light-cone string field theory, we constructed the spin vertex, which is the weak coupling counterpart of the string vertex for all sectors at tree level. Another formulation which is related to the form factor boostrap program in integrable field theory is reviewed. At weak coupling, we study the finite volume dependence of a special type of three-point functions which are related to the diagonal form factors.

Acknowledgements

First of all, I am deeply indebted to my supervisors, Ivan Kostov and Didina Serban. In the past three years, we worked like a family. I'm very grateful for all the patient guidance, wise advices both in science and in life, countless help in many aspects and the great tradition of the group, namely the progress cakes. They are the best supervisors that I can imagine.

I would like to thank the members of the defense committee: Jesper Jacobsen, Romuald Janik, Jean-Michel Maillet and Kostya Zarembo, especially Romuald and Kostya who read the manuscript carefully and offered valuable comments.

None of the works presented in this dissertation will be possible without my collaborators, from whom I've learned a lot: Omar Foda, Laszlo Hollo, Shota Komatsu, Florian Loebbert, Andrei Petrovskii and Gang Yang.

I'm lucky enough to be surrounded by a brilliant community. I benefited from numerous discussions with Benjamin Basso, Betrand Eynard, Jesper Jacobsen, Volodya Kazakov, Grisha Korchemsky, Vincent Pasquier, Sylvain Ribault and Hubert Saleur in the parisien area and Zoltan Bajnok, Joao Caetano, Kolya Gromov, Romuald Janik, Shota Komatsu, Sergei Lukyanov, Yutaka Matsuo, Tristan Mclaughlin, Pedro Vieira and Kostya Zarembo outside France. Special thanks to Kolya and Pedro who organize the nice Mathematica school every year, which helped me a lot.

I have to thank the French administration system which never fails to surprise me and has made me a much more patient person than before. However, the kind help from the secretaries of Saclay made things much easier for me and I thank them sincerely.

Very special thanks go to Professor Édouard Brézin, Professor Jean Illiopoulos and Professor Nikolas Sourlas who made everything possible for me to come to France at the very beginning.

Last but definitely not least, I am indebted to my best friend, my love of life and my dear wife Lu Yang, to whom this dissertation is dedicated, for coming to a foreign country because of me, for tolerating my endless travels, for taking care of our family and for the understanding and support all these years.

Yunfeng Jiang
Paris, July 2015.

Contents

Dedicated to Lu Yang

Chapter 1

Introduction

1.1 Integrability in gauge-string duality

The relation between gauge theory and string theory is an old and new idea. Its origin goes back to the 1970s when 'tHooft [1] observed that $U(N_c)$ gauge theory in the large N_c limit exhibits a genus-like expansion, which resembles very much the expansion in string theory. Although intuitively quite tempting, it turns out to be highly non-trivial to formulate such duality exactly. The first exact formulation was given by the so-called AdS/CFT correspondence proposed in 1998 by Maldacena [2], see also [3, 4]. Since then, it has been one of the main topics in theoretical physics for the last two decades and continue being one of the most important and active research area. The reason is that the two sides of the duality are related to two 'holy grail' in modern theoretical physics. On the gauge theory side it is the understanding of quantum field theory at strong coupling when the perturbation theory fails. On the string theory side is the quantization of strings on a curved background, which is related to the quantum theory of gravity. Since this duality is of strong/weak type, namely the strongly interacting regime on one side is dual to the weakly interacting regime on the other, the hope is to apply the ideas and technique on one side to tackle the problem of the other side.

On the other hand, the fact that the AdS/CFT duality is of strong/weak type makes it notoriously hard to prove. Around 2002, two breakthough were made which improved the situation largely. The first one was made by Berenstein, Maldacena and Nastase [5] who proposed to study a special limit of the duality. On the string theory side, this corresponds to taking the Penrose limit of the worldsheet theory. The quantization of superstring theory can be performed readily in the resultant pp-wave background. On the gauge theory side, it amounts to study operators with large R-charges. This shifts the attention from computing symmetry protected quantities which do not receive

quantum corrections to the study of non-protected and more interesting quantities. At the same time, Minahan and Zarembo [6] found that the one-loop dilatation operator in the scalar sector of the planar $\mathcal{N} = 4$ SYM theory is equivalent to an integrable spin chain. This discovery creates the possibility of applying powerful integrability methods such as Bethe ansatz to study the spectrum of the quantum field theory. The integrability method is particularly powerful for the heavy operators when direct diagonalization of mixing matrix becomes impossible. The discovery of Minhan and Zarembo was quickly generalized to larger sectors and higher loops. It is also shown that its AdS dual, the Type-IIB superstring theory in $\mathrm{AdS}_5 \times S^5$ is also integrable [7]. After a decade of intense investigation, it is now commonly accepted that the planar $\mathcal{N} = 4$ SYM theory is integrable. We will give a brief review of the ideas and history of the solution of spectral problem in section 1.2.

Integrability essentially implies the theory is completely solvable. In our context, 'solving' the theory does not necessary mean that we can write any quantity of the theory in terms of elementary functions, not even special functions. It means that we can write down a set of compact algebraic or functional equations, the solution of which gives the quantity at any coupling. Integrability usually occurs for models in 1 or 2 dimensions, such as 1 dimensional quantum spin chains and 2 dimensional quantum field theories. The secret behind the integrability of $\mathcal{N} = 4$ SYM, which is a 4-dimensional interacting quantum field theory lies in its huge amount of symmetry. Apart from the usual Lorentz invariance, this theory is also conformal invariant even at the quantum level due to the vanishing of its beta function. The theory also enjoys maximal supersymmetry in 4 dimensions, which uplifts the conformal symmetry into superconformal symmetry. In this sense, it is the simplest possible non-trivial 4d QFT which serve as an excellent playground for theoretical studies. Impressive progress in recent years have strengthened the hope to solve this theory completely and compute all the interesting quantities in a non-perturbative way. Like the solution of harnomic oscillators and 2d Ising model it is expected the solution of $\mathcal{N} = 4$ SYM theory will lead to new insights and understandings about QFT in 4d and the AdS/CFT in an exact manner.

It is also worth mentioning that the benefit is mutual. The study of $\mathcal{N} = 4$ SYM raises many intriguing questions for integrability itself. In the AdS/CFT context, usually we compute quantities on both the gauge theory side and the string theory side. The gauge theory side involves technique from quantum integrable system which are usually discrete. On the other hand, the computation done in the string theory side is mostly based on the technique of integrable field theory and depend heavily on classical integrability. Amazingly, these two seemingly very different methods gives the result of the same object at the weak and strong coupling limit. This reveals a deep relation between different integrable systems. In addition, many integrable system manifest themselves in certain

limit of $\mathcal{N}=4$ SYM theory, such as the Heisenberg spin chain, the Hubbard model, the non-linear sigma models, just to name a few. The $\mathcal{N}=4$ SYM theory seems to be a theory which encompass many known integrable systems and probably some unknown ones. Many of the hidden integrable structures are still await to be disclosed.

1.2 The spectral problem

In this section we offer a brief review of the developments in spectral problem of planar $\mathcal{N}=4$ SYM theory for the last decade. Apart from the original research papers, many useful reviews exists in the literature. The collection of review articles [8] is the most comprehensive one, for a shorter version, one can read Serban [9]. Beisert's Ph.D thesis [10] contains many early developments which are still valuable and important. For the string theory side, we refer to the review by Arutyunov and Frolov [11]. In order to simplify the narration, we divide the development of spectral problem into two stages [1]: **1**. The asymptotic Bethe ansatz (ABA), **2**. The finite volume corrections.

1.2.1 The asymptotic Bethe ansatz

After the work of Minahan and Zarembo who realized that the one-loop dilatation operator in the $SO(6)$ sector can be mapped to the $SO(6)$ integrable spin chain, the mapping was soon generalized to the whole sectors [12, 13] and higher loops in some sectors. Beisert, Kristjansen and Staudacher conjectured that the dilatation operator is integrable for all loops [14]. The interaction range of the dilatation operator grows order by order in perturbation theory and its explicit form soon becomes too complicated to compute or even write down. In addition, at higher loops the dilatation operator changes the length of the spin chain, which is a novel feature of spin chains in AdS/CFT. Trying to establish integrability for all loops by determining dilatation operator seems to be extremely hard. On the other hand, integrability basically allows one to write down a set of algebraic equations, called the Bethe equations. The solution of Bethe equations give the spectrum of the spin chain. Therefore, as long as the spectrum is concerned, it is most relevant to find the Bethe equations. If there's alternative way to determine the correct set of Bethe equations, it is not compulsory to find the dilatation operator itself.

The all-loop Bethe ansatz equation. In this direction, Staudacher [15] emphasis the relation between the asymptotic Bethe ansatz and the scattering matrix (S-matrix) of

1. Of course such a division is very rough and reveals my personal point of view. I apologize if I overlooked the contributions of some colleagues, which are mainly due to the limited volume of the thesis and my ignorance.

the magnons. The S-matrix bootstrap method is a well established subject in integrable theories starting from the work by Zamolodchikov [16], it is powerful enough to determine the S-matrix non-perturbatively up to some scalar factor.

In the meantime, Kazakov, Minahan, Marshakov and Zarembo [17] studied the continuum limit of the Bethe ansatz equations and write it in the language of algebraic curves. The similar algebraic curve structure appeared at strong coupling for the finite gap solution of $\mathfrak{su}(2)$ principle chiral field theory. Written in the same language, it becomes possible to compare directly the predictions of sigma models and spin chains. The results was soon extended to other sectors [18, 19] and finally to the full $\mathfrak{psu}(2,2|4)$ sector [20].

Bases on the work of KMMZ, the all-loop Bethe equation was first conjectured by Beisert, Dippel and Staudacher [21] and Beisert and Staudacher [22]. In the seminal paper [23], Beisert managed to derive the all-loop Bethe ansatz equation by assuming the excitations of the spin chain is subjected to the centrally extended $\mathfrak{su}(2|2) \times \mathfrak{su}(2|2)$ symmetry, which shows the power of the symmetry.

The determination of dressing phase. It was found by Arutyunov, Frolov, Staudacher that the BDS proposal is not the complete answer [24], the scalar factor alluded before plays an important role. This factor is called the *dressing phase*. The dressing phase was first determined at strong coupling at the leading order by the same authors [24]. The complete one-loop answer was derived by Hernandez and Lopez [25]. Notably, by using the method from quantum group, Janik [26] found an equation that should be satisfied by the dressing phase, which is equivalent to the crossing equations in the 2d relativistic quantum field theories. A solution of the crossing equation was found by Beisert, Hernandez and Lopez, which was confirmed at weak coupling by Beisert, Eden, Staudacher in [27] and at strong coupling by Dorey, Hofman and Maldacena [28]. Later, Volin found the minimal solution to the crossing equation and found that it indeed reproduces the BHL/BES dressing phase. The dressing phase was confirmed by many non-trivial checks, among which the most impressive is the interpolation of cusp anomalous dimension from weak coupling to strong coupling [29, 30]. The BDS all-loop Bethe ansatz equation, together with the dressing phase provide us the so-called asymptotic Bethe ansatz.

The finite volume corrections. Although impressive and powerful, the asymptotic Bethe ansatz has its limits. It inherently depends on the picture of particle scattering and hence is subjected to the limit where the length of the spin chain is very large. Since the so-called wrapping corrections have to be taken into account, the ABA fails to provide the correct result. It is thus an important question to find methods to take

into account the finite size corrections. Fortunately, this problem has been studied in the theory of integrable models before, which offers valuable guiding principles.

One method which is able to take into account the first few wrapping corrections was proposed by Lüscher [31]. This method was generalized to AdS/CFT by Janik and Lukowski [32]. The simplest operator in computing finite size corrections is the Konishi operator, whose anomalous dimension is determined by Lüscher's method up to five loops [33, 34]. The comparison to traditional Feynmann diagram computations confirms the correctness of the result.

Another commonly used method is the Thermodynamic Bethe Ansatz (TBA) initiated by Al. Zamolodchikov [35]. This method proves to be more powerful in AdS/CFT. It is first suggested by Ambjorn, Janik and Kristjansen [36] that TBA could be used to compute the finite volume corrections in AdS/CFT. The systematic implementation was done by several groups [37–40]. This method allows to reproduce the 4 [39] and 5 [41, 42] loop anomalous dimensions for the Konishi operator which was obtained by Lüscher's method before.

In principle, all the finite volume dependence of anomalous dimension is encoded in the set of TBA equations. However, the TBA is a set of coupled integral equations which is highly involved to solve even numerically. The equation cannot be solved in general and only for several cases one is able to find a solution. It takes serious effort to reduce the TBA into a compact and elegant form. In [39], it was already realized that the TBA equations are equivalent to a set of non-linear functional equations called the Y-system, together with the analytic constraints from original TBA [43]. Later, Gromov, Kazakov, Leurent and Volin explored the relation between Y-system and the bilinear Hirota equations [44], which are themselves integrable. This enables them to reduce the infinite set of equations into a finite set of non-linear integral equations (FiNLIE). The most advanced and elegant rewritting is the recently proposed Quantum Sepectral Curve (QSC)[45, 46], or the $\mathbf{P}\mu$ system, which reduced the infinite set of TBA equations to a Riemann-Hilbert problem of eight Q functions. The QSC method can be seen as an elegant generalization of the classical spectral curve in the sense that the latter is a WKB like approximation of the former. The QSC is not only elegant but is also powerful enough to obtain all the aforementioned results with much less effort. It depends on ones perspective to call QSC a final solution of the spectral problem in $\mathrm{AdS}_5/\mathrm{CFT}_4$, but it is fair to say that the spectral problem is well understood and we have readily available numerical results.

1.3 The three-point function

In this section, we review the recent developments in three-point functions of the same theory. There are several reasons for the interest in three-point functions. $\mathcal{N} = 4$ SYM is a conformal field theory. It is widely known that a CFT is specified by the set of conformal data, which includes the spectrum of the operators and the operator product expansion (OPE) coefficients of the operators. The spectrum contains kinematic data of of the theory while the OPE coefficients contains the dynamic data of the theory. In order to solve the $\mathcal{N} = 4$ SYM theory, it is crucial to determine the OPE coefficients, which are encoded in the three-point functions. In order to understand the 'interactions' between the operators or the strings, it is indispensable to study the three-point functions. The aim of this thesis is to compute three-point functions in $\mathcal{N} = 4$ SYM from integrability, so the review will put more emphasis at weak coupling side and especially from the point of view of integrability.

Compared to the spectral problem, the three-point function or the OPE coefficient problem is much less understood. However, substantial progress has been made in recently years.

Weak coupling At weak coupling, based on the earlier works of [47] and [48], Escobedo, Gromov, Sever and Vieira (EGSV), proposed a systematic method to compute the three-point functions at tree level by using Bethe ansatz [49]. The method involves a cutting and pasting procedure and is thus termed 'tailoring'. The authors computed a special configuration in the $\mathfrak{su}(2)$ sector. Foda realized that the EGSV result can actually be simplified largely and written in terms of a determinant [50]. Foda's method applies the map between Heisenberg spin chain and the 6-vertex model and involves a ingenuous use of the inhomogeneities, which was termed 'freezing'. The tailoring and freezing method were extended to the higher rank $\mathfrak{su}(3)$ sector which involves three complex scalars [51]. The tailoring method was later generalized to both non-compact $\mathfrak{sl}(2)$ sector [52] and the supersymmetric $\mathfrak{su}(1|1)$ sector [53].

The one loop computation of three-point functions were first investigated by [47] and [54] who used Feynamann diagrams to compute the quantum corrections for operators in the $\mathfrak{so}(6)$ sector. The crucial observation was that the quantum corrections manifest themselves as operator insertions at the splitting points (see Chapter 7 for more detail). Gromov and Vieira computed the three-point functions in $\mathfrak{su}(2)$ sector at one-loop, taking into account both the operator insertions of three-point functions and one-loop corrections of the operators by diagonalizing the two-loop dilatation operator. In [55], we computed the same three-point function by a different method which uses the relation

between long-range interacting spin chains and inhomogeneous $XXX_{1/2}$ spin chains. This method leads to a much more compact result and enable us to take the semi-classical limit. The one-loop result of $\mathfrak{su}(1|1)$ sector was obtained by Caetano and Fleury [53] who computed both the operator insertions by Feynamann diagrams and the three-point function by diagonalizing the higher loop dilatation operators. The one-loop result for $\mathfrak{sl}(2)$ sector was conjectured by Vieira and Wang [52], which passed through many non-trivial checks.

Strong coupling. At strong coupling, the string theory description and classical integrability plays an important role. It proves to be useful to characterize the operators or string states into three rough classes, namely the heavy, medium and light operators. From string theory side, the heavy states are classical string solutions with large quantum number; light operators are typically protected and are supergravity modes; the medium states corresponds to short massive string solutions.

The three-point functions with two heavy and one light operator (HHL) were investigated in several papers [56–58]. The idea is to describe the heavy operators by classical string solutions and the light operator by a vertex operator. By integrating the vertex operator over the world-sheet of the classical string solution, one obtain the corresponding three-point function. Recently, [59] pointed out that the original recipes of [56] was not adequate and suggested that one should also integrate over the moduli space of the classical solution. The three-point functions with three heavy operators (HHH) were investigated in the series paper by Janik, Surówka and Wereszczynski [60, 61], and Kazama, Komatsu [62–64] by using the algebraic curve and Pohlmeyer reduction technique, which was close to method of computing scattering amplitudes at strong coupling. The complete results were obtained for large spin GKP string [62, 63] in $\mathfrak{sl}(2)$ sector and the $\mathfrak{so}(4)$ sector [64]. The case with three operators were computed in the papers [65] and [66].

Semi-classical limit. One of the most interesting regimes of the three-point functions is the semi-classical limit. This is the same limit which allows algebraic curve description on both sides of the duality in the spectral problem. It is the comparison of the algebraic curves that leads to the proposal of all-loop Bethe equations. For the same reason, it is tempting to see whether the three-point functions in the same limit exhibit a similar feature and can be written in terms of algebraic curves.

One way is to use the coherent state approximation. In this approximation, one can describe the spin chain dynamics by the Landau-Ginzburg non-linear sigma model. The

works in this direction includes [67–69]. The coherent state approximation method is valid for the HHL case and is not yet applied to the HHH case.

For the latter case, the idea is to first obtain a compact exact result and then take the semi-classical limit. The first work was done in [70] where the authors studied the semi-classical three-point function with two BPS operators and one non-BPS operator using the saddle-point method in the $\mathfrak{su}(2)$ sector. This configuration was later generalized to three non-BPS operators by Kostov [71, 72] where the semi-classical limit is given in an elegant form written in terms of pseudomomenta of three operators. The semi-classical limit at one-loop was obtained in [55] and compared to the strong coupling limit. In the Frolov-Tesytlin limit it gives a match.

New formulations. The results obtained in various sectors and limits shows some hint of a unifying picture and the attempt of setting up a unifying framework that is applicable to all sectors and all loops have started. The three-point function problem in AdS/CFT context is intimately related to the problem of computing matrix elements, or correlation functions in the integrable systems where many well-established tools exists. One important and powerful method for computing correlation functions in integrable field theories is the form factor bootstrap program. The idea is to write down a very general set of axioms for the form factors of the theory. The solution of these set of equations determines the form factor non-perturbatively. Once these form factors are known, one can put them together to compute correlation functions. The new formulations of three-point functions in AdS/CFT are all related to the form factor approach in one way or another.

The first direct generalization of the form factor method to AdS/CFT context was proposed by Klose and McLoughlin in [73, 74] who wrote down a set of axioms for the world-sheet form factors. There are certain new features of the axioms in this context. The 2d integrable field theories are usually relativistic and Lorentz invariance was crucial there. However, superstring theory in the light-cone gauge is not Lorentz invariant. Therefore the generalization is not trivial.

Another method, inspired from the light-cone string field theory and the form factor bootstrap program was advocated in [75]. The idea is first take the decompactification limit of the two out of the three operators and define the three-point function in this limit as a generalized form factor. Then one can formulate a set of axioms for the generalized form factor. In principle, the solution of these axioms will give the result for three-point functions non-perturbatively. They also proposed the way of taking into account the finite size corrections up to wrapping order following the work of [76, 77]. In general,

solving form factor bootstrap axioms are highly demanding. It is simpler to work on some special cases, for example the diagonal form factors and check the proposal.

At weak coupling, the idea of string field theory leads to the proposal of the spin vertex formalism. In [78], we constructed the spin vertex, which is the weak coupling counterpart of the spin vertex for all sectors at tree level. In the BMN limit, the spin vertex coincide with the string vertex at the leading order of large curvature expansion.

Very recently, a new breakthrough was made by Basso, Komatsu and Vieira who proposed a powerful all-loop formalism for the three-point function [79]. The method was inspired from the OPE method of Wilson loops and uses the symmetry of the three-point function. In the decompactification limit, one can cut the three-point functions into two 'hexagons'. Each hexagon can be computed non-perturbatively using the form factor bootstrap method and the symmetry of the theory. By subsequently gluing together the two hexagons, one obtain the result of three-point function non-perturbatively.

1.4 Structure of the thesis

The structure of the thesis is given in Fig.(1.4.1).

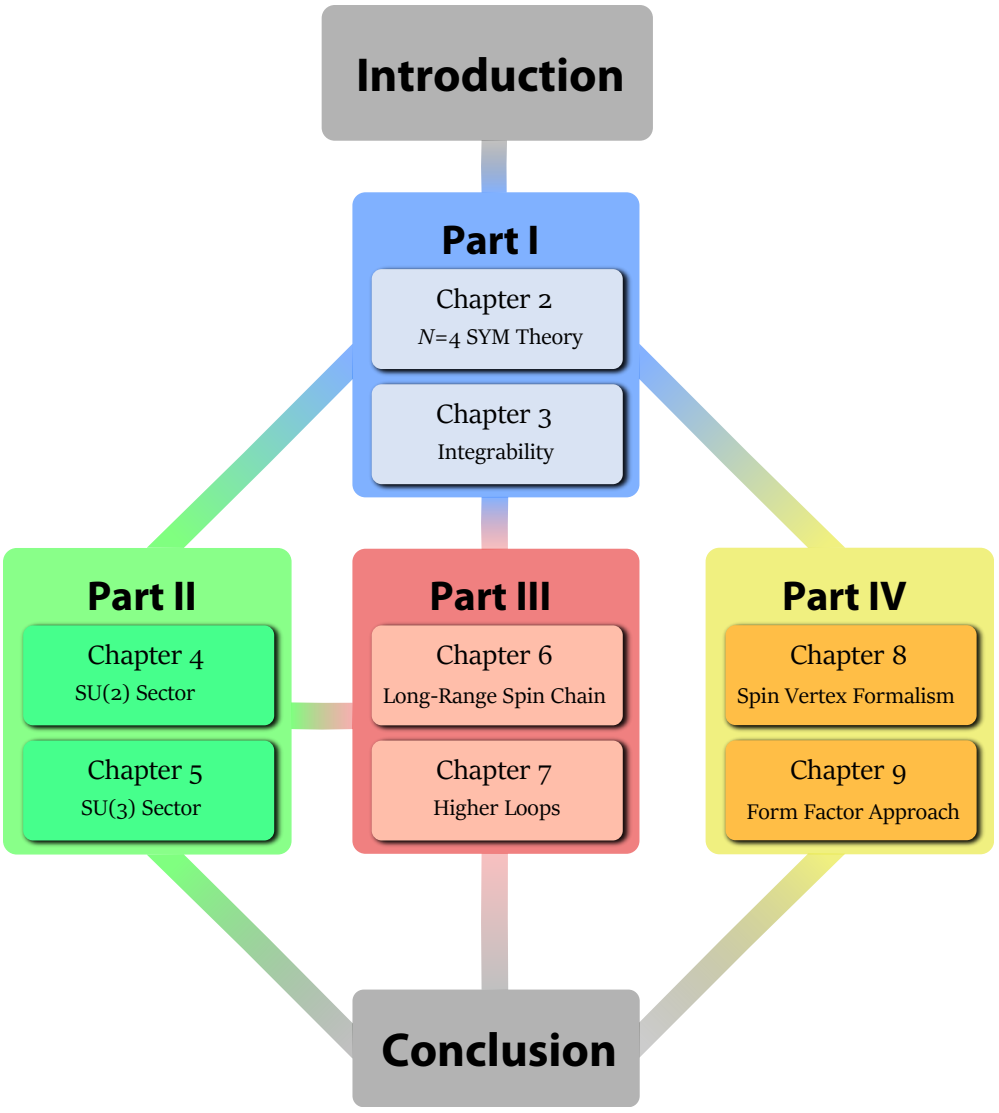

FIGURE 1.4.1: Structure of this thesis.

Apart from the introduction and conclusion, the main body of the dissertation can be divided into four groups, each group contains two chapters.

The first two chapters, namely chapter 2 and 3 contains introduction to the planar $\mathcal{N} = 4$ SYM theory and the fundamentals of integrability. The results of these two chapters are needed for all the later chapters.

Chapter 4 and 5 deal with tailoring and freezing method. In chapter 2, we review the results in the $\mathfrak{su}(2)$ sector and in chapter 3 we generalize the results to the higher rank $\mathfrak{su}(3)$ sector.

Chapter 6 and 7 contains the discussions on higher loop three-point functions. In section 6, we introduce the long-range interacting BDS spin chain and discuss the diagonalization of this spin chain by exploring its relation with the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain. In

section 7, we compute explicitly the scheme independent structure constant at one-loop and consider its semi-classical limit.

The last two chapters of the main body are related to the new formulations of three-point functions. In section 8, we present the spin vertex formalism and show that the BMN limit of the spin vertex reproduces the string vertex in the same limit at the leading order of large curvature expansion. In section 9, we study the finite volume dependence of a special kind of HHL correction function at weak coupling, which confirms a conjecture that is suppose to work at all loop.

Finally in chapter 10, we conclude this dissertation and provide an outlook for the future directions.

This dissertation is partly based on the following papers of the author:

1. O. Foda, **Y. Jiang**, I. Kostov and D. Serban, *A Tree Level 3-Point Function in the $su(3)$-sector of Planar $\mathcal{N} = 4$ SYM*, *JHEP* 1310 (2013) 138, arXiv: 1302.3539,
2. **Y. Jiang**, F. Loebbert, I. Kostov and D. Serban, *Fixing the Quantum Three-Point Function*, *JHEP* 04 (2014) 019, arXiv: 1401.0384,
3. **Y. Jiang**, I. Kostov, A. Petrovskii and D. Serban, *String Bits and the Spin Vertex*, *Nucl. Phys.* B (2015) 374-404, arXiv: 1410.8860,
4. **Y. Jiang**, A. Petrovskii, *From Spin Vertex to String Vertex*, arXiv: 1412.2256.
5. L. Hollo, **Y. Jiang**, A. Petrovskii, *Diagonal Form Factors and Heavy-Heavy-Light Three-Point Functions at Weak Coupling*, arXiv: 1504.07133

Chapter 2

$\mathcal{N} = 4$ Super-Yang-Mills Theory

In this chapter, we present the fundamentals of the $\mathcal{N} = 4$ SYM theory, including its action, symmetries, closed sectors and observables, especially the correlation functions.

2.1 The action

The $\mathcal{N} = 4$ SYM theory is a 4d quantum field theory. Its action can be most easily obtained from a dimensional reduction of the $\mathcal{N} = 1$ gauge theory action in 9+1 dimensions

$$S = \int d^{10}x \left(\frac{1}{4} \mathrm{Tr}\, F_{MN} F^{MN} + \frac{1}{2} \mathrm{Tr}\, \psi \Gamma^M \mathcal{D}_M \psi \right) \tag{2.1.1}$$

where $\mathcal{D}_M$ is the 10d covariant derivative

$$\mathcal{D}_M \star = \partial_M \star - i g_{\mathrm{YM}} [A_M, \star] \tag{2.1.2}$$

Here A_M, $(M = 0, \cdots, 10)$ is the 10d vector and ψ_A $(A = 1, \cdots, 16)$ is the 10d Majarana-Weyl spinor with 16 components. Upon dimensional reduction, different components of these fields becomes the various field in the 4d theory. More explicitly, we have

– The first 4 components of A_M, $(M = 0, 1, 2, 3)$ become the 4d vector A_μ $(\mu = 0, 1, 2, 3)$,
– The rest 6 components of A_M $(M = 4, \cdots, 9)$ become 6 scalar fields Φ_i $(i = 1, \cdots, 6)$,
– The 10d fermionic fields ψ_A become 4 copies of two-component Weyl spinors in 4d $\bar{\psi}^a_{\dot{\alpha}}, \psi_{a,\alpha}$, $a = 1, 2, 3, 4$, $\alpha, \dot{\alpha} = 1, 2$.

The resultant 4d action reads

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathcal{D}_\mu \Phi_i \mathcal{D}^\mu \Phi^i - \frac{g_{\rm YM}^2}{4} [\Phi_i, \Phi_j][\Phi^i, \Phi^j] \right. \\ \left. + \mathrm{Tr}\, \bar{\psi}^a \sigma^\mu \mathcal{D}_\mu \psi_a - \frac{i\, g_{\rm YM}}{2} \mathrm{Tr}\, \sigma_i^{ab} \psi_a [\Phi^i, \psi_b] - \frac{i\, g_{\rm YM}}{2} \mathrm{Tr}\, \sigma^i_{ab} \bar{\psi}^a [\Phi_i, \bar{\psi}^b] \right) \tag{2.1.3}$$

Here the matrices σ^μ and σ^i are the chiral projections of the gamma matrices in four and six dimensions, respectively. Note that all the fields are $N_c \times N_c$ matrices

$$A_\mu = A_\mu^a T^a, \quad \Phi_i = \Phi^a T^a, \quad \Psi = \Psi^a T^a, \tag{2.1.4}$$

where T^a are the generators of the gauge group $U(N_c)$ normalized to be $\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}$. The theory has two parameters, the gauge coupling constant $g_{\rm YM}$ and the rank of the gauge group N_c. In the planar limit, we have perturbative expansion in terms of the 'tHooft coupling constant $\lambda = g_{\rm YM}^2 N_c$. Later we also use the coupling constant $g^2 \equiv \frac{\lambda}{16\pi^2}$ below.

2.2 Symmetries

It is shown in [80–82] that the β function of $\mathcal{N} = 4$ SYM theory is zero at all orders. Therefore the theory is conformal invariant even after quantization. The conformal group in 4d is $SO(2,4) \approx SU(2,2)$. This group contains the usual Poincaré group with generators L_β^α, $\bar{L}_{\dot\beta}^{\dot\alpha}$ and translation P_μ, together with special conformal transformation K_μ and the dilatation operator D. The $SO(6)$ symmetry from rotating the rest 6 components in 10d spacetime now becomes the R-symmetry of the internal space time $SO(6) \approx SU_R(4)$. The bosonic part of the symmetry group is thus $SU(2,2) \times SU(4)$. This should be augmented by the fermionic part of the symmetry group, which contains super translations $Q_\alpha^a, \bar{Q}_{\dot\alpha a}$ and super special conformal transformation $S_a^\alpha, \bar{S}^{\dot\alpha a}$. The total symmetry group is $PSU(2,2|4)$. This is the $\mathcal{N} = 4$ superconformal group. The generators can be organized into the following form

$$\begin{pmatrix} L, \bar{L}, P, K, D & Q, \bar{S} \\ \bar{Q}, S & R \end{pmatrix} \tag{2.2.5}$$

where the generators in the diagonal elements are bosonic and the anti-diagonal ones are fermionic. It is useful to introduce the spinorial notation, which translates a vector into bi-spinor, as the following

$$\mathcal{D}_{\alpha\dot\beta} = \mathcal{D}_\mu \sigma^\mu_{\alpha\dot\beta}, \qquad \Phi^{ab} = \Phi^i \sigma_i^{ab}. \tag{2.2.6}$$

In terms of the spinorial representations, the commutation relations of the $\mathfrak{psu}(2,2|4)$ algebra reads

$$\begin{aligned}
&[S^\alpha_a, P_{\gamma\dot\beta}] = \delta^\alpha_\gamma \bar Q_{\dot\beta a}, && [K^{\alpha\dot\beta}, \bar Q_{\dot\gamma a}] = \delta^{\dot\beta}_{\dot\alpha} S^\alpha_a \\
&[\bar S^{\dot\alpha a}, P_{\gamma\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, && [K^{\alpha\dot\beta}, Q^a_\gamma] = \delta^\alpha_\gamma \bar S^{\dot\beta a} \\
&\{S^\beta_b, \bar S^{\dot\alpha a}\} = \delta^a_b K^{\beta\dot\alpha}, && \{Q^b_\beta, \bar Q_{\dot\alpha a}\} = \delta^b_a P_{\beta\dot\alpha}
\end{aligned} \tag{2.2.7}$$

and

$$\begin{aligned}
[K^{\alpha\dot\beta}, P_{\beta\dot\alpha}] &= \delta^\alpha_\beta\, \bar L^{\dot\beta}_{\dot\alpha} + \delta^{\dot\beta}_{\dot\alpha}\, L^\alpha_\beta + \delta^\alpha_\beta\, \delta^{\dot\beta}_{\dot\alpha} D, \\
\{S^\alpha_a, Q^b_\beta\} &= \delta^a_b\, L^\alpha_\beta + \delta^\alpha_\beta\, R^b_a + \frac{1}{2}\delta^a_b\, \delta^\alpha_\beta\, D, \\
\{\bar S^{\dot\alpha a}, \bar Q_{\dot\beta b}\} &= \delta^a_b\, \bar L^{\dot\alpha}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta}\, R^b_a + \frac{1}{2}\delta^a_b\, \delta^{\dot\alpha}_{\dot\beta}\, D.
\end{aligned} \tag{2.2.8}$$

2.3 The subsectors

Since $\mathcal{N} = 4$ SYM theory has many fundamental fields, it is usually easier to restrict oneself in some smaller sectors when studying various quantities such as correlation functions. There are sectors of the theory which are closed in perturbation theory [10]. We list them below

– $SU(2)$ sector: This sector contains two complex scalar fields, it is considered to be the simplest subsector. The states are of the type

$$|ZXZX\cdots ZZ\rangle \tag{2.3.9}$$

– $SL(2)$ sector: This sector contains one complex scalar field and the covariant derivatives, it is the simplest non-compact sector. The states are of the type

$$|\mathcal{D}^M ZZZ\cdots Z\rangle. \tag{2.3.10}$$

– $SU(1|1)$ sector: This sector contains one complex scalar field and one fermionic field, it is the simplest sector that contains fermions. The states are of the type

$$|Z\Psi Z\Psi\cdots ZZ\rangle, \qquad (\text{eg. } \Psi = \psi_{31}). \tag{2.3.11}$$

– $SU(3|2)$ sector: This sector contains three complex scalars fields and two fermionic fields. This is the smallest sector that allows length changing processes. The states are

of the type

$$|Z\Psi_1\Psi_2 Z\cdots XYZZ\rangle. \tag{2.3.12}$$

The length changing process corresponds to $|\cdots XYZ\cdots\rangle \to |\cdots\Psi_1\Psi_2\cdots\rangle$. This transformation conserves the $PSU(2,2|4)$ charges but reduces the length of the spin chain by 1.

– $PSU(1,1|2)$ sector: This sector contains two scalar fields, a fermionic field and its conjugate and the covariant derivative. The states are of the type

$$|\mathcal{D}^M Z\Psi_1\overline{\Psi}_1\cdots XZZZ\rangle. \tag{2.3.13}$$

There is another sector, which is close only at one-loop level but appears frequently. This is the scalar sector, or the $SO(6)$ sector which contains all the 6 scalar fields of the theory. It is conventional to take the following combinations

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}(\Phi_1 + i\,\Phi_2), \qquad \bar{X} = \frac{1}{\sqrt{2}}(\Phi_1 - i\,\Phi_2), \\ Y &= \frac{1}{\sqrt{2}}(\Phi_3 + i\,\Phi_4), \qquad \bar{Y} = \frac{1}{\sqrt{2}}(\Phi_3 - i\,\Phi_4), \\ Z &= \frac{1}{\sqrt{2}}(\Phi_5 + i\,\Phi_6), \qquad \bar{Z} = \frac{1}{\sqrt{2}}(\Phi_5 - i\,\Phi_6). \end{aligned} \tag{2.3.14}$$

Any of the three scalars which do not involve the complex scalar field and its conjugate at the same time form an $SU(3)$ sector, such as $\{X, Y, Z\}$.

2.4 Observables

First of all, planar $\mathcal{N} = 4$ SYM is not really a physical theory in the sense that it can be tested against experiments. What we call observables are the physically interesting quantities, such as scattering amplitudes, Wilson loops and correlation functions. There are also many exciting developments in scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ SYM theory using integrability in recent years. For these developments we refer to [83–86] and [87, 88] and references therein. In this dissertation, we focus on the correlation functions of local operators.

The gauge invariant local operators are the traces of sequences of fundamental fields of the type $\mathrm{Tr}\,(\star\star\star_1)\mathrm{Tr}\,(\star\star\star_2)\cdots\mathrm{Tr}\,(\star\star\star_n)$. In the planar limit, we only focus on the single trace operators since the multi-trace operators are suppressed. The most important correlation functions are the first few point correlation functions, namely the two, three and four-point functions. Conformal symmetry puts a strict constraint on the form of these correlation functions (see for example [89]).

Let us define the correlation functions of n local single trace operators

$$G_n(x_1, \cdots, x_n) \equiv \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle. \tag{2.4.15}$$

For non-extremal correlation functions, the $1/N_c$ dependence of the n-point functions are given by

$$G_n(x_1, \cdots, x_n) \propto \frac{1}{N_c^{n-2}}. \tag{2.4.16}$$

Two-point function. The form of two-point functions in a CFT is completely fixed by conformal symmetry and is given by

$$G_2(x_1, x_2) = \langle \mathcal{O}_i(x_1) \overline{\mathcal{O}}_j(x_2) \rangle = \mathcal{N}_i \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}} \tag{2.4.17}$$

where $x_{ij} = x_i - x_j$ and Δ_i is the conformal dimension or scaling dimension of the operator $\mathcal{O}_i$. Therefore, knowing the scaling dimensions amounts to knowing the two-point functions in a CFT. Here $\mathcal{N}_i$ is a normalization of the operator.

Three-point function For *scalar operators*, namely the operators with zero Lorentz spin, the three-point functions are fixed by the conformal symmetry up to a constant, called the *structure constant*. The value of the structure constant depends on the normalization of the three states. Let us take the normalization of (2.4.17) and take into account the large N_c counting, we have

$$G_3(x_1, x_2, x_3) = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{1}{N_c} \frac{\sqrt{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3} C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{2.4.18}$$

Here C_{123} is the structure constant, which will be the main focus of most part of the dissertation. Let us notice that if we multiply the operator by a phase factor, this will not change the normalization of the operators, but it will leads to a phase factor for the structure constant. Therefore the structure constant is not un-ambiguously defined, but the absolute value $|C_{123}|$ is defined without ambiguity.

For the operators with non-zero Lorentz spins, which corresponds to local operators that involve fermionic fields and covariant derivatives, the structure of the three-point functions is more complicated. If one of the operators is with non-zero Lorentz spin while the other two are still scalar, the space-time dependence is still fixed by conformal symmetry, but it takes a slightly different from from (2.4.18). If two or more operators have non-zero Lorentz spin, the conformal symmetry no longer fix the spacetime dependence of the three-point function. Instead, it fixes the spacetime dependence up to some

building blocks called *tensor structures*. Schematically,

$$\langle \mathcal{O}_1^{S_1}(x_1)\mathcal{O}_2^{S_2}(x_2)\mathcal{O}_3^{S_3}(x_3)\rangle = \sum_{r=1}^{n_r} C_{123}^r\, T_r(x_1,x_2,x_3) \tag{2.4.19}$$

where $T_r(x_1,x_2,x_3)$ are the tensor structures and C_{123}^r is a set of structure constants. The number of the tensor structures n_r grows with the Lorentz spins of the operators S_i, $(i=1,2,3)$. The classification and explicit forms of the tensor structures can be found in the literature, see for example [90] and references therein.

Four-point functions. Contrary to the two- and three-point functions, the four-point function is not fixed by conformal symmetry. The four point function of four scalar primary operators $\mathcal{O}_i$ with scaling dimensions Δ_i is fixed to the following form by conformal symmetry

$$\begin{aligned} G_4(x_1,x_2,x_3,x_4) &\equiv \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \\ &= \frac{1}{N_c^2}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} \frac{\mathcal{G}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}. \end{aligned} \tag{2.4.20}$$

where $\Delta_{ij}=\Delta_i-\Delta_j$ and $\mathcal{G}(u,v)$ is a function of the conformal invariant *cross ratios*

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}. \tag{2.4.21}$$

In general, the function $\mathcal{G}(u,v)$ can be expanded in a basis of functions called the *conformal blocks*. In order to see this, we first perform the operator product expansion (OPE) of two of the operators inside the four-point functions. For example, take the OPE of $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_\alpha C_{12\alpha}\,\hat{D}(x_{12},\partial_{x_2})^{a_1\cdots a_l}\mathcal{O}_{\alpha,a_1\cdots a_l}(x_2). \tag{2.4.22}$$

Here the sum is over all primary operators $\mathcal{O}_\alpha$. For scalar operators, these are all symmetric traceless tensors of an arbitrary spin l. The differential operator $\hat{D}$ is fixed by conformal symmetry and the scaling dimensions Δ_1,Δ_2. $C_{12\Delta}$ is a numerical factor which characterizes the OPE and is called the *OPE coefficient*. The OPE coefficient $C_{12\alpha}$ is contained in the three-point function $\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_\alpha(x_3)\rangle$. We can perform the OPE for the rest two operators and plug in the four-point function, which leads to

$$G_4(x_1,x_2,x_3,x_4) = \sum_\alpha C_{12\Delta}\,C_{34\Delta}\,W_\alpha(x_1,x_2,x_3,x_4) \tag{2.4.23}$$

where $W_\alpha(x_1, x_2, x_3, x_4)$ is called the *conformal partial wave* and can be written as

$$\begin{aligned} W_\alpha &= \hat{D}(x_{12}, \partial_{x_2})^{a_1 \cdots a_l}\, \hat{D}(x_{34}, \partial_{x_4})^{b_1 \cdots b_l} \langle \mathcal{O}_{a_1 \cdots a_l}(x_2) \mathcal{O}_{b_1 \cdots b_l}(x_4) \rangle \\ &= = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} \frac{\mathcal{G}_\alpha(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}. \end{aligned} \tag{2.4.24}$$

This defines the so-called *conformal block* $\mathcal{G}_\alpha(u, v)$ which is also fixed by conformal symmetry. Comparing to (2.4.20), we have

$$\mathcal{G}(u, v) = \sum_\alpha C_{12\alpha}\, C_{34\alpha}\, \mathcal{G}_\alpha(u, v). \tag{2.4.25}$$

Note that in (2.4.25), the left-hand side contains all information about the four-point function while the right-hand side depends only on the structure constants and the scaling dimensions of the operators, the rest is fixed by conformal symmetry. A similar procedure can be applied for higher point functions by using OPE. This is called the conformal bootstrap. The scaling dimensions and the OPE coefficients are called the *conformal data*, which specifies a CFT. The scaling dimensions and OPE coefficients are contained in the two- and three-point functions respectively, which explains their fundamental importance.

When we expand four-point function in terms of conformal blocks, we can also perform the OPE of $\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)$ and $\mathcal{O}_1(x_1)\mathcal{O}_4(x_4)$ which results in a different expansion. These two expansions should give the same four-point function, as is shown in Fig.(2.4.1). This requirement is the so-called *crossing symmetry* and imposes non-trivial constraints

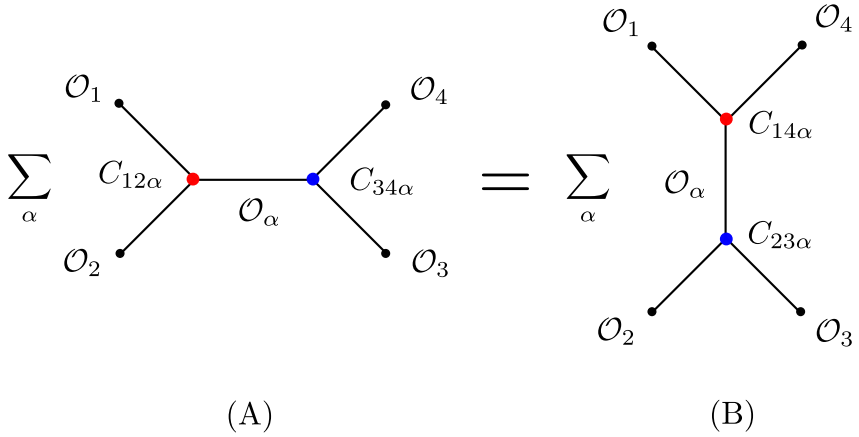

FIGURE 2.4.1: OPE expansions in different channels.

on the conformal data. The implication of crossing symmetries on conformal field theories in higher dimensions is now under intense exploration starting from the seminal works [91, 92].

Chapter 3

Fundamentals of Integrability

In this chapter, we introduce the fundamental ideas and technique of quantum integrability, which include coordinate and algebraic Bethe ansatz, the scalar products between Bethe states and the semi-classical limit of the scalar products. They will play an essential role in the following chapters in computing correlation functions in $\mathcal{N} = 4$ SYM theory. We mainly focus on the Bethe ansatz for $SU(2)$ $\mathrm{XXX}_{1/2}$ spin chain. Other integrability techniques such as the nested Bethe ansatz and the solution of inverse scattering problem will be introduced later when needed.

A spin chain of length L is an one dimensional lattice model with L sites. Each site is associated with a "spin" which has different polarizations. Let us consider the simplest case where the spin has two polarizations, that is to say at each site we have two possible states $|\uparrow\rangle$ and $|\downarrow\rangle$, as is shown in Fig.(3.0.1). The spins at different sites can interact with

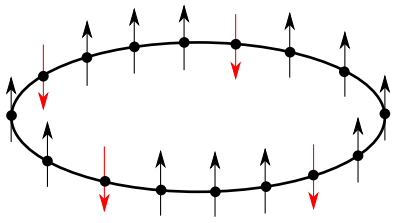

FIGURE 3.0.1: A periodic spin chain. Each site can have two possible spins.

each other. In this chapter, we consider the nearest neighboring interaction described by the following hamiltonian

$$H = g^2 \sum_{k=1}^{L} (1 - 4\vec{S}_k \cdot \vec{S}_{k+1}), \qquad \vec{S}_k = \frac{1}{2}\vec{\sigma}_k \tag{3.0.1}$$

where $g^2 = \frac{\lambda}{16\pi^2}$ and $\vec{\sigma}_k$ are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.0.2}$$

We impose the periodic boundary condition $\vec{S}_{L+1} = \vec{S}_1$. This Hamiltonian is $SU(2)$ invariant

$$[H, \vec{S}] = 0, \quad \vec{S} = \sum_{k=1}^{L} \vec{S}_k. \tag{3.0.3}$$

Our aim of the first two sections is to solve the model (3.0.1) by two different but intimately related methods under the name of coordinate and algebraic Bethe ansatz. Solving the model means two things in this chapter: find the *spectrum* and construct the corresponding *eigenstates*.

3.1 Coordinate Bethe Ansatz

The idea of H. Bethe in the seminal paper [93] is to propose an ansatz for the eigenstate of the Hamiltonian (3.0.1). It is easy to see that the state with all spin up $|\uparrow^L\rangle$ is an eigenstate with vanishing eigenvalue. This state is called *pseudovacuum* and is denoted by $|\Omega\rangle$. One can create excitations called *magnons* on top of the pseudovacuum by flipping spins. The spins can be flipped by the local spin operators $\sigma_k^{\pm}$ $(k = 1, \cdots, L)$ where

$$\sigma^-|\uparrow\rangle = |\downarrow\rangle, \quad \sigma^+|\downarrow\rangle = |\uparrow\rangle, \quad \sigma_k^{\pm} = \frac{1}{2}(\sigma_k^1 \pm i\sigma_k^2). \tag{3.1.4}$$

Since the Hamiltonian commute with S^z it can be diagonalized on the subspaces with given number of magnons. Let us analyze the case with one, two and three magnons and then we will see the pattern of the general N-magnon case.

One magnon states. Since the Hamiltonian is translational invariant, one can imagine that the magnon propagates freely with momentum p on the spin chain. The wave function is described by a plane wave. Hence we can propose the following eigenstate

$$|p\rangle = \sum_{n=1}^{L} e^{ipn}\, \sigma_n^-|\Omega\rangle \tag{3.1.5}$$

This is indeed an eigenstate of the Hamiltonian, with the eigenvalue

$$\epsilon(p) = 2g^2 \sin^2 \frac{p}{2} \tag{3.1.6}$$

Two magnon states. For the two magnon states, let us imagine two magnons moving along the spin chain with momenta p_1 and p_2. Now the situation is more interesting because the two magnons can interact with each other. One can propose the following ansatz

$$|p_1, p_2\rangle = \sum_{1 \le n_1 < n_2 \le L} \left(e^{ip_1 n_2 + ip_2 n_2} + S_{21} e^{ip_2 n_1 + ip_1 n_2} \right) \sigma^-_{n_1} \sigma^-_{n_2} |\Omega\rangle \tag{3.1.7}$$

where $S_{21} \equiv S(p_2, p_1)$ describes the scattering between the magnons and is called the scattering matrix. Acting the Hamiltonian on the proposed ansatz (3.1.7) and requiring the state to be an eigenstate, we fix the form of the scattering matrix to be

$$S(p_a, p_b) = \frac{\frac{1}{2}\cot\frac{p_a}{2} - \frac{1}{2}\cot\frac{p_b}{2} + i}{\frac{1}{2}\cot\frac{p_a}{2} - \frac{1}{2}\cot\frac{p_b}{2} - i}. \tag{3.1.8}$$

The corresponding eigenvalue is given by

$$E(p_1, p_2) = \epsilon(p_1) + \epsilon(p_2), \tag{3.1.9}$$

which is the sum of the energy of the individual magnons.

Three magnon states. For three magnons, there are more possibilities for interactions between the magnons. The key assumption of Bethe is that the three magnon or more scattering matrix are in fact *factorized* into the two magnon scattering matrices (3.1.8). This implies that the underlying dynamics is actually very simple. The N-body interactions can always be reduced to the two-body interactions, which is the simplest possible interaction apart from free theory. This is the key property underlying integrability. With this assumption, the three magnon state is given by

$$|p_1, p_2, p_3\rangle = \sum_{1 \le n_1 < n_2 < n_3 \le L} \psi(n_1, n_2, n_3) \sigma^-_{n_1} \sigma^-_{n_2} \sigma^-_{n_3} |\Omega\rangle \tag{3.1.10}$$

where now the wave function takes the following form

$$\psi(n_1, n_2, n_3) = e^{ip_1 n_1 + ip_2 n_2 + ip_3 n_3} + S_{213} e^{ip_2 n_1 + ip_1 n_2 + ip_3 n_3} + S_{231} e^{ip_2 n_1 + ip_3 n_2 + ip_1 n_3} + \cdots \tag{3.1.11}$$

There are $3! = 6$ terms in total, corresponding to all possible permutations of $\{p_1, p_2, p_3\}$. The S_{abc} is a shorthand notation for the three particle scattering matrix

$$S_{abc} = S(p_a, p_b, p_c), \quad a, b, c = 1, 2, 3. \tag{3.1.12}$$

By the factorized scattering assumption, these scattering matrices are the product of the two particle scattering matrices $S(p_a, p_b)$. The rules are the following. We take $S(p_1, p_2, p_3) = 1$ and each time we permute two momenta, we generate a scattering matrix. So for example

$$S_{213} = S(p_2, p_1), \quad S_{231} = S(p_2, p_1)S(p_3, p_1). \tag{3.1.13}$$

One can again check that the states generated in this way is an eigenstate, with the energy

$$E(p_1, p_2, p_3) = \sum_{k=1}^{3} \epsilon(p_k). \tag{3.1.14}$$

The general case. It is not hard to generalize our considerations to the general N-magnon case. With the factorized scattering assumption, we have the following ansatz

$$|p_1, \cdots, p_N\rangle = \sum_{1 \le n_1 < \cdots < n_N \le L} \psi(n_1, \cdots, n_N) \sigma^-_{n_1} \cdots \sigma^-_{n_N} |\Omega\rangle \tag{3.1.15}$$

where the wave function is given by the following

$$\psi(n_1, \cdots, n_N) = \sum_{\sigma} S_\sigma \prod_{j=1}^{N} e^{ip_{\sigma_j} n_j}. \tag{3.1.16}$$

Here the summation runs over all permutations of $\{1, 2, \cdots, N\}$, denoted by σ. The N magnon scattering matrices S_σ is again products of the two magnon scattering matrices with the rules

$$S_{12\cdots N} = 1, \quad S_{\cdots ab \cdots} = S(p_b, p_a) S_{\cdots ba \cdots}. \tag{3.1.17}$$

In general the wave function is the sum of $N!$ terms, which becomes quite complicated when the number of magnons N grows. However, the rules to construct the wave function is indeed simple and intuitively clear. The eigenvalue of the state is always the sum of the individual magnons

$$E = \sum_{k=1}^{N} \epsilon(p_k). \tag{3.1.18}$$

Since we are considering a periodic chain, the momenta have to be quantized. The quantization condition can be obtained by requiring the wave function to be periodic, which reads

$$e^{ip_k L}\prod_{j\neq k} S(p_j,p_k)=1, \quad k=1,\cdots,N. \tag{3.1.19}$$

Intuitively, these set of equations can be understood as the following. We pick a magnon of momenta p_k and move it along the spin chain until it comes back. During its journey, it will interact with all the other magnons and pick up the scattering phases $S(p_j,p_k)$. Also when it goes along the spin chain, it picks up a phase shift $e^{ip_k L}$. Since our wave function is periodic, the two factors should cancel each other so that the wave function is unchanged. This is depicted in Fig.(3.1.2) The equations (3.1.19) are the famous *Bethe*

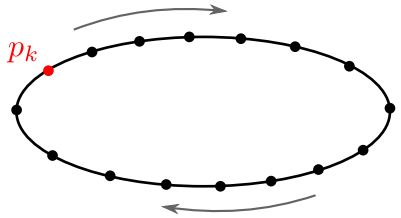

FIGURE 3.1.2: The intuitive understanding of Bethe ansatz equation.

ansatz equations (BEA). It is convenient to introduce another set of variables called *rapidities* which are defined as

$$u_k=\frac{1}{2}\cot\frac{p_k}{2}, \qquad e^{ip_k}=\frac{u_k+i/2}{u_k-i/2}. \tag{3.1.20}$$

In terms of rapidities, the Bethe ansatz equations take a simple form of a set of algebraic equations

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{N}\frac{u_k-u_j+i}{u_k-u_j-i}, \qquad k=1,\cdots,N. \tag{3.1.21}$$

The energy of the N-magnon state is written as

$$E=2g^2\sum_{k=1}^{N}\frac{1}{u_k^2+1/4}. \tag{3.1.22}$$

To summarize, the Heisenberg spin chain are solved by the Bethe ansatz (3.1.15) with the corresponding eigenvalues given in (3.1.22). In order to find the values of the momenta or rapidities, we need to solve the BAE (3.1.21). Originally, the problem of solving the model is equivalent to diagonalizing an $2^L\times 2^L$ matrix, when the length of the spin

chain gets larger, the diagonalization soon becomes impossible. Using Bethe ansatz, the problem is reduced to solving an N coupled algebraic equations which can be done readily for relatively large number of magnons and lengths. The main idea of the ansatz is the factorized scattering assumption. The success of Bethe's ansatz validates this assumption. However, although one can check explicitly that Bethe ansatz works for a few magnon states, it is not obvious how to prove that it is true in general. In the next section, we use another method which will enable us to prove rigorously that Bethe ansatz works for the N-magnon case.

3.2 Algebraic Bethe ansatz

In this section, we use another method called *algebraic Bethe Ansatz (ABA)* or the *quantum inverse scattering method (QISM)* to solve the Heisenberg spin chain. In order to apply the algebraic Bethe ansatz, it is useful to rewrite the Hamiltonian in a different way

$$H = 2g^2 \sum_{k=1}^{L} (\mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1}) \tag{3.2.23}$$

where $\mathrm{I}_{k,k+1}$ is the identity and $\mathrm{P}_{k,k+1}$ is the permutation operator acting on site k and $k+1$

$$\mathrm{P}_{k,k+1}|a\rangle_k \otimes |b\rangle_{k+1} = |b\rangle_k \otimes |a\rangle_{k+1} \tag{3.2.24}$$

The permutation operator is related to the local spin operators as

$$\mathrm{P}_{k,k+1} = \frac{1}{2}\left(\mathrm{I}_k \otimes \mathrm{I}_{k+1} + \sum_{\alpha=1}^{3} \sigma_k^\alpha \otimes \sigma_{k+1}^\alpha\right) \tag{3.2.25}$$

where I_k and σ_k^α are identity and Pauli sigma matrices at site k. The idea of the algebraic Bethe ansatz is to construct a generating object called the *monodromy matrix*. The elements of the monodromy matrix satisfy a quadratic algebra, which is the origin for the name *algebraic* Bethe ansatz. The eigenvectors can be constructed by the elements of the monodromy matrix, the conserved charges (including the Hamiltonian) can be generated by the trace of the monodromy matrix. Before we define the monodromy matrix and write down the quadratic algebra, we need to introduce some new objects. The central object of the algebraic Bethe ansatz is the R-matrix, which plays the role of "structure constants" for the quadratic algebra. The structure constants which determines the Lie algebra satisfy some consistency relations, namely the Jacobi identities. Similarly, the R-matrix has to satisfy the consistency condition called the *Yang-Baxter equations (YBE)*.

The R-matrix is an operator $R_{ab}(u)$ acting on the tensor product of two spaces $V_a \otimes V_b$ and depends on a variable called the *spectral parameter* u. The Yang-Baxter equation is given by

$$R_{ab}(u)R_{ac}(u+v)R_{bc}(v) = R_{bc}(v)R_{ac}(u+v)R_{ab}(u) \tag{3.2.26}$$

It is helpful to represent the YBE by diagrams in Fig.(3.2.3). For a specific integrable

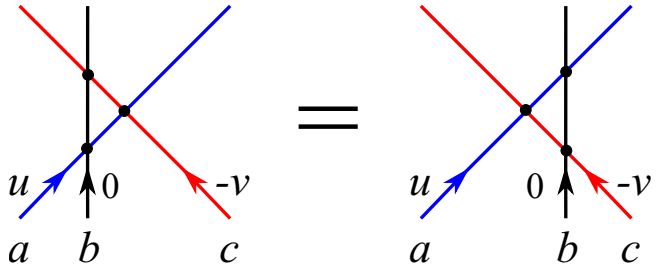

FIGURE 3.2.3: Diagrammatic representation of Yang-Baxter equation. Each black dot represent an R-matrix and Yang-Baxter equation states that we can move the black line from the left to the right.

model, the R-matrix is given. In our case, the R-matrix of the $SU(2)$ invariant Heisenberg $\mathrm{XXX}_{1/2}$ spin chain is given by

$$R_{ab}(u) = u\,\mathrm{I}_{ab} + i\,\mathrm{P}_{ab}. \tag{3.2.27}$$

It is straightforward to check that the R-matrix satisfies the YBE (3.2.3).

Now come back to our spin chain, for each site on the spin chain, we associate it with an operator called the *Lax matrix* $L_{an}(u)$ which acts on the tensor product of two spaces $V_a \otimes h_n$ and depends on the spectral parameter u. The space V_a is called the *auxiliary space* and h_n is called the *quantum space*. In our case they are both the two dimensional complex space $\mathbb{C}^2$. The Lax operator must satisfy the following important identity

$$R_{ab}(u-v)L_{an}(u)L_{bn}(v) = L_{bn}(v)L_{an}(u)R_{ab}(u-v) \tag{3.2.28}$$

where $R_{ab}(u)$ is the R-matrix defined in (3.2.27). One notice that the equation (3.2.28) takes a similar form as the YBE (3.2.3). This in fact offers us a way to construct the Lax matrix, we can simply define the Lax matrix as

$$L_{an}(u) = R_{an}(u - \frac{i}{2}) = (u - \frac{i}{2})\,\mathrm{I}_{an} + i\,\mathrm{P}_{an}. \tag{3.2.29}$$

The Lax matrix (3.2.29) can be written in terms of local spin operators

$$L_{an}(u) = u\,\mathrm{I}_a \otimes \mathrm{I}_n + i\sum_{\alpha=1}^{3} S_n^\alpha \otimes \sigma_a^\alpha, \quad S_n^\alpha = \frac{1}{2}\sigma_n^\alpha \tag{3.2.30}$$

Alternatively this can be written as a 2×2 matrix in the auxiliary space

$$L_{an}(u) = \begin{pmatrix} u + iS_n^3 & iS_n^- \\ iS_n^+ & u - iS_n^3 \end{pmatrix}_a \tag{3.2.31}$$

where the elements of the matrix are operators acting on the local quantum space h_n. More explicitly, we have

$$\begin{aligned} L_{an}(u)|\uparrow\rangle_n &= \begin{pmatrix} (u+i/2)|\uparrow\rangle_n & i\,|\downarrow\rangle_n \\ 0 & (u-i/2)|\uparrow\rangle_n\,, \end{pmatrix}_a \\ L_{an}(u)|\downarrow\rangle_n &= \begin{pmatrix} (u-i/2)|\downarrow\rangle_n & 0 \\ i\,|\uparrow\rangle_n & (u+i/2)|\downarrow\rangle_n \end{pmatrix}_a . \end{aligned} \tag{3.2.32}$$

If we regard the R-matrix as the structure constant, then the relation (3.2.28) is a quadratic algebra for the Lax operator. However, this is only defined for one site on the spin chain. In order to obtain an object which act on the whole spin chain, we take the product of the Lax matrix and define the aforementioned monodromy matrix

$$T_a(u) = \prod_{n=1}^{L} L_{an}(u) \tag{3.2.33}$$

where we take the same auxiliary space and the same spectral parameter for each Lax matrix. From the relation (3.2.28), one can deduce that the monodromy matrix also satisfies a similar relation called the *RTT relation* given by

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v). \tag{3.2.34}$$

This relation can be proved by successive use of (3.2.28), as is depicted in Fig.(3.2.4). The monodromy matrix can also be written as a 2×2 matrix in the auxiliary space

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \tag{3.2.35}$$

where the elements are operators acting on the whole quantum space of the spin chain $h_1 \otimes h_2 \otimes \cdots \otimes h_L$. In general these operators are very complicated functions of the local spin operators S_n^α and the spectral parameter u, but their explicit form is not important in most cases. What's crucial is the *algebra* between A, B, C, D operators. This is in fact

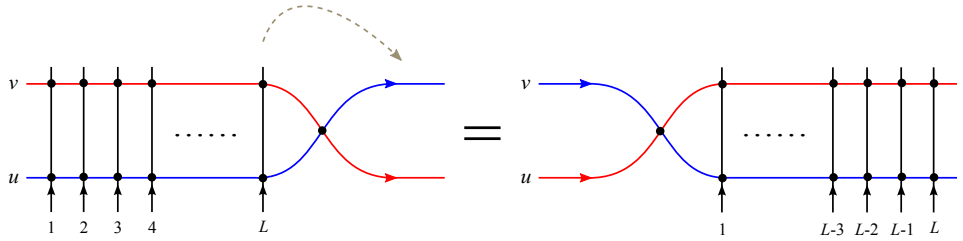

FIGURE 3.2.4: The proof of RTT relation. Each time one move an vertical line from the left to the right due to the RLL relation at each site.

the element form of the RTT relation. Let us write the RTT relation in a more explicit way in the auxiliary space $V_a \otimes V_b$. We take the basis $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ to span this space where the identity and permutation operator can be represented as 4×4 matrices

$$\mathrm{I}_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab}, \qquad \mathrm{P}_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab} \tag{3.2.36}$$

so the R-matrix takes the form

$$R_{ab}(u-v) = \begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & u-v & i & 0 \\ 0 & i & u-v & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix}_{ab} \tag{3.2.37}$$

The monodromy matrices can also be written as 4×4 matrices in the auxiliary space

$$T_a(u) = T_a(u) \otimes \mathrm{I}_b = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}_{ab} \tag{3.2.38}$$

$$T_b(v) = \mathrm{I}_a \otimes T_b(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}_{ab}$$

Plugging the explicit form (3.2.37) and (3.2.38) into the RTT relation and perform the matrix product, we obtain a matrix identity which contain 16 relations in terms of

elements. The algebra is summarized as

$$\begin{aligned}
A(v)B(u) &= f(u-v)B(u)A(v) + g(v-u)B(v)A(u) \\
B(v)A(u) &= f(u-v)A(u)B(v) + g(v-u)A(v)B(u) \\
D(v)B(u) &= f(v-u)B(u)D(v) + g(u-v)B(v)D(u) \\
B(v)D(u) &= f(v-u)D(u)B(v) + g(u-v)D(v)B(u) \\
C(v)A(u) &= f(v-u)A(u)C(v) + g(u-v)A(v)C(u) \\
A(v)C(u) &= f(v-u)C(u)A(v) + g(u-v)C(v)A(u) \\
C(v)D(u) &= f(u-v)D(u)C(v) + g(v-u)D(v)C(u) \\
D(v)C(u) &= f(u-v)C(u)D(v) + g(v-u)C(v)D(u)
\end{aligned} \tag{3.2.39}$$

$$\begin{aligned}
[C(v), B(u)] &= g(u-v)[A(v)D(u) - A(u)D(v)] = g(u-v)[D(u)A(v) - D(v)A(u)] \\
[D(v), A(u)] &= g(u-v)[B(v)C(u) - B(u)C(v)] = g(u-v)[C(u)B(v) - C(v)B(u)]
\end{aligned} \tag{3.2.40}$$

$$[A(u), A(v)] = [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0 \tag{3.2.41}$$

where

$$f(u) = 1 + \frac{i}{u}, \qquad g(u) = \frac{i}{u}. \tag{3.2.42}$$

Now let us construct the eigenvector of the Hamiltonian. An integrable system contains infinitely many commuting conserved charges [1]. The Hamiltonian is one of the conserved charges. The eigenstats that we are going to construct diagonalize all the conserved charges simultaneously. To this end, we first introduce the *transfer matrix*, which is the trace of the monodromy matrix in the auxiliary space

$$\mathcal{T}(u) = \mathrm{tr}_a\, T_a(u) = A(u) + D(u). \tag{3.2.43}$$

The transfer matrices commute with each other due to the RTT relation

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0. \tag{3.2.44}$$

1. For spin chain, the number of conserved charges equals the length of the spin chain.

By construction, $\mathcal{T}(u)$ is a polynomial of spectral parameter u of order L

$$\mathcal{T}(u) = 2u^L + \sum_{l=0}^{L-1} \mathrm{Q}_l \, u^l. \tag{3.2.45}$$

Due to the commutativity of the transfer matrix, the coefficients Q_l commute with each other. If one can show that the Hamiltonian is one of the coefficients, then these coefficients Q_l are identified with the conserved charges of the spin chain. This is indeed the case. Using the fact

$$\frac{d}{du}\mathcal{T}^{-1}(u)\mathcal{T}(u)\Big|_{u=i/2} = \frac{d}{du}\log\mathcal{T}(u)\Big|_{u=i/2} = -i\sum_{k=1}^{L}\mathrm{P}_{k,k+1}, \tag{3.2.46}$$

the Hamiltonian can be expressed in terms of the transfer matrix as

$$H = -2ig^2\frac{d}{du}\log\mathcal{T}(u) + 2g^2L. \tag{3.2.47}$$

This implies we can construct the eigenstates of the transfer matrix, which will automatically diagonalize the Hamiltonian and the other conserved charges. The construction again starts with the pseudovacuum $|\Omega\rangle$ defined in section 3.1. From (3.2.32),

$$T_a(u)|\Omega\rangle = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}|\Omega\rangle = \begin{pmatrix} a(u)|\Omega\rangle & * \\ 0 & d(u)|\Omega\rangle \end{pmatrix} \tag{3.2.48}$$

where

$$a(u) = (u + i/2)^L, \quad d(u) = (u - i/2)^L. \tag{3.2.49}$$

The relation (3.2.48) shows that the pseudovacuum diagonalizes the diagonal operators $A(u)$ and $D(u)$ and is annihilated by the operator $C(u)$. What about the action of $B(u)$ operator on the pseudovacuum ? The action of each B-operator on the pseudovacuum creates one magnon. Let us denote the action of various B-operators on the pseudovacuum by

$$|\mathbf{u}\rangle = B(u_1)\cdots B(u_N)|\Omega\rangle. \tag{3.2.50}$$

This state is called a Bethe state. Now we act the transfer matrix on the state (3.2.50). Using the algebra in (3.2.39), one can show that

$$A(v)|\mathbf{u}\rangle = a(v)\prod_{k=1}\frac{v-u_k-i}{v-u_k}|\mathbf{u}\rangle + \sum_{k=1}^{N}\mathrm{M}_k(v)B(u_1)\cdots\hat{B}(u_k)\cdots B(u_N)B(v)|\Omega\rangle \tag{3.2.51}$$

$$D(v)|\mathbf{u}\rangle = d(v)\prod_{k=1}\frac{v-u_k+i}{v-u_k}|\mathbf{u}\rangle + \sum_{k=1}^{N}\mathrm{N}_k(v)B(u_1)\cdots\hat{B}(u_k)\cdots B(u_N)B(v)|\Omega\rangle$$

where a hat on the B operator means the operator is missing and $M_k^{\mathbf{u}}$ and $N_k^{\mathbf{u}}$ are

$$\begin{aligned}\mathrm{M}_k(v) &= \frac{ia(u_k)}{v-u_k}\prod_{j\neq k}\frac{u_k-u_j-i}{u_k-u_j}\\ \mathrm{N}_k(v) &= \frac{id(u_k)}{u_k-v}\prod_{j\neq k}\frac{u_k-u_j+i}{u_k-u_j}.\end{aligned} \tag{3.2.52}$$

From the relation (3.2.51), we see that the first terms on both lines are the Bethe state $|\mathbf{u}\rangle$ multiplied with some factor. These terms take the form of eigenstates and are called *wanted terms*. The rest terms are a sum of states which replaces one of the spectral parameters u_k by the spectral parameter v and are called *unwanted terms*. Since the transfer matrix is the sum $A(u)+D(u)$, we can hope that the unwanted terms from the action of A and D operators cancel each other so that $|\mathbf{u}\rangle$ is the eigenstate of the transfer matrix. This is true if $\mathrm{M}_k(v)+\mathrm{N}_k(v)=0$, $(k=1,\cdots,N)$. Written explicitly, these conditions read

$$\frac{a(u_k)}{d(u_k)} = \prod_{j\neq k}\frac{u_k-u_j+i}{u_k-u_j-i}, \quad k=1,\cdots,N. \tag{3.2.53}$$

which is nothing but the Bethe ansatz equation (3.1.19) and we see that the spectral parameters plays the role of rapidities. If the rapidities satisfy the BAE, the state $|\mathbf{u}\rangle$ is called the *on-shell* Bethe state, otherwise it is called *off-shell* Bethe state. We have thus proved that the on-shell Bethe state diagonalizes the transfer matrix $\mathcal{T}(u)$

$$\mathcal{T}(u)|\mathbf{u}\rangle = t_{\mathbf{u}}(u)|\mathbf{u}\rangle \tag{3.2.54}$$

for *any* u with the eigenvalue

$$t_{\mathbf{u}}(u) = a(u)\prod_{k=1}^{N}\frac{u-u_k-i}{u-u_k} + d(u)\prod_{k=1}^{N}\frac{u-u_k+i}{u-u_k}. \tag{3.2.55}$$

The eigenvalue of the Hamiltonian is given by

$$E_{\mathbf{u}} = -2ig^2 \frac{d}{du} \log t_{\mathbf{u}}(u)\Big|_{u=i/2} + 2g^2 L = g^2 \sum_{k=1}^{N} \frac{1}{u_k^2 + 1/4}, \tag{3.2.56}$$

which agrees with (3.1.22).

Before ending the section, we remark that although the construction of Bethe states in CBA and ABA seems quite different, the resultant Bethe states are in fact proportional to each other

$$|\mathbf{u}\rangle^{\text{ABA}} = \prod_{k=1}^{N} \frac{i\, d(u_k)}{u_k + i/2} \prod_{j<k} \frac{u_j - u_k + i}{u_j - u_k} |\mathbf{u}\rangle^{\text{CBA}}. \tag{3.2.57}$$

3.3 Scalar products between Bethe states

For an integrable system, after obtaining the spectrum and eigenstates of the Hamiltonian, the most important question is to construct *manageable* expressions for physical observables such as form factors and correlation functions. This problem turns out to be more challenging than the spectral problem. The problem of computing three-point functions in the planar $\mathcal{N} = 4$ SYM theory is intimately related to the computation of form factors and correlation functions [2] of quantum spin chain. In order to compute the these quantities, the most important ingredient is the scalar product between Bethe states, which we will discuss in this section.

3.3.1 Inhomogeneous $\text{XXX}_{1/2}$ spin chain

Before doing that, let us introduce the inhomogeneous Heisenberg $\text{XXX}_{1/2}$ spin chain. We will encounter this important model again in the following chapters, with slightly different normalizations and emphasis on different aspects of the model. It is most easily defined in the framework of algebraic Bethe ansatz. The monodromy matrix for the inhomogeneous spin chain is a generalization of (3.2.33)

$$T_a(u; \boldsymbol{\theta}) = \prod_{n=1}^{L} L_{an}(u - \theta_n). \tag{3.3.58}$$

The parameters $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_L\}$ are called inhomogeneities or impurities. Taking all inhomogeneities to zero, we recover the monodromy matrix for homogeneous Heisenberg spin chain (3.2.33). The RTT relation for the inhomogeneous monodromy matrix is

2. We call both quantities matrix elements in below

the same as the homogeneous case and the algebra between A, B, C, D operators are unchanged. The Bethe states are generated in the same way

$$|\mathbf{u};\boldsymbol{\theta}\rangle = B(u_1;\boldsymbol{\theta})\cdots B(u_N;\boldsymbol{\theta})|\Omega\rangle. \tag{3.3.59}$$

In the inhomogeneous case, the pseudovacuum also diagonalize A and D operators, but with different eigenvalues

$$A(u;\boldsymbol{\theta})|\Omega\rangle = a(u;\boldsymbol{\theta})|\Omega\rangle, \quad D(u;\boldsymbol{\theta})|\Omega\rangle = d(u;\boldsymbol{\theta})|\Omega\rangle. \tag{3.3.60}$$

where

$$a(u;\boldsymbol{\theta}) = \prod_{n=1}^{L}(u-\theta_n+i/2), \quad d(u;\boldsymbol{\theta}) = \prod_{n=1}^{L}(u-\theta_n+i/2). \tag{3.3.61}$$

The corresponding BAE reads

$$\frac{a(u_k;\boldsymbol{\theta})}{d(u_k;\boldsymbol{\theta})} = \prod_{j\neq k}^{N}\frac{u_k-u_j+i}{u_k-u_j-i}, \qquad k=1,\cdots,N. \tag{3.3.62}$$

In what follows, we will not write the inhomogeneities explicitly for simplicity.

3.3.2 Definition of scalar products

The *dual Bethe state* is defined as

$$\langle\mathbf{u}| \equiv \langle\Omega|\prod_{n=1}^{N}C(u_n), \qquad \langle\Omega| = \langle\uparrow^L|. \tag{3.3.63}$$

We want to remark here that the dual Bethe state is *not* exactly the adjoint conjugation of the Bethe state $|\mathbf{u}\rangle$, the later is in fact

$$\dagger: \quad |\mathbf{u}\rangle \longrightarrow (-1)^N\langle\Omega|C(u_1^*)\cdots C(u_N^*) = (-1)^N\langle\mathbf{u}^*| \tag{3.3.64}$$

where we have used the property

$$B^\dagger(u) = -C(u^*). \tag{3.3.65}$$

Here u^* denotes the complex conjugate of u. If the rapidities $\mathbf{u}$ is complex invariant as a set, which is true for most of the cases that we are interested in, then the dual Bethe state coincides with the adjoint conjugation of the Bethe state up to the phase $(-1)^N$. One can define two kinds of overlaps between Bethe states. The *scalar product* is

defined as the overlap between a Bethe state and a *conjugated Bethe state* and the *inner product* is defined as the overlap of a Bethe state and a *dual Bethe state*. We denote the scalar product and the inner product of Bethe states $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ by $\langle\mathbf{v}|\mathbf{u}\rangle$ and $\langle\mathbf{v},\mathbf{u}\rangle$, respectively. From (3.3.64), we see that the two kinds of overlaps are related by

$$\langle\mathbf{v}|\mathbf{u}\rangle = (-1)^N\langle\mathbf{v}^*,\mathbf{u}\rangle. \tag{3.3.66}$$

In most cases that we are interested in, the set of rapidities is invariant under complex conjugation so the difference between scalar product and inner product is simply a phase factor. In addition, we are only interested in the absolute value of the structure constant so the phase factor is irrelevant. Therefore we will not make the careful distinction between the scalar product and inner product in this dissertation and use them interchangeably.

The scalar product between two Bethe states is defined as

$$S_N(\mathbf{v};\mathbf{u}) = \langle\mathbf{v}|\mathbf{u}\rangle = \langle\Omega|\prod_{n=1}^{N} C(v_n)\prod_{n=1}^{N} B(u_n)|\Omega\rangle. \tag{3.3.67}$$

In order to have non-vanishing results, the number of B operators should equal to the number of C operators. The scalar product can be computed in principle by using the algebra (3.2.39). One can start with one of the C operators and commute it through the B operators until it hits the pseudovacuum $|\Omega\rangle$ which gives zero. The commutators of C and B operators will generate A and D operators, one can further commute these operators to the right most until they hit the pseudovacuum and give the eigenvalues. Then one can start with another C operator and repeat the process. This procedure is straightforward but becomes very complicated as the number of magnons grows. The computation of the scalar product turns out to be a highly non-trivial problem. For generic values of the rapidities $\mathbf{u}$ and $\mathbf{v}$, there exists an explicit expression for the scalar product which is written as a sum of 2^N terms. Although explicit, this expression is too complicated to be useful in practice. The much more interesting case is when one set of the rapidities, say $\mathbf{u}$ satisfy the BAE, namely $|\mathbf{u}\rangle$ is on-shell, then the scalar product can be written in terms of a determinant. This result is called the Slavnov theorem and it simplifies tremendously many computations of matrix elements of Heisenberg spin chain.

3.3.3 Slavnov determinant and Gaudin norm

In what follows, we will present the Slavnov determinant formula. The proof is rather technical and will not be presented here. There exist several different proofs of the

theorem and we refer the interested readers to [94–96]. The Slavnov theorem states that, if $|\mathbf{u}\rangle$ is on-shell, the scalar product of $|\mathbf{u}\rangle$ with an off-shell state $\langle\mathbf{v}|$ takes the following form [3]

$$\langle\mathbf{v}|\mathbf{u}\rangle = \prod_{j=1}^{N} a(v_j)d(u_j)\mathscr{S}_{\mathbf{u},\mathbf{v}} \tag{3.3.68}$$

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{1}{\prod_{j=1}^{N} a(v_j)} \frac{\det_{jk} \frac{\partial}{\partial u_j} t_{\mathbf{u}}(v_k)}{\det_{jk} \frac{1}{u_j - v_k}}. \tag{3.3.69}$$

Here $t_{\mathbf{u}}(v_k)$ is the eigenvalue of the transfer matrix

$$t_{\mathbf{u}}(u) = a(u)\frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + d(u)\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)} \tag{3.3.70}$$

where we have defined the so-called *Baxter polynomial*

$$Q_{\mathbf{u}}(u) \equiv \prod_{k=1}^{N}(u - u_k). \tag{3.3.71}$$

In (3.3.69), the determinant in the denominator is singular when $u_k = v_j$. One can write another expression for the scalar product without this problem. Following [72], we have

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{\det_{jk} \Omega(u_j, v_k)}{\det_{jk} \frac{1}{u_j - v_k + i}} \tag{3.3.72}$$

where the kernel $\Omega(u, v)$ is defined as in [72]

$$\Omega(u,v) = t(u-v) - e^{2ip_{\mathbf{u}}(v)}t(v-u), \qquad t(u) = \frac{1}{u} - \frac{1}{u+i}. \tag{3.3.73}$$

The quantity $p_{\mathbf{u}}(v)$ is called *pseudo-momentum*, which is defined as

$$e^{2ip_{\mathbf{u}}} \equiv \frac{d(u)}{a(u)}\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u-i)}. \tag{3.3.74}$$

In terms of the pseudo-momentum, the BAE takes a very simple form

$$e^{2ip_{\mathbf{u}}(u_k)} = -1, \qquad k = 1, \cdots, N. \tag{3.3.75}$$

Gaudin norm When two sets of rapidities coincide $\mathbf{u} = \mathbf{v}$, the scalar product computes the square of the norm of the on-shell Bethe state $|\mathbf{u}\rangle$. This quantity is given by the *Gaudin formula* which was first conjectured by Gaudin [97] and later proved by

3. Our conventions and notations follow [72].

Korepin [98]. The Gaudin formula can be obtained from the Slavnov determinant by taking carefully the limit $\mathbf{v} \to \mathbf{u}$, which leads to

$$\langle \mathbf{u}|\mathbf{u}\rangle = \prod_{j=1}^{N} a(u_j)d(u_j) \frac{\det_{jk} \frac{\partial}{\partial u_j}\Phi_k}{\det_{jk} \frac{1}{u_j - u_k + i}} \tag{3.3.76}$$

where

$$\Phi_k \equiv \log\left(\frac{d(u_k)}{a(u_k)} \frac{Q_{\mathbf{u}}(u_k + i)}{Q_{\mathbf{u}}(u_k - i)}\right), \quad k = 1, \cdots, N. \tag{3.3.77}$$

More explicitly, we can calculate that

$$\frac{\partial}{\partial u_j}\Phi_k = \frac{2i}{(u_j - u_k)^2 + 1} + i\delta_{jk}\left(\sum_{m=1}^{L} \frac{1}{(u_k - \theta_m)^2 + \frac{1}{4}} - \sum_{l=1}^{N} \frac{1}{(u_k - u_l)^2 + 1}\right) \tag{3.3.78}$$

Using the Cauchy determinant formula, the denominator can be written as

$$\det_{jk} \frac{1}{u_j - u_k + i} = (-i)^N \prod_{j \neq k} \frac{u_j - u_k}{u_j - u_k + i}. \tag{3.3.79}$$

which is simply a product.

3.3.4 DWPF, pDWPF and the $\mathscr{A}$-functional

We introduce another important special kind of scalar product which can be written in terms of determinants.

DWPF. The following scalar product is called the *domain wall partition function* (DWPF)

$$Z_L(\mathbf{u}|\boldsymbol{\theta}) \equiv \langle \downarrow^L | B(u_1) \cdots B(u_L) | \uparrow^L \rangle. \tag{3.3.80}$$

It evaluates the partition function of the six vertex model with domain wall boundary conditions on a $L \times L$ square lattice. Note that in order to have non-zero result, the number of magnons should equal the length of the spin chain. The determinant representation of this quantity was first obtained by Izergin [99, 100] and is called the Izergin determinant which reads

$$Z_L(\mathbf{u}|\boldsymbol{\theta}) = \prod_{k=1}^{L} Q_{\boldsymbol{\theta}}^{-}(u_k)\, \mathscr{Z}_{\mathbf{u},\boldsymbol{\theta}} \tag{3.3.81}$$

where

$$\mathscr{Z}_{\mathbf{u},\boldsymbol{\theta}} = \frac{\det_{jk} t(u_j - \theta_k - i/2)}{\det_{jk} \frac{1}{u_j - \theta_k + i/2}}, \qquad t(u) = \frac{1}{u} - \frac{1}{u+i}. \tag{3.3.82}$$

Note that for DWPF, the determinant representation does not require that the rapidities $\mathbf{u} = \{u_1, \cdots, u_L\}$ are on-shell.

pDWPF. If in (3.3.80) the number of mangons $N < L$, in order to have non-zero result we can complement it by the action of S^-. This defines the *partial domain wall partition function* (pDWPF)

$$\hat{Z}_L(\mathbf{u}|\boldsymbol{\theta}) \equiv \langle \downarrow^L | (S^-)^{L-N} B(u_1) \cdots B(u_N) | \uparrow^L \rangle. \tag{3.3.83}$$

Using the relation

$$\lim_{u\to\infty} \frac{B(u)}{i\, u^{L-1}} = S^- \tag{3.3.84}$$

we can obtain the pDWPF from DWPF as

$$\hat{Z}_L(\mathbf{u}|\boldsymbol{\theta}) = \lim_{\mathbf{w}\to\infty} \left(\frac{Z_L(\mathbf{u}\cup\mathbf{w}|\boldsymbol{\theta})}{i w_1^{L-1} \cdots i w_{L-N}^{L-1}} \right). \tag{3.3.85}$$

The $\mathscr{A}$-functional. It was found by Kostov and Matsuo [72, 101] that DWPF and pDWPF have a different determinant representation by using a quantity called the $\mathscr{A}$-functional. It turns out the $\mathscr{A}$-functional is of fundamental importance, especially in the semi-classical limit as we will discuss below. For any set of points $\mathbf{u} = \{u_1, \cdots, u_N\}$ in the complex plane and for any complex function $f(u)$, the $\mathscr{A}$-function is define by

$$\mathscr{A}^{\pm}_{\mathbf{u}} \equiv \frac{1}{\Delta_{\mathbf{u}}} \prod_{u\in\mathbf{u}} \left(1 - f(u) e^{\pm i\partial/\partial u}\right) \Delta_{\mathbf{u}} \tag{3.3.86}$$

where $\Delta_{\mathbf{u}}$ is the Vandermonde determinant

$$\Delta_{\mathbf{u}} = \det_{jk} \left(u_j^{k-1}\right) = \prod_{1\le j<k\le N} (u_j - u_k). \tag{3.3.87}$$

The $\mathscr{A}$-functional can be represented by a ratio of determinants

$$\mathscr{A}^{\pm}_{\mathbf{u}}[f] = \frac{\det_{jk} \left(u_j^{k-1} - f(u_j)(u_j \pm i)^{k-1}\right)}{\det_{jk} \left(u_j^{k-1}\right)}. \tag{3.3.88}$$

The Izergin-Gaudin determinant can be written in terms of the $\mathscr{A}$-functional as

$$\mathscr{Z}_{\mathbf{u},\boldsymbol{\theta}} = (-1)^N \mathscr{A}^-_{\mathbf{u}} \left[\frac{Q^+_{\boldsymbol{\theta}}(u)}{Q^-_{\boldsymbol{\theta}}(u)} \right] \tag{3.3.89}$$

We will discuss the properties of $\mathscr{A}$-functional and its semi-classical limit in section 3.4.2

3.4 Semi-classical limit of scalar product

In the following chapters, the study of semi-classical limit of the structure constant will be one of the most important subject. In order to take the semi-classical limit of the structure constant, we need to learn how to take the semi-classical limit of the scalar product. This problem was first investigated in [70] for a special case and later studied in the papers [72, 101, 102] more generally. We will review the semi-classical limit of the scalar product in this section.

The semi-classical limit, or the Sutherland limit is the limit when the number of rapidities and the length of the spin chain are both very large $N, L \to \infty$ with their ratio kept fixed $\alpha = N/L < 1$. This limit has been studied in condensed matter physics by Sutherland [103] and Dhar and Shastry [104]. In the semi-classical limit, the Bethe roots typically arrange themselves on umbrella-shaped arcs which become cuts. Two examples are given in (3.4.5) The size of the arc is of the scale L. In this limit, the rapidities scales as $u_k \sim L$

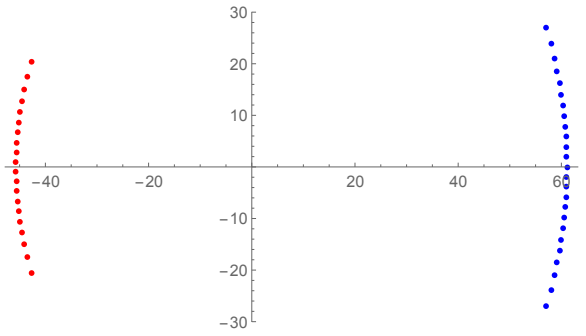

FIGURE 3.4.5: The cuts of Bethe roots. On the left is the one-cut solution for $N = 20, L = 300$ with mode number $n_0 = -1$; on the right is the one-cut solution for $N = 25$, $L = 400$ with mode number $n_0 = 1$.

and the energy scales $E \sim 1/L$. By AdS/CFT correspondence, Bethe states in this regime are dual to classical string solutions with large quantum numbers. The spectral problem in this limit was described by the finite-gap solution in [17–19, 105]. Here we study the behavior of scalar products in this limit.

The idea of [72] is to write the Slavnov determinant in an operator factorized form in terms $\mathscr{A}$-functionals. By taking the semi-classical limit of the $\mathscr{A}$-functional, we obtain the semi-classical limit of the Slavnov determinant.

3.4.1 Operator factorization formula

Let us introduce the shift operator $e^{a\,\partial/\partial u}$ where a is a constant. It acts on a function $f(u)$ of variable u as

$$e^{a\partial/\partial u} f(u) = f(u+a), \quad e^{a\partial/\partial u}\, 1 = 1. \tag{3.4.90}$$

This relation can be proved easily by expanding the shift operator on the l.h.s.

$$e^{a\partial/\partial u} = 1 + \sum_{n=1}^{\infty} \frac{a^n}{n!} \frac{\partial^n}{\partial u^n} \tag{3.4.91}$$

and compare with the Taylor expansion of the r.h.s. at $u = a$. Using the shift operator, the kernel of the Slavnov determinant $\Omega(u, v)$ can be written as

$$\Omega(u,v) = \left(1 - e^{2ip_{\mathbf{u}}(v)} e^{i\partial/\partial v}\right) \left(e^{-i\partial/\partial u} - 1\right) \frac{1}{u - v + i}. \tag{3.4.92}$$

The Slavnov determinant can be written as a result of acting N pairs of difference operators on the Cauchy determinant

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{\prod_{v\in\mathbf{v}} \left(1 - e^{2ip_{\mathbf{u}}(v)} e^{i\partial/\partial v}\right) \prod_{u\in\mathbf{u}} \left(e^{-i\partial/\partial u} - 1\right) \det_{jk} \frac{1}{u_j - v_k + i}}{\det_{jk} \frac{1}{u_j - v_k + i}} \tag{3.4.93}$$

Notice that the operators belonging to the same block commute while the ones belonging to two different blocks do not. The Cauchy determinant can be computed readily

$$\det_{jk} \frac{1}{u_j - v_k + i} = \frac{\prod_{j<k}(u_j - u_k) \prod_{j<k}(v_k - v_j)}{\prod_{j,k=1}^{N}(u_j - v_k + i)} \equiv \frac{\Delta_{\mathbf{u}} \Delta_{-\mathbf{v}}}{\Pi_{\mathbf{u},\mathbf{v}}}. \tag{3.4.94}$$

We can move the shift operators from left to right across the Cauchy determinant using the following identities

$$\begin{aligned} e^{-i\partial/\partial u} \frac{1}{\Pi_{\mathbf{u},\mathbf{v}}} &= E^{+}_{\mathbf{v}}(u) \frac{1}{\Pi_{\mathbf{u},\mathbf{v}}} e^{-i\partial/\partial u}, \quad u \in \mathbf{u}, \\ e^{i\partial/\partial v} \frac{1}{\Pi_{\mathbf{u},\mathbf{v}}} &= E^{-}_{\mathbf{u}}(v) \frac{1}{\Pi_{\mathbf{u},\mathbf{v}}} e^{i\partial/\partial v}, \quad v \in \mathbf{v} \end{aligned} \tag{3.4.95}$$

where $E^{\pm}_{\mathbf{u}}$ is defined as

$$E_{\mathbf{u}}(u) = \frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}. \tag{3.4.96}$$

The Slavnov determinant can be written in the following factorized form

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = (-1)^{|\mathbf{u}|}\frac{1}{\Delta_{\mathbf{v}}}\prod_{v\in\mathbf{v}}\left(1-e^{2ip_{\mathbf{u}}(v)}E^{-}_{\mathbf{u}}(v)e^{i\partial/\partial v}\right)\Delta_{\mathbf{v}}\cdot\frac{1}{\Delta_{\mathbf{u}}}\prod_{u\in\mathbf{u}}\left(1-E^{+}_{\mathbf{v}}(u)e^{-i\partial/\partial u}\right)\Delta_{\mathbf{u}}. \tag{3.4.97}$$

where $|\mathbf{u}|$ denotes the cardinality of the set of rapidities $\mathbf{u}$. Each part of the factorized form can be written in terms the $\mathscr{A}$-functional (3.3.86).

3.4.2 Semi-classical limit of the $\mathscr{A}$-functional

From (3.4.97) the Slavnov determinant can be written in terms of the $\mathscr{A}$-functionals. In order to take the semi-classical limit of the Slavnov determinant, it is enough to take the semi-classical limit of the $\mathscr{A}$-functional, which was done in [72]. Before taking the semi-classical limit, we list below the important properties of the $\mathscr{A}$-functional without proof. For more detail, the reader is referred to the original paper [72].

Expansion. The functions $\mathscr{A}^{\pm}_{\mathbf{u}}[f]$ are completely symmetric polynomials of the N variables $f(u_1),\cdots,f(u_N)$. They have the following expansion

$$\mathscr{A}^{\pm}_{\mathbf{u}}[f] = \sum_{\mathbf{u}'\cup\mathbf{u}''=\mathbf{u}}(-1)^{|\mathbf{u}'|}\left(\prod_{u'\in\mathbf{u}'}f(u')\right)\frac{1}{\Delta_{\mathbf{u}}}\prod_{u'\in\mathbf{u}'}e^{\pm i\partial/\partial u'}\Delta_{\mathbf{u}}. \tag{3.4.98}$$

The last factor can be written alternatively as

$$\frac{1}{\Delta_{\mathbf{u}}}\prod_{u'\in\mathbf{u}'}e^{\pm i\partial/\partial u'}\Delta_{\mathbf{u}} = \prod_{u'\in\mathbf{u}',u''\in\mathbf{u}''}\frac{u'-u''\pm i}{u'-u''}. \tag{3.4.99}$$

One important observation by the authors in [70] is that when $f(u)$ is a constant κ, the functional takes a very simple form which does not depend on $\mathbf{u}$ at all

$$\mathscr{A}^{\pm}_{\mathbf{u}}[\kappa] = (1-\kappa)^N = \exp\left(-N\sum_{n=1}^{\infty}\frac{\kappa^n}{n}\right). \tag{3.4.100}$$

This observation is crucial in order to find the semi-classical limit of the $\mathscr{A}$-functional.

Linear term in f as a contour integral The linear term in f in the expansion of $\mathscr{A}^{\pm}_{\mathbf{u}}[f]$ can be written as a contour integral

$$\begin{aligned}\mathscr{A}^{\pm}_{\mathbf{u}}[f] &= 1 - \sum_{j=1}^{N} f(u_j) \prod_{k\neq j} \frac{u_j - u_k \pm i}{u_j - u_k} + \mathcal{O}[f^2] \\ &= 1 \pm \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} f(u) E^{\pm}_{\mathbf{u}}(u) + \mathcal{O}[f^2].\end{aligned} \tag{3.4.101}$$

The integration contour encircles all points of the set $\mathbf{u}$ and leaves the other possible singularities of $f(u)$ outside.

The functional identity The two functionals $\mathscr{A}^{+}$ and $\mathscr{A}^{-}$ are related by the following functional identity

$$\mathscr{A}^{\pm}_{\mathbf{u}}[1/f] = (-1)^N \frac{\mathscr{A}^{\mp}_{\mathbf{u}}[f]}{\prod_{j=1}^{N} f(u_j)}. \tag{3.4.102}$$

In the semi-classical limit, the Bethe roots condense into cuts, it is useful to define the resolvent function

$$G_{\mathbf{u}}(u) = \sum_{k=1}^{N} \frac{1}{u - u_k}. \tag{3.4.103}$$

We now start to take the semi-classical limit of the $\mathscr{A}$-functional. We assume in the derivation that the Bethe roots accumulate along a contour $C_{\mathbf{u}}$ for simplicity but our discussion is valid for the case of multiple cuts. The strategy is to study the asymptotic behavior of $\mathscr{A}$-functional in the limit when f is very small and very large. Then use the special value when f is constant (3.4.100) to fix the whole functional.

When the functional argument f is very small, according to (3.4.101) we have

$$\log \mathscr{A}^{\pm}_{\mathbf{u}}[f] = \pm \oint_{A_{\mathbf{u}}} \frac{dz}{2\pi} e^{iq^{\pm}(z)} + O[f^2], \tag{3.4.104}$$

where the contour $A_{\mathbf{u}}$ encircles the cut $C_{\mathbf{u}}$ counterclockwise and the function $q^{\pm}(u)$ is given by

$$q^{\pm}(u) = -i \log[f(u)] \pm G_{\mathbf{u}}(u). \tag{3.4.105}$$

When f is very large, we can apply the functional identity (3.4.102) to write the functional in terms $\mathscr{A}^{\pm}_{\mathbf{u}}[1/f]$. Using also the relation

$$\begin{aligned}\log\left[(-1)^N\prod_{j=1}^{N}f(u_j)\right] &= \oint_{A_{\mathbf{u}}}\frac{du}{2\pi i}G_{\mathbf{u}}(u)\left(\log[f(u)]+i\pi\right)\\ &= \oint_{A_{\mathbf{u}}}\frac{du}{2\pi}\left[\frac{1}{2}q^2_{\pm}(u)+i\pi q_{\pm}(u)\right],\end{aligned} \tag{3.4.106}$$

we find the large f asymptotic

$$\log\mathscr{A}^{\pm}_{\mathbf{u}}[f]\simeq\oint_{A_{\mathbf{u}}}\frac{du}{2\pi}\left(\pm\frac{1}{2}[q_{\pm}(u)+i\pi]^2\mp e^{-iq^{\pm}(u)}\right)+O[f^{-2}]. \tag{3.4.107}$$

Assume that

$$\log\mathscr{A}^{\pm}_{\mathbf{u}}[f]=\oint_{A_{\mathbf{u}}}\frac{du}{2\pi}F^{\pm}\left(e^{iq^{\pm}(u)}\right). \tag{3.4.108}$$

We need to find the meromorphic functions $F^{\pm}(\omega)$ with the following asymptotic

$$F^{\pm}(\omega)=\begin{cases}\pm\omega+O(\omega^2), & |\omega|\ll 1,\\ \mp\log(-\omega)^2\mp 1/\omega+O(1/\omega^2), & |\omega|\gg 1.\end{cases} \tag{3.4.109}$$

It turns out that the functions $F^{\pm}(\omega)$ can be fixed completely by comparing the ansatz (3.4.108) and the special case where f is a constant (3.4.100). Let us assume that near the vicinity of $\omega=0$, $F^{\pm}(\omega)$ can be Taylor expanded

$$F^{\pm}(\omega)=\sum_{n=1}^{\infty}F^{\pm}_n\omega^n. \tag{3.4.110}$$

For $f(u)=\kappa$, we have $q^{\pm}(u)=-i\log\kappa\pm G_{\mathbf{u}}(u)$. The contour integral can be performed by expanding the contour to infinity

$$\sum_n F^{\pm}_n\oint\frac{du}{2\pi}e^{inq^{\pm}(u)}=\sum_n F^{\pm}_n\oint\frac{du}{2\pi}\kappa^n(1\pm i\frac{N}{u}+\cdots)^n=\mp\sum_n F^{\pm}_n nN\kappa^n. \tag{3.4.111}$$

Comparing this expansion with (3.4.100), we can fix all the coefficients $F^{\pm}_n$ as

$$F^{\pm}_n=\pm\frac{1}{n^2} \tag{3.4.112}$$

and find that the function $F^{\pm}(\omega)$ are in fact dilogarithm functions

$$F^{\pm}(\omega)=\pm\sum_{n=1}^{\infty}\frac{\omega^n}{n^2}=\pm\mathrm{Li}_2(\omega). \tag{3.4.113}$$

Finally we obtain the semi-classical limit of the $\mathscr{A}$-functional in a very compact form

$$\log \mathscr{A}_{\mathbf{u}}[f] \simeq \pm \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(f(u) e^{\pm i G_{\mathbf{u}}(u)} \right) \tag{3.4.114}$$

If the Bethe roots condense into more than one cut $C^1_{\mathbf{u}}, \cdots, C^n_{\mathbf{u}}$, then the contour $A_{\mathbf{u}}$ is the union of the disjoint contours $A^1_{\mathbf{u}}, \cdots, A^n_{\mathbf{u}}$ where $A^k_{\mathbf{u}}$ encircles $C^k_{\mathbf{u}}$ counterclockwise.

3.4.3 Semi-classical limit of the Slavnov determinant

We obtained the semi-classical limit in the last subsection, now we can apply the result to obtain the semi-classical limit of the Slavnov determinat (3.4.97). The Slavnov determinant can be written in the following way

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = (-1)^N \frac{(\mathbf{v}|\mathscr{A}^+_{\mathbf{v}}[U]\mathscr{A}^-_{\mathbf{u}}[V]|\mathbf{u})}{(\mathbf{v}|\mathbf{u})}, \tag{3.4.115}$$

where the operators $U(v)$ and $V(u)$ satisfy the following algebra

$$U(v)V(u) = V(u)U(v)\left(1 - \frac{1}{(u-v)^2+1}\right) \tag{3.4.116}$$

and act on the vector as

$$U(v)|\mathbf{u}) = e^{2ip_{\mathbf{u}}(v)} E^-_{\mathbf{u}}(v)|\mathbf{u}), \quad (\mathbf{v}|V(u) = E^+_{\mathbf{v}}(\mathbf{v}|. \tag{3.4.117}$$

Notice that in the semi-classical limit, the rapidities u and v scales as $u \sim v \sim L \to \infty$ in (3.4.116) so $U(v)$ and $V(u)$ commute and we have a complete factorization into two $\mathscr{A}$-functionals. In the semi-classical limit, the various functions can be written in terms of the resolvents (3.4.103)

$$\begin{aligned} p_{\mathbf{u}}(u) =& \frac{1}{2i}\log\left(\frac{d(u)}{a(u)}\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u-i)}\right) \\ =& \frac{1}{2i}\sum_{n=1}^{L}\left(\log(u-\theta_n - i/2) - \log(u-\theta_n+i/2)\right) \\ &+ \frac{1}{2i}\sum_{k=1}^{N}\left(\log(u-u_k+i) - \log(u-u_k-i)\right) \\ \simeq& -\frac{1}{2}\sum_{n=1}^{N}\frac{1}{u-\theta_n} + \sum_{k=1}^{N}\frac{1}{u-u_k} \\ =& G_{\mathbf{u}}(u) - \frac{1}{2}G_{\boldsymbol{\theta}}(u). \end{aligned} \tag{3.4.118}$$

where we have used the fact that the rapidities scale like $u_k \sim L$ in the semi-classical limit. Similarly, we have

$$\log E^{\pm}_{\mathbf{u}}(u) = \pm i G_{\mathbf{u}}(u). \tag{3.4.119}$$

Therefore, in the semi-classical limit, we have

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = (-1)^N \mathscr{A}^{+}_{\mathbf{v}} \left[e^{iG_{\mathbf{u}}(u) - iG_{\boldsymbol{\theta}}(u)} \right] \mathscr{A}^{-}_{\mathbf{u}} \left[e^{iG_{\mathbf{v}}(u)} \right]. \tag{3.4.120}$$

Using the semi-classical limit of $\mathscr{A}$-functional (3.4.114), we find finally the semi-classical limit of the Slavnov determinant

$$\log \mathscr{S}_{\mathbf{u},\mathbf{v}} = i\pi N + \oint_{A_{\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u) - iG_{\boldsymbol{\theta}}(u)} \right) - \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{-iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u)} \right). \tag{3.4.121}$$

This the main result for this section. If we take the limit $\mathbf{v} = \mathbf{u}$, we obtain the semi-classical limit of the Gaudin norm

$$\log \mathscr{S}_{\mathbf{u},\mathbf{u}} = i\pi N + \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{u}}(u)} \right). \tag{3.4.122}$$

The semi-classical limit of Slavnov determinant (3.4.120) and Gaudin norm (3.4.122) are all the ingredient that we need to obtain the semi-classical limit of the structure constant in the determinant representation.

Chapter 4

The $\mathfrak{su}(2)$ Sector

Starting from this chapter, we will compute three-point functions in the planar $\mathcal{N} = 4$ SYM theory. In order to compute three-point functions, we need to start with three local operators. The choice of the operators is infinite since we have infinitely many gauge invariant local operators. The strategy to simplify the problem is to restrict ourselves in certain sector of the theory and compute three-point functions within the sector. The simplest sector is the $SU(2)$ sector which contains two complex scalar fields. In this chapter we will consider the three-point functions in the $SU(2)$ sector. It is the best understood sector for the moment and it provides us with intuitions and techniques that are also useful in other sectors.

We start by explaining how the computation of three-point functions can be recast in the language of the spin chain in section 4.1. Then we review the so called *tailoring* method [49], which works for the generic configurations but the results are usual very complicated in section 4.2. In section 4.3, we introduce the *freezing* method [50] which enables us to find the determinant representation for a special configuration. In section 4.4, we take the semi-classical limit of the structure constant by using the determinant representation.

4.1 Set-up of the problem

The three operators under consideration are in the $\mathfrak{su}(2)$ sector and are composed of the following complex scalar fields

$$\mathcal{O}_1 : \{Z, X\}, \quad \mathcal{O}_2 : \{\bar{Z}, \bar{X}\}, \quad \mathcal{O}_3 : \{Z, \bar{X}\}. \tag{4.1.1}$$

The complex scalar fields are defined in (2.3.14). The operators take the form of traces of strings of these fields, for example

$$\mathcal{O}_1 = \mathrm{Tr}\,(ZXZZZXZ\cdots) + \cdots . \tag{4.1.2}$$

It is clear that such operators can be mapped to states of the $\mathrm{XXX}_{1/2}$ spin chain by mapping $Z, \bar{Z}$ to spin up and $X, \bar{X}$ to spin down. In what follows, the scalar fields $Z, \bar{Z}$ are regarded as 'vacuum' and $X, \bar{X}$ are regarded as 'excitations'. For example,

$$\mathrm{Tr}\,(ZXZZZXZ) \longleftrightarrow |\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\rangle. \tag{4.1.3}$$

In the $SU(2)$ sector, the three operators are scalar operators, i.e. with zero Lorentz spin. In a conformal field theory, we are mainly interested in operators with definite scaling dimension, whose two-point function is fixed by conformal symmetry up to normalization

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}}_j(y)\rangle = \mathcal{N}_i \frac{\delta_{ij}}{(x-y)^{2\Delta_i}}. \tag{4.1.4}$$

The scaling dimension depends on the coupling constant $g = \frac{\sqrt{\lambda}}{4\pi}$ and can be expanded as

$$\Delta_i(g^2) = L_i + \gamma_i(g^2) \tag{4.1.5}$$

where L_i is the length of the operator and $\gamma_i(g^2)$ is the anomalous dimension. Consider three such operators with definite scaling dimension, the three-point function is again fixed by conformal symmetry up to a constant called the *structure constant*

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{1}{N_c}\frac{\sqrt{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3}\,C_{123}}{x_{12}^{2\Delta_{12}}x_{23}^{2\Delta_{23}}x_{13}^{2\Delta_{13}}} \tag{4.1.6}$$

where

$$\begin{aligned} \Delta_{ij} &= \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k), \\ x_{ij}^2 &= (x_i - x_j)^2, \qquad\qquad i,j,k = 1,2,3. \end{aligned} \tag{4.1.7}$$

and Δ_i $(i = 1,2,3)$ are the scaling dimensions of the three operators. The structure constant depends on the coupling constant g and has the following expansion in perturbation theory

$$C_{123}(g^2) = C_{123}^{(0)} + g^2 C_{123}^{(1)} + \cdots \tag{4.1.8}$$

In this chapter, we will compute the structure constant at tree level, namely $C^{(0)}_{123}$. The computation of one-loop structure constant will be discussed in Chapter 7. At tree level, the scaling dimension equals the length of the operator and hence operators with the same length have the same scaling dimension. This huge degeneracy can be lifted at one-loop. Following [49], we will consider the operators with definite one-loop anomalous dimension. This implies the operators diagonalize the one-loop dilatation operator in the $\mathfrak{su}(2)$ sector, which is nothing but the $\mathfrak{su}(2)$ invariant Heisenberg $\mathrm{XXX}_{1/2}$ spin chain. In Chapter 3, we learned how to construct the eigenstates of the Heisenberg spin chain using coordinate and algebraic Bethe ansatz. By the mapping from operators to spin chain states, the three operators correspond to three on-shell Bethe states.

Now we discuss how to compute the three-point function in principle. At tree-level, the computation amounts to planar Wick contractions between the operators, which sounds simple. However, the on-shell Bethe states are highly non-trivial linear combinations of the basis states and the number of terms grows exponentially with the number of magnons. Therefore, although the Wick contractions for each term are simple, there are many terms to sum up and it is not clear a priori that the sum will give rise to a simple result. The computation at tree level is a combinatorics problem and the scalar product of Bethe states turns out to be a powerful tool.

4.2 Tailoring three-point function

The planar Wick contractions of scalar fields can be mimicked by the scalar products of the spin chain states. For our configuration (4.1.1), the structure of Wick contraction is given as in Fig.(4.2.1). As we consider only the planar Wick contractions, each spin chain can be split into two parts which contract with two other spin chains. It is thus natural to 'cut' the closed spin chain into two open subchains, called *left* and *right* subchains. The method contains mainly three steps, namely *cutting*, *flipping* and *sewing*, the meaning of which will be made more precise in what follows.

Consider a spin chain of length L, we take the first L_l sites as the *left* subchain and the rest $L_r = L - L_l$ sites as the right subchain. A generic state of the original spin chain can be written as entangled states of the two subchains. When the state under consideration is a Bethe state, the states of the subchains also take the form of Bethe states. This can be seen most easily by the so-called *generalized two-component model* [106, 107] within the framework of algebraic Bethe ansatz. The monodromy matrix of the original spin chain can be written as the product of the monodromy matrices of the two subchains

$$T_a(u) = \left(L_{1,a}(u)\cdots L_{L_l,a}(u)\right)\left(L_{L_l+1,a}(u)\cdots L_{L_r,a}(u)\right) \equiv T^l_a(u)\, T^r_a(u). \tag{4.2.9}$$

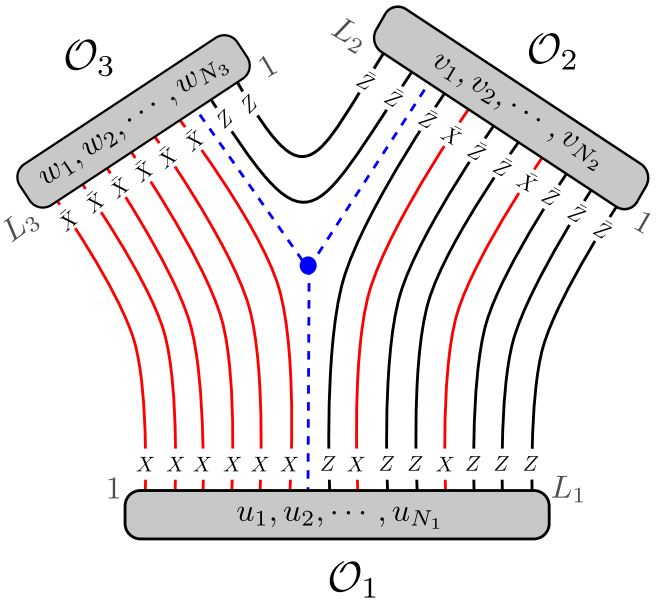

FIGURE 4.2.1: Wick contraction of the EGSV configuration. The black lines represent Wick contraction of scalar fields $Z, \bar{Z}$ and the red lines represent the Wick contraction of $X, \bar{X}$.

In terms of components, we have the following relation between the operators A, B, C, D of the original spin chain and the subchains

$$\begin{aligned} A(u) &= A_l(u)A_r(u) + B_l(u)C_r(u) \\ B(u) &= A_l(u)B_r(u) + B_l(u)D_r(u) \\ C(u) &= C_l(u)A_r(u) + D_l(u)C_r(u) \\ D(u) &= C_l(u)B_r(u) + D_l(u)D_r(u). \end{aligned} \tag{4.2.10}$$

The Bethe state can be written as

$$|\mathbf{u}\rangle = \prod_{i=1}^{N} B(u_i)|\Omega\rangle = \prod_{i=1}^{N} \left(A_l(u_i)B_r(u_i) + B_l(u_i)D_r(u_i)\right)|\Omega_l\rangle \otimes |\Omega_r\rangle. \tag{4.2.11}$$

where $|\Omega_l\rangle = |\uparrow^{L_l}\rangle$, $|\Omega_r\rangle = |\uparrow^{L_r}\rangle$ are the pseudovacua of left and right subchains, respectively. The operators with the index 'l' acts only on the left subchain and the ones with index 'r' act only on the right subchain. After expand the product in (4.2.11), we can use the algebra of the components of the monodromy matrices to move all the A_l, D_l and A_r, D_r operators to the rightmost and act on the pseudovacua, keeping in mind that operators with different indices commute. In this way, we are are left with only B_l and B_r operators acting on the pseudovacua, which are (off-shell) Bethe states.

It is straightforward to see that we have the following result

$$|\mathbf{u}\rangle = \sum_{\mathbf{u}'\cup\mathbf{u}''=\mathbf{u}} \mathrm{H}_{\mathbf{u},\mathbf{u}''}|\mathbf{u}'\rangle \otimes |\mathbf{u}''\rangle \tag{4.2.12}$$

where the sum is over all the possible partitions of rapidities $\mathbf{u}', \mathbf{u}''$ such that $\mathbf{u}' \cup \mathbf{u}'' = \mathbf{u}$ and $\mathbf{u}' \cap \mathbf{u}'' = \emptyset$. For example, for a two magnon state $\{u_1, u_2\}$, there are 4 possible partitions $\{u_1, u_2\}\cup\emptyset$, $\{u_1\}\cup\{u_2\}$, $\{u_2\}\cup\{u_1\}$ and $\emptyset\cup\{u_1, u_2\}$. The coefficients $\mathrm{H}_{\mathbf{u}',\mathbf{u}''}$ measure the 'entanglement' between the two substates $|\mathbf{u}'\rangle$ and $|\mathbf{u}''\rangle$ and takes the following form

$$\mathrm{H}_{\mathbf{u}',\mathbf{u}''} = \prod_{\substack{u_j\in\mathbf{u}'\\ \bar{u}_k\in\mathbf{u}''}} f(u_j,\bar{u}_k) d_r(u_j) a_l(\bar{u}_k) \tag{4.2.13}$$

where

$$f(u,v) = \frac{u-v+i}{u-v}, \quad a_r(u) = (u+i/2)^{L_r}, \quad d_l(u) = (u-i/2)^{L_l}. \tag{4.2.14}$$

As we mentioned before, we want to use scalar products between Bethe states to take into account the Wick contractions. To this end, we need a ket state and a bra state. After we cut each spin chain state, we obtain two ket states. We need to map one of the ket states into a bra state, such an operation is called *flipping*. It is worth mentioning that the flipping operation is not the same as taking the conjugate of the state. A conjugation of the state corresponds to change the order of fundamental fields inside the single trace spin chain as well as taking the charge conjugation. However, the flipped state represent the same operator, but in the dual space. In algebraic Bethe ansatz, the flipped state is closely related to the conjugated state. Let us use $\mathcal{F}$ to denote the flipping operation, we have

$$\mathcal{F}|\mathbf{u}\rangle = (-1)^N \langle \mathbf{u}^*| \tag{4.2.15}$$

where $\{\mathbf{u}^*\} = \{u_1^*, \cdots, u_N^*\}$ means taking the complex conjugate of all the rapidities. The final result does not depend on which substate we flip, so as a convention we choose always to flip the right substate. After cutting and flipping, we have the same number of ket states and bra states and thus can take the scalar product, which is called *sewing*. In order to obtain a result which does not depend on the normalization of the three operators, we need to divide the result by the product of the norms of the three Bethe states.

To summarize, the three operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ of length L_1, L_2, L_3 are represented by three on-shell Bethe states $|\mathbf{u}\rangle$, $|\mathbf{v}\rangle$, $|\mathbf{w}\rangle$ with number of magnons N_1, N_2, N_3. The result

for the structure constant is given by

$$C_{123} = \sqrt{L_1 L_2 L_3} \sum_{\mathbf{u}'\cup\mathbf{u}''=\mathbf{u}} \sum_{\mathbf{v}'\cup\mathbf{v}''=\mathbf{v}} \sum_{\mathbf{w}'\cup\mathbf{w}''=\mathbf{w}} \times \mathrm{H}_{\mathbf{u}',\mathbf{u}''}\mathrm{H}_{\mathbf{v}',\mathbf{v}''}\mathrm{H}_{\mathbf{w}',\mathbf{w}''} \frac{\langle \mathbf{u}^{*''}|\mathbf{v}'\rangle\langle \mathbf{v}^{*''}|\mathbf{w}'\rangle\langle \mathbf{w}^{*''}|\mathbf{u}'\rangle}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle\langle \mathbf{v}|\mathbf{v}\rangle\langle \mathbf{w}|\mathbf{w}\rangle}} \tag{4.2.16}$$

The following comments are in order. The tailoring method, compared to the brute force computation is already a big progress both conceptually and computationally. This procedure is rather general and can be generalized to other sectors. However, once we cut the on-shell Bethe states, the resulting substates are no longer on-shell on the subchains. This means all the scalar products in the numerator in (4.2.16) are of the type off-shell/off-shell and we cannot apply the Slavnov determinant formula. The result for each scalar product is not simple. What's more, after computing the scalar products and multiply them, we need to take the sum over partitions of the rapidities, which involves $2^{N_1+N_2+N_3}$ terms. As the number of magnons increases, this sum becomes complicated quickly. The two difficulties of the tailoring method, namely a compact formula for the off-shell/off-shell scalar product and performing the sum over partitions of rapidities remains or get even worse for other sectors.

In some special cases, the general formula (4.2.16) can be largely simplified and gives rise to a compact result. This is the case for the configuration (4.1.1) and we will discuss it in the next section.

4.3 The determinant representation

In this section, we show that the configuration in (4.1.1) can be recast in terms a determinant. First of all, let us notice that in Fig. (4.2.1) there are two pieces of Wick contractions which are trivial. The contractions between $\mathcal{O}_2$ and $\mathcal{O}_3$ are are all of the type $\langle Z\bar{Z}\rangle$ and the contractions between $\mathcal{O}_3$ and $\mathcal{O}_1$ are all of the type $\langle X\bar{X}\rangle$. The only non-trivial piece is the contractions between $\mathcal{O}_1$ and $\mathcal{O}_2$ where we can have both kinds of contractions. We emphasis that the trivial contractions originate from our choice of fields for the three operators and in general we do not have this simplification.

It is convenient for us to generalize the problem slightly by introducing the inhomogeneous Heisenberg spin chain. In the case of three-point function, the inhomogeneities of the three spin chains are not independent because the inhomogeneities associated with the subchains whose fundamental fields are contracted should match. The independent inhomogeneities associated with the contractions between the i-th left subchain and the

j-th right subchain are denoted by $\boldsymbol{\theta}^{(ij)}$. The cardinality of the set $\boldsymbol{\theta}^{(ij)}$ is L_{ij}. In this notation

$$\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(12)} \cup \boldsymbol{\theta}^{(13)}, \quad \boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(21)} \cup \boldsymbol{\theta}^{(23)}, \quad \boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(31)} \cup \boldsymbol{\theta}^{(32)}. \tag{4.3.17}$$

The three-point function is given by the product of two factors

– The probability to find the component $\mathrm{Tr}\,(Z^{L_{23}}\bar{X}^{L_{13}})$ in the state $|\mathbf{w}\rangle$,

– The remaining contractions can be recast as the scalar product of an on-shell state of rapidity $\mathbf{u}$ and an off-shell state of rapidities $\mathbf{v}$, in the spin chain of length L_1.

It turns out both factors can be written as determinants, the derivation of which we will present below. It is very helpful to perform the derivation in the language of 6-vertex model. The mapping between the Heisenberg spin chain and the 6-vertex model is well-known in condensed matter physics. The 6-vertex model is a 2d lattice model with 6 possible configurations of arrows at each point, as depicted in Fig.(4.3.2). There are three

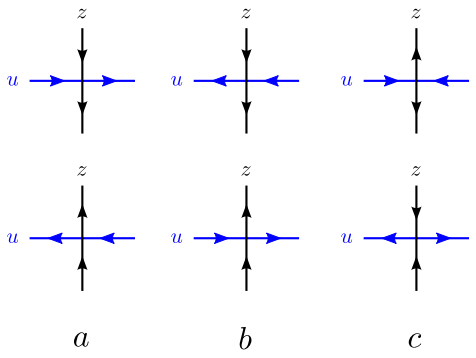

FIGURE 4.3.2: The three types of vertices of 6-vertex model.

types of vertices called type a, b and c, respectively. The vertices with all the arrows flipped simultaneously are of the same type. The three types of vertices are associated with the following weights

$$a(u-z) = \frac{u-z+i}{u-z}, \quad b(u-z) = 1, \quad c(u-z) = \frac{i}{u-z}, \quad z = \theta + i/2. \tag{4.3.18}$$

The important condition about the six vertex model vertices is that the number of arrows flowing in and out the vertices are always equal. In what follows we shall use another representation of the 6 vertices. Let us denote the right arrow and down arrow by a think line and the rest two kinds of arrow with a thin line, which are given in Fig.(4.3.3). We associate each thin line with index "1" and each thick line with index "2". The three types of weights corresponds to the three kinds of non-zero elements of the Lax operator

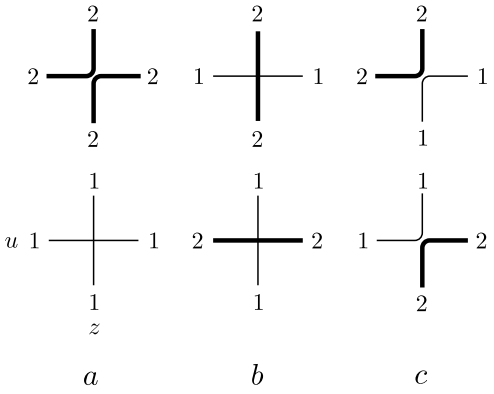

FIGURE 4.3.3: Another representation of the 6 vertex model. The down arrows and right arrows are represented by a thick line and the left arrows and up arrows are represented by a thin line.

of the Heisenberg spin chain [1]

$$L_{an}(u-z) = \mathrm{I}_{an} + \frac{i}{u-z}\mathrm{P}_{an} = \begin{pmatrix} a(u-z) & 0 & 0 & 0 \\ 0 & b(u-z) & c(u-z) & 0 \\ 0 & c(u-z) & b(u-z) & 0 \\ 0 & 0 & 0 & a(u-z) \end{pmatrix}. \tag{4.3.19}$$

An interesting and important observation is that the type b vertices do not depend on the value of rapidities and if we choose the rapidity u such that $u = z - i$, then the three weights become simply

$$a(-i) = 0, \quad b(-i) = 1, \quad c(-i) = -1. \tag{4.3.20}$$

In terms of the 6-vertex model language, a Bethe state can be represented by the following diagram Fig.(4.3.4) Each horizontal line corresponds to a B-operator which creates

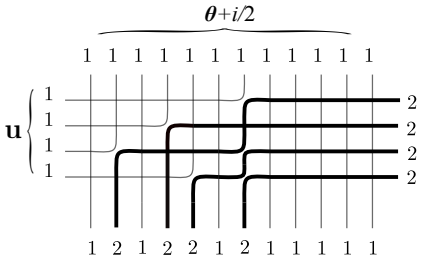

FIGURE 4.3.4: Representation of a coefficient of Bethe state in the 6-vertex model. Each horizontal line corresponds to a B-operator, the action of which creates an index '2'.

1. We choose a slightly different normalization for the Lax operator from the one in Chapter 2.

an excitation with index '2'. For any term of the Bethe state, the flow of thick lines always comes in from the right side and go out from the bottom. The number of indices of each type are conserved. We can take the overlap of a Bethe state and a dual Bethe state by identifying the indices. Therefore a term in the scalar product between Bethe states looks like the diagram in Fig.(4.3.5). In the diagram for a scalar product, the flow

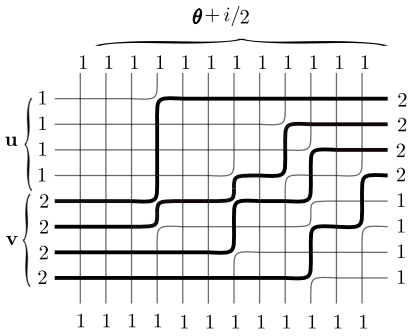

FIGURE 4.3.5: Graphic representation of a term in the scalar product between Bethe states, the indices and the inhomogeneities are identified.

of thick lines comes in from the upper right side and go out from the left bottom side. The flow of indices is again conserved.

Let us now compute the structure constant. We first write the structure constant in the following form

$$C^{(0)}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle, \tag{4.3.21}$$

where $\mathcal{N}_i$ is the norm of the Bethe state corresponding to $\mathcal{O}_i$, $i = 1, 2, 3$ which can be computed easily by the Gaudin norm formula. The non-trivial part $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ is called the *cubic vertex*. It can be evaluated using the fact that it gives partition function of the 6-vertex model on a lattice by gluing three rectangular lattices with dimension $L_1 \times N_1$, $L_2 \times N_2$ and $L_3 \times N_3$ as shown in Fig.(4.3.6). The indices 1 and 2 are identified with Z and X or their complex conjugates, depending on the operator under consideration. First we notice that in the part of lattice that has vertical lines labeled by $\boldsymbol{\theta}^{(23)}$, represented by the shaded area of Fig.(4.3.6), there is only one 6-vertex configuration, and therefore its contribution to the cubic vertex factorizes out. The factor is a pure phase if the sets $\mathbf{v}$ and $\boldsymbol{\theta}^{(23)}$ are symmetric under complex conjugation. We will assume that this is the case will ignore the phase factor. Therefore we can delete this part of the lattice.

Next, we observe that the sub-lattice associated with the operator $\mathcal{O}_3$ factorizes because all lines that connect it with the rest of the lattice are of type 2. These operations are schematically represented in Fig.(4.3.7). The problem boils down to the calculation of

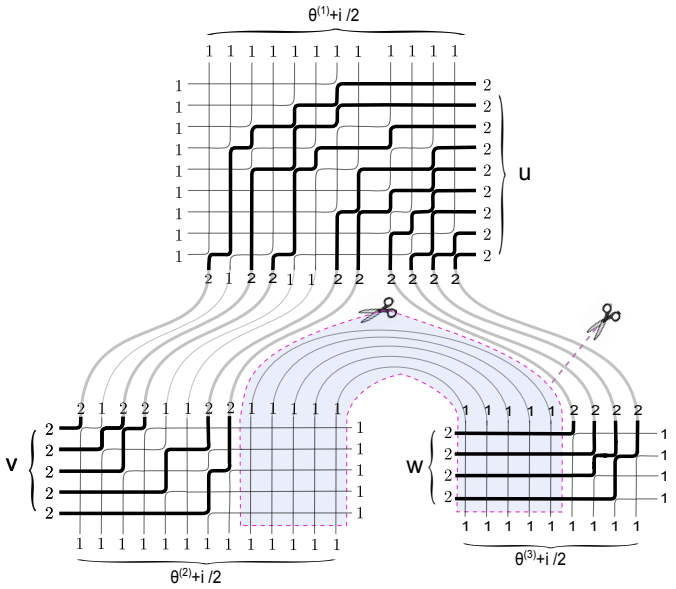

FIGURE 4.3.6: The cubic vertex in terms of the 6-vertex model configurations. The shaded area factorizes and can be cut out. Furthermore, the right piece is connected to the rest only with type-2 contractions and factorizes out. The lattice split into two pieces which can be evaluated separately.

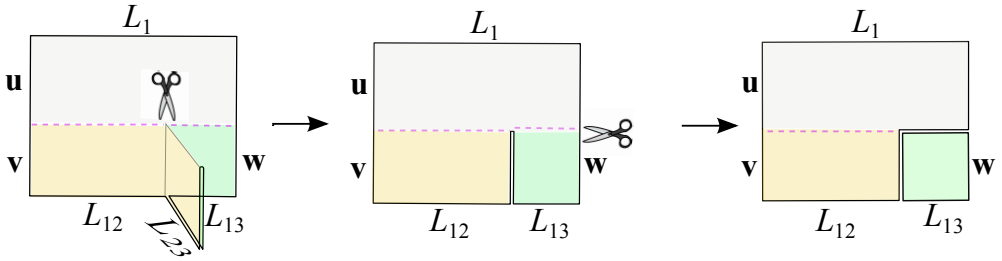

FIGURE 4.3.7: Schematic representation of the lattice obtained by gluing the rectangular lattices corresponding to the states $\langle \mathbf{u}|$, $|\mathbf{v}\rangle$, $|\mathbf{w}\rangle$, with subsequence removal of the redundant piece and separating the two non-interacting sub-lattices. After the removal of the redundant pieces, the states $|\mathbf{v}\rangle$ and $|\mathbf{w}\rangle$ are no more Bethe eigenstates, because the chains are shortened ($L_1 \to L_{12}$ and $L_3 \to L_{23}$).

two independent 6-vertex partition functions, which gives two non-trivial factors in the structure constant. These two factors will be computed by the *freezing procedure*.

The freezing procedure for the first factor works as the follows. One starts from a rectangular lattice corresponding to the scalar product $\langle \tilde{\mathbf{v}}|\mathbf{u}\rangle$. Both sets of rapidities have cardinality N_1. The first N_2 rapidities coincide with the rapidities $\mathbf{v}$ characterizing $\mathcal{O}_2$, the rest $N_3 = N_1 - N_2$ rapidities $\tilde{\mathbf{v}}$ will be denoted by $\tilde{v}_{N_2+1} = \tilde{z}_{L_{12}+1}, \cdots, \tilde{v}_{N_1} = \tilde{z}_{L_1}$. Symbolically we denote $\tilde{\mathbf{v}} = \mathbf{v} \cup \tilde{\mathbf{z}}$.

If we adjust the rapidity of the last magnon to the value of the last inhomogeneity, $\tilde{z}_{L_1} = \theta_{L_1} - i/2$, then according to (4.3.20) the vertex at the lower right corner is

necessary of the type c. Then the only possibilities for the rest of the vertices on the last row and the last column is that they are of type b. This is what we called "freezing". Hence the last row and the last column form a hooked index line carrying index 2, as is shown in the left figure of Fig.(4.3.8). This procedure is repeated N_3 times, the rapidities

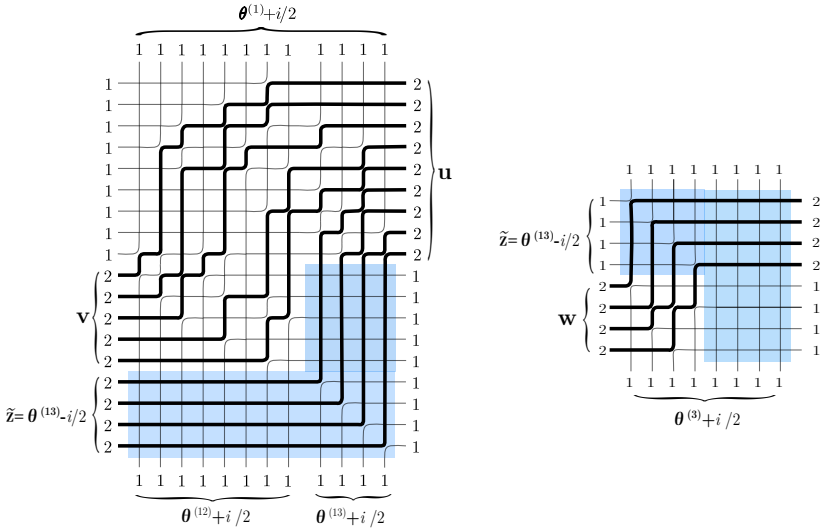

FIGURE 4.3.8: The freezing procedure for the two factors in $C^{(0)}_{123}$.

of the lowest N_3 rows are fixed to $\tilde{z} = \boldsymbol{\theta}^{(13)} - i/2$. The result is that the rightmost N_3 indices below the lowest horizontal u-line are fixed to have value 2. After removing the frozen part of the lattice, shaded in blue in the figure, we obtain the first factor in the cubic vertex equals the scalar product $\langle \mathbf{v} \cup \tilde{\mathbf{z}} | \mathbf{u} \rangle$. The contribution of the frozen part of the lattice is the product of all type c vertices on the diagonal, which equals $(-1)^{L_{13}}$, a factor that we will ignore. In terms of the algebraic Bethe ansatz, the freezing procedure is equivalent to

$$B(\tilde{z}_{L_{12}+1}) \cdots B(\tilde{z}_{L_1}) |\Omega\rangle = (-1)^{L_{13}} | \uparrow^{L_{12}} \downarrow^{L_{13}} \rangle. \tag{4.3.22}$$

In a similar way, we compute the second factor. The freezing procedure is shown in the right of Fig.(4.3.8). We start with a scalar product $\langle \mathbf{w}, \tilde{\mathbf{z}} \rangle$ for a chain of length $L_3 = N_3 + L_{23}$. We freeze the rapidities of the bra state to $\tilde{\mathbf{z}}$. The frozen area gives a contribution which is a pure phase if both sets of $\mathbf{w}$ and $\boldsymbol{\theta}^{(23)}$ are symmetric under complex conjugation. We will assume that this is the case and will ignore this factor. The rest of the lattice gives the second expression for the cubic vertex. We thus found that

$$\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v} \cup \tilde{\mathbf{z}}, \mathbf{u} \rangle_{\boldsymbol{\theta}^{(1)}} \langle \mathbf{w}, \tilde{\mathbf{z}} \rangle_{\boldsymbol{\theta}^{(3)}}. \tag{4.3.23}$$

The cubic vertex factorizes into the product of two scalar products of Bethe states. What's more, both scalar products can be evaluated by the Slavnov determinant formula since $|\mathbf{u}\rangle$ and $\langle\mathbf{w}|$ are on-shell. Now the structure constant reads

$$C_{123}^{(0)} = \sqrt{L_1 L_2 L_3}\frac{\langle \mathbf{v}\cup\tilde{\mathbf{z}},\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}}\langle \mathbf{w},\tilde{\mathbf{z}}\rangle_{\boldsymbol{\theta}^{(3)}}}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}}\langle \mathbf{v}|\mathbf{v}\rangle_{\boldsymbol{\theta}^{(2)}}\langle \mathbf{w}|\mathbf{w}\rangle_{\boldsymbol{\theta}^{(3)}}}}. \tag{4.3.24}$$

We need only compute five determinants and there is no sum over partitions of rapidities anymore. The result can be computed with arbitrary value of inhomogeneities. Then we can take the homogeneous limit, putting all the inhomogeneities to zero we obtain the result for the structure constant. The determinant representation for the three-point function is much simpler compared to the result from tailoring method. The formula also facilitates the study of semi-classical limit and the one-loop structure constant.

4.4 Semi-classical limit

We obtained a compact expression (4.3.24) in the last section. In this section we study the semi-classical limit of the structure constant. This same limit for the scalar product was reviewed in section 4.4. The result (4.3.24) contains five scalar products of the type studied in section 4.4 and it is straightforward to take the semi-classical limit of it.

Let us recall the form of structure constant in the determinant representation (4.3.24). We neglect the trivial factor $\sqrt{L_1 L_2 L_3}$ and take the semi-classical limit of the rest part. The dependence on $a(u)$ and $d(u)$ from the numerator and the denominator cancel each other and we replace the scalar product by the Slavnov determinant.

$$\begin{aligned}\log \mathscr{S}_{\mathbf{u},\mathbf{v}\cup\mathbf{z}} &= i\pi N_1 + \oint_{A_{\mathbf{v}}}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{iG_{\mathbf{u}}+iG_{\mathbf{v}}+iG_{\mathbf{z}}-iG_{\boldsymbol{\theta}}}\right) - \oint_{A_{\mathbf{u}}}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{-iG_{\mathbf{u}}+iG_{\mathbf{v}}+iG_{\mathbf{z}}}\right) \\ &= i\pi N + \oint_{A_{\mathbf{u}\cup\mathbf{v}}}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{iG_{\mathbf{u}}+iG_{\mathbf{v}}+iG_{\mathbf{z}}-iG_{\boldsymbol{\theta}}}\right) \\ &= i\pi N + \oint_{A_{\mathbf{u}\cup\mathbf{v}}}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{iG_{\mathbf{u}}+iG_{\mathbf{v}}-iG_{\boldsymbol{\theta}^{(12)}}}\right)\end{aligned} \tag{4.4.25}$$

where in the second line we have used the Bethe ansatz equation in the semi-classical limit

$$2\phi_{\mathbf{u}}(u) - G_{\boldsymbol{\theta}}(u) = 0, \quad u\in C_{\mathbf{u}}. \tag{4.4.26}$$

Here $2\not{G}_{\mathbf{u}}(u) = G_{\mathbf{u}}(u+i0) + G_{\mathbf{u}}(u-i0)$ is the principal part of the resolvent. The other factor in the numerator reads

$$\log \mathscr{S}_{\mathbf{w},\mathbf{z}} = i\pi N_3 + \oint \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{iG_{\mathbf{w}} - iG_{\boldsymbol{\theta}^{(23)}}} \right). \tag{4.4.27}$$

Let us introduce the three pseudo-momenta

$$p_{\mathbf{u}} = G_{\mathbf{u}} - \frac{1}{2} G_{\boldsymbol{\theta}^{(1)}}, \quad p_{\mathbf{v}} = G_{\mathbf{v}} - \frac{1}{2} G_{\boldsymbol{\theta}^{(2)}}, \quad p_{\mathbf{w}} = G_{\mathbf{w}} - \frac{1}{2} G_{\boldsymbol{\theta}^{(3)}}. \tag{4.4.28}$$

The semi-classical limit of the structure constant reads

$$\begin{aligned} \log C^{(0)}_{123} \simeq & \oint_{A_{\mathbf{u}\cup\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{i(p_{\mathbf{u}} + p_{\mathbf{v}} + \frac{1}{2} G_{\boldsymbol{\theta}^{(3)}})} \right) + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{i(p_{\mathbf{w}} + \frac{1}{2} G_{\boldsymbol{\theta}^{(2)}} - \frac{1}{2} G_{\boldsymbol{\theta}^{(1)}})} \right) \\ & - \frac{1}{2} \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{u}}} \right) - \frac{1}{2} \oint_{A_{\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{v}}} \right) - \frac{1}{2} \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{w}}} \right). \end{aligned} \tag{4.4.29}$$

The tree-level structure constant is obtained by sending all inhomogeneities to zero

$$\begin{aligned} \log C^{(0)}_{123} \simeq & \oint_{A_{\mathbf{u}\cup\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_{\mathbf{u}} + ip_{\mathbf{v}} + iL_3/2u} \right) + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_{\mathbf{w}} + i(L_2 - L_1)/2u} \right) \\ & - \frac{1}{2} \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{u}}} \right) - \frac{1}{2} \oint_{A_{\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{v}}} \right) - \frac{1}{2} \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{w}}} \right). \end{aligned} \tag{4.4.30}$$

In Chapter 7, we will see that the semi-classical limit of the same structure constant at one-loop takes the same form as (4.4.29), but with the inhomogeneities fixed to some specific value called the BDS values. We will perform the comparison with the strong coupling computation then.

Chapter 5

Towards All Sectors

In the last chapter, we computed three-point functions of planar $\mathcal{N} = 4$ SYM theory in the $\mathfrak{su}(2)$ sector at tree level. Our ultimate goal is to compute all OPE coefficients to all orders. Therefore the natural next steps are investigating other sectors and higher loops. In this chapter, we will discuss the results in $\mathfrak{su}(3)$ sector at tree level.

The computation of three-point functions at tree level for other sectors are also of combinatorics nature, but usually more complicated than the simplest $\mathfrak{su}(2)$ sector. The difficulties are due to several different reasons in different sectors. For the compact higher rank sectors like $\mathfrak{su}(3)$ sector, there are more possibilities for the choice of fundamental fields for the three operators and we need to compute more kinds of three-point functions. The operators are mapped to eigenvectors of higher rank spin chain Hamiltonian which can be constructed by the nested Bethe ansatz. However, the nested Bethe ansatz is more complicated than $\mathfrak{su}(2)$ case and the computation of scalar product between the Bethe states are also much harder. At present, there is no Slavnov determinant like formula for the $\mathfrak{su}(3)$ scalar products of on-shell/off-shell type in general. Only in some very special cases which we will discuss below the result can be written as two factorized $\mathfrak{su}(2)$ Slavnov determinant. Apart from the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ spin chains, little is known about the scalar product of Bethe states for higher rank spin chains.

In the non-compact sectors like $\mathfrak{sl}(2)$ sector and supersymmetric sectors like $\mathfrak{su}(1|1)$ sector, there is another complication. For these cases, the operators are no longer scalar operators but carry non-zero Lorentz spins. As we discussed in Chapter 2, the conformal symmetry fixes the form of the three-point functions of spinning operators up to a set of conformally covariant building blocks called tensor structures. Therefore we need to compute a set of structure constants, each one corresponds to one tensor structure. At present, there is no systematic way to determine all the structure constants. The

commonly used trick is to consider certain special configurations where there is only one tensor structure [52, 53] or all the tensor structures coincide [108].

The structure of this chapter is as the follows. We will start by introducing the nested coordinate and algebraic Bethe ansatz for the $\mathfrak{su}(3)$ spin chain in section 5.1. In section 5.2, we classify the $\mathfrak{su}(3)$ three-point functions and give prescriptions for the tailoring method. Section 5.3 is devoted to the special case where freezing method applies. In section 5.4 we take the semi-classical limit of the structure constant obtained in the determinant representation.

5.1 The nested Bethe ansatz

In this section, we introduce the nested Bethe ansatz for the $\mathfrak{su}(3)$ spin chain. Most results discussed below can be generalized readily to the $\mathfrak{su}(N)$ spin chain for $N > 3$. We will discuss both the nested coordinate Bethe ansatz and the algebraic Bethe ansatz.

5.1.1 Nested coordinate Bethe ansatz

The Hamiltonian of the $\mathfrak{su}(3)$ invariant Heisenberg spin chain takes the same form as the $\mathfrak{su}(2)$ case

$$H_{\mathfrak{su}(3)} = 2g^2 \sum_{k=1}^{L} (\mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1}) \tag{5.1.1}$$

where I and P are identity and permutation operators, respectively. At each site, the 'spin' has three different polarizations and the Hilbert space is $\mathbb{C}^3$. We denote the three polarization states by $|1\rangle, |2\rangle$ and $|3\rangle$. A generic state is a linear combination of the basis states $|s_1, s_2, \cdots, s_L\rangle$ where $s_k = 1, 2, 3$ $(k = 1, \cdots, L)$. We can view the state $|1\rangle$ as the 'vacuum' and the rest two polarizations as 'excitations'. It proves to be useful to think of the excitations in a nested fashion. For example, the state $|1, 2, 2, 3, 2, 3, 1\rangle$ can be generated by two steps. First, on top of the vacuum state $|1\rangle$ one creates only the states $|2\rangle$. Then on top of the excited states $|2\rangle$, one creates only the excitations $|3\rangle$

$$|1, 1, 1, 1, 1, 1, 1\rangle \to |1, 2, 2, 2, 2, 2, 1\rangle \to |1, 2, 2, 3, 2, 3, 1\rangle \tag{5.1.2}$$

Note that in this process, the excitations are generated level by level. The point of doing so is that at each level, we have only the vacuum and *one* kind of excitation. This is similar to the situation of $\mathfrak{su}(2)$ case and we can recycle the Bethe ansatz technique from the $\mathfrak{su}(2)$ spin chain. Before going to the construction of the Bethe state for $\mathfrak{su}(3)$

spin chain, we first discuss the coordinate Bethe ansatz for the *inhomogeneous* $\mathfrak{su}(2)$ spin chain. This is necessary because the 'vacuum' at each level is the 'excitation' of the previous level and comes with the corresponding rapdities u_k, which makes the vacuum inhomogeneous. The inhomogeneities are the rapidities of the previous level. Therefore, at each level except the first, we are in the same situation as the inhomogeneous $\mathfrak{su}(2)$ spin chain. We have already encountered this model in the framework of algebraic Bethe ansatz. For the coordinate Bethe ansatz, the recipe to take into account the inhomogeneities is simply replacing the phase factor e^{ipn} by

$$e^{ip_k n} \equiv \left(\frac{u_k + i/2}{u_k - i/2}\right)^n \to \left(\prod_{l=1}^{n-1} \frac{u - \theta_l + i/2}{u - \theta_l - i/2}\right) \cdot \frac{1}{u - \theta_n - i/2}. \tag{5.1.3}$$

The wave function $\psi(n_1, \cdots, n_N)$ corresponding to the ket state $\sigma_{n_1}^- \cdots \sigma_{n_N}^- |\Omega\rangle$ can be constructed straightforwardly. The first few terms read

$$\begin{aligned}\psi(n_1, \cdots, n_N|\boldsymbol{\theta}) = &\left(\prod_{k=1}^{n_1} \frac{(u_1 - \theta_k + i/2)^{\delta_{k\neq n_1}}}{u_1 - \theta_k - i/2}\right)\left(\prod_{k=1}^{n_2} \frac{(u_2 - \theta_k + i/2)^{\delta_{k\neq n_1}}}{u_2 - \theta_k - i/2}\right)\cdots \\ &\times S(u_2, u_1)\left(\prod_{k=1}^{n_1} \frac{(u_2 - \theta_k + i/2)^{\delta_{k\neq n_1}}}{u_2 - \theta_k - i/2}\right)\left(\prod_{k=1}^{n_2} \frac{(u_1 - \theta_k + i/2)^{\delta_{k\neq n_1}}}{u_1 - \theta_k - i/2}\right)\cdots + \cdots\end{aligned}$$

The Bethe equations are the same as derived before from ABA

$$\prod_{a=1}^{L} \frac{u_k - \theta_a + i/2}{u_k - \theta_a - i/2} = \prod_{j\neq k} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad k = 1, \cdots, N. \tag{5.1.4}$$

With these preparations, we are now ready to construct the Bethe states for the $\mathfrak{su}(3)$ spin chain. The Bethe state is a linear combination of the basis states

$$|\Psi\rangle = \sum_{s_1,\cdots,s_L} \psi_{s_1,\cdots,s_L} |s_1, \cdots, s_L\rangle, \quad s_k = 1, 2, 3. \tag{5.1.5}$$

We need an algorithm to construct the coefficients $\psi_{s_1,\cdots,s_L}$. From the nested point of view, we regard $|1\rangle$ as the vacuum, $|2\rangle$ as the first level excitation and $|3\rangle$ as the second level excitation. We can use two sets of integers to label the basis as

$$|\{n_{1,1}, \cdots, n_{1,N}\}, \{n_{2,1}, \cdots, n_{2,M}\}\rangle. \tag{5.1.6}$$

The first set of integers $\{n_{1,1}, \cdots, n_{1,N}\}$ labels the positions of the first level excitations on the *original* spin chain. The second set of integers $\{n_{2,1}, \cdots, n_{2,M}\}$ label the positions of the second level excitations on the *reduced* spin chain formed by the first excitations.

It is obvious that $M \leq N$. Our example in (5.1.2) can be labeled by

$$|1,2,2,3,2,3,1\rangle = |\{2,3,4,5,6\},\{3,5\}\rangle. \tag{5.1.7}$$

We denote the first and second level excitations by $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively. For a given set of $\{n_{1,1},\cdots,n_{1,N}\}$, the set of rapidities can have different permutations, denoted by σ. Each permutation gives rise to an reduced $\mathfrak{su}(2)$ spin chain with the inhomogeneities $\{u_{1,\sigma_1},\cdots,u_{1,\sigma_N}\}$. The wave function can be constructed level by level. For the first level, by the recipe of the $\mathfrak{su}(2)$ coordinate Bethe ansatz

$$\psi(\mathbf{n}_1;\mathbf{n}_2) = \sum_{\sigma} S_1(\sigma) \prod_{j=1}^{N} \left(\frac{u_{1,\sigma_j}+i/2}{u_{1,\sigma_j}-i/2}\right)^{n_{1,j}} \psi_2(\mathbf{n}_2|\sigma) \tag{5.1.8}$$

where $S_1(\sigma)$ has the usual property

$$S_1(1,\cdots,N)=1, \quad \frac{S_1(\cdots,j,k,\cdots)}{S_1(\cdots,k,j,\cdots)} = \frac{u_j-u_k+i}{u_j-u_k-i}. \tag{5.1.9}$$

Apart from the factor in red color the wave function (5.1.8) is exactly the coordinate Bethe ansatz for the $\mathfrak{su}(2)$ spin chain. The extra factor can again be constructed by the coordinate Bethe ansatz of the inhomogeneous spin chain

$$\psi_2(\mathbf{n}_2|\sigma) = \sum_{\pi} S_2(\pi) \prod_{j=1}^{M} \left(\prod_{a=1}^{n_{2,j}} \frac{(u_{2,j}-u_{1,a}+i/2)^{\delta_{a\neq n_j^{(2)}}}}{u_{2,j}-u_{1,a}-i/2} \right) \tag{5.1.10}$$

where π denotes the set of permutations of the second set of rapidities $\mathbf{u}_2$. Combining (5.1.8) and (5.1.10) we have a well defined procedure to construct the wave function $\psi(\mathbf{n}_1;\mathbf{n}_2)$. Multiply this wave function with the corresponding ket state $|\mathbf{n}_1,\mathbf{n}_2\rangle$ and sum over all possible sets of $\mathbf{n}_1$ and $\mathbf{n}_2$ we obtain the corresponding Bethe state with $M-N$ excitations of type $|2\rangle$ and M excitations of type $|3\rangle$

$$|\Psi\rangle = \sum_{\mathbf{n}_1}\sum_{\mathbf{n}_2} \psi(\mathbf{n}_1;\mathbf{n}_2)|\mathbf{n}_1,\mathbf{n}_2\rangle. \tag{5.1.11}$$

The periodicity of the wave function leads to the Bethe equations. Now that we have two sets of rapidities, we also have two sets of Bethe equations

$$\begin{aligned}
&\left(\frac{u_{1,j}+i/2}{u_{1,j}-i/2}\right)^{L} = \prod_{k\neq j}^{N} \frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i} \prod_{k=1}^{M} \frac{u_{1,j}-u_{2,k}-i/2}{u_{1,j}-u_{2,k}+i/2}, \\
&\prod_{k=1}^{N} \frac{u_{1,j}-u_{2,k}+i/2}{u_{1,j}-u_{2,k}-i/2} = \prod_{k\neq j}^{M} \frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i}.
\end{aligned} \tag{5.1.12}$$

We recognize that the second set of Bethe equations are nothing but the Bethe equation of the $\mathfrak{su}(2)$ inhomogeneous chain. The corresponding eigenvalue of the Bethe state is given by

$$E = 2g^2 \sum_{k=1}^{N} \frac{1}{u_{1,k}^2 + 1/4} \tag{5.1.13}$$

which involves only the rapidities $\mathbf{u}_1$. The procedure we described above can be generalized to the more general $\mathfrak{su}(N)$ and $\mathfrak{su}(M|N)$ spin chain. The only complication is that there are more levels of nesting and different levels might contain different kind of rank 1 spin chains for the $\mathfrak{su}(M|N)$ case. For a useful introduction, we refer to [109]. From the explicit construction of the Bethe states, we see that the nested Bethe states are considerably more complicated compared to the $\mathfrak{su}(2)$ Bethe state. As a result, the scalar products between the Bethe states are also much more involved.

5.1.2 Nested algebraic Bethe ansatz

In this subsection, we describe the construction of algebraic Bethe ansatz for the $\mathfrak{su}(3)$ spin chain. Again, the discussions below can be generalized to the $\mathfrak{su}(N)$ and $\mathfrak{su}(M|N)$ case. The nested Bethe ansatz for the $\mathfrak{su}(N)$ spin chain was developed in [110]. For a more transparent explanation, we refer to [111] and [112]. The starting point of the algebraic Bethe ansatz is the $\mathfrak{su}(3)$ invariant R-matrix. It is an operator acting on the tensor product of two spaces $\mathbb{C}^3_a \otimes \mathbb{C}^3_b$ given by

$$R_{ab}(u) = \mathrm{I}_{ab} + \frac{i}{u}\mathrm{P}_{ab} \tag{5.1.14}$$

where I_{ab} and P_{ab} are identity and permutation operators respectively. The permutation operator can be written in terms of the basis matrices E^{ij} as

$$\mathrm{P}_{ab} = \mathrm{E}^{ij}_a \otimes \mathrm{E}^{ji}_b, \quad i,j = 1,2,3 \tag{5.1.15}$$

where a sum over repeated indices is understood. The basis matrix E^{ij} is the matrix with the element at position (i,j) being 1 and the rest elements being zero, namely $(\mathrm{E}^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Using the properties of the permutation operator, one can check that the R-matrix (5.1.14) satisfies the Yang-Baxter equation

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \tag{5.1.16}$$

The monodromy matrix is defined as

$$T_a(u) = \prod_{n=1}^{L} R_{an}(u - \theta_n) \tag{5.1.17}$$

where $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_L\}$ are the inhomogeneities. The auxiliary space is chosen to be $\mathbb{C}^3$ and the corresponding monodromy matrix is 3×3 given by

$$T_a^{(1)}(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) & t_{13}(u) \\ t_{21}(u) & t_{22}(u) & t_{23}(u) \\ t_{31}(u) & t_{32}(u) & t_{33}(u) \end{pmatrix}. \tag{5.1.18}$$

The nested picture is still valid in ABA, in a slightly different way. The index '(1)' on the monodromy matrix indicates that it is the monodromy matrix of the first level. The monodromy matrix satisfies the RTT relation

$$R_{ab}^{(1)}(u - v) T_a^{(1)}(u) T_b^{(1)}(v) = T_b^{(1)}(v) T_a^{(1)}(u) R_{ab}^{(1)}(u - v). \tag{5.1.19}$$

In terms of elements, (5.1.19) leads to the quadratic algebra between the elements of monodromy matrix.

The next step is to construct the Bethe states using the elements of the monodromy matrix. We need to start with the pseudovacuum state $|\Omega\rangle$. In the $\mathfrak{su}(2)$ case, it is the state which is annihilated by the C operator and is the eigenstate of the diagonal A and D operators. These conditions are easily generalized to the $\mathfrak{su}(3)$ and the pseudovacuum state is defined as

$$t_{ii}(u)|\Omega\rangle = a_i(u)|\Omega\rangle, \quad t_{ij}|\Omega\rangle = 0, \quad i, j = 1, 2, 3 \quad \text{and} \quad i > j, \tag{5.1.20}$$

which states that the pseudovacuum is the eigenstate of all diagonal elements and is annihilated by all the lower triangular elements. For the $\mathfrak{su}(3)$ spin chain, we can choose the pseudovacuum to be the state $|\Omega\rangle \equiv |1^L\rangle$, which means at every site we have state $|1\rangle$. In order to make the nested structure manifest, we introduce the following decomposition of the monodromy matrix

$$B_a^{(1)} = \begin{pmatrix} t_{12}(u) & t_{13}(u) \end{pmatrix}, \quad C_a^{(1)} = \begin{pmatrix} t_{21}(u) \\ t_{31}(u) \end{pmatrix}, \tag{5.1.21}$$

$$A^{(1)} = t_{11}(u), \qquad D_a^{(1)} = \begin{pmatrix} t_{22}(u) & t_{23}(u) \\ t_{32}(u) & t_{33}(u) \end{pmatrix}.$$

Here the index 'a' is an abstract index indicating the auxiliary space and is not the index that labels the elements. Later we shall see that the 2×2 matrix $D_a(u)$ can be

regarded as the monodromy matrix of the reduced $\mathfrak{su}(2)$ spin chain at the second level. The second level R-matrix $R^{(2)}(u)$ is the R-matrix for the $\mathfrak{su}(2)$ spin chain

$$R^{(2)}_{ab}(u) = \mathrm{I}^{(2)} + \frac{i}{u}\mathrm{P}^{(2)}_{ab} = \begin{pmatrix} f(u) & 0 & 0 & 0 \\ 0 & 1 & g(u) & 0 \\ 0 & g(u) & 1 & 0 \\ 0 & 0 & 0 & f(u) \end{pmatrix} \tag{5.1.22}$$

where

$$f(u) = 1 + \frac{i}{u}, \qquad g(u) = \frac{i}{u}. \tag{5.1.23}$$

The operators $B^{(1)}_a(u)$ are the creation operators of the first level. $t_{12}(u)$ and $t_{13}(u)$ create excitations $|2\rangle$ and $|3\rangle$ from the vacuum state $|1\rangle$, respectively. The $C^{(1)}_a$ are the corresponding annihilation operators. The creation operator of the second level is $t_{23}(u)$, which creates the second level excitation $|3\rangle$ from the second level vacuum $|2\rangle$. The algebra between of the elements of the monodromy matrix can be given in terms of the decomposed operators. We list some of the relevant relations below

$$B^{(1)}_a(u)B^{(1)}_b(v) = B_b(v)^{(1)}B_a(u)^{(1)}\mathbb{R}^{(2)}_{ab}(u-v) \tag{5.1.24}$$

$$A^{(1)}(u)B^{(1)}_a(v) = f(u-v)B^{(1)}_a(v)A^{(1)}(u) - g(u-v)B^{(1)}_a(u)A^{(1)}(v) \tag{5.1.25}$$

$$D^{(1)}_a(u)B^{(1)}_b(v) = f(u-v)B^{(1)}_b(v)D^{(1)}_a(u)\mathbb{R}^{(2)}_{ab}(u-v) - g(u-v)B^{(1)}_b(u)D^{(1)}_a(v)\mathbb{R}^{(2)}_{ab}. \tag{5.1.26}$$

$$R^{(2)}_{ab}(u-v)D^{(1)}_a(u)D^{(1)}_b(v) = D^{(1)}_b(v)D^{(1)}_a(u)R^{(2)}_{ab}(u-v) \tag{5.1.27}$$

where $\mathbb{R}^{(2)}_{ab}(u) = R^{(2)}_{ab}/f(u)$. From (5.1.27) it is obvious that $D_a(u)$ satisfies the RTT relation of the $\mathfrak{su}(2)$ spin chain, so it can be seen as the monodromy matrix of an $\mathfrak{su}(2)$ spin chain, which is the reduced spin chain in the nested CBA.

The transfer matrix is defined by

$$\mathcal{T}(u) = \mathrm{tr}_a\, T^{(1)}_a(u) = t_{11}(u) + t_{22}(u) + t_{33}(u) = A^{(1)}(u) + \mathrm{tr}_a\, D^{(1)}_a(u). \tag{5.1.28}$$

The eigenstates of the transfer matrix can be constructed by acting the creation operators on the pseudovacuum. In the $\mathfrak{su}(3)$ case, we have two choices of the creation operators. Let us denote $B_1(u) = t_{12}(u)$ and $B_2(u) = t_{13}(u)$. An eigenstate can be

written as the following linear combination

$$|\Psi\rangle = \sum_{i_1,\cdots,i_N} \psi_{i_1,\cdots,i_N} B_{i_1}(u_1)\cdots B_{i_N}(u_N)|\Omega\rangle, \quad i_k = 1,2. \tag{5.1.29}$$

The question is to determine the appropriate coefficients $\psi_{i_1,\cdots,i_N}$. Recall that in the $\mathfrak{su}(2)$ spin chain, for a spin chain of length N, the Bethe state takes the following form

$$|\Psi\rangle_{\mathfrak{su}(2)} = \sum_{s_1,\cdots,s_N} \psi^{\mathfrak{su}(2)}_{s_1,\cdots,s_N} |s_1,\cdots,s_N\rangle \tag{5.1.30}$$

where the coefficient $\psi^{\mathfrak{su}(2)}_{s_1,\cdots,s_N}$ can be constructed by Bethe ansatz. The crucial idea here is to identify the indices of the creation operators $\{i_1,\cdots,i_N\}$ with an $\mathfrak{su}(2)$ spin chain state $|i_1,\cdots,i_N\rangle$ and the coefficients $\psi_{i_1,\cdots,i_N}$ in (5.1.29) with the one $\psi^{\mathfrak{su}(2)}_{s_1,\cdots,s_N}$ in (5.1.30). Let us denote the creation operator of the second level as $B^{(2)} = t_{23}(u)$, we have

$$B^{(2)}(v_1)\cdots B^{(2)}(v_M)|\uparrow^N\rangle = \sum_{i_1,\cdots,i_N} \psi_{i_1,\cdots,i_N}|i_1,\cdots,i_N\rangle, \quad M \leq N. \tag{5.1.31}$$

In the nested Bethe ansatz, the ket state $|i_1,\cdots,i_N\rangle$ in fact corresponds to the choice of first level creation operators. For example, the state $|\uparrow\downarrow\uparrow\rangle$ corresponds to the first level ket state $B_1(u_1)B_2(u_2)B_1(u_3)|\Omega\rangle$. Multiplying the first level state with the corresponding coefficient $\psi_{i_1,\cdots,i_N}$ and sum over all possible terms, we obtain the Bethe state. To sum up, The $\mathfrak{su}(3)$ off-shell Bethe state can be written as

$$|\Psi\rangle = B_{i_1}(u_1)\cdots B_{i_N}(u_N)|\Omega\rangle \otimes B^{(2)}(v_1)\cdots B^{(2)}(v_M)|\uparrow^N\rangle. \tag{5.1.32}$$

In order to see how this formula works and its exact meaning, we give a simple example. Consider the Bethe state

$$|\Psi\rangle = B_{i_1}(u_1)B_{i_2}(u_2)|\Omega\rangle \otimes B^{(2)}(v)|\uparrow\uparrow\rangle \tag{5.1.33}$$

We first evaluate the second level, which gives

$$B^{(2)}(v)|\uparrow\uparrow\rangle = c_1|\downarrow\uparrow\rangle + c_2|\uparrow\downarrow\rangle. \tag{5.1.34}$$

where c_1 and c_2 can be determined easily. Each ket state specifies a choice of first level creation operators and the weight for this term is given by the corresponding coefficient.

By this rule the real meaning of (5.1.33) is that

$$\begin{aligned}|\Psi\rangle &= B_{i_1}(u_1)B_{i_2}(u_2)|\Omega\rangle \otimes B^{(2)}(v)|\uparrow\uparrow\rangle \\ &= c_1 B_2(u_1)B_1(u_2)|\Omega\rangle + c_2 B_1(u_1)B_2(u_2)|\Omega\rangle.\end{aligned} \tag{5.1.35}$$

In order for Bethe states to be on-shell, the rapidities need to satisfy the Bethe ansatz equations. In the framework of ABA, they can be worked out by using the commutation relations between the elements of monodromy matrix (5.1.24),(5.1.25),(5.1.26) and requiring that the 'unwanted terms' vanish, as in the case of the $\mathfrak{su}(2)$ algebraic Bethe ansatz. The derivation can be found for example in [110] and [112]. We present the equations following the convention of [112]

$$\begin{aligned}\frac{a_1(u_{1,j})}{a_2(u_{1,j})} &= \prod_{k\neq j}^{N}\frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i}\prod_{k=1}^{M} f(u_{2,k}-u_{1,j}), \\ \frac{a_2(u_{2,j})}{a_3(u_{2,j})} &= \prod_{k\neq j}^{M}\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i}\prod_{k=1}^{N}\frac{1}{f(u_{2,j}-u_{1,k})}.\end{aligned} \tag{5.1.36}$$

where $f(u)$ is defined in (5.1.23).

5.2 Three-Point functions in the $\mathfrak{su}(3)$ sector

In this section, we consider three-point functions in the $\mathfrak{su}(3)$ sector. We start by classifying all possible configurations of three-point functions when at least one of the operators is in the $\mathfrak{su}(3)$ sector and with definite R-charges. Then we give the prescription for the computation using tailoring method for all the configurations. The freezing method does not work for the general three-point functions in $\mathfrak{su}(3)$, but only for one configuration. We will then focus on this case and compute the three-point function using freezing method. In the special limit when the second set of rapidities go to infinity, the scalar product of $\mathfrak{su}(3)$ Bethe states can be written as product of determinants. Applying the result in the $\mathfrak{su}(2)$ case, it is straightforward to take the semi-classical limit.

5.2.1 Classification of $\mathfrak{su}(3)$ three-point functions

In the $\mathfrak{su}(2)$ sector, there is only one non-trivial configuration of three-point functions. In the presence of one or more operators from an $\mathfrak{su}(3)$ sector, the structure of the three-point functions becomes richer and we need to classify the set of possible non-trivial configurations of structure constants. An example of a set of planar contractions is given in Fig.(5.2.1). The contractions $\langle X\bar{X}\rangle$, $\langle Y\bar{Y}\rangle$ and $\langle Z\bar{Z}\rangle$ are represented respectively by

solid lines, red wavy lines and dashed lines. Let us introduce some conventions. There are

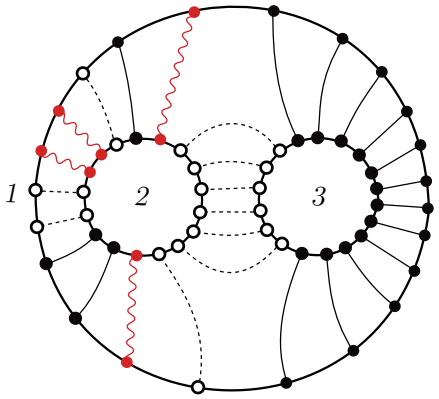

FIGURE 5.2.1: Planar contractions contributing to the tree-level 3-point function. The contractions $\langle X\bar{X}\rangle, \langle Y\bar{Y}\rangle$ and $\langle Z\bar{Z}\rangle$ are represented respectively by black solid lines, blue solid lines and dashedlines.

several possible choices of an $\mathfrak{su}(3)$ sector, which correspond to a choice of three distinct complex scalar fields X, Y, Z, $\bar{X}$, $\bar{Y}$, $\bar{Z}$, with pairs of mutually conjugated fields, like Z and $\bar{Z}$, excluded. When only two types of non-conjugate scalar fields are chosen, the composite operator belongs to an $\mathfrak{su}(2)$ sector. If $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ belong to $\mathfrak{su}(\alpha)$, $\mathfrak{su}(\beta)$ and $\mathfrak{su}(\gamma)$ sectors respectively, then the corresponding three-point function of type $\{\alpha, \beta, \gamma\}$ $(\alpha, \beta, \gamma = 2, 3)$. By permutation invariance, the order of α, β, γ is irrelevant.

We represent the different classes of correlation functions schematically by specifying the different types of Wick contractions between pairs of operators. For example, the correlation functions corresponding to Fig.(5.2.1) belong to the class $\{2, 3, 3\}$ in Fig.(5.2.2). We call the operator at the bottom $\mathcal{O}_1$, the one at right $\mathcal{O}_3$ and the one at left $\mathcal{O}_2$. Exchanging a scalar field and its complex conjugate in all the operators does not change the value of the structure constant. This enables us to choose $\mathcal{O}_1$ such that it contains only the scalar fields X, Y and Z. Since we are interested in the large N_c limit, only planar contractions are retained. We start by classifying the type-$\{3, 3, 3\}$ structure constants. In this case, there are two non-trivial inequivalent configurations, as is shown in Fig.(5.2.3). Deleting one line, that is, one type of Wick contractions, from each of these two configurations, one obtains type-$\{2, 3, 3\}$ structure constants. There are three such configurations, as shown in Fig.(5.2.2) and Fig.(5.2.4). Deleting one line from the configurations in Fig.(5.2.4), one obtains a type-$\{2, 2, 3\}$ or a type-$\{2, 2, 2\}$ 3-point functions. The latter is a pure $\mathfrak{su}(2)$ three-point function of the type studied before. There is one configuration of type-$\{2, 2, 3\}$, as in Fig.(5.2.5). To summarize, there are six non-trivial types of three-point functions with at least one $\mathfrak{su}(3)$ operator. We want to mention here that the operators under consideration are all with definite R-charge and no global rotation in the $\mathfrak{so}(6)$ internal space is considered.

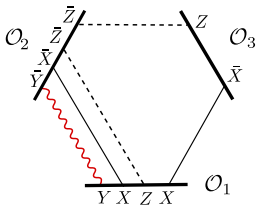

FIGURE 5.2.2: Schematic representation of the type-$\{2,3,3\}$ correlation function from Fig. 5.2.1.

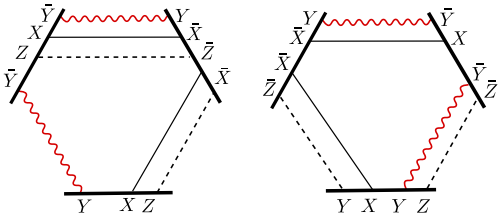

FIGURE 5.2.3: The two type-$\{3,3,3\}$ structure constants.

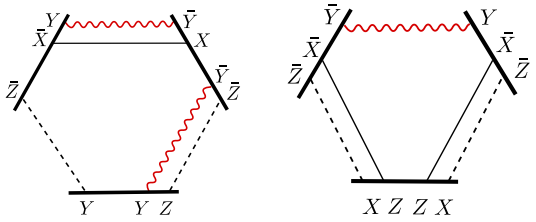

FIGURE 5.2.4: The remaining two type-$\{2,3,3\}$ structure constants.

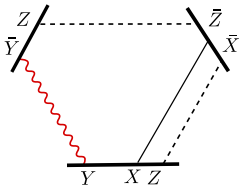

FIGURE 5.2.5: The type-$\{2,2,3\}$ structure constant.

5.2.2 Tailoring the tree-level three-point function

The framework of tailoring method is general enough to be applied to all the cases classified above. As in the $\mathfrak{su}(2)$ case, we construct the structure constant in three steps. **1.** We split the algebraic Bethe Ansatz representation of each spin chain into two: a left sub-chain and a right sub-chain. **2.** We flip each left sub-chain from an ket state to an bra state. **3.** We take the scalar products of the left sub-chain state of $\mathcal{O}_i$ with the right sub-chain state of $\mathcal{O}_{i+1 \bmod 3}$ $(i = 1, 2, 3)$. Finally, we normalize the three external states. In the first and second step, we will generate non-trivial factors which can be worked out explicitly. We present the result in the following. We give our setup data in the table below.

Operators	Length	Rapidities	No. of Rapidities	Partitions of Rapidities
$\mathcal{O}_1$	L_1	$\mathbf{u}_1, \mathbf{u}_2$	$\#\ \mathbf{u}_1{=}N_1,\ \#\ \mathbf{u}_2{=}M_1$	$\mathbf{u}'_1 \cup \mathbf{u}''_1 = \mathbf{u}_1,\ \mathbf{u}'_2 \cup \mathbf{u}''_2 = \mathbf{u}_2$
$\mathcal{O}_2$	L_2	$\mathbf{v}_1, \mathbf{v}_2$	$\#\ \mathbf{v}_1{=}N_2,\ \#\ \mathbf{v}_2{=}M_2$	$\mathbf{v}'_1 \cup \mathbf{v}''_1 = \mathbf{v}_1,\ \mathbf{v}'_2 \cup \mathbf{v}''_2 = \mathbf{v}_2$
$\mathcal{O}_3$	L_3	$\mathbf{w}_1, \mathbf{w}_2$	$\#\ \mathbf{w}_1{=}N_3,\ \#\ \mathbf{w}_2{=}M_3$	$\mathbf{w}'_1 \cup \mathbf{w}''_1 = \mathbf{w}_1,\ \mathbf{w}'_2 \cup \mathbf{w}''_2 = \mathbf{w}_2$

The lengths of the left subchains are

$$\begin{aligned} L_{13} &= \frac{1}{2}(L_1 + L_3 - L_2) \,, \\ L_{12} &= \frac{1}{2}(L_1 + L_2 - L_3) \,, \\ L_{23} &= \frac{1}{2}(L_2 + L_3 - L_1) \,. \end{aligned} \tag{5.2.37}$$

The structure constant reads

$$C_{123}^{(0)} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \sum_{\mathbf{u}',\mathbf{v}',\mathbf{w}'} \mathrm{H}_F^{\mathbf{u}} \, \mathrm{H}_F^{\mathbf{v}} \, \mathrm{H}_F^{\mathbf{w}} \, \langle \mathbf{u}''^* | \mathbf{v}' \rangle \, \langle \mathbf{v}''^* | \mathbf{w}' \rangle \, \langle \mathbf{w}''^* | \mathbf{u}' \rangle \,, \tag{5.2.38}$$

where $\mathcal{N}_i$ are the square of norms of the Bethe states [1]:

$$\mathcal{N}_1 = \langle \mathbf{u} | \mathbf{u} \rangle \,, \qquad \mathcal{N}_2 = \langle \mathbf{v} | \mathbf{v} \rangle \,, \qquad \mathcal{N}_3 = \langle \mathbf{w} | \mathbf{w} \rangle \,. \tag{5.2.39}$$

The H_F factors are given by

$$\begin{aligned} \mathrm{H}_F^{\mathbf{u}} &= S_1(\mathbf{u}_1'', \mathbf{u}_0) \; S_1(\mathbf{u}_2'', \mathbf{u}_1) S_2^{>}(\mathbf{u}_1, \mathbf{u}_1'') S_2^{>}(\mathbf{u}_2, \mathbf{u}_2'') \\ \mathrm{H}_F^{\mathbf{v}} &= S_1(\mathbf{v}_1'', \mathbf{v}_0) \; S_1(\mathbf{v}_2'', \mathbf{v}_1) \; S_2^{>}(\mathbf{v}_1, \mathbf{v}_1'') \; S_2^{>}(\mathbf{v}_2, \mathbf{v}_2'') \\ \mathrm{H}_F^{\mathbf{w}} &= S_1(\mathbf{w}_1'', \mathbf{w}_0) \; S_1(\mathbf{w}_2'', \mathbf{w}_1) \; S_2^{>}(\mathbf{w}_1, \mathbf{w}_1'') \; S_2^{>}(\mathbf{w}_2, \mathbf{w}_2'') \,. \end{aligned} \tag{5.2.40}$$

with $\mathbf{u}_0 = \{0^{L_1+1}\}$, $\mathbf{v}_0 = \{0^{L_2+1}\}$ and $\mathbf{w}_0 = \{0^{L_3+1}\}$. In the previous formula we used the following notations: we denote the scattering factors as

$$S_\sigma(u_{a,i}, u_{b,j}) = \frac{u_{a,i} - u_{b,j} + \frac{i}{2}\sigma}{u_{a,i} - u_{b,j} - \frac{i}{2}\sigma} \,, \qquad \sigma = 1, 2 \,, \tag{5.2.41}$$

and, given a function $F(x, y)$ and two sets of variables $\mathbf{u}$, $\mathbf{v}$, we define

$$F(\mathbf{u}, \mathbf{v}) \equiv \prod_{u_i \in \mathbf{u}, \; v_j \in \mathbf{v}} F(u_i, v_j), \quad F^{>}(\mathbf{u}, \mathbf{v}) \equiv \prod_{\substack{i>j \\ u_i \in \mathbf{u}, \; v_j \in \mathbf{v}}} F(u_i, v_j) \,. \tag{5.2.42}$$

The proportionality factor between ABA and CBA Bethe state: $|\mathbf{u}\rangle_{\text{alg}} = c_{\mathbf{u}} \, |\mathbf{u}\rangle_{\text{cor}}$ is given by

$$c_{\mathbf{u}} = i^{N+M} \prod_{a=1,2} \prod_{j<k} \frac{u_{a,j} - u_{a,k} + i}{u_{a,j} - u_{a,k}}. \tag{5.2.43}$$

While the formula (5.2.38) can be explicitly used for a small numbers of magnons, it is not adapted for taking the classical limit where the number of magnons is large. The

1. In this section all scalar products and norms are understood in the Coordinate Bethe Ansatz normalization.

main obstruction for taking the classical limit of (5.2.38) is that the scalar products involved are between off-shell states, and there is no closed form expression such as a determinant for this scalar product.

5.3 Determinant representation

In this sections, we restrict our attention to a particular situation where the three-point function can be written in terms of a scalar products of an off-shell state and an on-shell state. The configuration we shall focus on is the one in Fig.(5.2.2) and Fig.(5.2.1).

5.3.1 The $\mathfrak{su}(3)$ cubic vertex in terms of scalar products

We consider that the three operators, $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, are described by three sets of rapidities $\mathbf{u} = \{\mathbf{u}_1, \mathbf{u}_2\}, \mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{w} = \{\mathbf{w}_1, \mathbf{w}_2\}$ with cardinalities respectively $N_1 + M_1, N_2 + M_2$ and $N_3 + M_3$. In the configuration we are considering, $\mathbf{w}_2 = \emptyset$, since $\mathcal{O}_3$ is an $\mathfrak{su}(2)$ operator. We have two types of contributions to the correlation function:

– the contribution of the $\langle Z\bar{Z}\rangle$ contractions between the operators $\mathcal{O}_2$ and $\mathcal{O}_3$, through the factor $\langle \tilde{\mathbf{z}}_1, \mathbf{w}_1\rangle$, with $\tilde{\mathbf{z}}_1 = \boldsymbol{\theta}^{(13)} - i/2$ and $\tilde{\mathbf{z}}_1 = N_3$,

– the remaining contractions, which can be recast as the inner product $\langle \mathbf{v}\cup\tilde{\mathbf{z}}, \mathbf{u}\rangle$ between an on-shell vector of a spin chain with length L_1 and rapidities $\mathbf{u} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and an off-shell state with the same length and rapidities $(\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2\})\cup(\tilde{\mathbf{z}} = \{\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2\})$, with $\tilde{\mathbf{z}}_1 = \boldsymbol{\theta}^{(13)} - i/2$ and $\tilde{\mathbf{z}}_2 = \boldsymbol{\theta}^{(13)} - i$.

Below we evaluate, using the freezing argument, the $\{2, 3, 3\}$ type structure constant,

$$C^{(0)}_{123} = \frac{\langle \mathbf{u}, \mathbf{v}, \mathbf{w}_1\rangle^{\mathrm{su}(3)}}{\sqrt{\langle \mathbf{u}, \mathbf{u}\rangle^{\mathrm{su}(3)}\langle \mathbf{v}, \mathbf{v}\rangle^{\mathrm{su}(3)}\langle \mathbf{w}_1, \mathbf{w}_1\rangle}} \tag{5.3.44}$$

We will show that the corresponding cubic vertex is given by

$$\langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle^{\mathrm{su}(3)} = \langle \mathbf{v}\cup\tilde{\mathbf{z}}, \mathbf{u}\rangle^{\mathrm{su}(3)}_{\boldsymbol{\theta}^{(1)}}\langle \mathbf{w}_1, \tilde{\mathbf{z}}_1\rangle_{\boldsymbol{\theta}^{(3)}} \tag{5.3.45}$$

where

$$\mathbf{z} = \{\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2\} = \{\boldsymbol{\theta}^{(13)} - i/2, \boldsymbol{\theta}^{(13)} - i\}. \tag{5.3.46}$$

Here $\langle\cdot,\cdot\rangle$ denotes, as before, the $\mathfrak{su}(2)$ inner product, and $\langle\cdot,\cdot\rangle^{\mathrm{su}(3)}$ denotes the $\mathfrak{su}(3)$ inner product.

5.3.1.1 The $\mathfrak{su}(3)$ Bethe states in terms of the 15-vertex-model

In order to generalize the freezing procedure to $\mathfrak{su}(3)$, let us first show how to represent the components of the $\mathfrak{su}(3)$ Bethe vectors in terms of configurations of a 15-vertex model shown in Fig.(5.3.6). The vertices are similar to those from Fig.(4.3.2), with the difference that the indices carried by the lines can be now $1, 2$ or 3. We represent them graphically by thin, red and black lines, respectively. The weights are identical to those from the 6-vertex model corresponding to the $\mathfrak{su}(2)$ spin chain (4.3.20), depending on whether the indices carried by the lines are equal or different. The Bethe vector $|\mathbf{u}\rangle$ is

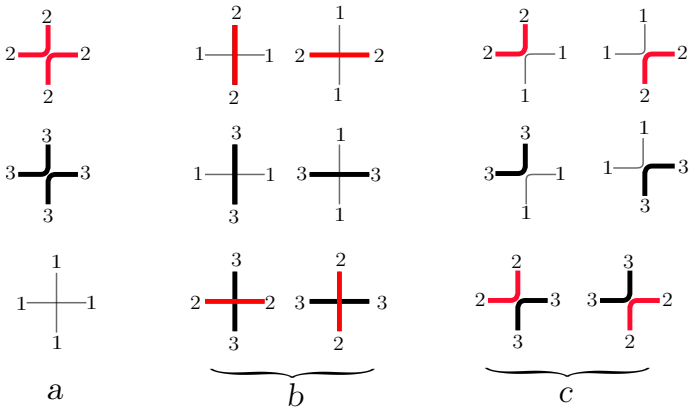

FIGURE 5.3.6: Graphical representation of the 15 non-zero elements of the $su(3)$ L -matrix, Eq. 3.2.27. The rapidities u and $z = \theta + i/2$ are associated with the horizontal and the vertical lines, respectively.

given by the expansion

$$|\mathbf{u}\rangle = \sum_{s_1,\ldots,s_L=1}^{3} \psi_{s_1,\ldots,s_L}(\mathbf{u})\ |s_1,\ldots,s_L\rangle \tag{5.3.47}$$

where $\psi_{s_1,\ldots,s_L}(\mathbf{u})$ is a sum over all the possible 15-vertex configurations on a rectangular lattice with L_1+N_1 vertical lines and N_1+M_1 horizontal lines, with the free spin indices equal to $s_1,\ldots,s_L$. An example for such a vertex configuration is given in Fig.(5.3.7). The first L_1 vertical lines carry rapidities $\theta_1^{(1)}+i/2,\ldots,\theta_{L_1}^{(1)}+i/2$ and spin indices 1 on the top, which correspond to the vacuum $|\Omega\rangle = |1^L\rangle \equiv |11\ldots1\rangle$. The right N_1 vertical lines carry rapidities $u_{1,1}+i\ldots u_{1,N_1}+i$ and have index 2 on the top. At the bottom, the first L_1 indices are free, and the last N_1 ones are fixed to 1. The lower N_1 horizontal line correspond to the first-level magnons and carry rapidities $u_{1,1},\ldots,u_{1,N_1}$. The higher M_1 horizontal lines represent the second-level magnons with rapidities $u_{2,1},\ldots,u_{2,M_1}$. Due to the particular spin and rapidity choices, the shaded regions are frozen to the particular configuration shown in the Figure. This diagram is equivalent to (a special

case of[2]) the one used by Reshetikhin in [113]. The structure constant factorizes as in

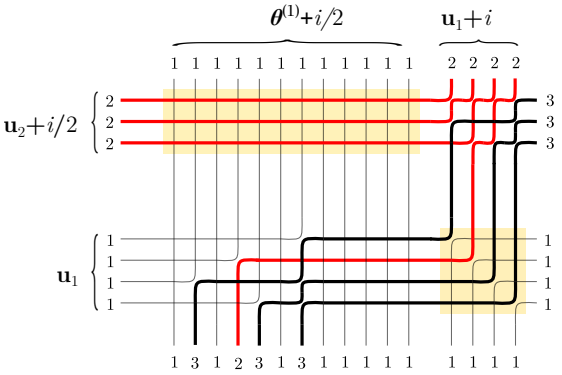

FIGURE 5.3.7: A configuration contributing to the coefficient $\psi_{131231311111}(\mathbf{u})$ of a $SU(3)$ Bethe vector (5.3.47) with $L = 12$, $N = 4$ and $M = 3$.

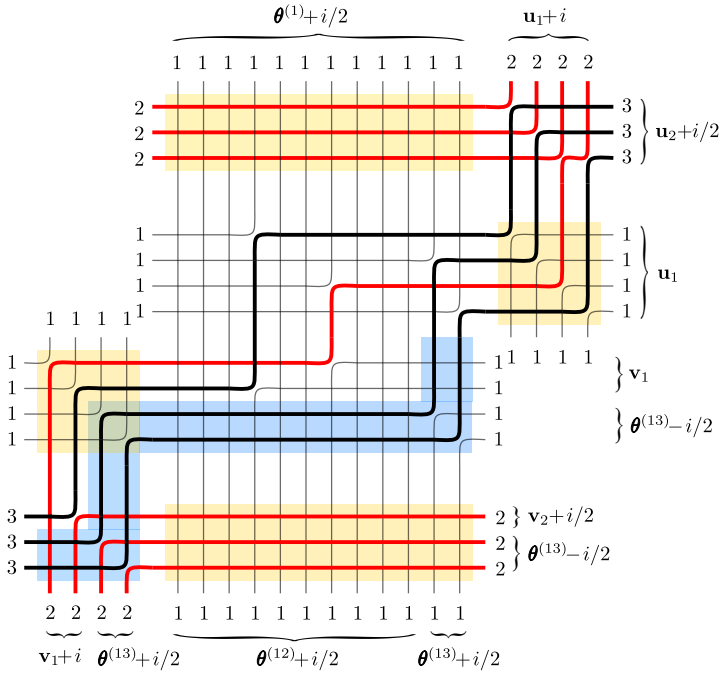

FIGURE 5.3.8: The inner product $\langle \tilde{\mathbf{v}}, \mathbf{u} \rangle$ and the freezing to $\langle \mathbf{v} \cup \boldsymbol{\theta}^{(13)} - i/2, \mathbf{u} \rangle$.

the $\mathfrak{su}(2)$ case. The two factors can be cast in the form of scalar products of an on-shell and an off-shell Bethe states by applying the freezing procedure.

5.3.1.2 The $\mathfrak{su}(3)$ freezing procedure

Consider the scalar product of two $\mathfrak{su}(3)$ states of the first chain, $\langle \tilde{\mathbf{v}}, \mathbf{u} \rangle = \langle \{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2\}, \{\mathbf{u}_1, \mathbf{u}_2\} \rangle$, as represented in Fig.(5.3.8). Our purpose is to freeze the rightmost N_3 indices to the

2. In [113], there are had two momentum-carrying nodes, while our spin chain has only one momentum-carrying node.

value 3, as imposed by the planarity of the contractions in the three point function. This can be done by setting the last N_3 rapidities of first level magnons at their freezing values,

$$\tilde{v}_{1,N_2+1} = \theta_1^{(13)} - i/2, \quad \dots, \quad \tilde{v}_{1,N_1} = \theta_{N_3}^{(13)} - i/2. \tag{5.3.48}$$

This will insure that the corresponding frozen region contains only red and black lines propagating from the top to the bottom of the diagram. The number of black or red lines is not fixed by the freezing, only their sum is fixed. In order to force all the lines in the frozen region to be black, we have to apply once again the freezing procedure to the magnons of second level, by fixing

$$\tilde{v}_{2,M_2+1} = \theta_1^{(13)} - i, \quad \dots \quad \tilde{v}_{2,M_1} = \theta_{N_3}^{(13)} - i. \tag{5.3.49}$$

The remaining magnons are set to the corresponding values in the state $|\mathbf{v}\rangle$,

$$\tilde{v}_{1,1} = v_{1,1}, \quad \dots, \quad \tilde{v}_{2,M_2} = v_{2,M_2}. \tag{5.3.50}$$

This gives us the first factor of the expression (5.3.45) for the cubic vertex. The second factor, $\langle \mathbf{z}_1, \mathbf{w}_1 \rangle$, is the same as in the $\mathfrak{su}(2)$ case.

5.3.2 The $\mathfrak{su}(3)$ structure constant in terms of $\mathscr{A}^{\pm}$ functionals

A generic Bethe state $|\mathbf{u}\rangle$ in an $\mathfrak{su}(3)$ sector is characterized by the rapidities $\mathbf{u} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and the inhomogeneity parameters $\boldsymbol{\theta}$ associated with the momentum-carrying node (1), where

$$\mathbf{u}_1 = \{u_{1,j}, \dots, u_{1,N}\}, \quad \mathbf{u}_2 = \{u_{2,1}, \dots, u_{2,M}\}, \quad \boldsymbol{\theta} = \{\theta_1, \dots, \theta_L\}. \tag{5.3.51}$$

The rapidities satisfy the nested Bethe wave functions for the $\mathfrak{su}(3)$ R-matrix given by (3.2.27):

$$\begin{aligned} \prod_{l=1}^{L} \frac{u_{1,j} - \theta_l + \frac{1}{2}i}{u_{1,j} - \theta_l - \frac{1}{2}i} &= -\prod_{n=1}^{N} \frac{u_{1,j} - u_{1,n} + i}{u_{1,j} - u_{1,n} - i} \prod_{m=1}^{M} \frac{u_{1,j} - u_{2,m} - \frac{1}{2}i}{u_{1,j} - u_{2,m} + \frac{1}{2}i} \\ 1 &= -\prod_{m=1}^{M} \frac{u_{2,j} - u_{2,m} + i}{u_{2,j} - u_{2,m} - i} \prod_{n=1}^{N} \frac{u_{2,j} - u_{1,n} - \frac{1}{2}i}{u_{2,j} - u_{1,n} + \frac{1}{2}i}. \end{aligned} \tag{5.3.52}$$

For given distribution of the roots $\mathbf{u}_1$ and $\mathbf{u}_2$, the pseudomomenta $p^i_{\mathbf{u}}(z)$ are defined modulo 2π by

$$e^{ip^1_{\mathbf{u}}} = \frac{Q^{++}_{\mathbf{u}_1}}{Q_{\mathbf{u}_1}}\frac{Q^-_{\boldsymbol{\theta}}}{Q^+_{\boldsymbol{\theta}}}, \quad e^{ip^2_{\mathbf{u}}} = \frac{Q^{--}_{\mathbf{u}_1}}{Q_{\mathbf{u}_1}}\frac{Q^+_{\mathbf{u}_2}}{Q^-_{\mathbf{u}_2}}, \quad e^{ip^3_{\mathbf{u}}} = \frac{Q^{---}_{\mathbf{u}_2}}{Q^-_{\mathbf{u}_2}} \tag{5.3.53}$$

see e.g. [114]. Here

$$Q^{\pm}(u) = Q(u \pm i/2), \quad Q^{\pm\pm} = Q(u \pm i), \quad Q^{[n]}(u) = Q(u + in/2). \tag{5.3.54}$$

In terms of the three pseudomomenta, the Bethe equations (3.1.19) read

$$\begin{aligned} e^{ip^1_{\mathbf{u}}(z) - ip^2_{\mathbf{u}}(z)} &= -1 \quad \text{if} \quad z \in \mathbf{u}_1; \\ e^{ip^2_{\mathbf{u}}(z) - ip^3_{\mathbf{u}}(z)} &= -1 \quad \text{if} \quad z - i/2 \in \mathbf{u}_2. \end{aligned} \tag{5.3.55}$$

It is convenient to introduce the functions $P^1_{\mathbf{u}}(z)$ and $P^1_{\mathbf{u}}(z)$, associated with the two nodes of the Dynkin graph of $\mathfrak{su}(3)$, and related to the quasimomenta $p^i_{\mathbf{u}}(z)$, $i = 1, 2, 3$, by

$$P^1_{\mathbf{u}}(z) = p^1_{\mathbf{u}}(z) - p^2_{\mathbf{u}}(z), \quad P^2_{\mathbf{u}}(z) = p^2_{\mathbf{u}}(z + i/2) - p^3_{\mathbf{u}}(z + i/2). \tag{5.3.56}$$

In terms of these functions, which we will also call pseudomomenta, the Bethe equations take the more standard form

$$e^{iP^a_{\mathbf{u}}(z)} = -1, \quad \text{if} \quad z \in \mathbf{u}_a \quad (a = 1, 2). \tag{5.3.57}$$

The functions P_1 and P_2 can be expressed in terms of the $\mathfrak{su}(3)$ Cartan matrix

$$M_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{5.3.58}$$

as

$$e^{iP^a_{\mathbf{u}}} = \left(\frac{Q^-_{\boldsymbol{\theta}}}{Q^+_{\boldsymbol{\theta}}}\right)^{\delta_{a,1}} \prod_{b=1,2} Q^{[M_{ab}]}_{\mathbf{u}_b} Q^{[-M_{ab}]}_{\mathbf{u}_b} \quad a = 1, 2. \tag{5.3.59}$$

Let us stress that the values of the local conserved charges are determined only by the level-1 roots $\mathbf{u}_1$. The duality transformations change the level-2 roots $\mathbf{u}_2$, but leave invariant the level-1 roots $\mathbf{u}_1$, which carry the physical information [114].

The norm of an on-shell Bethe state The squared norm of an on-shell Bethe state has been computed for the case of $\mathfrak{su}(3)$ by Reshetikhin [3] [113] and is expressed as the

3. A conjecture for $\mathfrak{su}(N)$ is proposed in [49].

determinant of the matrix of the derivatives of the two quasi-momenta: [4]

$$\langle \mathbf{u}, \mathbf{u} \rangle = c_{\mathbf{u}} \det \left[\partial_{u_{a,j}} P^b_{\mathbf{u}}(u_{b,k}) \right], \tag{5.3.60}$$

where the determinant is with respect to the double indices $A = \{a, j\}$ and $B = \{b, k\}$. The normalizationn factor $c_{\mathbf{u}}$ is given by (5.2.43). The matrix of the derivatives of the two quasimomenta is explicitly

$$\partial_{u_{a,j}} P^b_{\mathbf{u}}(u_{b,k}) = t_{ab}(u_{a,j} - u_{b,k}) + t_{ab}(-u_{a,j} + u_{b,k}) + i\,\delta_{a,b}\delta_{j,k} \frac{\partial P^a_{\mathbf{u}}(z)}{\partial z}\bigg|_{z=u_{a,j}} \tag{5.3.61}$$

where

$$t_{ab}(u) = \frac{1}{u} - \frac{1}{u + \frac{1}{2}M_{ab}}. \tag{5.3.62}$$

Instead of taking the derivatives, we will compute the norm as the limit of the determinant depending on two sets of rapidities, $\mathbf{u}$ and $\mathbf{v}$, which has the limit (5.3.61) when $\mathbf{v} \to \mathbf{u}$. We define the $(N + M) \times (N + M)$ square matrix $\Omega_{ab}(u_j, v_k)$, with

$$\Omega_{ab}(u, v) = t_{ab}(u - v) - e^{iP^a_{\mathbf{u}}(v)} t_{ab}(-u + v). \tag{5.3.63}$$

The expression for the norm, which we are going to evaluate in the classical limit, is

$$\langle \mathbf{u}, \mathbf{u} \rangle = c_{\mathbf{u}} \lim_{v_{a,j} \to u_{a,j}} \det \left[\Omega_{ab}(u_{a,j}, v_{b,k}) \right]. \tag{5.3.64}$$

5.3.3 The inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ in the limit $\mathbf{u}_2 \to \infty$

Unlike the $\mathfrak{su}(2)$ case, the inner product of an on-shell Bethe state with an of-shell Bethe state is not generically a determinant. Determinant representations exist in some particular cases [112, 115]. We will use the determinant expression obtained by Wheeler [112], when the rapidities of the second type of magnons of the Bethe eigenstate are sent to infinity. We assume that M is odd; then one can send to infinity the $\mathbf{u}_2$ roots one by one. As a result the second level Bethe equations become trivial and the first level Bethe equations take the same form as for $\mathfrak{su}(2)$. The inner product $\langle \mathbf{u}, \mathbf{v} \rangle^{\mathrm{su}(3)}_{\boldsymbol{\theta}}$ factorizes into

4. Here it is assumed that the set of the Bethe roots is symmetric under complex conjugation.

two $\mathfrak{su}(2)$ inner products [112]

$$\begin{aligned}\lim_{\mathbf{u}_2\to\infty}\langle\mathbf{v},\mathbf{u}\rangle^{\mathrm{su}(3)} &= \det_{jk}\left((v_{2,j})^{k-1}-(v_{2,j}+i)^{k-1}\frac{Q^-_{\mathbf{v}_1}(v_{2,j})}{Q^+_{\mathbf{v}_1}(v_{2,j})}\right)\\ &\times\det_{ij}\left(t(u_{1,j}-v_{1,k})-Q^-_{\boldsymbol{\theta}}(v_{1,k})Q^+_{\boldsymbol{\theta}}(v_{1,k})\frac{Q^{++}_{\mathbf{u}_1}(v_{1,k})}{Q^{--}_{\mathbf{u}_1}(v_{1,k})}\,t(-u_{1,j}+v_{1,k})\right)\\ &\times\frac{1}{\Delta[\mathbf{v}_1]\Delta[\mathbf{v}_2]\Delta[\mathbf{u}_1]}\times\prod_{j,k}(u_{1,j}-v_{1,k}+i)\\ &=\langle\mathbf{u}_1,\mathbf{v}_1\rangle^{\mathrm{su}(2)}_{\boldsymbol{\theta}}\langle\infty,\mathbf{v}_2\rangle^{\mathrm{su}(2)}_{\mathbf{v}_1}.\end{aligned}\tag{5.3.65}$$

Using (3.4.114), we write (5.3.65) in the form

$$\lim_{\mathbf{u}_2\to\infty}\langle\mathbf{v},\mathbf{u}\rangle_{\boldsymbol{\theta}}=\mathscr{A}^+_{\mathbf{u}_1\cup\mathbf{v}_1}\left[\frac{Q^-_{\boldsymbol{\theta}}}{Q^+_{\boldsymbol{\theta}}}\right]\mathscr{A}^+_{\mathbf{v}_2}\left[Q^-_{\mathbf{v}_1}Q^+_{\mathbf{v}_1}\right].\tag{5.3.66}$$

Combining (5.3.65) in (5.3.45), we get

$$\langle\mathbf{u},\mathbf{v},\mathbf{w}\rangle=\mathscr{A}^+_{\mathbf{w}_1}[Q^-_{\boldsymbol{\theta}^{(13)}}Q^+_{\boldsymbol{\theta}^{(13)}}]\,\mathscr{A}^+_{\mathbf{u}_1\cup\mathbf{v}_1}[\frac{Q^-_{\boldsymbol{\theta}^{(12)}}}{Q^+_{\boldsymbol{\theta}^{(12)}}}]\,\mathscr{A}^+_{\mathbf{v}_2}[Q^-_{\mathbf{v}_1}Q^+_{\mathbf{v}_1}].\tag{5.3.67}$$

5.4 The semi-classical limit of the $\mathfrak{su}(3)$ three-point function

5.4.0.1 The semi-classical limit

The semi-classical limit for $\mathfrak{su}(3)$ Bethe states is defined similarly as for the $\mathfrak{su}(2)$ Bethe states. It is attained for long spin chains ($L\gg 1$) with macroscopically many excitations $N,M\sim L$, and in the low energy regime ($E\sim 1/L$) [17, 20, 116]. In the semi-classical limit the roots are organized in several macroscopic strings, which condense into cuts in the complex rapidity plane. The three quasimomenta p_1,p_2,p_3 become the three branches of the same meromorphic function. The three sheets of the corresponding Riemann surface are joined among themselves along the cuts defined by the long Bethe strings. In the semi-classical limit, the Bethe state is characterised by the resolvents

$$G_{\mathbf{u}_1}(z)=\partial_z\log Q_{\mathbf{u}}(z),\quad G_{\mathbf{u}_2}(z)=\partial_z\log Q_{\mathbf{u}_2}(z),\tag{5.4.68}$$

as well as the resolvent for the inhomogeneities

$$G_{\boldsymbol{\theta}}(u)=\partial_u\log Q_{\boldsymbol{\theta}}(u).\tag{5.4.69}$$

The two resolvents, $G_{\mathbf{u}_1}$ and $G_{\mathbf{u}_2}$, can be expressed in terms of the three quasimomenta $p^1_{\mathbf{u}}, p^2_{\mathbf{u}}$ and $p^3_{\mathbf{u}}$, which become the three branches of a single meromorphic function on the tri-foliated Riemann surface,

$$p^1_{\mathbf{u}} = G_{\mathbf{u}_1} - G_{\boldsymbol{\theta}} \pmod{2\pi}, \tag{5.4.70}$$

$$p^2_{\mathbf{u}} = G_{\mathbf{u}_2} - G_{\boldsymbol{\theta}_1} \pmod{2\pi}, \tag{5.4.71}$$

$$p^3_{\mathbf{u}} = -G_{\mathbf{u}_2} \pmod{2\pi}. \tag{5.4.72}$$

or

$$P^1_{\mathbf{u}} = 2G_{\mathbf{u}_1} - G_{\mathbf{u}_2} - G_{\boldsymbol{\theta}} \pmod{2\pi}, \tag{5.4.73}$$

$$P^2_{\mathbf{u}} = 2G_{\mathbf{u}_2} - G_{\mathbf{u}_1} \pmod{2\pi}. \tag{5.4.74}$$

Let $\mathcal{C}^\alpha_{ij}$ be the cuts joining the i-th and the j-th sheets. Then the Bethe equations (5.3.55) become boundary conditions on these cuts, depending on the mode numbers n^α_{ij}:

$$2\pi n^\alpha_{12} = \not{p}_1 - \not{p}_2, \qquad z \in \mathcal{C}^\alpha_{12} \tag{5.4.75}$$

$$2\pi n^\alpha_{23} = \not{p}_2 - \not{p}_3, \qquad z \in \mathcal{C}^\alpha_{23} \tag{5.4.76}$$

where $\not{p}$ denotes the half-sum of the values of the function p on both sides of the cut.

5.4.0.2 Stacks

In addition, there is the possibility of configurations called stacks (bound states of rapidities associated with different nodes [117]), which represent pairs of roots belonging to the nodes 1 and 2 and at distance $O(1)$ from each other [19, 20, 114]. We can have macroscopic strings of stacks, which in the classical limit become two cuts that merge into one cut. Since the roots that form the string of stacks belong to two different nodes, they correspond to a cut type 1-2 and a cut type 2-3, where we understand that the cut of type i-j joins the i-th and the j-th sheets of the Riemann surface. The result of merging of the two cuts is a cut of the type 1-3. Therefore, in order to have a description of the generic Bethe state in the semi-classical limit, we must assume also the existence of cuts of type 1-3. The boundary condition on these cuts is obtained by taking the limit of (5.4.75) and has the form

$$2\pi n^\alpha_{13} = \not{p}_1 - \not{p}_3, \qquad z \in \mathcal{C}^\alpha_{13} \tag{5.4.77}$$

The bosonic duality transformations [114] in the classical limit corresponds simply to the exchange of the Riemann sheets 2 and 3.

5.4.0.3 The semi-classical norm

The determinant (5.3.64) can be computed in the classical limit under the assumption that there are only 1-2 and 2-3 type cuts, which are separated at macroscopic distance $\sim L$. With this assumption, the off diagonal elements of $\Omega(u,v) \sim (u-v)^{-2} \sim 1/L^2$, and the only matrix elements of order one are those in a strip of width $\sim 1/\sqrt{L}$ along the diagonal. As a consequence, the non-diagonal blocks do not contribute in the classical limit and the determinant is simply the product of the determinants of the diagonal blocks,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{u}\rangle^{\mathrm{su}(3)} &\simeq \det\left[\Omega_{11}(u_{1,j}, u_{1,k})\right] \det\left[\Omega_{22}(u_{2,j}, u_{2,k})\right] \\ &= \langle \mathbf{u}_1, \mathbf{u}_1\rangle^{\mathrm{su}(2)} \langle \mathbf{u}_2, \mathbf{u}_2\rangle^{\mathrm{su}(2)}.\end{aligned} \tag{5.4.78}$$

Let us evaluate the norm assuming that there there are no cuts relating the first and the third sheet of the Riemann surface. We can use the expression for the classical limit of the norm in the $\mathfrak{su}(2)$ sector:

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle_{\boldsymbol{\theta}} &= \log \mathscr{A}^{+}_{\mathbf{u}\cup\mathbf{v}}[\frac{Q^{-}_{\boldsymbol{\theta}}}{Q^{+}_{\boldsymbol{\theta}}}] \\ &= \oint_{C_{\mathbf{u}}\cup C_{\mathbf{v}}} \frac{dz}{2\pi} \mathrm{Li}_2[f(z)\, e^{iG_{\mathbf{u}}(z)+iG_{\mathbf{v}}(z)-iG_{\boldsymbol{\theta}}(z)}] + \mathcal{O}(\log L),\end{aligned} \tag{5.4.79}$$

with

$$G_{\mathbf{u}}(z) = \partial_z \log Q_{\boldsymbol{u}}(z), \quad G_{\boldsymbol{\theta}}(z) = \partial_z \log Q_{\boldsymbol{\theta}}(z). \tag{5.4.80}$$

The norm of the classical Bethe state is then [5]

$$\log\langle \mathbf{u}|\mathbf{u}\rangle = \oint_{\mathcal{C}_{\mathbf{u}_1}} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{2iG_{\mathbf{u}_1}(z)-iG_{\mathbf{u}_2}(z)-iG_{\boldsymbol{\theta}}(z)}\right) + \oint_{\mathcal{C}_{\mathbf{u}_2}} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{2iG_{\mathbf{u}_2}(z)-iG_{\mathbf{u}_1}(z)}\right). \tag{5.4.82}$$

5. It is likely that in the most general case, when some of the roots can form bound states ("stacks"), this logarithm of the norm is given by

$$\log\langle \mathbf{u}|\mathbf{u}\rangle = \sum_{\alpha<\beta} \oint_{\mathcal{C}_{\alpha\beta}} \frac{dz}{2\pi} \mathrm{Li}_2\left(e^{ip^{\alpha}_{\mathbf{u}}(z)-ip^{\beta}_{\mathbf{u}}(z)}\right) \tag{5.4.81}$$

where $\mathcal{C}_{ij}$ $(i,j=1,2,3)$ denote the contour (or contours) surrounding the cuts between the i-th and the j-th sheets.

5.4.1 Semi-classical limit of the structure constant

Substituting (5.4.79) in (5.3.67) we find for the structure constant in the classical limit

$$\log C^{(0)}_{123} = \oint_{\mathcal{C}_{\mathbf{u}_1\cup\mathbf{v}_1}} \frac{dz}{2\pi}\,\mathrm{Li}_2\left(e^{iG_{\mathbf{u}_1}(z)+iG_{\mathbf{v}_1}(z)-iG^{(12)}_{\boldsymbol{\theta}}(z)}\right) \tag{5.4.83}$$

$$+\oint_{\mathcal{C}_{\mathbf{v}_2}} \frac{dz}{2\pi}\,\mathrm{Li}_2\left(e^{iG_{\mathbf{v}_2}(z)-iG_{\mathbf{v}_1}(z)}\right)+\oint_{\mathcal{C}_{\mathbf{w}_1}} \frac{dz}{2\pi}\,\mathrm{Li}_2\left(e^{iG_{\mathbf{w}_1}(z)-iG_{\boldsymbol{\theta}^{(13)}}(z)}\right)$$

$$-\frac{1}{2}\log\langle\mathbf{u}|\mathbf{u}\rangle-\frac{1}{2}\log\langle\mathbf{v}|\mathbf{v}\rangle-\frac{1}{2}\log\langle\mathbf{w}|\mathbf{w}\rangle. \tag{5.4.84}$$

The last line is evaluated in the classical limit according to (5.4.82).

Chapter 6

Long-range Interacting Spin Chain

In this chapter and Chapter 7, we consider three-point functions at higher loops in perturbation theory. The main discussions are at one loop but some results can be generalized to higher loops. There are two new features at higher loops. Firstly, the dilatation operator is mapped to a long-range interacting spin chain instead of the nearest neighboring Heisenberg spin chain. One has to construct the eigenstates of the long range spin chain and compute their scalar products. Secondly, we need to take into account quantum corrections, which, at one-loop level, manifest themselves as operator insertions at the splitting points. These insertions have to be computed by Feynman diagrams for the moment and is only known for some sectors at one-loop. Even the loop insertions are known, it is non-trivial to take them into account in the spin chain language and obtain manageable results for three-point functions.

The current chapter is devoted to long-range interacting spin chain. In the $\mathfrak{su}(2)$ sector, the dilatation operator up to three-loop (before the dressing phase needs to be taken into account) is described by the so-called BDS spin chain [21]. This spin chain is long-range interacting and is integrable in a perturbative sense, the meaning of which will be made more precise in section 2. Although some special types of long-range interacting spin chains such as the Haldane-Shastry chain and Inozemtsev chain have been studied in the context of condensed matter physics, the BDS spin chain is of novel type and new methods are needed to construct its eigenstates. This problem is solved by the important observation [118, 119] that the BDS spin chain has the same spectrum as an inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain with the inhomogeneities fixed to a set of specific values. This implies the two models are related by an unitary transformation. Once this unitary transformation is worked out, we can construct the eigenstates of BDS spin

chain by first constructing the eigenstates of the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain and then apply the unitary transformation. We will present the explicit form of this unitary transformation and describe in detail how to construct the eigenstates of BDS spin chain.

This chapter is structured as the follows. In section 1, we will discuss the inhomogeneous $\mathrm{XXX}_{1/2}$ spin with an emphasis on its conserved charges. In section 2, we introduce the long-range interacting spin chain, in particular, the BDS spin chain. In section 3, we work out the unitary transformation and relate the two models.

6.1 Inhomogeneous $\mathrm{XXX}_{1/2}$ Spin Chain

We have encountered the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain several times before. In this section, our focus will be on the conserve charges and its relation to the long range interacting spin chain. To this end, we choose a slightly different normalization

$$R_{ab}(u) = \frac{u}{u+i}\,\mathrm{I}_{ab} + \frac{i}{u+i}\,\mathrm{P}_{ab}. \tag{6.1.1}$$

The corresponding monodromy matrix is defined by

$$T_a(u;\boldsymbol{\theta}) = \prod_{k=1}^{L} R_{ak}(u-\theta_k-\frac{i}{2}), \tag{6.1.2}$$

The normalization does not change the RTT relation. The only difference is that the eigenvalues of the A and D operator on the pseudovacuum, which reads

$$a(u) = 1, \qquad d(u) = \prod_{l=1}^{L} \frac{\left(u-\theta_l-\frac{i}{2}\right)}{\left(u-\theta_l+\frac{i}{2}\right)}. \tag{6.1.3}$$

The on-shell states are eigenstates of the transfer matrix $\mathcal{T}(u)$ with the eigenvalue now given by

$$t_{\mathbf{u}}(u) = \frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + \frac{d(u)}{a(u)}\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}. \tag{6.1.4}$$

6.1.1 Conserved charges

We define the r-th integrals of motion or conserved charges of the inhomogeneous spin chain conventionally as the logarithmic derivatives of the transfer matrix $\mathcal{T}(u)$ around

the point $u = i/2$:

$$\mathrm{Q}_r^\theta = \frac{1}{i(r-1)!}\frac{d^{r-1}}{du^{r-1}}\ln \mathcal{T}(u)\Big|_{u=i/2}. \tag{6.1.5}$$

Any combination of the above integrals of motion is an integral of motion. The definition given above is convenient if the values of the inhomogeneities are small, $\theta_k \sim g$, where g is a perturbation parameter and will be identified with $g = \frac{\sqrt{\lambda}}{4\pi}$.

Homogenous chains For the homogeneous case (*i.e.* $\theta_k = 0$), the first conserved quantity is the shift operator:

$$\mathrm{U}_0 \equiv \mathcal{T}_0(\frac{i}{2}) = \mathrm{Tr}_a \prod_{k=1}^{L} \mathrm{P}_{ak} = \mathrm{P}_{L-1,L}\mathrm{P}_{L-2,L-1}\ldots\mathrm{P}_{12}\ . \tag{6.1.6}$$

The homogeneous shift U_0 translates the chain by one lattice spacing, that is we have

$$\mathrm{U}_0\mathrm{P}_{k,k+1}\mathrm{U}_0^{-1} = \mathrm{P}_{k-1,k}. \tag{6.1.7}$$

Periodicity of the chain means that $\mathrm{U}_0^L = 1$. The first few homogeneous Hamiltonians take the form

$$\begin{aligned}
\mathrm{Q}_2^{\mathrm{SR}} &= \sum_{k=1}^{L}\mathrm{H}_k, \qquad (6.1.8)\\
\mathrm{Q}_3^{\mathrm{SR}} &= \frac{i}{2}\sum_{k=1}^{L}[\mathrm{H}]_{k-1},\\
\mathrm{Q}_4^{\mathrm{SR}} &= \frac{1}{3}\sum_{k=1}^{L}\Big(\llbracket \mathrm{H}\rrbracket_{k-1} + [\mathrm{H}]_k\,\mathrm{H}_{k-1} - [\mathrm{H}]_{k-1} - \mathrm{H}_k\Big),
\end{aligned}$$

where "SR" denotes "short-range" and we have introduced the compact recursive notation

$$\mathrm{H}_k = \mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1}, \qquad [\mathrm{H}]_k = [\mathrm{H}_k, \mathrm{H}_{k+1}], \qquad \llbracket \mathrm{H}\rrbracket_k = [[\mathrm{H}]_k, \mathrm{H}_{k+2}]. \tag{6.1.9}$$

For completeness we note that in terms of the R-matrix the nearest-neighbor Hamiltonian is given by

$$\mathrm{H}_k = -iR_{k,k+1}^{-1}(u)\frac{dR_{k,k+1}(u)}{du}\Big|_{u=0}, \tag{6.1.10}$$

and the homogeneous transfer matrix can be expressed in the convenient form

$$\mathcal{T}_0(u+\frac{i}{2}) = \mathrm{U}_0 \exp\Big[i\sum_{r=2}^{L} u^{r-1}\mathrm{Q}_r^{\mathrm{SR}}\Big]. \tag{6.1.11}$$

Inhomogenous chains In the inhomogeneous case the conserved quantities do not take the simple form (6.1.8). It is useful for later purposes to write them as an expansion in the value of the inhomogeneities. The momentum is no longer a conserved quantity, since the inhomogeneous chain is not translationally invariant. However, the periodicity condition $\mathrm{U}_0^L = 1$ still holds. The conserved quantity which replaces the shift U_0 is the operator

$$\mathrm{U}_\theta = \mathrm{Tr}_a \prod_{k=1}^{L} R_{ak}(-\theta_k) \,, \tag{6.1.12}$$

whose expansion in θ exponentiates to [1]

$$\mathrm{U}_\theta = \mathrm{U}_0 \exp\Big[-i \sum_k \theta_k \mathrm{H}_k - \frac{1}{2} \sum_k \theta_{k-1}\theta_k [\mathrm{H}]_{k-1} + \mathcal{O}(\theta^2) \Big] \,. \tag{6.1.13}$$

Note that for $\theta_k = -u$ the inhomogeneous shift U_θ gives back the homogeneous transfer matrix $\mathcal{T}_0(u+i/2)$ (6.1.11). The expansion of the inhomogeneous Hamiltonian takes the form

$$\mathrm{Q}_2^\theta = \sum_{k=1}^{L} \Big[\mathrm{H}_k - i\theta_k [\mathrm{H}]_{k-1} + \theta_k^2 \left(\mathrm{H}_k [\mathrm{H}]_{k-1} - [\mathrm{H}]_{k-1} - \mathrm{H}_k \right) + \theta_k \theta_{k+1} [\![\mathrm{H}]\!]_{k-1} \Big] + \mathcal{O}(\theta^3). \tag{6.1.14}$$

The M-magnon eigenvalues E_r of the conserved quantities Q_r are the sum over one-magnon eigenvalues

$$\mathrm{E}_r^\theta = \sum_{j=1}^{M} q_r(u_j) + \mathcal{O}(\theta^L), \tag{6.1.15}$$

where $q_r(u)$ takes the standard form of the XXX one-magnon eigenvalues

$$q_r(u) = \frac{i}{r-1} \left(\frac{1}{(u+\frac{i}{2})^{r-1}} - \frac{1}{(u-\frac{i}{2})^{r-1}} \right). \tag{6.1.16}$$

Here $u_1, \ldots, u_M$ are solutions of the BAEs and depend on the values of the inhomogeneities θ_k.

6.1.2 Corner Transfer Matrix

An interesting quantity with regard to the construction of integrals of motion is Baxter's corner transfer matrix (CTM) [120]. After a brief review of some aspects of the CTM

1. We assume periodic boundary conditions, $k + L \equiv k$.

for homogeneous spin chains, we define an inhomogeneous CTM that will be useful in the subsequent sections.

Homogeneous chains. Let us briefly review the definition of a *homogeneous* CTM and its relation to the so-called nearest-neighbor boost operator [120–122]. In the following we will assume to work on infinite chains ($L \to \infty$) or in the bulk of a periodic chain. [2] First we introduce a half-row matrix G_A ranging from site $A(< L)$ to site L:

$$\mathrm{G}_A(u) = \hat{\mathrm{R}}_{L-1,L}(u)\hat{\mathrm{R}}_{L-2,L-1}(u)\dots\hat{\mathrm{R}}_{A+1,A+2}(u)\hat{\mathrm{R}}_{A,A+1}(u). \tag{6.1.17}$$

Here we have defined the symbol $\hat{\mathrm{R}}(u)$ as the R-matrix times the permutation operator: [3]

$$\hat{\mathrm{R}}_{k,k+1}(u) = \mathrm{P}_{k,k+1}R_{k,k+1}(u). \tag{6.1.18}$$

Then we define the CTM as a stack of half-row matrices of different lengths according to Fig.(6.1.1)

$$\mathcal{A}(u) = \mathrm{G}_1(u)\dots\mathrm{G}_{L-2}(u)\mathrm{G}_{L-1}(u). \tag{6.1.19}$$

Note that the triangular definition of the CTM originates in the context of vertex models. In fact, this matrix can be defined for every quadrant of a square lattice of R-matrices (vertices). In the bulk the half-row matrix G_A has (up to the shift) the same structure as the parity inverted row-to-row transfer matrix $\mathcal{T}^{-1}(-u+\frac{i}{2})$ and consequently a similar expansion [4]

$$\mathrm{G}_A(u) = 1 + iu\sum_{k=A}^{L}\mathrm{H}_k + \mathcal{O}(u^2). \tag{6.1.20}$$

This form makes it clear that the CTM expands as

$$\mathcal{A}(u) = 1 + iu\mathcal{B}[\mathrm{Q}_2^{\mathrm{SR}}] + \mathcal{O}(u)^2, \tag{6.1.21}$$

where $\mathcal{B}[\mathrm{Q}_2^{\mathrm{SR}}]$ denotes the so-called *boost operator* of the nearest-neighbor Hamiltonian $\mathrm{Q}_2^{\mathrm{SR}} = \sum_k \mathrm{H}_k$. For a generic local operator $\mathcal{L}$ with local density $\mathcal{L}_k$, the boost is defined as

$$\mathcal{B}[\mathcal{L}] = \sum_k k\mathcal{L}_k. \tag{6.1.22}$$

It is well-known that the boost of the nearest-neighbor Hamiltonian allows to obtain higher integrable Hamiltonians of a short-range spin chain model based on a rational

2. Note that typically some spins on the edge of the CTM are fixed.
3. Usually the CTM is defined in terms of ordinary R-matrices or vertex weights and the ingoing site k is identified with the outgoing site $k+1$ when mapping the vertex model to a spin chain. Here it seems convenient to circumvent the vertex model interpretation to avoid confusion.
4. Note that $\mathrm{R}^{-1}(u) = \mathrm{R}(-u)$.

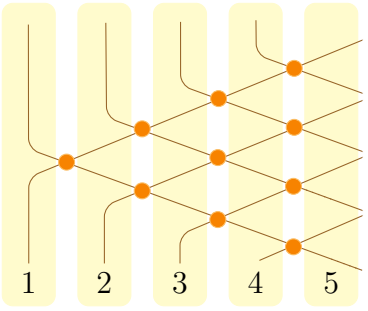

FIGURE 6.1.1: Corner transfer matrix (CTM) acting on a spin chain. We assume an infinite lattice on the right hand side. The small discs denote $\hat{\mathrm{R}}$-matrices defined in (6.1.18).

(or trigonometric) R-matrix [123]:

$$\mathrm{Q}^{\mathrm{SR}}_{r+1} = -\frac{i}{r}[\mathcal{B}[\mathrm{Q}^{\mathrm{SR}}_2], \mathrm{Q}^{\mathrm{SR}}_r]. \tag{6.1.23}$$

In fact, on infinite chains the homogeneous CTM can be expressed as the exponential of the nearest-neighbor boost operator as shown in [120, 121] for the XYZ model:

$$\mathcal{A}(u) = \exp\left(iu\mathcal{B}[\mathrm{Q}^{\mathrm{SR}}_2]\right). \tag{6.1.24}$$

Since the row-to-row transfer matrix $\mathcal{T}(u)$ is the generating function of the local integrals of motion, (6.1.23) is equivalent to the differential equation [121, 124]

$$\frac{d}{du}\mathcal{T}(u+\frac{i}{2}) = i[\mathcal{B}[\mathrm{Q}^{\mathrm{SR}}_2], \mathcal{T}(u+\frac{i}{2})], \qquad \mathcal{T}(\frac{i}{2}) = \mathrm{U}_0, \tag{6.1.25}$$

where we have fixed the initial value of the transfer matrix to be the homogeneous shift operator. This implies that a finite boost transformation corresponds to a shift of the rapidity parameter of the row-to-row transfer matrix:

$$\mathcal{A}^{-1}(u)\mathcal{T}(v)\mathcal{A}(u) = \mathcal{T}(u+v). \tag{6.1.26}$$

In particular, one can understand the row-to-row transfer matrix as being generated by the CTM through a transformation of the shift operator $\mathrm{U}_0 = \mathcal{T}(i/2)$:

$$\mathcal{T}(u+\frac{i}{2}) = \mathcal{A}^{-1}(u)\mathrm{U}_0\mathcal{A}(u). \tag{6.1.27}$$

Inhomogeneous chains. Now we would like to extend the above considerations to *inhomogeneous* spin chains. We define the inhomogeneous CTM as a stack of homogeneous

half-row matrices with different rapidity shifts: [5]

$$\mathcal{A}_\theta(u) = \mathrm{G}_1(u-\theta_1)\mathrm{G}_2(u-\theta_2)\dots\mathrm{G}_L(u-\theta_L). \tag{6.1.28}$$

Expanding this inhomogeneous CTM evaluated at $u=0$ in terms of the inhomogeneities θ we find

$$\mathcal{A}_\theta(0) = \exp\Big[i\sum_k \nu_k \mathrm{H}_k - \frac{1}{2}\sum_k \hat{\rho}_k[\mathrm{H}]_{k-1} + \mathcal{O}(\theta^3)\Big]. \tag{6.1.29}$$

where the coefficients ν_k and $\hat{\rho}_k$ are given by

$$\nu_k = -\sum_{x=1}^{k}\theta_x, \qquad \hat{\rho}_k = -\theta_k\nu_k - \sum_{x=1}^{k}\theta_x^2. \tag{6.1.30}$$

In analogy to (6.1.27) we may interpret the inhomogeneous shift operator as being generated by the operator $\mathcal{A}_\theta$ on infinite chains:

$$\mathrm{U}_\theta = \mathcal{A}_\theta^{-1}(0)\,\mathrm{U}_0\,\mathcal{A}_\theta(0). \tag{6.1.31}$$

While we have no proof for this transformation property in general, we have verified it up to order g^2. Similarly one can check that the inhomogeneous bulk Hamiltonian is generated according to $\mathrm{Q}_2^\theta = \mathcal{A}_\theta^{-1}(0)\mathrm{Q}_2^{\mathrm{SR}}\mathcal{A}_\theta(0)$, at least up to order g^2. In section 6.3 we will rediscover the inhomogeneous CTM in the context of a map between inhomogeneous and long-range spin chains.

6.2 Long-range interacting spin chain

Long-range interaction for a spin chain means the interactions are not restricted to the nearest neighboring, but can involve more sites or spins far away. It is a general feature of $\mathcal{N}=4$ SYM that at higher loops, the spin chain Hamiltonian which corresponds to the dilatation operators will become long-range interacting [10]. The interacting range grows with the order of perturbation.

Several different methods were employed to describe and solve long-range spin chains, at least partially. Historically, one of the first methods to completely solve a long-range system is based on so-called Dunkl operators [126], and it was used successfully for the Haldane-Shastry model, and for some aspects of the infinite length Inozemtsev model [127, 128]. The drawback of this method is that an explicit representation of the Dunkl operators is known only for a restricted class of models. Another restriction is that, with

5. In [125] it was speculated on the connection of the long-range deformations discussed in the subsequent sections to an inhomogeneous version of the CTM. We have not found any discussion of the inhomogeneous CTM defined in 6.1.28 in the literature.

the exception of the Haldane-Shastry model, the Dunkl operators cannot be rendered periodic on a finite lattice. The price to pay for rendering the lattice finite is to introduce a defect [129]. The advantage is that explicit exact expressions for the monodromy matrix can be obtained, and the scalar products are relatively straightforward to compute [128]. Another method to deform the $\mathrm{XXX}_{1/2}$ spin chain uses so-called boost and bilocal charges and was proposed in [118, 130]. This method works again fairly well for long spin chains, but does not include wrapping interactions.

As alluded before here we use yet another method, which is to map the inhomogeneous XXX model to a long-range model. The authors of [21] noticed that the BAE of the long-range spin chain they have proposed, can be obtained from those of an inhomogeneous spin chain by a judicious choice the inhomogeneities. This equivalence ceases to hold when wrapping interactions, *i.e.* interactions of range equal or greater than the length of the spin chain, are taken into account. However, the Hamiltonian of the inhomogeneous spin chain is not a homogeneous long-range spin chain, because it depends on inhomogeneities, which are site-dependent. The observation of BDS was taken further in [130], where it was noticed that if the two spin chains have the same spectrum, they should be related by a unitary transformation, which was computed up to two-impurity order (or two-loop order in $\mathcal{N}=4$ SYM terms). This unitary operator was not explicitly used before to construct the eigenfunctions of the long-range spin chain. Instead, the wave functions of long-range spin chains were constructed via another relation to inhomogeneous spin chains [131, 132] or by the relation to Dunkl operators [128, 129].

6.2.1 The BDS spin chain

By deforming the homogeneous short-range XXX model, one can obtain long-range spin chain models. One possibility is to define these models exactly, for any value of the deformation parameter and for any length of the chain. This is the case for the Inozemtsev model [133] whose Hamiltonian takes the form

$$\mathrm{H_I} = \prod_{\substack{k=1\\k\neq l}}^{L} \mathcal{P}_{L,i\pi/\kappa}(k-l)\,\mathrm{P}_{kl}. \tag{6.2.32}$$

Here $\mathcal{P}_{L,i\pi/\kappa}$ is the Weierstrass function with periods L and $i\pi/\kappa$. At $\kappa\to\infty$ this model gives back the short-range Heisenberg model. Another limiting case of this model is the $\kappa\to 0$ limit, which yields the Haldane-Shastry model [134, 135], and which was widely studied in connection with exclusion statistics. Another possibility to define long-range deformations is to define the model through a series expansion in the deformation parameter. This was done for example for the dilatation operator of $\mathcal{N}=4$ SYM theory

[14], which corresponds to an (asymptotically) integrable spin chain Hamiltonian. Integrability can then be defined perturbatively; for example if the deformed conserved charges are given by an expansion in the deformation parameter g of the form [6]

$$\mathrm{Q}_r(g) = \sum_{k\geq 0} \mathrm{Q}_r^{[k]} g^{2k}, \tag{6.2.33}$$

then the terms in the expansion can be computed order by order and to test integrability to order ℓ, one checks that

$$[\mathrm{Q}_r(g), \mathrm{Q}_s(g)] = \mathcal{O}(g^{2(\ell+1)}) \ . \tag{6.2.34}$$

We then say that the model is integrable up to ℓ-loop order. An important example of a long-range spin chain that we will use in this work is the BDS chain [21]. It was defined in the perturbative sense as a long-range spin chain whose first three orders coincide with the dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the $\mathfrak{su}(2)$ sector:

$$D(g) = L + 2\sum_{k\geq 1} g^{2k} D^{[k-1]} \ . \tag{6.2.35}$$

The first three non-trivial orders of the dilatation operator were computed by Beisert, Kristjansen and Staudacher [14] and they are given by

$$\begin{aligned} D^{[0]} &= \sum_{k=1}^{L} (1 - \mathrm{P}_{k,k+1}) \ , \\ D^{[1]} &= \sum_{k=1}^{L} (4\mathrm{P}_{k,k+1} - \mathrm{P}_{k,k+2} - 3) \ , \\ D^{[2]} &= \sum_{k=1}^{L} (-14\mathrm{P}_{k,k+1} + 4\mathrm{P}_{k,k+2} + 10 - \mathrm{P}_{k,k+3}\mathrm{P}_{k+1,k+2} + \mathrm{P}_{k,k+2}\mathrm{P}_{k+1,k+3}) \ . \end{aligned} \tag{6.2.36}$$

In the initial BDS paper [21], the model was defined beyond three-loop order by the BAE:

$$\left(\frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})}\right)^L = \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad e^{ip} = \frac{x(u + \frac{i}{2})}{x(u - \frac{i}{2})}, \tag{6.2.37}$$

6. Here we suppose that only even powers of g appear in the small g expansion, as it is the case for the $\mathcal{N} = 4$ SYM dilatation operator in the $\mathfrak{su}(2)$ sector.

with the rapidity map $x(u)$ and its inverse given by the Zhukovsky relation

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right), \quad u(x) = x + \frac{g^2}{x}. \tag{6.2.38}$$

In [136] it was shown that the Hamiltonian (6.2.35) and the Bethe ansatz (6.2.37) can be obtained by reducing the one-dimensional half-filled Hubbard model to the spin sector. In principle, the higher order terms in (6.2.35) can be computed from perturbation theory of the Hubbard model, and at increasing perturbative order they involve interactions connecting more and more spins. The difference between the Hubbard model prediction and the Bethe ansatz equations appears at order g^{2L} when wrapping interactions start to contribute.

Notably, the above Bethe equations for the BDS model equal the inhomogeneous Bethe equations up to wrapping, if the inhomogeneities are fixed to [21] [7]

$$\theta_k^{\text{BDS}} = 2g \sin \frac{2\pi k}{L}. \tag{6.2.39}$$

In consequence, the spectra of the two models are the same up to wrapping order and their Hamiltonians can be related by a similarity transformation [119]. In the subsequent sections we will pursue the investigation of this relation between the two spin chain models.

6.2.2 Boost Operators

In this section we review a general method for the construction of long-range spin chains using a deformation equation that preserves integrability [118, 130]. We then discuss the BDS spin chain in this context.

The starting point for these long-range deformations is a given short-range system with mutually commuting Hamiltonians Q_r^{SR}, $r = 2, 3, \ldots$, (*e.g.* generated through (6.1.23)) that act *locally* and *homogeneously* on a spin chain. The long-range charges $\mathrm{Q}_r(g)$ are then defined by the deformation equation

$$\frac{d}{dg}\mathrm{Q}_r(g) = i[\mathrm{X}(g), \mathrm{Q}_r(g)], \qquad \mathrm{Q}_r(0) \equiv \mathrm{Q}_r^{[0]} = \mathrm{Q}_r^{\text{SR}}, \tag{6.2.40}$$

7. For odd values of the length L one should add a twist to the inhomogeneities that we neglect here for simplicity [136].

This deformation equation is algebra-preserving. Let us for the moment consider a non-trivial algebra between the operator $\mathrm{Q}_r(g)$

$$[\mathrm{Q}_r(g), \mathrm{Q}_s(g)] = f_{rst}\, \mathrm{Q}_t(g). \tag{6.2.41}$$

By Jacobi identity we have

$$\frac{d}{dg}[\mathrm{Q}_r(g), \mathrm{Q}_s(g)] = i[\mathrm{X}(g), [\mathrm{Q}_r(g), \mathrm{Q}_s(g)]] \tag{6.2.42}$$

Substituting (6.2.41) into (6.2.42), we find that

$$\frac{d}{dg} f_{rst} = 0, \tag{6.2.43}$$

which means the structure constant and hence the algebra is preserved at any point of the moduli space. In particular, for conserved charges $f_{rst} = 0$. Therefore $\mathrm{Q}_r(g)$ commute with each other at any value of g.

The differential equation (6.2.40) can be solved perturbatively. First we integrate the equation

$$\mathrm{Q}_r(g) = \mathrm{Q}_r(0) + \int_0^g dg'\, i[\mathrm{X}(g'), \mathrm{Q}(g')]. \tag{6.2.44}$$

Expand both $\mathrm{Q}(g)$ and $\mathrm{X}(g)$ perturbatively in g

$$\begin{aligned} \mathrm{Q}_r(g) &= \mathrm{Q}_r^{(0)} + g^2\, \mathrm{Q}_r^{(1)} + g^4\, \mathrm{Q}_r^{(2)} + g^6\, \mathrm{Q}_r^{(3)} + \cdots \\ \mathrm{X}(g) &= \mathrm{X}^{(0)} + g^2\, \mathrm{X}^{(1)} + g^4\, \mathrm{X}^{(2)} + g^6\, \mathrm{X}^{(3)} + \cdots \end{aligned} \tag{6.2.45}$$

Matching the powers of g on both sides,

$$\begin{aligned} \mathrm{Q}_r^{(1)} &= i[\mathrm{X}^{(0)}, \mathrm{Q}^{(0)}], \\ \mathrm{Q}_r^{(2)} &= \frac{i}{2}[\mathrm{X}^{(1)}, \mathrm{Q}_r^{(0)}] + \frac{i}{2}[\mathrm{X}^{(1)}, \mathrm{Q}_r^{(0)}], \\ \mathrm{Q}_r^{(3)} &= \cdots \end{aligned} \tag{6.2.46}$$

This shows that the higher orders of charges are determined by lower order charges in an iterative way. The equation defines an integrable system recursively.

The generators of long-range deformations $\mathrm{X}(g)$ are constrained by the requirement that the $\mathrm{Q}_r(g)$ are *local* and *homogeneous* operators. In [118, 130] two main classes of

generators X were identified and their physical interpretation was studied: [8]

$$\text{Boost charges:} \qquad \mathrm{X} = \mathcal{B}[\mathrm{Q}_r] = [\mathrm{I}|\mathrm{Q}_r] \qquad \text{(rapidity map)} \tag{6.2.47}$$

$$\text{Bilocal charges:} \qquad \mathrm{X} = [\mathrm{Q}_r|\mathrm{Q}_s] \qquad \text{(dressing phase)} \tag{6.2.48}$$

Here the bilocal composition of two local operators $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$[\mathcal{L}_1|\mathcal{L}_2] = \sum_{k<\ell} \mathcal{L}_{1,k}\, \mathcal{L}_{2,\ell}. \tag{6.2.49}$$

Furthermore one may deform the charges by local operators $\mathrm{X} = \mathcal{L}$ which amounts to a similarity transformation. Deformations with local conserved charges $\mathrm{X} = \mathrm{Q}_r$ are trivial. The basis of local charges can be transformed without spoiling integrability. Typically the initial basis of short-range Hamiltonians is chosen in such a way that the charge $\mathrm{Q}_r(0)$ acts on at most r neighboring spin chain sites at the same time.

Let us note that the boost operator (6.1.22) transforms under translations as

$$\mathrm{U}_0\, \mathcal{B}[\mathcal{L}]\, \mathrm{U}_0^{-1} = \mathcal{B}[\mathcal{L}] + \mathcal{L}, \tag{6.2.50}$$

and is therefore not well-defined globally, since it is not compatible with the periodicity condition $\mathrm{U}_0^L = 1$. However, if $\mathcal{L}$ is a conserved charge, the above boost recursions (6.1.23),(6.2.40) are well-defined locally, since the defining relations yield a local homogeneous operator. The fact that the boost is not well-defined globally insures that the deformation (6.2.40) is not just a similarity transformation and that the spectrum of the deformed model is different from the spectrum of the undeformed model. Similar arguments apply to deformations with bilocal charges.

The BDS spin chain introduced in the previous sections is obtained from a specific combination of the above boost deformations and basis transformations. Therefore we will here focus on boost deformations (6.2.47) of the XXX chain and leave the study of bilocal deformations in this context for future work. In order to obtain the full integrable model describing the asymptotic $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM theory (including the dressing phase contributions), also the bilocal charges (6.2.48) have to be switched on. The BDS and the full $\mathcal{N} = 4$ SYM theory chain in the $\mathfrak{su}(2)$ sector correspond to a specific choice of parameters in the large class of different long-range models that can be generated by the above method.

It is important to note that generically the interaction range of the solutions $\mathrm{Q}_r(g)$ to (6.2.40) increases with each order of the coupling parameter g. This implies that for a

8. Note that more types of generators can be specified depending on the deformed short-range model (see for instance the discussions of open boundary conditions [137] or the XXZ model [138]).

given spin chain, the range of the charge $\mathrm{Q}_r(g)$ exceeds the length of the chain from a given perturbative order in g. It is not known how to define the action of the charges beyond this so-called *wrapping order*. Hence, the validity of the considered long-range model is limited to the *asymptotic* regime of long states.

Generic boost deformations induced by the deformation equation [9]

$$\frac{d}{dg}\mathrm{Q}_r(g) = i\sum_{k=3}^{\infty}\tau_k\,[\mathcal{B}[\mathrm{Q}_k(g)],\mathrm{Q}_r(g)], \tag{6.2.51}$$

result in models that are diagonalized by the asymptotic BAE

$$\left(\frac{f(u+\frac{i}{2})}{f(u-\frac{i}{2})}\right)^L = \prod_{\substack{i=1\\ i\neq k}}^{M}\frac{u_j-u_k+i}{u_j-u_k-i}, \qquad e^{ip} = \frac{f(u+\frac{i}{2})}{f(u-\frac{i}{2})}. \tag{6.2.52}$$

Here the rapidity map $f(u)$ is related to τ_k by

$$\frac{df(u)}{dg} = -\sum_{k=3}^{\infty}\frac{\tau_k}{(k-1)}\frac{1}{f^{k-2}}. \tag{6.2.53}$$

When expressed as functions of the rapidity u, the charge eigenvalues for the pure boost deformations take the ordinary short-range form (6.1.16) and the deformation enters only via the Bethe equations. Note that the equation (6.2.51) for the charges $\mathrm{Q}_r(g)$ is non-linear. Let us now compare the Bethe Ansatz equations for the deformed spin chain (6.2.52) with those for the inhomogeneous spin chain. We notice that they look similar, up to terms of order θ^L at least, if we write

$$\frac{d\ln f(u)}{du} = \frac{1}{L}\sum_{k=0}^{\infty}\frac{\sigma_k}{u^{k+1}}, \quad \text{with} \quad \sigma_k = \sum_{i=1}^{L}\theta_i^k, \tag{6.2.54}$$

and we relate τ_k to the symmetric sums σ_k as prescribed by the relations (6.2.53) and (6.2.54). Since the functional form of their conserved charges can also be chosen to be identical, we conclude that the spectra of the inhomogeneous model and the corresponding deformed model are the same. Because the spectrum depends only on the value of the symmetric sums, any permutation of the values of the impurities gives a model with the same spectrum (but not the same Hamiltonian). One may therefore suspect that the two types of models are mutually related by a unitary transformation [119]. In the next sections, this transformation is defined, and determined explicitly for the first two orders in perturbation. The values of the symmetric sums σ_k in (6.2.54) can be translated into

9. For a more detailed discussion of the relation between these deformations and the Bethe ansatz see [118, 130]. For comparison to the notation used in [119] we note that $\tau_k = \Pi_k/dg$, where Π_k is a one-form defined in that paper.

values of the coupling constants τ_k for the long-range deformations. These coupling constants define a whole family of long-range integrable models, since the values of the first L symmetric sums can be tuned independently. Among these models we are particularly interested in the BDS model.

The BDS spin chain. Let us consider the recursive definition of the BDS chain in some more detail. In this case the rapidity map and its inverse are given by $f^{\rm BDS}(u) \equiv x(u)$, see (6.2.38). We may use this explicit form to evaluate (6.2.53) according to

$$-\frac{dx(u)}{dg} = \frac{du(x)}{dg}\Big/\frac{du(x)}{dx} = \frac{2gx}{x^2-g^2} = \sum_{k=3}^{\infty}\frac{\tau_k^{\rm BDS}}{(k-1)x^{k-2}}\,, \tag{6.2.55}$$

such that writing the left hand side as a series we find the BDS expressions

$$\tau_{2k}^{\rm BDS} = 0, \qquad \tau_{2k+1}^{\rm BDS} = 4k\, g^{2k-1}. \tag{6.2.56}$$

Note that the pure boost deformations (6.2.51) merely deform the map between the momentum p and the rapidity u while the functional form (6.1.16) of the charge eigenvalues $q_r(u,g^2) = q_r^{\rm SR}(u)$ remains unchanged. These boost deformed charges have the same eigenvalues as those of the inhomogeneous models [10] with the values of the symmetric sums given by

$$\sigma_{2k+1}^{\rm BDS} = 0\,, \qquad \sigma_{2k}^{\rm BDS} = Lg^{2k}\frac{(2k)!}{(k!)^2}\,. \tag{6.2.57}$$

In particular this is true for the model with the inhomogeneities given in formula (6.2.39).

6.3 Map from Long-Range to Inhomogeneous Models

In this section we elaborate on the relation between long-range and inhomogeneous spin chains. After studying the unitary transformation S that relates the charge operators of the two models at leading orders, we argue that the operator S originates from a combination of boost operators and an inhomogeneous version of Baxter's corner transfer matrix discussed above. Finally we comment on the morphism defined by the S-operator and the scalar products between BDS Bethe states.

10. Let us emphasize that two inhomogeneous models obtained from one another by permutation of inhomogeneities have the same spectrum but different conserved charges, so they can be considered as being different.

6.3.1 S-Operator

As we have shown in the previous section, the BDS spin chain is related to the inhomogeneous spin chain by a unitary transformation. In particular, the conserved charges of the two models, Q_r^θ (inhomogeneous) and $\mathrm{Q}_r(g)$ (long range) are related by the same similarity transformation

$$\mathrm{Q}_r(g) = \mathrm{S}\,\mathrm{Q}_r^\theta\,\mathrm{S}^{-1}. \tag{6.3.58}$$

The l.h.s. of (6.3.58) can be obtained from the recursive procedure (6.2.46) to all orders. On the other hand, Q_r^θ is defined by the expansion of inhomogeneous transfer matrix (6.1.5) and can also be worked out to all orders. Therefore, we can use (6.3.58) as an all-loop definition of the S-operator. By proposing a proper ansatz for S and comparing both sides of (6.3.58), we can work out S order by order.

Definition of the S-operator. The similarity transformation S is the same for any conserved charges. Computationally, it is simplest to consider the shift operator

$$\mathrm{U}(g) = \mathrm{S}\,\mathrm{U}_\theta\,\mathrm{S}^{-1}. \tag{6.3.59}$$

Since both shift operators $\mathrm{U}(g)$ and U_θ are defined to all orders, this yields an all-order definition of the S-operator in the parameter g. On the one hand, the inhomogeneous shift operator is defined by equation (6.1.12) from which we can read off its expansion ($\theta \sim g$):

$$\mathrm{U}_\theta = \mathrm{U}_0\left[1 - i\sum_{k=1}^{L}\theta_k\mathrm{H}_k - \frac{1}{2}\sum_{k,l=1}^{L}\theta_k\theta_l\mathrm{H}_k\mathrm{H}_l - \frac{1}{2}\sum_{k=1}^{L}\theta_{k-1}\theta_k\,[\mathrm{H}]_{k-1}\right] + \mathcal{O}(g^3). \tag{6.3.60}$$

For the boost induced long-range models on the other hand, we may apply the deformation equation (6.2.51) to the shift operator in analogy to deforming the local charges:

$$\frac{d}{dg}\mathrm{U}(g) = i\sum_{k=3}^{\infty}\tau_k[\mathcal{B}[\mathrm{Q}_k(g)],\mathrm{U}(g)], \qquad \mathrm{U}(0) = \mathrm{U}_0. \tag{6.3.61}$$

Here τ_k is defined by the rapidity map $f(u)$ (6.2.53).

Let us assume that the expansion of τ_{2k+1} starts at g^{2k-1} and that $\tau_{2k} = 0$.[11] Then we can use the shift property (6.2.50) of the boost charges ($[\mathcal{B}[\mathcal{L}],\mathrm{U}_0] = -\mathrm{U}_0\mathcal{L}$) to write

11. The former assumption is motivated by the interaction range of the local charges being constrained by the gauge theory. The latter assumption corresponds to a parity conserving model. Both assumptions are satisfied for the BDS chain.

down the first two non-trivial orders of $\mathrm{U}(g^2)$:

$$\mathrm{U}(g) = \mathrm{U}_0\Big[1 - i\bar{\tau}_3\,\mathrm{Q}_3^{\mathrm{SR}} + \bar{\tau}_3^2\Big([\mathcal{B}[\mathrm{Q}_3^{\mathrm{SR}}], \mathrm{Q}_3^{\mathrm{SR}}] - \frac{1}{2}\big(\mathrm{Q}_3^{\mathrm{SR}}\big)^2\Big) - i(\bar{\tau}_5 + \bar{\tau}_3^2)\mathrm{Q}_5^{\mathrm{SR}}\Big] + \mathcal{O}(g^6). \quad (6.3.62)$$

Here we have defined [12]

$$\bar{\tau}_k = \int_0^g \tau_k(g')dg'. \quad (6.3.63)$$

Thus, both shift operators U_θ and $\mathrm{U}(g)$ are defined to all orders in g and (6.3.59) furnishes an all-order definition of the operator S. In the following we will elaborate more on the generic structure of the S-operator.

Let us now explicitly derive the perturbative expression for the unitary transformation that relates the two models up to order g^2. We make the same ansatz as in [119], namely

$$\mathrm{S} = \exp i \sum_k \Big[\nu_k \mathrm{H}_k + \frac{i}{2}\rho_k[\mathrm{H}]_{k-1} + \mathcal{O}(g^3)\Big], \quad (6.3.64)$$

where ν_k and ρ_k are unknown functions of inhomogeneities θ_k to be determined. Here we assume that $\theta_k \sim g$, $\nu_k \sim g$ and $\rho_k \sim g^2$. We can now compare the two shift operators (6.3.60) and (6.3.62) and derive the constraints following from (6.3.59).

First order. We apply the ansatz (6.3.64) for the S-operator to the inhomogeneous shift and evaluate the expression at order g:

$$\mathrm{S}\,\mathrm{U}_\theta\,\mathrm{S}^{-1} = \mathrm{U}_0\Big[1 - i\sum_{k=1}^{L}\mathrm{H}_k(\nu_k - \nu_{k-1} + \theta_k) + (\nu_L - \nu_0)\mathrm{H}_1\Big] + \mathcal{O}(g^2). \quad (6.3.65)$$

Here we have used that $\mathrm{H}_k\mathrm{U}_0 = \mathrm{U}_0\mathrm{H}_{k+1}$. Since the long-range shift operator has no contribution at order g^1, (6.3.59) yields the constraints

$$\nu_k - \nu_{k-1} + \theta_k = 0, \qquad \nu_L - \nu_0 = 0. \quad (6.3.66)$$

These equations are solved by the explicit expression

$$\nu_k = \nu_0 - \sum_{x=1}^{k}\theta_x, \quad (6.3.67)$$

and the periodicity condition for the first-order parameters yields

$$\nu_{k+L} = \nu_k \qquad \Rightarrow \qquad \sum_{x=1}^{L}\theta_x = 0. \quad (6.3.68)$$

12. For instance we have $\bar{\tau}_{2k+1}^{\mathrm{BDS}} = 2g^{2k}$.

The latter condition guarantees that the operator S is periodic, *i.e.* it represents a well-defined transformation on a *periodic* spin chain at the considered order.

Second order. Proceeding to terms at order g^2 in (6.3.59) we assume that the above constraints (6.3.66) hold. Also at this order we require S to be periodic which amounts to $\rho_{k+L} = \rho_k$. We may again evaluate the right hand side of (6.3.59) and after some manipulations we arrive at

$$\mathrm{S}\,\mathrm{U}_\theta\,\mathrm{S}^{-1} = \mathrm{U}_0\Big[1 + \frac{1}{2}\sum_{k=1}^{L}\big(\rho_k - \rho_{k-1} + \nu_{k-1}(\theta_k - \theta_{k-1})\big)[\mathrm{H}]_{k-1}\Big] + \mathcal{O}(g^3). \tag{6.3.69}$$

We may now compare this expression to the long-range shift operator (6.3.62) which gives the constraint equation for the second order parameters ρ_k:

$$\rho_k - \rho_{k-1} = \bar{\tau}_3 - \nu_{k-1}(\theta_k - \theta_{k-1}). \tag{6.3.70}$$

This equation is solved by (here we assume for simplicity $\nu_0 = 0$)

$$\rho_k = \rho_0 + \bar{\tau}_3 k - \sum_{x=1}^{k}\nu_{x-1}(\theta_x - \theta_{x-1}) = \rho_0 + \bar{\tau}_3 k - \theta_k\nu_k - \sum_{x=1}^{k}\theta_x^2, \tag{6.3.71}$$

and periodicity for the second order parameter yields

$$\rho_{k+L} = \rho_k \qquad \Rightarrow \qquad \sum_{x=1}^{L}\theta_x^2 = \sigma_2 = \bar{\tau}_3 L. \tag{6.3.72}$$

General structure Assembling the results, we obtain the S-operator

$$\mathrm{S} = \exp i\Big[\bar{\tau}_3\mathcal{B}[\mathrm{Q}_3^{\mathrm{SR}}] + \sum_{k=1}^{L}\big(\nu_k\mathrm{H}_k + \frac{i}{2}\hat{\rho}_k[\mathrm{H}]_{k-1}\big)\Big] + \mathcal{O}(g^3), \tag{6.3.73}$$

Notably the S-operator can be split into two contributions

$$\mathrm{S} = \mathrm{S}_\mathcal{B} \times \mathrm{S}_\theta^{-1}, \quad \mathrm{S}_\mathcal{B} = \exp i\Phi, \quad \mathrm{S}_\theta^{-1} = \exp i\Theta. \tag{6.3.74}$$

Here the boost and inhomogeneous generator are given by

$$\Phi = \bar{\tau}_3\mathcal{B}[\mathrm{Q}_3^{\mathrm{SR}}] + \mathcal{O}(g^4), \qquad \Theta = \sum_{k=1}^{L}\big(\nu_k\mathrm{H}_k + \frac{i}{2}\hat{\rho}_k[\mathrm{H}]_{k-1}\big) + \mathcal{O}(g^3). \tag{6.3.75}$$

and we have defined the inhomogeneous parameter $\hat{\rho}_k$ to separate the boost and inhomogeneous piece:

$$\hat{\rho}_k = \rho_0 - \theta_k \nu_k - \sum_{x=1}^{k} \theta_x^2. \tag{6.3.76}$$

This splitting into a boost and an inhomogeneous piece is natural knowing that the boost deformations generate the long-range model from the short-range (here Heisenberg) model (6.2.2). In particular this implies two important features for the inhomogeneous part S_θ^{-1} of the S-operator:

– In the bulk, S_θ^{-1} sets all inhomogeneities to zero and hence represents the generator of the inhomogeneous rapidity shift as indicated in (6.1.2).
– At the boundary, S_θ^{-1} completes $\mathrm{S}_\mathcal{B}$ to a periodic operator S.

Remarkably, the expression for S_θ^{-1} in (6.3.74) agrees with the expansion of the inhomogeneous corner transfer matrix $\mathcal{A}_\theta$ (6.1.29). That is to say that the parameters ν_k and $\hat{\rho}_k$ are the same functions of θ as defined in (6.1.30) (for $\nu_0 = 0$ and $\rho_0 = 0$). We have thus found that the expansion of S_θ^{-1} is identical with the expansion of the inhomogeneous CTM at first orders:

$$\mathrm{S}_\theta^{-1} = \mathcal{A}_\theta(0) + \mathcal{O}(\theta^3). \tag{6.3.77}$$

Assuming that the map between S_θ and $\mathcal{A}_\theta$ holds at higher orders, it seems natural to use the CTM to define the operator S_θ. In fact, the inhomogeneous CTM is defined to all orders in θ according to (6.1.28) in terms of R-matrices. Together with the boost deformations discussed in the previous sections this could furnish an explicit definition of the complete S-operator. Note that at higher orders it remains to be shown that an operator S_θ defined in this way has the desired property to combine with the boost part into the transformation translating between long-range and inhomogeneous spin chains.

6.4 Morphism Property and Scalar Products

In the previous chapter we have shown how to obtain the integrals of motion for the long-range (LR) model by transforming the inhomogeneous integrals of motion with the unitary operator S. The same unitary transformation can be applied to the monodromy matrix as well,

$$T^{\mathrm{LR}}(u) = \mathrm{S}\, T(u;\boldsymbol{\theta})\, \mathrm{S}^{-1}\,, \tag{6.4.78}$$

where the values of $\boldsymbol{\theta}$ are chosen as explained in (6.2.54). It is straightforward to show that the monodromy matrix of the long-range model $T^{\mathrm{LR}}(u)$ obeys the Yang-Baxter equation, and that its matrix elements obey the same algebra as the inhomogeneous (or homogeneous) ones. The unitary transformation is therefore a *morphism* of the Yangian

algebra,

$$T_a^{\mathrm{LR}}(u)\, T_b^{\mathrm{LR}}(v) = \mathrm{S}\, T_a(u;\boldsymbol{\theta})\, T_b(v;\boldsymbol{\theta})\, \mathrm{S}^{-1}\ , \tag{6.4.79}$$

for any spaces a and b. It is important to note that this morphism works for periodic chains of arbitrary length, up to wrapping order g^{2L}. This is in contradistinction to the morphism considered in [128, 129], based on the Dunkl operators, where a defect was added at the point where the chain closes. Of course, the difference between the two is small for large chains. Let us explore the consequences of this morphism. First, the Bethe vectors for the long-range model, on-shell or of-shell, can be written simply as

$$|\mathbf{u}\rangle_{\mathrm{LR}} = \mathrm{S}\,|\mathbf{u};\boldsymbol{\theta}\rangle\ , \qquad {}_{\mathrm{LR}}\langle\mathbf{u}| = \langle\mathbf{u};\boldsymbol{\theta}|\,\mathrm{S}^{-1}\ . \tag{6.4.80}$$

This means that the scalar products, including the norms, are the same for the long-range model and the corresponding inhomogeneous model,

$${}_{\mathrm{LR}}\langle\mathbf{v}|\mathbf{u}\rangle_{\mathrm{LR}} = \langle\mathbf{v};\boldsymbol{\theta}|\mathbf{u};\boldsymbol{\theta}\rangle\ . \tag{6.4.81}$$

The evaluation of the scalar products in the long-range model, up to wrapping order, is then straightforward. Let us emphasize that we do not need to know the explicit form of the operator S in order to compute the scalar products of the long-range spin chain. The above formulas are readily adapted for going to the semiclassical limit where L and N are large.

The dressing phase and the inhomogeneities. The inhomogeneities can also be used to emulate the effect of the dressing phase, provided that we allow their value to depend on the value of the rapidities. This amounts to allowing the symmetric sums to be symmetric functions of the rapidities $\mathbf{u}$ ($\sigma_k = \sigma_k(\mathbf{u})$) so that we have for the eigenstates of the model with the dressing phase, for example with the BES phase [139],

$$|\mathbf{u}\rangle_{\mathrm{BES}} = \mathrm{S}(\mathbf{u})|\mathbf{u};\boldsymbol{\theta}(\mathbf{u})\rangle. \tag{6.4.82}$$

Since the operator $\mathrm{S}(\mathbf{u})$ depends now on the state on which it acts, we cannot compute the scalar products in the same straightforward manner, but at least we can compute the norms of the Bethe ansatz vectors,

$${}_{\mathrm{BES}}\langle\mathbf{u}|\mathbf{u}\rangle_{\mathrm{BES}} = \langle\mathbf{u};\boldsymbol{\theta}(\mathbf{u})|\mathbf{u};\boldsymbol{\theta}(\mathbf{u})\rangle = \lim_{\mathbf{v}\to\mathbf{u}}\langle\mathbf{v};\boldsymbol{\theta}(\mathbf{v})|\mathbf{u};\boldsymbol{\theta}(\mathbf{u})\rangle \tag{6.4.83}$$

The above scalar product can be computed as a usual scalar product in the inhomogeneous model, in particular the last expression, before taking the limit, is a usual scalar

product with the one of vectors on-shell and the other off-shell, which can be computed using (3.3.69).

Chapter 7

Higher Loops

In this chapter, we compute the three-point function in the $\mathfrak{su}(2)$ sector at one-loop. We have shown how to construct the eigenstates of the BDS spin chain in Chapter 6. In this chapter we will combine this result with the other new features at one loop and write down the three-point in a compact form.

The chapter is structured as the follows. First we will define the scheme independent structure constant at one loop and the one-loop operator insertion in section 1. In section 2, we will introduce the permutation-derivative relation (PD relation) which will play an important role in the computation of three-point functions. In section 3 we compute the three-point function in detail and write the final result in a compact form in terms of the $\mathscr{A}$-functionals. Finally in section 4, we take the semi-classical limit of our result and compare with the result at strong coupling obtained by Kazama and Komatsu [64]. We find an agreement in the Frolov-Tseytlin limit.

7.1 The one-loop operator insertion

In the computation of correlation functions at higher loops there will be divergences and one needs to regularize the result. Let us consider three scalar operators $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The two-point functions at one-loop reads

$$\langle \overline{\mathcal{O}}_i(x_1)\mathcal{O}_i(x_2)\rangle = \frac{\mathcal{N}_i}{x_{12}^{2\Delta_{0,i}}}\left(1+2g^2a_i-\gamma_i\,\log\left(\frac{x_{12}^2}{\varepsilon^2}\right)\right), \quad i=1,2,3. \tag{7.1.1}$$

where $\Delta_{0,i}$ and γ_i are tree level scaling dimensions and the one-loop anomalous dimensions, respectively. Here ε is the regulator and a_i is a *scheme dependent* constant. It is easy to see that by shifting $\varepsilon \to e^{\alpha}\varepsilon$, the constant a_i is shifted by $a_i \to a_i + \alpha\,\gamma_i/g^2$. The one-loop three-point function takes the following form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\sqrt{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3}}{x_{12}^{\Delta_{0,12}}x_{13}^{\Delta_{0,13}}x_{23}^{\Delta_{0,23}}}\,C^{(0)}_{123}\times \tag{7.1.2}$$
$$\left(1+g^2(c_{123}+a_1+a_2+a_3)-\frac{\gamma_1}{2}\log\left(\frac{x_{12}^2\,x_{13}^2}{x_{23}^2\,\varepsilon^2}\right)-\frac{\gamma_2}{2}\log\left(\frac{x_{12}^2\,x_{23}^2}{x_{13}^2\,\varepsilon^2}\right)-\frac{\gamma_3}{2}\log\left(\frac{x_{13}^2\,x_{23}^2}{x_{12}^2\,\varepsilon^2}\right)\right)$$

where $C^{(0)}_{123}$ is the tree-level structure constant which we computed in the previous chapters and

$$\Delta_{0,ij} = \frac{1}{2}\left(\Delta_{0,i}+\Delta_{0,j}-\Delta_{0,k}\right), \quad i,j,k=1,2,3. \tag{7.1.3}$$

In the second line, $g^2(c_{123}+a_1+a_2+a_3)$ is scheme dependent. We choose to include the scheme dependence in the constants a_i $(i=1,2,3)$. The well-defined quantity at one-loop is the regularization scheme independent structure constant. This quantity can be extracted by dividing the three-point function by the proper combinations of two-point functions, which cancels the scheme dependent constants a_i. To be more specific, we define

$$N(x_1,x_2,x_3) = \prod_{i=1}^{3}\sqrt{\frac{\langle\overline{\mathcal{O}}_i(x_i)\mathcal{O}_i(x_j)\rangle\langle\overline{\mathcal{O}}_i(x_i)\mathcal{O}_i(x_k)\rangle}{\langle\overline{\mathcal{O}}_i(x_j)\mathcal{O}_i(x_k)\rangle}}, \quad i\neq j\neq k. \tag{7.1.4}$$

Up to one loop,

$$N(x_1,x_2,x_3) = \frac{\sqrt{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3}}{x_{12}^{\Delta_{0,12}}x_{13}^{\Delta_{0,13}}x_{23}^{\Delta_{0,23}}} \tag{7.1.5}$$
$$\left(1+g^2(a_1+a_2+a_3)-\frac{\gamma_1}{2}\log\left(\frac{x_{12}^2\,x_{13}^2}{x_{23}^2\,\varepsilon^2}\right)-\frac{\gamma_2}{2}\log\left(\frac{x_{12}^2\,x_{23}^2}{x_{13}^2\,\varepsilon^2}\right)-\frac{\gamma_3}{2}\log\left(\frac{x_{13}^2\,x_{23}^2}{x_{12}^2\,\varepsilon^2}\right)\right)$$

The scheme independent structure constant is defined to be

$$\begin{aligned} C_{123}(g^2) \equiv& \frac{\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle}{N(x_1,x_2,x_3)} \\ =& C^{(0)}_{123}+g^2C^{(1)}_{123}+\mathcal{O}(g^3). \end{aligned} \tag{7.1.6}$$

When we compute the three-point functions at one-loop, the quantum corrections have to be taken into account by computing Feynamnn diagrams, see [47, 54, 140]. For a three-point function, there are two different types of Feynmann diagrams. The first kind involves only two operators and is called the *two-body* process, which is given in Fig.(7.1.1). This kind of Feynmann diagrams also appears in the calculation of higher loop two-point

functions and contribute to the correction of scaling dimensions. Another type of cor-

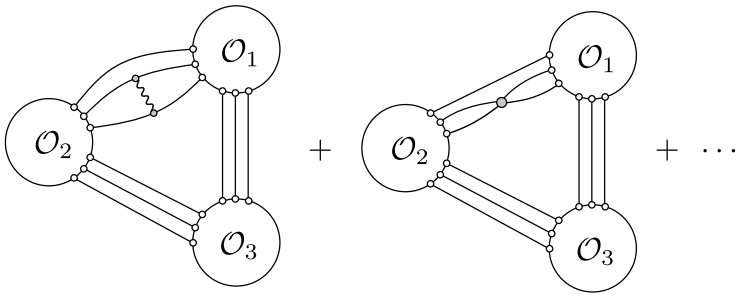

FIGURE 7.1.1: Examples of the two-body process. This kind of processes involves only two operators. In our example, it involves only $\mathcal{O}_1$ and $\mathcal{O}_2$. This kind of diagram also appears in the loop calculation of two-point functions.

rection involves three operators at the same time, which is depicted in Fig.(7.1.2). In

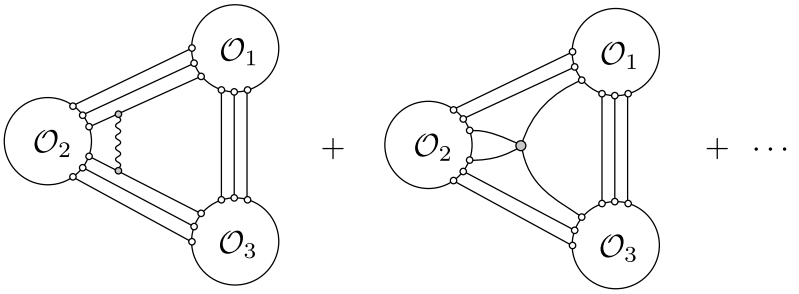

FIGURE 7.1.2: Examples of the three-body process. This kind of process is new in the three-point function case since it involves three operators at the same time.

the computation of the structure constant, these two kinds of processes contribute in different ways. The two-body process, which is depicted in Fig(7.1.3), cancels the same

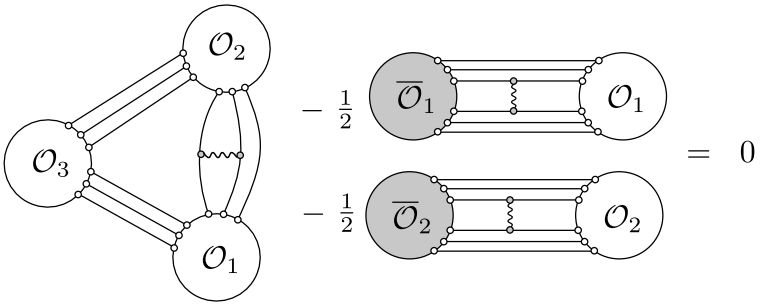

FIGURE 7.1.3: The two-body process of the scheme independent structure constant. The contribution of the three-point function cancels the contribution of the two-point functions and we can neglect this kinds of diagrams.

contributions from the two point functions and can be neglected. The other process which gives non-trivial contribution is depicted in Fig.(7.1.4). It is obvious that the non-trivial contribution can only occur at the points where the spin chain splits. There

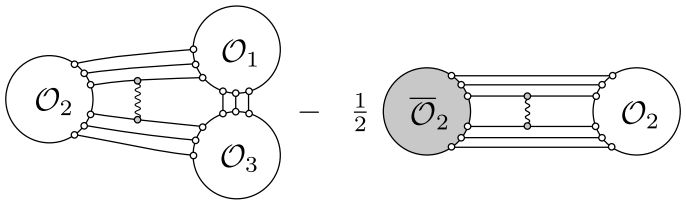

FIGURE 7.1.4: The three-body process of the scheme independent structure constant which gives non-trivial contribution.

are 6 such points, called the *splitting points*. The explicit computation of the Feynmann diagrams corresponding to the above process has been carried out in the compact $\mathfrak{so}(6)$ sector [47, 54] and the supersymmetric $\mathfrak{su}(1|1)$ sector [140]. We focus on the $\mathfrak{so}(6)$ sector in what follows.

Given three operators in the $\mathfrak{so}(6)$ sector

$$\begin{aligned}\mathcal{O}_1 &= \frac{1}{\sqrt{N_c^{L_1}}}\mathrm{Tr}\,\left(\phi^{i_1}\cdots\phi^{i_{L_1}}\right),\\ \mathcal{O}_2 &= \frac{1}{\sqrt{N_c^{L_2}}}\mathrm{Tr}\,\left(\phi^{j_1}\cdots\phi^{j_{L_2}}\right),\\ \mathcal{O}_3 &= \frac{1}{\sqrt{N_c^{L_3}}}\mathrm{Tr}\,\left(\phi^{k_1}\cdots\phi^{k_{L_3}}\right).\end{aligned} \tag{7.1.7}$$

The one-loop structure constant $C_{123}^{(1)}$ reads

$$C_{123}^{(1)} = \sum_{a,b,c}\mathrm{H}^{i_a\, i_{a+1}}_{j_{b+1}\, k_c}\,\mathrm{I} + \sum_{a,b,c}\mathrm{H}^{j_b\, j_{b+1}}_{k_{c+1}\, i_a}\,\mathrm{I} + \sum_{a,b,c}\mathrm{H}^{k_c\, k_{c+1}}_{i_{a+1}\, j_b}\,\mathrm{I} \tag{7.1.8}$$

Where H is the $\mathfrak{so}(6)$ spin chain Hamiltonian density

$$\mathrm{H}^{ij}_{kl} = 2\delta^j_k\,\delta^i_l - 2\delta^i_k\delta^j_l - \delta^{ij}\delta_{kl} = 2\mathrm{P} - 2\mathrm{I} - \mathrm{K}. \tag{7.1.9}$$

Here P and K are the permutation and trace operator, respectively. Each summation in (7.1.8) has two terms, which corresponds to the two splitting points. For example the two terms of the first summation corresponds to $a = 1, b = L_2, c = 1$ and $a = L_{12}-1, b = L_{12}, c = L_{13} - 1$, where

$$L_{ij} = \frac{1}{2}(L_i + L_j - L_k), \quad i, j, k = 1, 2, 3. \tag{7.1.10}$$

These insertions are depicted in Fig.(7.1.5). Apart from the insertions at the splitting points, the other parts are still given by free Wick contractions, which is the meaning of the identity in (7.1.8).

Let us summarize what we have discussed so far. The quantum corrections manifest

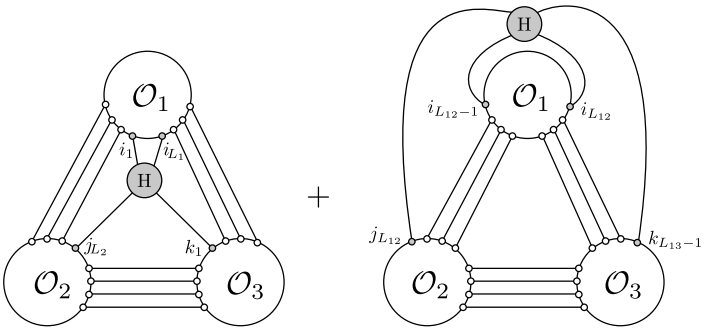

FIGURE 7.1.5: The one-loop insertions for operator $\mathcal{O}_1$ at the two splitting points.

themselves in the scheme independent structure constant as operator insertions at the 6 splitting points. The operators insertions for the $\mathfrak{so}(6)$ takes the form of Hamiltonian density of the $\mathfrak{so}(6)$ spin chain. In the following sections, we will take the insertions into account and compute the scheme independent structure constant at one loop for the three operators with definition two-loop anomalous dimensions.

7.2 The permutation-derivative relation

In this section, we explain how to convert the action of any permutation operators on Bethe states into derivatives with respect to impurities. We shall call this kind of relations permutation-derivative relations or PD relations for short. We derive the PD relations both in the bulk and at the boundary.

7.2.1 PD Relations in the bulk

We choose the normalization of the R-matrix to be

$$R_{\alpha n}(u) = \mathrm{I}_{an} + \frac{i}{u}\mathrm{P}_{an}, \quad n = 1, \ldots, L \tag{7.2.11}$$

and define the monodromy matrix as usual

$$T_a(u, \boldsymbol{\theta}) \equiv \prod_{n=1}^{L} R_{an}(u - \theta_n - i/2). \tag{7.2.12}$$

In the homogeneous limit it becomes

$$T_a(u) = \prod_{n=1}^{L} R_{an}(u - i/2)\ . \tag{7.2.13}$$

The authors in [132] found the following relation

$$[\mathrm{P}_{k,k+1}, T_a(u)] = i(\partial_k - \partial_{k+1})T_a(u,\boldsymbol{\theta})|_{\boldsymbol{\theta}=0} \ , \quad \partial_k \equiv \frac{\partial}{\partial\theta_k} \ , \tag{7.2.14}$$

where in the r.h.s. one first takes the derivatives with respect to the impurities and then sends all impurities to zero. For simplicity, we will denote the r.h.s. of (7.2.14) by $i(\partial_k - \partial_{k+1})T_a(u)$ and adopt the same convention for all PD relations. For later convenience, we introduce the following notations

$$\mathrm{D}_k \equiv i(\partial_k - \partial_{k+1}), \qquad \mathrm{D}_L = i(\partial_L - \partial_1) \ . \tag{7.2.15}$$

We will generalize (7.2.14) to the case when several permutations act on the monodromy matrix. To this end, we first notice that if the action of permutation and derivatives do not overlap, they will act independently. This means, for example

$$\partial_j[\mathrm{P}_{k,k+1}, T_a(u)] = \partial_j \mathrm{D}_k \, T_a(u), \quad \text{if } j \neq k, k+1 \ . \tag{7.2.16}$$

The case where permutations and derivatives overlap needs to be considered more carefully. From the definition of monodromy matrix, one can derive the following relations

$$\begin{aligned}
\partial_k^n[\mathrm{P}_{k,k+1}, T_a(u)] &= -(\partial_k^n - \partial_{k+1}^n)T_a(u)\mathrm{P}_{k,k+1} + \frac{1}{n+1}(i\partial_k^{n+1} - i\partial_{k+1}^{n+1})T_a(u) \qquad (7.2.17)\\
\partial_{k+1}^n[\mathrm{P}_{k,k+1}, T_a(u)] &= (\partial_k^n - \partial_{k+1}^n)T_a(u)\mathrm{P}_{k,k+1} + \frac{1}{n+1}(i\partial_k^{n+1} - i\partial_{k+1}^{n+1})T_a(u)\\
\partial_k^m\partial_{k+1}^n[\mathrm{P}_{k,k+1}, T_a(u)] &= \frac{m!n!}{(m+n+1)!}(i\partial_k^{m+n+1} - i\partial_{k+1}^{m+n+1})T_a(u)
\end{aligned}$$

for any $m, n \in \mathbb{N}$. Relations (7.2.17) can also be written as

$$\begin{aligned}
\mathrm{P}_{k,k+1}\partial_{k+1}^n T_a(u) &= \partial_k^n T_a(u)\mathrm{P}_{k,k+1} + \frac{1}{n+1}(i\partial_k^{n+1} - i\partial_{k+1}^{n+1})T_a(u) \qquad (7.2.18)\\
\mathrm{P}_{k,k+1}\partial_k^n T_a(u) &= \partial_{k+1}^n T_a(u)\mathrm{P}_{k,k+1} + \frac{1}{n+1}(i\partial_k^{n+1} - i\partial_{k+1}^{n+1})T_a(u)\\
\mathrm{P}_{k,k+1}\partial_k^m\partial_{k+1}^n T_a(u) &= \frac{m!n!}{(m+n+1)!}(i\partial_k^{m+n+1} - i\partial_{k+1}^{m+n+1})T_a(u) + \partial_k^m\partial_{k+1}^n T_a(u)\mathrm{P}_{k,k+1} \ .
\end{aligned}$$

By the help of (7.2.18), we can derive the general PD relation. To see how this works, let us consider the following example

$$[\mathrm{P}_{k,k-1}\mathrm{P}_{k,k+1}, T_a(u)] = [\mathrm{P}_{k,k-1}, T_a(u)]\mathrm{P}_{k,k+1} + \mathrm{P}_{k,k-1}[\mathrm{P}_{k,k+1}, T_a(u)] \tag{7.2.19}$$

$$
\begin{aligned}
&= \mathrm{D}_{k-1}T_a(u)\mathrm{P}_{k,k+1} + i\mathrm{P}_{k,k-1}(\partial_k - \partial_{k+1})T_a(u) \\
&= \mathrm{D}_{k-1}T_a(u)\mathrm{P}_{k,k+1} + \frac{i^2}{2}(\partial_{k-1}^2 - \partial_k^2)T_a(u) + i\partial_{k-1}T_a(u)\mathrm{P}_{k,k-1} \\
&- i\partial_{k+1}T_a(u)\mathrm{P}_{k,k-1} - i^2\partial_{k+1}(\partial_{k-1} - \partial_k)T_a(u) \\
&= \frac{1}{2}(\mathrm{D}_{k-1}^2 + 2\mathrm{D}_{k-1}\mathrm{D}_k)T_a(u) + \mathrm{D}_{k-1}T_a(u)\mathrm{P}_{k,k+1} + (\mathrm{D}_{k-1} + \mathrm{D}_k)T_a(u)\mathrm{P}_{k,k-1} \ .
\end{aligned}
$$

Similarly, we can derive

$$
\begin{aligned}
[\mathrm{P}_{k,k+1}\mathrm{P}_{k,k-1}, T_a(u)] &= \qquad (7.2.20) \\
&= \frac{1}{2}(\mathrm{D}_k^2 + 2\mathrm{D}_{k-1}\mathrm{D}_k)T_a(u) + \mathrm{D}_k T_a(u)\mathrm{P}_{k,k-1} + (\mathrm{D}_{k-1} + \mathrm{D}_k)T_a(u)\mathrm{P}_{k,k+1} \ .
\end{aligned}
$$

It is straightforward to generalize this calculation to $[\mathcal{P}, T_a(u)]$ where $\mathcal{P}$ is a product of $\mathrm{P}_{k,k+1}$. In order to apply PD relation on a Bethe state instead of monodromy matrix, one also need to show the PD relation has morphism property. This means, given two functions of the monodromy matrix $\mathrm{X}(u)$ and $\mathrm{Y}(u)$, we have

$$
[\mathrm{P}_{k,k-1}\mathrm{P}_{k,k+1}, \mathrm{XY}] = \frac{1}{2}(\mathrm{D}_k^2 + 2\mathrm{D}_{k-1}\mathrm{D}_k)(\mathrm{XY}) + \mathrm{D}_k(\mathrm{XY})\mathrm{P}_{k,k-1} + (\mathrm{D}_{k-1} + \mathrm{D}_k)(\mathrm{XY})\mathrm{P}_{k,k+1} \ .
$$

One can show this is true by explicit calculation. Using PD relation and morphism property we can derive the following relations, which will be useful in later discussion

$$
\begin{aligned}
\mathrm{H}_{k-1}\mathrm{H}_k|\mathbf{u}\rangle &= [\mathrm{P}_{k,k-1}, [\mathrm{P}_{k,k+1}, B(\mathbf{u})]]|\Omega\rangle \qquad (7.2.21) \\
&= [\mathrm{P}_{k,k-1}\mathrm{P}_{k,k+1}, B(\mathbf{u})]|\Omega\rangle - [\mathrm{P}_{k,k+1}, B(\mathbf{u})]|\Omega\rangle - [\mathrm{P}_{k,k-1}, B(\mathbf{u})]|\Omega\rangle \\
&= \frac{1}{2}(\mathrm{D}_{k-1}^2 + 2\mathrm{D}_k\mathrm{D}_{k-1})|\mathbf{u}\rangle + \mathrm{D}_{k-1}|\mathbf{u}\rangle \ ,
\end{aligned}
$$

where we use the shorthand notation $B(\mathbf{u}) \equiv B(u_1)\cdots B(u_N)$. Similarly, we have

$$
\mathrm{H}_k\mathrm{H}_{k-1}|\mathbf{u}\rangle = \tfrac{1}{2}(\mathrm{D}_k^2 + 2\mathrm{D}_k\mathrm{D}_{k-1})|\mathbf{u}\rangle + \mathrm{D}_k|\mathbf{u}\rangle \ . \qquad (7.2.22)
$$

Taking the sum and difference of (7.2.21) and (7.2.22), we obtain

$$
[\mathrm{H}_{k-1}, \mathrm{H}_k]|\mathbf{u}\rangle = [\mathrm{H}]_{k-1}|\mathbf{u}\rangle = \left(\tfrac{1}{2}(\mathrm{D}_{k-1}^2 - \mathrm{D}_k^2) + \mathrm{D}_{k-1} - \mathrm{D}_k\right)|\mathbf{u}\rangle \qquad (7.2.23)
$$

$$
\{\mathrm{H}_{k-1}, \mathrm{H}_k\}|\mathbf{u}\rangle = \left(\tfrac{1}{2}(\mathrm{D}_{k-1}^2 + \mathrm{D}_k^2) + \mathrm{D}_{k-1} + \mathrm{D}_k + 2\mathrm{D}_k\mathrm{D}_{k-1}\right)|\mathbf{u}\rangle \ . \qquad (7.2.24)
$$

Higher order PD relations can be determined along the same lines.

7.2.2 PD Relations at the Boundary

The PD relations at the boundary are more subtle. In this section, we will derive the boundary PD relations for one and two overlapping permutations, at least one of them involving the bond $1L$. The key observation is that D_k should satisfy the following trivial constraint

$$\sum_{k=1}^{L} \mathrm{D}_k = 0 \,. \tag{7.2.25}$$

At first order, we have

$$\mathrm{D}_L|\mathbf{u}\rangle = -\sum_{k=1}^{L-1} \mathrm{D}_k|\mathbf{u}\rangle = \sum_{k=1}^{L-1} \mathrm{H}_k|\mathbf{u}\rangle = \mathrm{E}_2|\mathbf{u}\rangle - \mathrm{H}_L|\mathbf{u}\rangle \,, \tag{7.2.26}$$

where E_2 is the energy of the state. We find the boundary PD relation at first order,

$$\mathrm{H}_L|\mathbf{u}\rangle = -\mathrm{D}_L|\mathbf{u}\rangle + \mathrm{E}_2|\mathbf{u}\rangle \,. \tag{7.2.27}$$

We consider now the square,

$$\mathrm{D}_L^2 = (\mathrm{D}_1 + \cdots \mathrm{D}_{L-1})^2 \tag{7.2.28}$$

such that

$$\begin{aligned} \frac{1}{2}(\mathrm{D}_L^2 - \mathrm{D}_1^2)|\mathbf{u}\rangle = {} & \tfrac{1}{2}(\mathrm{D}_2^2 + 2\mathrm{D}_1\mathrm{D}_2)|\mathbf{u}\rangle + \cdots + \tfrac{1}{2}(\mathrm{D}_{L-1}^2 + 2\mathrm{D}_{L-2}\mathrm{D}_{L-1})|\mathbf{u}\rangle \\ & + \text{non-connected terms} \end{aligned} \tag{7.2.29}$$

where "non-connected terms" are the terms $2\mathrm{D}_j\mathrm{D}_k|\mathbf{u}\rangle$ with $|j-k| \geq 2$. Using (7.2.22),

$$\frac{1}{2}(\mathrm{D}_k^2 + 2\mathrm{D}_{k-1}\mathrm{D}_k)|\mathbf{u}\rangle = \mathrm{H}_k\mathrm{H}_{k-1}|\mathbf{u}\rangle - \mathrm{D}_k|\mathbf{u}\rangle \tag{7.2.30}$$

we have

$$\begin{aligned} \frac{1}{2}(\mathrm{D}_L^2 - \mathrm{D}_1^2)|\mathbf{u}\rangle = {} & (\mathrm{H}_2\mathrm{H}_1 + \cdots + \mathrm{H}_{L-1}\mathrm{H}_{L-2})|\mathbf{u}\rangle - (\mathrm{D}_2 + \cdots \mathrm{D}_{L-1})|\mathbf{u}\rangle \\ & + \text{non-connected terms} \,, \end{aligned} \tag{7.2.31}$$

which is the same as

$$\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_1^2)+(\mathrm{D}_L-\mathrm{D}_1)\right)|\mathbf{u}\rangle=\sum_{k=2}^{L-1}\mathrm{H}_k\mathrm{H}_{k-1}|\mathbf{u}\rangle+2\mathrm{D}_L|\mathbf{u}\rangle \\ +\text{non-connected terms}\,. \tag{7.2.32}$$

Similarly, using (7.2.21) we have

$$\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_{L-1}^2)+(\mathrm{D}_L-\mathrm{D}_{L-1})\right)|\mathbf{u}\rangle=\sum_{k=2}^{L-1}\mathrm{H}_{k-1}\mathrm{H}_k|\mathbf{u}\rangle+2\mathrm{D}_L|\mathbf{u}\rangle \\ +\text{non-connected terms}\,. \tag{7.2.33}$$

Taking the difference of (7.2.32) and (7.2.33),

$$\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_1^2)+(\mathrm{D}_L-\mathrm{D}_1)\right)|\mathbf{u}\rangle-\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_{L-1}^2)+(\mathrm{D}_L-\mathrm{D}_{L-1})\right)|\mathbf{u}\rangle \tag{7.2.34}$$
$$=-\sum_{k=2}^{L-1}[\mathrm{H}_{k-1},\mathrm{H}_k]|\mathbf{u}\rangle=(2i\mathrm{Q}_3+[\mathrm{H}_{L-1},\mathrm{H}_L]+[\mathrm{H}_L,\mathrm{H}_1])|\mathbf{u}\rangle=(2i\mathrm{Q}_3+[\mathrm{H}]_{L-1}+[\mathrm{H}]_L)|\mathbf{u}\rangle\,.$$

where Q_3 is the third conserved charge. Taking the sum of (7.2.32) and (7.2.33),

$$\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_1^2)+(\mathrm{D}_L-\mathrm{D}_1)\right)|\mathbf{u}\rangle+\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_{L-1}^2)+(\mathrm{D}_L-\mathrm{D}_{L-1})\right)|\mathbf{u}\rangle \tag{7.2.35}$$
$$=\sum_{k=2}^{L-1}\{\mathrm{H}_{k-1},\mathrm{H}_k\}|\mathbf{u}\rangle+4\mathrm{D}_L|\mathbf{u}\rangle+\text{cross terms}=\left(\sum_{k=1}^{L-1}\mathrm{H}_k\right)^2|\mathbf{u}\rangle+2\mathrm{D}_L|\mathbf{u}\rangle\,,$$

where we have used the fact that

$$\sum_{k=1}^{L-1}\mathrm{H}_k^2|\mathbf{u}\rangle=2\sum_{k=1}^{L-1}\mathrm{H}_k|\mathbf{u}\rangle=-2\sum_{k=1}^{L-1}\mathrm{D}_k|\mathbf{u}\rangle=2\mathrm{D}_L|\mathbf{u}\rangle\,. \tag{7.2.36}$$

Now we plug $\sum_{k=1}^{L-1}\mathrm{H}_k=\mathrm{Q}_2-\mathrm{H}_L$ into (7.2.35),

$$\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_1^2\right)+(\mathrm{D}_L-\mathrm{D}_1))|\mathbf{u}\rangle+\left(\frac{1}{2}(\mathrm{D}_L^2-\mathrm{D}_{L-1}^2)+(\mathrm{D}_L-\mathrm{D}_{L-1})\right)|\mathbf{u}\rangle= \tag{7.2.37}$$
$$=(\mathrm{Q}_2-\mathrm{H}_L)^2|\mathbf{u}\rangle+2\mathrm{D}_L|\mathbf{u}\rangle=(\mathrm{E}_2^2+2\mathrm{E}_2-2\mathrm{E}_2\mathrm{H}_L)|\mathbf{u}\rangle-[\mathrm{Q}_2,\mathrm{H}_L]|\mathbf{u}\rangle$$
$$=(\mathrm{E}_2^2+2\mathrm{E}_2-2\mathrm{E}_2\mathrm{H}_L)|\mathbf{u}\rangle-[\mathrm{H}]_{L-1}|\mathbf{u}\rangle+[\mathrm{H}]_L|\mathbf{u}\rangle\,,$$

where we have used (7.2.27). Taking the sum of (7.2.34) and (7.2.37), we find that

$$[\mathrm{H}]_L|\mathbf{u}\rangle = \left(\frac{1}{2}(\mathrm{D}_L^2 - \mathrm{D}_1^2) + (\mathrm{D}_L - \mathrm{D}_1)\right)|\mathbf{u}\rangle + \mathrm{E}_2\mathrm{H}_L|\mathbf{u}\rangle - C(\mathbf{u})|\mathbf{u}\rangle \ , \tag{7.2.38}$$
$$[\mathrm{H}]_{L-1}|\mathbf{u}\rangle = \left(\frac{1}{2}(\mathrm{D}_{L-1}^2 - \mathrm{D}_L^2) + (\mathrm{D}_L - \mathrm{D}_1)\right)|\mathbf{u}\rangle - \mathrm{E}_2\mathrm{H}_L|\mathbf{u}\rangle + C(\mathbf{u})|\mathbf{u}\rangle - 2i\mathrm{E}_3|\mathbf{u}\rangle \ .$$

where $C(\mathbf{u})$ is a function of rapidities defined by

$$C(\mathbf{u}) = \frac{1}{2}[\mathrm{E}_2^2(\mathbf{u}) + 2\mathrm{E}_2(\mathbf{u}) + 2i\mathrm{E}_3(\mathbf{u})] \ . \tag{7.2.39}$$

In the derivation above, we used the fact that the second and third conserved charges are

$$\mathrm{Q}_2 = \sum_{k=1}^{L} \mathrm{H}_k, \quad \mathrm{Q}_3 = \frac{i}{2}\sum_{k=1}^{L}[\mathrm{H}]_k \tag{7.2.40}$$

and $\mathrm{Q}_r|\mathbf{u}\rangle = \mathrm{E}_r|\mathbf{u}\rangle$ when $|\mathbf{u}\rangle$ is on-shell.

Using the PD relation and the S-operator, we can derive the theta-morphism proposed by Gromov and Vieira [132], this is given in Appendix A.

7.3 Calculation of Three-Point Function

In this section, we perform the detail calculation of the three-point function. The set-up is the same as the one (4.1.1) considered in Chapter 4.

The S-operator for $\mathcal{O}_i$ is denoted by S_i. We decompose the S_i operator into bulk and boundary parts, as the following

$$\mathrm{S}_i = \mathrm{S}_{ij}\,\mathrm{S}_{ik}\,\epsilon_i, \quad i \neq j,k \tag{7.3.41}$$

We take S_1 for an example,

$$\mathrm{S}_1 = \exp\left(-\sum_{k=1}^{L_1} i\mu_k\mathrm{H}_k + \frac{1}{2}\rho_k[\mathrm{H}]_{k-1}\right) \tag{7.3.42}$$
$$= \exp\left(-\sum_{k=1}^{L_{12}-1} i\mu_k\mathrm{H}_k + \frac{1}{2}\rho_k[\mathrm{H}]_{k-1}\right) \times \exp\left(-\sum_{k=L_{12}+1}^{L_1} i\mu_k\mathrm{H}_k + \frac{1}{2}\rho_k[\mathrm{H}]_{k-1}\right) \times \epsilon_1 + \mathcal{O}(g^3)$$

where

$$\epsilon_1 = \exp\left(-i\mu_{L_{12}}\mathrm{H}_{L_{12}} - i\mu_{L_1}\mathrm{H}_{L_1} - \frac{1}{2}\rho_{L_{12}}[\mathrm{H}]_{L_{12}} - \frac{1}{2}\rho_{L_1}[\mathrm{H}]_{L_1}\right) \tag{7.3.43}$$

by our choice of the parameters, $\mu_{L_{12}} = \mu_{L_1} = 0$ and $\rho_{L_{12}} = \rho_{L_1} = 2g^2$, we have

$$\epsilon_1 = \exp\left(-g^2[\mathrm{H}]_{L_{12}} - g^2[\mathrm{H}]_{L_1}\right) = 1 - g^2\delta_1 \tag{7.3.44}$$

Similar decompositions can be made for S_2 and S_3.

In what follows, we adopt the formalism in [132] for the three-point function.

$$C_{123} = \textbf{norms} \times \textbf{simple} \times \textbf{involved} \tag{7.3.45}$$

where

$$\begin{aligned}
\textbf{norms} &= \sqrt{L_1L_2L_3}/\sqrt{\langle\mathbf{1}|\mathbf{1}\rangle\langle\mathbf{2}|\mathbf{2}\rangle\langle\mathbf{3}|\mathbf{3}\rangle} \qquad (7.3.46)\\
\textbf{simple} &= \langle\underbrace{\downarrow\cdots\downarrow}_{L_{13}}\underbrace{\uparrow\cdots\uparrow}_{L_{23}}|1 - g^2\mathrm{H}_{L_{13}} - g^2\mathrm{H}_{L_3}|\mathbf{3}\rangle = \langle\mathrm{vac}|1 - g^2\mathbb{I}^{(3)}|\mathbf{3}\rangle\\
\textbf{involved} &= \langle\mathbf{1}|1 - g^2\mathrm{H}_{L_{12}} - g^2\mathrm{H}_{L_1}|i_1\cdots i_{L_{12}}\underbrace{\downarrow\cdots\downarrow}_{L_{13}}\rangle\\
&\quad\langle i_1\cdots i_{L_{12}}\underbrace{\uparrow\cdots\uparrow}_{L_{23}}|1 - g^2\mathrm{H}_{L_{12}} - g^2\mathrm{H}_{L_3}|\mathbf{2}\rangle = \langle\mathbf{1}|(1 - g^2\mathbb{I}^{(1)})\hat{O}_{12}(1 - g^2\mathbb{I}^{(2)})|\mathbf{2}\rangle
\end{aligned}$$

For later convenience, let us introduce the following notations

$$\begin{aligned}
\mathbb{I}^{(1)} &= \mathrm{H}_{L_1} + \mathrm{H}_{L_{12}}, \quad \delta_1 = [\mathrm{H}]_{L_1} + [\mathrm{H}]_{L_{12}} \qquad (7.3.47)\\
\mathbb{I}^{(2)} &= \mathrm{H}_{L_2} + \mathrm{H}_{L_{12}}, \quad \delta_2 = [\mathrm{H}]_{L_2} + [\mathrm{H}]_{L_{12}}\\
\mathbb{I}^{(3)} &= \mathrm{H}_{L_3} + \mathrm{H}_{L_{13}}, \quad \delta_3 = [\mathrm{H}]_{L_3} + [\mathrm{H}]_{L_{13}}
\end{aligned}$$

and differential operators

$$\begin{aligned}
\mathcal{D}_B^{(1)} &= \frac{1}{2}(\mathrm{D}_{L_1}^{(1)2} + 2\mathrm{D}_{L_1}^{(1)}) + \frac{1}{2}(\mathrm{D}_{L_{12}}^{(1)2} + 2\mathrm{D}_{L_{12}}^{(1)}) - \frac{1}{2}(\mathrm{D}_1^{(1)2} + 2\mathrm{D}_1^{(1)}) - \frac{1}{2}(\mathrm{D}_{L_{12}+1}^{(1)2} + 2\mathrm{D}_{L_{12}+1}^{(1)})\\
\mathcal{D}_B^{(2)} &= \frac{1}{2}(\mathrm{D}_{L_2}^{(2)2} + 2\mathrm{D}_{L_2}^{(2)}) + \frac{1}{2}(\mathrm{D}_{L_{12}}^{(2)2} + 2\mathrm{D}_{L_{12}}^{(2)}) - \frac{1}{2}(\mathrm{D}_1^{(2)2} + 2\mathrm{D}_1^{(2)}) - \frac{1}{2}(\mathrm{D}_{L_{12}+1}^{(2)2} + 2\mathrm{D}_{L_{12}+1}^{(2)})\\
\mathcal{D}_B^{(3)} &= \frac{1}{2}(\mathrm{D}_{L_3}^{(3)2} + 2\mathrm{D}_{L_3}^{(3)}) + \frac{1}{2}(\mathrm{D}_{L_{13}}^{(3)2} + 2\mathrm{D}_{L_{13}}^{(3)}) - \frac{1}{2}(\mathrm{D}_1^{(3)2} + 2\mathrm{D}_1^{(3)}) - \frac{1}{2}(\mathrm{D}_{L_{13}+1}^{(3)2} + 2\mathrm{D}_{L_{13}+1}^{(3)})
\end{aligned}$$

7.3.1 Calculation of "simple"

Let us start with **simple**.

$$\begin{aligned}
\textbf{simple} &= \langle\mathrm{vac}|1 - g^2\mathbb{I}^{(3)}\mathrm{S}_3|\mathbf{w};\theta^{(3)}\rangle \qquad (7.3.48)\\
&= \langle\mathrm{vac}|\mathrm{S}_3|\mathbf{w};\theta^{(3)}\rangle - g^2\langle\mathrm{vac}|\mathbb{I}^{(3)}|\mathbf{w}\rangle\\
&= \langle\mathrm{vac}|\mathbf{w};\theta^{(3)}\rangle - g^2\langle\mathrm{vac}|\mathbb{I}^{(3)}|\mathbf{w}\rangle - g^2\langle\mathrm{vac}|\delta_3|\mathbf{w}\rangle
\end{aligned}$$

Using the PD relation at the splitting points, we have

$$\delta_i|\mathbf{w}\rangle = \mathcal{D}_B^{(i)}|\mathbf{w}\rangle + \mathrm{E}_2(\mathbf{w})\mathrm{H}_{L_i}|\mathbf{u}\rangle + C(\mathbf{w})|\mathbf{w}\rangle \tag{7.3.49}$$

Taking the Hermitian conjugate, we obtain

$$\langle\mathbf{w}|\delta_i^\dagger = \mathcal{D}_B^{(i)\dagger}(\langle\mathbf{w}|) + \mathrm{E}_2(\mathbf{w})\langle\mathbf{w}|\mathrm{H}_{L_i} + C^*(\mathbf{w})\langle\mathbf{w}| \tag{7.3.50}$$

Projecting (7.3.49) on the vacuum, we obtain

$$\langle\mathrm{vac}|\delta_3|\mathbf{w}\rangle = \mathcal{D}_B^{(3)}(\langle\mathrm{vac}|\mathbf{w};\theta\rangle) + \mathrm{E}_2(\mathbf{w})\langle\mathrm{vac}|\mathrm{H}_{L_3}|\mathbf{w}\rangle + C(\mathbf{w})\langle\mathrm{vac}|\mathbf{w}\rangle \tag{7.3.51}$$

Let us now compute the first term. It is useful to split the differential operator into two parts, namely the quadratic part and the linear part,

$$\mathcal{D}_B^{(3)} = \mathcal{D}_{Bq}^{(3)} + \mathcal{D}_{Bl}^{(3)} \tag{7.3.52}$$

where

$$\begin{aligned}\mathcal{D}_{Bq}^{(3)} &= \frac{1}{2}(\mathrm{D}_{L_3}^{(3)2} - \mathrm{D}_1^{(3)2} + \mathrm{D}_{L_{13}}^{(3)2} - \mathrm{D}_{L_{13}+1}^{(3)2}) \\ \mathcal{D}_{Bl}^{(3)} &= \mathrm{D}_{L_3}^{(3)} - \mathrm{D}_1^{(3)} + \mathrm{D}_{L_{13}}^{(3)} - \mathrm{D}_{L_{13}+1}^{(3)}\end{aligned} \tag{7.3.53}$$

We can first calculate the linear derivatives

$$\begin{aligned}\langle\mathrm{vac}|\mathcal{D}_{Bl}^{(3)}|\mathbf{w};\theta^{(3)}\rangle &= \langle\mathrm{vac}| - \mathrm{D}_1^{(3)} - \mathrm{D}_{L_{13}+1}^{(3)}|\mathbf{w};\theta^{(3)}\rangle \\ &\quad + \langle\mathrm{vac}| - (i\partial_1^{(3)} - i\partial_{L_{13}}^{(3)}) - (i\partial_{L_{13}+1}^{(3)} - i\partial_{L_3}^{(3)})|\mathbf{w};\theta^{(3)}\rangle \\ &= \langle\mathrm{vac}|\mathrm{H}_1 + \mathrm{H}_{L_{13}+1}|\mathbf{w}\rangle + \langle\mathrm{vac}|\sum_{k=1}^{L_{13}-1}\mathrm{H}_k + \sum_{k=L_{13}+1}^{L_3-1}\mathrm{H}_k|\mathbf{w}\rangle \\ &= \mathrm{E}_2(\mathbf{w})\langle\mathrm{vac}|\mathbf{w}\rangle - \langle\mathrm{vac}|\mathbb{I}^{(3)}|\mathbf{w}\rangle\end{aligned} \tag{7.3.54}$$

As we can see, it produces a term which cancels exactly the insertion. This is also the case for the computation of **involved**.

The quadratic part can also be simplified. Notice that the scalar product $\langle\mathrm{vac}|\mathbf{w};\theta^{(3)}\rangle$ does not depend on the impurities $\theta^{(23)}$, any derivative with respect to these impurities vanish, which leads to

$$\begin{aligned}\mathcal{D}_{Bq}\langle\mathrm{vac}|\mathbf{w};\theta^{(3)}\rangle &= -\frac{1}{2}(\partial_1^{(3)2} + \partial_{L_{12}}^{(3)2} - \partial_1^{(3)2} - \partial_2^{(3)2} - 2\partial_1^{(3)}\partial_2^{(3)})\langle\mathrm{vac}|\mathbf{w};\theta^{(3)}\rangle \\ &= \partial_1^{(3)}\partial_2^{(3)}\langle\mathrm{vac}|\mathbf{w};\theta^{(3)}\rangle\end{aligned} \tag{7.3.55}$$

where in the second line we used the fact that $\langle \text{vac}|\mathbf{w};\theta^{(3)}\rangle$ is totally symmetric with respect to $\theta^{(13)}$. Gathering the pieces together, we obtain

$$\begin{aligned}\textbf{simple} &= \langle \text{vac}|\mathbf{w};\theta^{(3)}\rangle - g^2\partial_1^{(3)}\partial_2^{(3)}\langle \text{vac}|\mathbf{w};\theta^{(3)}\rangle \\ &+ g^2(i\mathrm{E}_3(\mathbf{w}) - \frac{1}{2}\mathrm{E}_2^2(\mathbf{w}))\langle \text{vac}|\mathbf{w}\rangle - g^2\mathrm{E}_2(\mathbf{w})\langle \text{vac}|\mathrm{H}_{1,L}|\mathbf{w}\rangle\end{aligned} \tag{7.3.56}$$

The scalar product can be written as an $\mathscr{A}$-functional,

$$\langle \text{vac}|\mathbf{w}\rangle_{\boldsymbol{\theta}^{(13)}} \equiv \mathscr{A}^{\text{BDS}}_{\mathbf{w},\boldsymbol{\theta}^{(13)}} \tag{7.3.57}$$

where we have defined a shorthand notation for the $\mathscr{A}$-functional defined in (3.3.86)

$$\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}} \equiv \mathscr{A}^{+}_{\mathbf{w}}\left[\frac{Q^{-}_{\boldsymbol{\theta}}(u)}{Q^{+}_{\boldsymbol{\theta}}(u)}\right]. \tag{7.3.58}$$

We have the final result for **simple**

$$\textbf{simple} \simeq \mathscr{A}^{\text{BDS}}_{\mathbf{w},\boldsymbol{\theta}^{(13)}} - g^2\left(\partial_1^{(3)}\partial_2^{(3)} - i\mathrm{E}_2(\mathbf{w})\partial_1^{(3)} + \frac{1}{2}\mathrm{E}_2^2(\mathbf{w})\right)\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}^{(13)}} \tag{7.3.59}$$

where in the first term we fix the impurities to their BDS value while in the second term we take the homogeneous limit after taking the derivatives. We have neglected the pure imaginary term, which does not contribute to the absolute value of the structure constant.

7.3.2 Calculation of "involved"

Now we compute the piece **involved**.

$$\begin{aligned}\textbf{involved} &= \langle \mathbf{u};\theta^{(1)}|S_1^{\dagger}(1-g^2\mathbb{I}^{(1)})\hat{O}_{12}(1-g^2\mathbb{I}^{(2)})S_2|\mathbf{v};\theta^{(2)}\rangle \\ &= \langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}|\mathbf{v};\theta^{(2)}\rangle - g^2(\langle \mathbf{u}|\mathbb{I}^{(1)}\hat{O}_{12}|\mathbf{v}\rangle + \langle \mathbf{u}|\hat{O}_{12}\mathbb{I}^{(2)}|\mathbf{v}\rangle) \\ &- g^2(\langle \mathbf{u}|\delta_1^{\dagger}\hat{O}_{12}|\mathbf{v}\rangle + \langle \mathbf{u}|\hat{O}_{12}\delta_2|\mathbf{v}\rangle)\end{aligned} \tag{7.3.60}$$

We again use the PD relations at the boundary.

$$\begin{aligned}\langle \mathbf{u}|\delta_1^{\dagger}\hat{O}_{12}|\mathbf{v}\rangle &= \mathcal{D}_B^{(1)\dagger}(\langle \mathbf{u};\theta^{(1)}|)\hat{O}_{12}|\mathbf{v}\rangle + \mathrm{E}(\mathbf{u})\langle \mathbf{u}|\mathrm{H}_{L_1}\hat{O}_{12}|\mathbf{v}\rangle + C^*(\mathbf{u})\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle \\ \langle \mathbf{u}|\hat{O}_{12}\delta_2|\mathbf{v}\rangle &= \langle \mathbf{u}|\hat{O}_{12}\mathcal{D}_B^{(2)}(|\mathbf{v};\theta^{(2)}\rangle) + \mathrm{E}(\mathbf{v})\langle \mathbf{u}|\hat{O}_{12}\mathrm{H}_{L_2}|\mathbf{v}\rangle + C(\mathbf{v})\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle\end{aligned} \tag{7.3.61}$$

As a next step, we simplify the sum of the two differential terms, i.e.

$$\mathcal{D}_B^{(1)\dagger}(\langle \mathbf{u};\theta^{(1)}|)\hat{O}_{12}|\mathbf{v}\rangle + \langle \mathbf{u}|\hat{O}_{12}\mathcal{D}_B^{(2)}(|\mathbf{v};\theta^{(2)}\rangle) \tag{7.3.62}$$

We consider linear and quadratic derivatives in the operator $\mathcal{D}_B$ separately

$$\mathcal{D}_B^{(i)} = \mathcal{D}_{Bl}^{(i)} + \mathcal{D}_{Bq}^{(i)} \tag{7.3.63}$$

where

$$\begin{aligned} \mathcal{D}_{Bl}^{(i)} &= \mathrm{D}_{L_i} - \mathrm{D}_1 + \mathrm{D}_{L_{12}} - \mathrm{D}_{L_{12}+1} \\ \mathcal{D}_{Bq}^{(i)} &= \frac{1}{2}(\mathrm{D}_{L_i}^2 - \mathrm{D}_1^2 + \mathrm{D}_{L_{12}}^2 - \mathrm{D}_{L_{12}+1}^2) \end{aligned} \tag{7.3.64}$$

Here we omit the upper indices for simplicity. We have

$$\begin{aligned} &(\mathrm{D}_{L_2} - \mathrm{D}_1 + \mathrm{D}_{L_{12}} - \mathrm{D}_{L_{12}+1})|\mathbf{v};\theta^{(2)}\rangle \\ =&(\mathrm{H}_1 + \mathrm{H}_{L_{12}+1})|\mathbf{v}\rangle + (-(i\partial_1 - i\partial_{L_{12}}) - (i\partial_{L_{12}+1} - i\partial_{L_2}))|\mathbf{v};\theta^{(2)}\rangle \\ =&(\mathrm{H}_1 + \mathrm{H}_{L_{12}+1})|\mathbf{v}\rangle + \left(\sum_{k=1}^{L_{12}-1} \mathrm{H}_k + \sum_{k=L_{12}+1}^{L_2-1} \mathrm{H}_k\right)|\mathbf{v}\rangle \\ =&(\mathrm{H}_1 + \mathrm{H}_{L_{12}+1})|\mathbf{v}\rangle + (\mathrm{E}_2(\mathbf{v}) - \mathbb{I}^{(2)})|\mathbf{v}\rangle \end{aligned} \tag{7.3.65}$$

Now that

$$\hat{O}_{12} = \sum_{i_1,\cdots,i_{L_{12}}=\uparrow,\downarrow} |i_1 \cdots i_{L_{12}} \uparrow \cdots \uparrow\rangle\langle i_1 \cdots i_{L_{12}} \downarrow \cdots \downarrow| \tag{7.3.66}$$

We have immediately $\hat{O}_{12}\mathrm{H}_{L_{12}+1} = 0$, hence we have

$$\hat{O}_{12}\mathcal{D}_{Bl}|\mathbf{v};\theta^{(2)}\rangle = \hat{O}_{12}\mathrm{H}_1|\mathbf{v}\rangle + \mathrm{E}_2(\mathbf{v})\hat{O}_{12}|\mathbf{v}\rangle - \hat{O}_{12}\mathbb{I}^{(2)}|\mathbf{v}\rangle \tag{7.3.67}$$

Similarly, we can show that

$$\mathcal{D}_{Bl}^{\dagger}(\langle\mathbf{u};\theta^{(1)}|)\hat{O}_{12} = \mathrm{E}_2(\mathbf{u})\langle\mathbf{u}|\hat{O}_{12} - \langle\mathbf{u}|\mathbb{I}^{(1)}\hat{O}_{12} + \langle\mathbf{u}|\mathrm{H}_1\hat{O}_{12} \tag{7.3.68}$$

Therefore we have

$$\begin{aligned} &\langle\mathbf{u}|\hat{O}_{12}\mathcal{D}_{Bl}(|\mathbf{v};\theta^{(2)}\rangle) + \mathcal{D}_{Bl}^{\dagger}(\langle\mathbf{u};\theta^{(1)}|)\hat{O}_{12}|\mathbf{v}\rangle \\ =&(\mathrm{E}(\mathbf{u}) + \mathrm{E}(\mathbf{v}))\langle\mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle - (\langle\mathbf{u}|\mathbb{I}^{(1)}\hat{O}_{12}|\mathbf{v}\rangle + \langle\mathbf{u}|\hat{O}_{12}\mathbb{I}^{(2)}|\mathbf{v}\rangle) + 2\langle\mathbf{u}|\hat{O}_{12}\mathrm{H}_1|\mathbf{v}\rangle \end{aligned} \tag{7.3.69}$$

where the colored terms will cancel with contributions from the other terms. Now we compute the quadratic derivatives

$$\begin{aligned}
&\langle \mathbf{u}|\hat{O}_{12}\mathcal{D}_{Bq}(|\mathbf{v};\theta^{(2)}\rangle) + \mathcal{D}^{\dagger}_{Bq}(\langle \mathbf{u};\theta^{(1)}|)\hat{O}_{12}|\mathbf{v}\rangle \qquad (7.3.70)\\
=&\mathcal{D}_{Bq}\big(\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}|\mathbf{v};\theta^{(2)}\rangle\big) - \mathrm{D}_{L_1}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_1}|\mathbf{v};\theta^{(2)}\rangle\\
&+\mathrm{D}_1\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_1|\mathbf{v};\theta^{(2)}\rangle - \mathrm{D}_{L_{12}}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_{12}}|\mathbf{v};\theta^{(2)}\rangle + \mathrm{D}_{L_{12}+1}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_{12}+1}|\mathbf{v};\theta^{(2)}\rangle
\end{aligned}$$

The four crossing terms are calculated as the following:

– First term

$$\begin{aligned}
&-\mathrm{D}_{L_1}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_1}|\mathbf{v};\theta^{(2)}\rangle \qquad (7.3.71)\\
&= -\mathrm{D}_{L_1}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_2}|\mathbf{v};\theta^{(2)}\rangle\\
&= \langle \mathbf{u}|(-\mathrm{H}_{L_1} + \mathrm{E}_2(\mathbf{u}))\hat{O}_{12}(-\mathrm{H}_{L_2} + \mathrm{E}_2(\mathbf{v}))|\mathbf{v}\rangle\\
&= \mathrm{E}_2(\mathbf{u})\mathrm{E}_2(\mathbf{v})\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle - \mathrm{E}_2(\mathbf{u})\langle \mathbf{u}|\hat{O}_{12}\mathrm{H}_{L_2}|\mathbf{v}\rangle - \mathrm{E}_2(\mathbf{v})\langle \mathbf{u}|\mathrm{H}_{L_1}\hat{O}_{12}|\mathbf{v}\rangle
\end{aligned}$$

– Second term

$$\mathrm{D}_1\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_1|\mathbf{v};\theta^{(2)}\rangle = -\langle \mathbf{u}|\mathrm{H}_1\hat{O}_{12}\mathrm{H}_1|\mathbf{u}\rangle = -2\langle \mathbf{u}|\hat{O}_{12}\mathrm{H}_1|\mathbf{v}\rangle \qquad (7.3.72)$$

This term will cancel the blue color term in (7.3.69).

– Third term

$$-\mathrm{D}_{L_{12}}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_{12}}|\mathbf{v};\theta^{(2)}\rangle = \langle \mathbf{u}|\mathrm{H}_{L_{12}}\hat{O}_{12}\mathrm{H}_{L_{12}}|\mathbf{v}\rangle = 0 \qquad (7.3.73)$$

This is due to the special configuration of $\hat{O}_{12}$. If $i_{L_{12}} = \uparrow$, we have $\mathrm{H}_{L_{12}}\hat{O}_{12} = 0$; if $i_{L_{12}} = \downarrow$, we have $\hat{O}_{12}\mathrm{H}_{L_{12}} = 0$. Therefore we always have $\mathrm{H}_{L_{12}}\hat{O}_{12}\mathrm{H}_{L_{12}} = 0$.

– Last term

$$\mathrm{D}_{L_{12}+1}\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}\mathrm{D}_{L_{12}+1}|\mathbf{v};\theta^{(2)}\rangle = -\langle \mathbf{u}|\mathrm{H}_{L_{12}+1}\hat{O}_{12}\mathrm{H}_{L_{12}+1}\rangle = 0 \qquad (7.3.74)$$

This is simply because $\mathrm{H}_{L_{12}+1}\hat{O}_{12} = \hat{O}_{12}\mathrm{H}_{L_{12}+1} = 0$.

Now we combine the results of the linear contribution (7.3.69) and the quadratic contribution (7.3.70),

$$\begin{aligned}
&\mathcal{D}^{\dagger}_{B}(\langle \mathbf{u};\theta^{(1)}|)\hat{O}_{12}|\mathbf{v}\rangle + \langle \mathbf{u}|\hat{O}_{12}\mathcal{D}_{B}(|\mathbf{v};\theta^{(2)}\rangle) \qquad (7.3.75)\\
=&\mathcal{D}_{Bq}(\langle \mathbf{u};\theta^{(1)}|\hat{O}_{12}|\mathbf{v};\theta^{(2)}\rangle) + (\mathrm{E}_2(\mathbf{u}) + \mathrm{E}_2(\mathbf{v}))\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle - (\langle \mathbf{u}|\mathbb{I}^{(1)}\hat{O}_{12}|\mathbf{v}\rangle + \langle \mathbf{u}|\hat{O}_{12}\mathbb{I}^{(2)}|\mathbf{v}\rangle)\\
&+\mathrm{E}_2(\mathbf{u})\mathrm{E}_2(\mathbf{v})\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle + \mathrm{E}_2(\mathbf{u})\langle \mathbf{u}|\hat{O}_{12}\mathrm{H}_{L_2}|\mathbf{v}\rangle + \mathrm{E}_2(\mathbf{v})\langle \mathbf{u}|\mathrm{H}_{L_1}\hat{O}_{12}|\mathbf{v}\rangle
\end{aligned}$$

Assembling the pieces, we obtain

$$\begin{aligned}\textbf{involved} = &\langle \mathbf{u}; \theta^{(1)}|\hat{O}_{12}|\mathbf{v}; \theta^{(2)}\rangle - g^2\mathcal{D}_{Bq}(\langle \mathbf{u}; \theta^{(1)}|\hat{O}_{12}|\mathbf{v}; \theta^{(2)}\rangle) \\ &+ \frac{g^2}{2}(\mathrm{E}_2(\mathbf{u}) - \mathrm{E}_2(\mathbf{v}))^2\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle \\ &+ g^2(\mathrm{E}_2(\mathbf{v}) - \mathrm{E}_2(\mathbf{u}))\langle \mathbf{u}|\mathrm{H}_{L_1}\hat{O}_{12} - \hat{O}_{12}\mathrm{H}_{L_2}|\mathbf{v}\rangle \\ &- i(\mathrm{E}_3(\mathbf{u}) - \mathrm{E}_3(\mathbf{v}))\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle\end{aligned} \tag{7.3.76}$$

Using the fact

$$\langle \mathbf{u}|\mathrm{H}_{L_1}\hat{O}_{12} - \hat{O}_{12}\mathrm{H}_{L_2}|\mathbf{v}\rangle = [\mathrm{D}_{L_{12}} - (\mathrm{E}_2(\mathbf{v}) - \mathrm{E}_2(\mathbf{u}))]\langle \mathbf{u}; \theta^{(1)}|\hat{O}_{12}|\mathbf{v}; \theta^{(2)}\rangle, \tag{7.3.77}$$

we can combine the second line and the third line in (7.3.76),

$$\begin{aligned}&\frac{g^2}{2}(\mathrm{E}_2(\mathbf{u}) - \mathrm{E}_2(\mathbf{v}))^2\langle \mathbf{u}|\hat{O}_{12}|\mathbf{v}\rangle + g^2(\mathrm{E}_2(\mathbf{v}) - \mathrm{E}_2(\mathbf{u}))\langle \mathbf{u}|\mathrm{H}_{1,L_1}\hat{O}_{12} - \hat{O}_{12}\mathrm{H}_{1,L_2}|\mathbf{v}\rangle \\ =&g^2(\mathrm{E}_2(\mathbf{v}) - \mathrm{E}_2(\mathbf{u}))\left[\mathrm{D}^{(1)}_{L_{12}} - \frac{1}{2}(\mathrm{E}_2(\mathbf{v}) - \mathrm{E}_2(\mathbf{u}))\right]\langle \mathbf{u}; \theta^{(1)}|\hat{O}_{12}|\mathbf{v}; \theta^{(2)}\rangle\end{aligned} \tag{7.3.78}$$

This term is purely imaginary (see [132] Appendix E.1). We also notice that the last line in (7.3.76) is also purely imaginary. Hence these terms do not contribute to $|C_{123}|$. The scalar product in **involved** can also be written in terms of an $\mathscr{A}$-functional, which is quite similar to the one of **simple**

$$\langle \mathbf{u}; \boldsymbol{\theta}^{(1)}|\hat{O}_{12}|\mathbf{v}; \boldsymbol{\theta}^{(2)}\rangle \equiv \mathscr{A}^{\mathrm{BDS}}_{\mathbf{u}\cup\mathbf{v}, \boldsymbol{\theta}^{(12)}} \tag{7.3.79}$$

Then the **involved** is simply

$$\textbf{involved} \simeq \mathscr{A}^{\mathrm{BDS}}_{\mathbf{u}\cup\mathbf{v}, \boldsymbol{\theta}^{(12)}} - g^2\partial_1^{(1)}\partial_2^{(1)}\mathscr{A}_{\mathbf{u}\cup\mathbf{v}, \boldsymbol{\theta}^{(12)}} \tag{7.3.80}$$

Combining the result from **simple** and **involved**, we can write down the final result for $|C_{123}(g^2)|$

$$|C_{123}(g^2)| = C^{\mathrm{BDS}}_{123} + g^2 c_{123} + \mathcal{O}(g^4) \tag{7.3.81}$$

Note that the first term contains higher order terms in g^2 and one should expand it for the first order. The explicit form for first term is

$$|C^{\mathrm{BDS}}_{123}| = \frac{\mathscr{A}^{\mathrm{BDS}}_{\mathbf{w}, \boldsymbol{\theta}^{(13)}}\mathscr{A}^{\mathrm{BDS}}_{\mathbf{u}\cup\mathbf{v}, \boldsymbol{\theta}^{(12)}}}{\sqrt{\langle \mathbf{u}; \theta^{(1)}_{\mathrm{BDS}}|\mathbf{u}; \theta^{(1)}_{\mathrm{BDS}}\rangle\langle \mathbf{v}; \theta^{(2)}_{\mathrm{BDS}}|\mathbf{v}; \theta^{(2)}_{\mathrm{BDS}}\rangle\langle \mathbf{w}; \theta^{(3)}_{\mathrm{BDS}}|\mathbf{w}; \theta^{(3)}_{\mathrm{BDS}}\rangle}} \tag{7.3.82}$$

The second term reads

$$g^2 c_{123} = -g^2 \frac{\left(\partial_1^{(3)}\partial_2^{(3)} - i\mathrm{E}_2(\mathbf{w})\partial_1^{(3)} + \frac{1}{2}\mathrm{E}_2^2(\mathbf{w})\right)\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}^{(13)}}\mathscr{A}_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}^{(12)}}}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle\langle\mathbf{v}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle}} - g^2 \frac{\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}^{(13)}}\partial_1^{(1)}\partial_2^{(1)}\mathscr{A}_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}^{(12)}}}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle\langle\mathbf{v}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle}} \tag{7.3.83}$$

From (7.3.83) it is easy to see that the term $g^2 c_{123}$ is $1/L^2$ suppressed compared to C_{123}^{BDS}, hence it is negligible in the semi-classical limit $N, L \to \infty$. It is interesting to notice that this term mainly comes from the contribution from the splitting points, it is reasonable to suspect that the same thing happens for the higher loop corrections. Then the only piece that matters is the term C_{123}^{BDS}, whose classical limit is known.

7.4 Three-Point Functions in the Semi-Classical Limit

In this section we take the semi-classical limit of the one-loop structure constant (7.3.81). The semi-classical limit of tree level structure constant was obtained in Chapter 4.

7.4.1 Scalar Products and Norms in the Semi-Classical Limit

Let us recall that an N-magnon Bethe state with magnon rapidities $\mathbf{u} = \{u_1, \ldots, u_N\}$ is characterized by its quasi-momentum $p(u)$, which is defined modulo π by (4.4.28). The pseudomomentum of an on-shell Bethe state satisfies N conditions

$$e^{2ip(u)}\Big|_{u=u_j} = -1, \quad j = 1, 2, , \ldots, N, \tag{7.4.84}$$

which are equivalent to the Bethe equations for the roots $\mathbf{u}$. In the semi-classical limit the pseudomomentum is given by

$$p(u) \simeq G_{\mathbf{u}}(u) - \frac{1}{2}G_{\boldsymbol{\theta}}(u) + \pi n, \tag{7.4.85}$$

with $G_{\mathbf{u}}$ and $G_{\boldsymbol{\theta}}$ being the resolvents for the magnon rapidities and the inhomogeneities,

$$G_{\mathbf{u}}(u) = \partial \log Q_{\mathbf{u}}(u), \qquad G_{\boldsymbol{\theta}}(u) = \partial \log Q_{\boldsymbol{\theta}}(u). \tag{7.4.86}$$

The resolvent corresponding to the distribution of the rapidities (6.2.39) is

$$G_{\boldsymbol{\theta}}(u) = \frac{L}{\sqrt{u^2 - 4g^2}}, \tag{7.4.87}$$

Let us recall the semi-classical limit of the $\mathscr{A}$-functional from Chapter 3,

$$\log \mathscr{A}^{\pm}_{\mathbf{u},\boldsymbol{\theta}} = \pm \oint_{\mathcal{C}} \frac{du}{2\pi} \mathrm{Li}_2\big(e^{\pm iG_{\mathbf{u}}(u) \mp iG_{\boldsymbol{\theta}}(u)}\big) + \mathcal{O}(1), \qquad L \to \infty, N/L \sim 1, \tag{7.4.88}$$

where the contour $\mathcal{C}$ surrounds the rapidities $\mathbf{u}$ and leaves outside $\boldsymbol{\theta}$. As a consequence, the scalar product $\langle \mathbf{u}^{(1)}; \boldsymbol{\theta} | \mathbf{u}^{(2)}; \boldsymbol{\theta} \rangle$ is expressed through the sum of the two pseudomomenta:

$$\log \langle \mathbf{u}^{(1)}; \boldsymbol{\theta} | \mathbf{u}^{(2)}; \boldsymbol{\theta} \rangle = \oint_{\mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}} \frac{du}{2\pi} \mathrm{Li}_2\big(e^{ip^{(1)}(u) + ip^{(2)}(u)}\big), \tag{7.4.89}$$

where the contour $\mathcal{C}^{(a)}$ surrounds the set of rapidities $\mathbf{u}^{(a)}$ and leaves outside the set of the inhomogeneities $\boldsymbol{\theta}$. In the classical limit the derivative of the pseudomomentum $p^{(a)}$ is defined on a four-sheeted Riemann surface and the discrete set of points $\mathbf{u}^{(a)}$ condense in a set of cuts on the main sheet. Similarly for the set $\boldsymbol{\theta}^{(a)}$.

The norm of a Bethe eigenstate in the semi-classical limit reads:

$$\log \langle \mathbf{u}; \boldsymbol{\theta} | \mathbf{u}; \boldsymbol{\theta} \rangle = \oint_{\mathcal{C}} \frac{du}{2\pi} \mathrm{Li}_2\big(e^{2ip(u)}\big), \tag{7.4.90}$$

where the contour of integration surrounds the rapidities $\mathbf{u}$ and leaves outside $\boldsymbol{\theta}$.

The determination of the contour is a subtle issue because of the logarithmic branch cuts starting at the points where the argument of the dilogarithm equals 1. The contour must avoid these cuts and its choice depends on the analytic properties of the pseudomomenta.

7.4.2 One-Loop Three-Point Function in the Classical Limit

By the computation of the previous subsection, the structure constant up to two-loop corrections is given by

$$\langle \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)} \rangle \equiv e^{F_{123}} = \left(1 + g^2 \hat{\Delta}\right) e^{F_{123}(\boldsymbol{\theta})} + \mathcal{O}(g^4), \tag{7.4.91}$$

$$F_{123}(\boldsymbol{\theta}) \equiv \log \mathscr{A}_{\mathbf{u}^{(2)} \cup \mathbf{u}^{(1)}, \boldsymbol{\theta}^{(12)}} \ + \log \mathscr{A}_{\mathbf{u}^{(3)}, \boldsymbol{\theta}^{(13)}}. \tag{7.4.92}$$

where in order to compare with the strong coupling result, we have changed the notation slightly, representing $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ by $\mathbf{u}^{(1)}$, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(1)}$, respectively. The differential operator $\hat{\Delta}$ is defined as ($\delta \mathrm{E}_r = \mathrm{E}_r^{(1)} - \mathrm{E}_r^{(2)}$)

$$\begin{aligned} \hat{\Delta} = &\left(\partial_1^{(3)} \partial_2^{(3)} - i\mathrm{E}_2^{(3)} \partial_1^{(3)} + i\mathrm{E}_3^{(3)} - \frac{1}{2}\mathrm{E}_2^{(3)2}\right) \\ &+ \left(\partial_1^{(1)} \partial_2^{(1)} - i\delta\mathrm{E}_2 \partial_1^{(1)} + i\delta\mathrm{E}_3 - \frac{1}{2}\delta\mathrm{E}_2^2\right). \end{aligned} \tag{7.4.93}$$

Thus the one-loop result for the structure constant is expressed in terms of the tree-level quasiclassical expression with the inhomogeneities entering as free parameters (7.4.89). Using the quasiclassical formula (7.4.88), one obtains in the semi-classical limit

$$F_{123}(\boldsymbol{\theta}) \simeq \oint_{\mathcal{C}^{(1)}\cup\mathcal{C}^{(2)}} \frac{du}{2\pi}\mathrm{Li}_2\big(e^{ip^{(1)}(u)+ip^{(2)}(u)-iq^{(3)}(u)}\big) + \oint_{\mathcal{C}^{(3)}} \frac{du}{2\pi}\mathrm{Li}_2\big(e^{ip^{(3)}(u)+iq^{(1)}(u)-iq^{(2)}(u)}\big) \tag{7.4.94}$$

Here $p^{(a)}$ are the three quasimomenta and $q^{(a)}$ are their singular parts:

$$p^{(a)} = G_{\mathbf{u}^{(a)}} + q^{(a)} \qquad q^{(a)} = -\tfrac{1}{2}G_{\boldsymbol{\theta}^{(a)}} \qquad (a=1,2,3). \tag{7.4.95}$$

For the complete phase in (7.4.91) we obtain

$$F_{123} = F_{123}(\boldsymbol{\theta}) + g^2\,\delta F_{123} + \mathcal{O}(g^4), \tag{7.4.96}$$

where the inhomogeneities in the first term on the r.h.s. are fixed to their BDS values, and the second term

$$\delta F_{123} = e^{-F_{123}(\boldsymbol{\theta})}\,\hat{\Delta}\,e^{F_{123}(\boldsymbol{\theta})}\Big|_{\boldsymbol{\theta}=0} \tag{7.4.97}$$

will be computed below. The first term $F_{123}(\boldsymbol{\theta})$ is an infinite series in g^2 from which only the $\mathcal{O}(g^0)$ and the $\mathcal{O}(g^2)$ terms should be retained. In order to evaluate δF_{123} one should compute the derivatives in $\theta_{1,2}$ of the phase $F_{123}(\boldsymbol{\theta})$. The computation of the derivatives in $\theta_{1,2}$ is done using the representation (7.4.88) of the $\mathscr{A}$-functional:

$$\frac{\partial}{\partial\theta_1}\log\mathscr{A}_{\mathbf{u},\boldsymbol{\theta}} = -i\oint_{\mathcal{C}}\frac{du}{2\pi i}\,\frac{1}{u^2}\,\log\left(1-e^{iG_{\mathbf{u}}-iG_{\boldsymbol{\theta}}}\right), \tag{7.4.98}$$

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\log\mathscr{A}_{\mathbf{u},\boldsymbol{\theta}} = -\oint_{\mathcal{C}}\frac{du}{2\pi i}\,\frac{1}{u^4}\,\frac{1}{1-e^{iG_{\mathbf{u}}-iG_{\boldsymbol{\theta}}}}. \tag{7.4.99}$$

Below we will neglect the term with the second derivative, which is of order $1/L$ compared to the other terms. Then we have

$$\begin{aligned}\delta F_{123} \;=\;& i\mathrm{E}_3^{(3)} - \frac{1}{2}\mathrm{E}_2^{(3)2} + i\mathrm{E}_3^{(1)} - i\mathrm{E}_3^{(2)} - \frac{1}{2}(\mathrm{E}_3^{(1)}-\mathrm{E}_3^{(2)})^2 \\ & -(\mathrm{E}_2^{(1)}-\mathrm{E}_2^{(2)})\oint_{\mathcal{C}^{(1)}\cup\mathcal{C}^{(2)}}\frac{du/u^2}{2\pi i}\,\log\left(1-e^{ip^{(1)}+ip^{(2)}-iq^{(3)}}\right) \\ & -\mathrm{E}_2^{(3)}\oint_{\mathcal{C}^{(3)}}\frac{du/u^2}{2\pi i}\,\log\left(1-e^{ip^{(3)}+iq^{(1)}-iq^{(2)}}\right) \\ & -\left[\oint_{\mathcal{C}^{(3)}}\frac{du/u^2}{2\pi i}\,\log\left(1-e^{ip^{(3)}+iq^{(1)}-iq^{(2)}}\right)\right]^2 \\ & -\left[\oint_{\mathcal{C}^{(1)}\cup\mathcal{C}^{(3)}}\frac{du/u^2}{2\pi i}\,\log\left(1-e^{ip^{(1)}+ip^{(2)}-iq^{(3)}}\right)\right]^2.\end{aligned} \tag{7.4.100}$$

The complete result for $\log C_{123}$ is obtained by subtracting from $F_{123} + \delta F_{123}$ the logarithms of the norms of the three states, given by the contour integrals (7.4.90).

As we mentioned earlier, the choice of the integration contours is a non-trivial problem. The heuristic derivations of the quasiclassical limit in [141, 142] require that the contour of integration $\mathcal{C}^{(a)}$ encircles the cuts $\mathbf{u}^{(a)}$ and leaves outside the the $\boldsymbol{\theta}$-cut. However this prescription does not determine the contours completely because it says nothing about the logarithmic singularities of the integrand at the points where the argument of the dilogarithm takes value 1. A necessary condition on the integration contours is that they should not cross any of the cuts produced by these singularities. In the contour integral along $\mathcal{C}^{(a)} \cup \mathcal{C}^{(b)}$ the positions of the singularities depend on the analytic properties of both $p^{(a)}$ and $p^{(b)}$. Let us denote by $\mathcal{C}^{(ab|c)}$ the contour which encircles the cuts $\mathbf{u}^{(a)}$ and $\mathbf{u}^{(b)}$, leaves outside the $\boldsymbol{\theta}$-cut and does not cross any of the logarithmic cuts ending at the other singularities of the integrand.

$$\mathcal{C}^{(a)} \cup \mathcal{C}^{(b)} \to \mathcal{C}^{(ab|c)} \,.$$

In order to determine the contour of integration $\mathcal{C}^{(ab|c)}$, one can consider a family of solutions characterised by their global filling fractions $\alpha^{(a)} = N^{(a)}/L^{(a)}$, solve for the singular points in the limit $\alpha^{(a)} \ll 1$ $(a = 1, 2, 3)$ and place the contours $\mathcal{C}^{(ab|c)}$ so that they return to the same sheet. When $\alpha^{(i)}$ increases, the contour deforms in a continuous way.

The above rule works only if the logarithmic singularities at the points where the argument of the dilogarithm equals 1 are macroscopically far from the cuts formed by condensation of Bethe roots. If a singular point gets close or crosses such a cut, the integration contour should be closed on the second sheet, possibly through infinity, as in the example considered in [142].

7.4.3 Comparison with the String Theory Results

The semiclassical limit of the one-loop result in the SYM theory is expected to match the strong coupling result in the Frolov–Tseytlin [143] limit, where the gauge coupling g is large, but the typical length L is even larger, so that the effectve coupling $g' = g/L$ is small. This is however not obvious because of the order-of-limits problem [144, 145].

The hope that such a comparison is meaningful is based on the observation that the first two orders of the expansion in $g'^2 = g^2/L^2$ of the anomalous dimension of a heavy operator in the weakly coupled gauge theory, and of the energy of the corresponding classical string state, coincide. Since the computation of the correlation function requires

the knowledge of the wave functions one order beyond, it is reasonable to expect that for the three-point functions the match is to the linear order in g'^2.

A string theory computation of the three-point function at strong coupling was carried out by Kazama and Komatsu [64]. They expressed the three-point function in terms of the pseudo-momenta $p^{(1)}, p^{(2)}, p^{(3)}$ obtained from the monodromy matrix for a solution of the $\mathfrak{so}(4)$ sigma model at strong coupling. They obtained for the logarithm of the structure constant an expression in terms of contour integrals, very similar to (7.4.94). The arguments of the dilogarithm functions are $p^{(a)} + p^{(b)} - p^{(c)}$ for $a, b, c \in \{1, 2, 3\}$, as well as $p^{(1)} + p^{(2)} + p^{(3)}$, and the expression is symmetric in the permutations of the three operators.

Here we will compare the Frolov–Tseytlin limit of the strong-coupling answer of [64] with the semi-classical limit of our solution (7.4.100) to the linear order in g'^2, assuming that the integration contours coincide, which is very likely to be the case.

Let L be the length scale such that $L^{(a)}/L \sim 1$ for $a = 1, 2, 3$. The operators $\mathcal{O}^{(a)}$ correspond to solutions of the Bethe equations consisting of a few macroscopic Bethe strings. Since the typical distance between the roots forming such a string is ~ 1, the spectral parameter scales as $u \sim L$, which implies for the conserved charges $\mathrm{E}_r \sim L^{1-r}$ (r=1,2, ...). As a consequence, the correction δF_{123} to the phase (7.4.96) scales as [1]

$$g^2\, \delta F_{123} \sim g'^2. \tag{7.4.101}$$

On the other hand, the one-loop correction in $F_{123}(\boldsymbol{\theta})$ due to the inhomogeneities, which comes from replacing $L/u \to G_{\boldsymbol{\theta}} = L/u + 2Lg^2/u^3 + \ldots$, scales as $L \times Lg^2/L^3 = Lg'^2$. Therefore the correction $g^2\delta F_{123}/F_{123} \sim 1/L$ can be neglected in the Frolov-Tseytlin limit and our one-loop result reads simply

$$\langle \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)} \rangle \simeq \exp F_{123}(\boldsymbol{\theta}). \tag{7.4.102}$$

In the Frolov–Tseytlin limit the result of Kazama and Komatsu for F_{123} (section 7.5 of [64]) consists of four terms,

$$F_{123}^{\mathrm{KK}} \simeq \oint \mathrm{Li}_2\big(e^{ip^{(1)}+ip^{(2)}-ip^{(3)}}\big) + \oint \mathrm{Li}_2\big(e^{ip^{(3)}+ip^{(1)}-ip^{(2)}}\big) \tag{7.4.103}$$

$$+ \oint \mathrm{Li}_2\big(e^{ip^{(2)}+ip^{(3)}-ip^{(1)}}\big) + \oint \mathrm{Li}_2\big(e^{ip^{(2)}+ip^{(3)}+ip^{(1)}}\big)\,. \tag{7.4.104}$$

1. The fact that δF_{123} does not contain a factor of L in the Frolov-Tseytlin limit is not trivial in our computation, because Hamiltonian insertions scale as Lg'^2 and δS also contains terms that scale as Lg'^2. These two contributions nicely cancel each other leaving us with a net result scaling as g'^2. We suspect that similar cancellations will happen also at higher loop orders.

Comparing this with (7.4.94), we see that the first two terms resemble the two terms of (7.4.94), while the last two terms do not have counterparts in the weak coupling result. The correspondence with the gauge theory requires that the last two terms vanish. It is found by Kazama and Komatsu [146] that one can deform the contour in such a way that the last two terms vanish.

We will compare the first two terms (7.4.103) with the one-loop result (7.4.94). We will give an interpretation of the exponent in (7.4.102) in terms of the complex curves of the three heavy fields. Obviously the asymmetric form of the tree-level expression (7.4.94) is a consequence of the specific choice of the $\mathfrak{su}(2)$ sectors for the three operators ($\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{su}(2)_R$ and $\mathcal{O}_2 \in \mathfrak{su}(2)_L$). Since the left and the right $\mathfrak{su}(2)$ sectors do not talk to each other perturbatively, the dependence on the third operator factors out. This factorisation is accidental and is a consequence of the choice of the three $\mathfrak{su}(2)$ sectors and the weak coupling limit. At strong coupling, there is no reason to expect that the three-point function factorises.

Below we are going to show that the arguments of the dilogarithm function in (7.4.94) are the $g/L \to 0$ limit of symmetric combinations of the three quasimomenta, *e.g.* $p^{(3)} + q^{(1)} - q^{(2)}$ is obtained as a limit of $p^{(3)} + p^{(1)} - p^{(2)}$. For that we assume that the three operators are on-shell Bethe states from the $\mathfrak{so}(4)$ sector. This makes sense at strong coupling when the $\mathfrak{so}(4)$ sector is closed. [2] Then the linear combination of the three quasimomenta is a meromorphic function with a four-sheeted Riemann surface as the one depicted in Fig.(7.4.6).

The natural parametrization of the momenta in the strong coupling limit is by the Zhukovsky variable x defined in (6.2.38). The a-th quasimomentum is determined by the set of $N^{(a)}$rapidities $\mathbf{x}^{(a)} = \{x_1^{(a)}, \dots, x_{N^{(a)}}^{(a)}\}$, which are related to the rapidities $\mathbf{u}^{(a)}$ by the Zhukovsky map (6.2.38). Instead of (7.4.95), we have

$$p(x) = \mathcal{G}(x) - \frac{\Delta/2}{x - g^2/x}, \tag{7.4.105}$$

2. The $\mathfrak{so}(4)$ sectors at weak and at strong coupling have different nature and the comparison should be taken with caution, see the discussion in [147]. In the XXX spin chain (with or without inhomogeneities) the length of the chain $L = \Delta|_{g=0}$ is expressed in terms of the two conserved R-charges. At perturbative level the length of an operator is conserved, since the dimension Δ of the states that contain n pairs $X\bar{X}$ and have the same R-charges is separated by a gap $2n$ from the states belonging to the $\mathfrak{su}(2)$ sector and are unreachable perturbatively. On the contrary, in the sigma model there is no such gap and to the states of given charge one can add X and $\bar{X}$ as constituent fields, since this combination has zero total charge. The length of a state is not a conserved charge and it is not defined at strong coupling. The $\mathfrak{so}(4)$ sector is therefore not stable for finite g, but in the limit $g \to \infty$ it becomes stable again, as the $\mathfrak{so}(4)$ sigma model is classically stable.

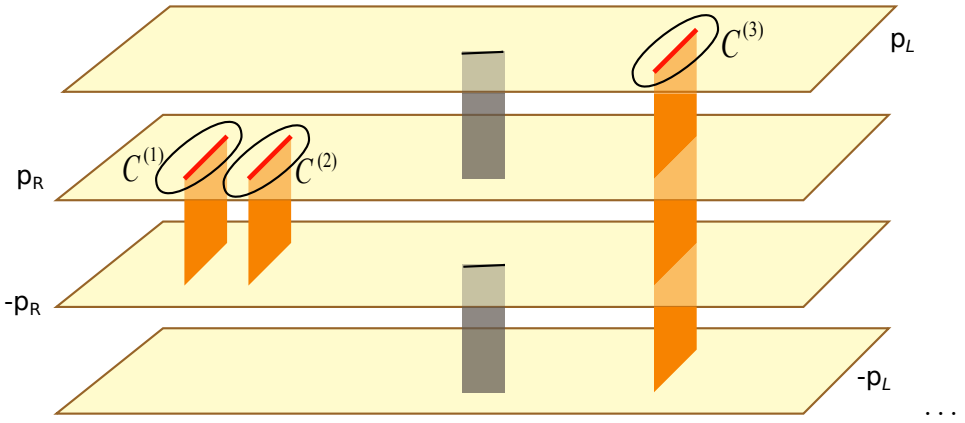

FIGURE 7.4.6: The Riemann surface for the three quasimomenta in the u-parametrization. For simplicity we assumed one-cut solutions. The left (sheets 1,4) and the right (sheets 2,3) sectors are connected by Zhukowsky cuts. In the limit $g \to 0$ the Zhukovsky cuts shrink to points and the $\mathfrak{so}(4)$ Riemann surface decomposes into two disconnected two-sheet Riemann surfaces describing the $\mathfrak{su}(2)_R$ and the $\mathfrak{su}(2)_L$ sectors.

where $\Delta = L + \delta$ is the conformal dimension and the resolvent

$$\mathcal{G}(x) = \sum_j \frac{x'_j}{x - x_j}, \qquad x'_j \equiv \frac{1}{1 - g^2/x_j^2}\,, \tag{7.4.106}$$

is related to the resolvent in the u-plane by

$$G(u) = \mathcal{G}(x) + \mathcal{G}(g^2/x) - \mathcal{G}(0). \tag{7.4.107}$$

The left and the right $\mathfrak{su}(2)$ sectors in $\mathfrak{so}(4)$ are related by the inversion symmetry $x \leftrightarrow g^2/x$, which exchanges right and left pseudomomenta, p_R and p_L [148, 149]:

$$p_R(x) = -p_L(g^2/x) - 2\pi m, \quad m \in \mathbb{Z}. \tag{7.4.108}$$

This allows to go from the four-sheeted Riemann surface in the u-parametrization to a two-sheet Riemann surface in the x-parametrization. We will use the convention

$$p_R(u) = p(x)\Big|_{|x|>g}, \qquad p_L(x) = p(x)\Big|_{|x|<g}. \tag{7.4.109}$$

With this convention the left and right pseudomomenta are assembled into a single pseudomomenta $p(x)$ without inversion symmetry, defined on the whole x-plane [148]. The pseudomomenta $p(x)$ is thus an analytic function defined on a hyper-elliptic Riemann surface, with poles at $x = 0, x = \infty$ and at the fixed points of the inversion symmetry $x = \pm g$. The behavior of the pseudomomenta near these poles is [148] [3]

3. In our convention the pseudomomentum has negative sign compared to [148].

$$p(x) \simeq \begin{cases} (N - \frac{1}{2}L)/x & (x \to \infty); \\ -\frac{1}{2}\Delta/(x - g^2/x) & (x \to \pm g); \\ 2\pi m + \frac{1}{2}L\, x/g^2 & (x \to 0). \end{cases} \tag{7.4.110}$$

For the problem we are interested in, $p^{(1)}$ and $p^{(2)}$ belong to the $\mathfrak{su}(2)_R$ sector, while $p^{(3)}$ belongs to the $\mathfrak{su}(2)_L$ sector. Therefore the linear combinations of the type $p^{(1)}+p^{(2)}-p^{(3)}$ should be understood as

$$p^{(1)} + p^{(2)} - p^{(3)} \quad \to \quad p^{(1)}(x) + p^{(2)}(x) + p^{(3)}(g^2/x); \tag{7.4.111}$$

$$p^{(3)} + p^{(1)} - p^{(2)} \quad \to \quad p^{(3)}(x) - p^{(1)}(g^2/x) + p^{(2)}(g^2/x). \tag{7.4.112}$$

In the limit $g^2 \to 0$, as it is clear from the asymptotics (7.4.110) of the pseudomomenta at the origin, we obtain exactly the combination that appeared in the arguments of the dilogarithm in (7.4.94)! Since the quasimomentum appears only in the exponent, the term $2\pi m$ can be neglected.

Now let us see if the the r.h.s. of (7.4.111) and the arguments of the dilogarithm in (7.4.94) match at linear order in $g'^2 = g^2/L^2$. This will be the case if the function $p(g^2/x) + q(x)$ vanishes up to g'^4. We have from (7.4.105)

$$\begin{aligned} p(g^2/x) + q(x) &= \sum_{j=1}^{N} \frac{x'_j}{g^2/x - x_j} + \frac{\Delta/2}{x - g^2/x} - \frac{L/2}{x - g^2/x} \\ &= 2\pi m + g^4 \left(\frac{\mathrm{E}_2}{x^3} - \frac{2\mathrm{E}_3}{x^2} \right) + \mathcal{O}(g^6). \end{aligned} \tag{7.4.113}$$

Therefore, if the second two terms in (7.4.103) can be ignored, the Frolov–Tseytlin limit the strong coupling result from the string theory side matches, up to the subtleties related to the choice of the contour, with the one-loop level result from the SYM side at one loop order in g^2/L^2. In any case, if the results match at tree level, they will match also at one loop. Note that if the Hamiltonian insertions at two loops are located only at the splitting points, there will be disagreement at two-loop order in the Frolov–Tseytlin limit.

We also see that the factorisation of the structure constant into two pieces, the first depending on $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ and the second depending on $\mathbf{u}^{(3)}$, takes place only in the weak coupling limit and it is a consequence of the fact that at $g \to 0$ the spectral curve for the $\mathfrak{so}(4)$ sector splits into two components connected by a vanishing cycle (the

Zhukowsky circle $|x| = g$). Returning to the u-parametrization, the three operators are defined on the Riemann surface for the $\mathfrak{so}(4)$ sector sketched in (7.4.6). The Riemann surface splits into two disjoined hyperelliptic surfaces in the limit $g \to 0$, when the two Zhukovsky cuts disappear.

Chapter 8

Spin Vertex Formalism

In this chapter, we introduce a new formulation for computing the three-point functions in $\mathcal{N} = 4$ SYM theory. The main idea is to construct an entangled state called the *spin vertex* and the three-point function is given by projecting the three states on the spin vertex. Conceptually this is similar to the light-cone string field theory, which was used to compute the three-point functions of BMN strings. In fact, as we shall show below that in certain sectors in the BMN limit, the spin vertex coincide with the spin vertex at the leading order. We hope this formulation can be generalized to higher loops but for present such a generalization seems to be plagued with several technical difficulties. We therefore focus on the tree level formulation.

This chapter is structured as the follows. In section 1, we introduce the spin vertex in the $\mathfrak{so}(4)$ sector. We will discuss in detail the construction and properties of the spin vertex in this sector and show how to use it to compute the three-point functions. In order to generalize the construction to all sectors, we introduce the oscillator representation of the $\mathfrak{psu}(2,2|4)$ algebra at tree level in section 2. Then we construct the spin vertex in all sectors and discuss its properties in section 3. In order to make contact with the strong coupling computation, we take the BMN limit of the spin vertex in the scalar $\mathfrak{so}(6)$ sector in section 4 and show that it reproduces the bosonic part of the string vertex in light-cone string field theory at the leading order.

8.1 Spin Vertex in the $\mathfrak{so}(4)$ sector

In this section, we restrict ourselves to the $\mathfrak{so}(4) = \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ sector. This sector contains four scalar fields $\{Z, \bar{Z}, X, \bar{X}\}$ and was the main focus in the proceeding chapters. Following [150], we introduce the double-spin notation for the four scalar fields. It will be clear in section 3 that the origin of the double spin notation is that one use two fermionic oscillators to represent a scalar field. The double-spin notation for the fields reads

$$\begin{aligned} Z &\mapsto |\uparrow\rangle_L \otimes |\uparrow\rangle_R, & X &\mapsto |\uparrow\rangle_L \otimes |\downarrow\rangle_R \\ \bar{Z} &\mapsto |\downarrow\rangle_L \otimes |\downarrow\rangle_R, & -\bar{X} &\mapsto |\downarrow\rangle_L \otimes |\uparrow\rangle_R. \end{aligned} \tag{8.1.1}$$

The doublet $(|\uparrow\rangle_L, |\downarrow\rangle_L)$ and $(|\uparrow\rangle_R, |\downarrow\rangle_R)$ form the fundamental representations of $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$, respectively. In this language, the left and right sector factorizes and the local operators can be excited on either sector. The operators $\mathrm{Tr}\, XZ\cdots ZX$ with definite one-loop anomalous dimension can now be represented as

$$\mathcal{O}_R = |\uparrow^L\rangle_L \otimes B(u_1)\cdots B(u_N)|\uparrow^L\rangle_R. \tag{8.1.2}$$

The operators $\mathrm{Tr}\, Z\bar{X}\cdots\bar{X}Z$ can be represented by

$$\mathcal{O}_L = B(u_1)\cdots B(u_N)|\uparrow^L\rangle_L \otimes |\uparrow^L\rangle_R. \tag{8.1.3}$$

In the computation of correlation functions, we can compute the left sector and the right sector separately.

8.1.1 Construction of the spin vertex

At tree-level, the computation of correlation functions is reduced to Wick contractions. In order to mimic the Wick contraction, let us first focus on the left sector and define the object

$$\langle v_{12}| = \epsilon_{ab1}\langle a| \otimes {}_2\langle b| = {}_1\langle\uparrow| \otimes {}_2\langle\downarrow| - {}_1\langle\downarrow| \otimes {}_2\langle\uparrow| \tag{8.1.4}$$

where $a, b = 0, 1$ and $\langle 0| = \langle\uparrow|$, $\langle 1| = \langle\downarrow|$. We have the convention that $\langle a|b\rangle = \delta_{ab}$. Then $\langle v_{12}|$ defines a scalar product in the space $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$ with the following property

$$\begin{aligned} \langle v_{12}|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 &= 1, & \langle v_{12}|\downarrow\rangle_1 \otimes |\uparrow\rangle_2 &= -1, \\ \langle v_{12}|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 &= 0, & \langle v_{12}|\downarrow\rangle_1 \otimes |\downarrow\rangle_2 &= 0. \end{aligned} \tag{8.1.5}$$

We can define the same object in the right sector. Combining both sectors we can define

$$\begin{aligned}\langle \mathrm{v}_{12}| &= {}_L\langle v_{12}| \otimes {}_R\langle v_{12}| \\ &= {}_1\langle Z| \otimes {}_2\langle \bar{Z}| + {}_1\langle \bar{Z}| \otimes {}_2\langle Z| + {}_1\langle X| \otimes {}_2\langle \bar{X}| + {}_1\langle \bar{X}| \otimes {}_2\langle X|.\end{aligned} \tag{8.1.6}$$

The scalar product defined by $\langle \mathrm{v}_{12}|$ can be used to mimic Wick contraction. We have

$$\langle \mathrm{v}_{12}|\Phi\rangle_1 \otimes |\bar{\Phi}\rangle_2 = 1, \qquad \Phi = X, Z, \bar{Z}, \bar{X}. \tag{8.1.7}$$

and the other projections gives zero.

We can depict the object $\langle v_{12}|$ as an arc with one end on space 1 and the other end on space 2, as is shown in Fig.(8.1.1) By taking the tensor products of L copies of $\langle v_{12}|$'s as

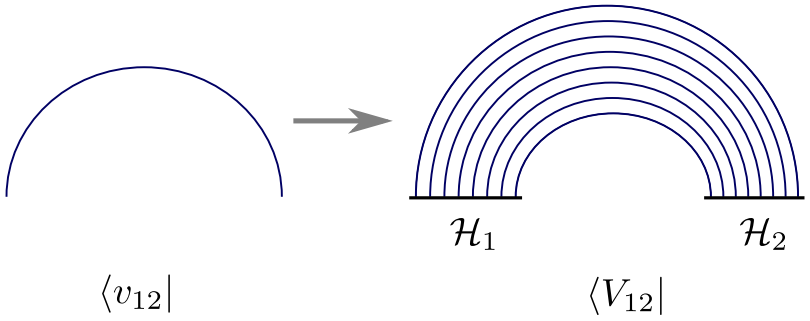

FIGURE 8.1.1: Construction of the two-point spin vertex by taking tensor products of $\langle v_{12}|$.

is shown in Fig.(8.1.1), we define the *two-point spin vertex*. The two-point spin vertex is an entangled state in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_i$, $(i = 1, 2)$ is the Hilbert space of the i-th spin chain. Similarly, by taking the tensor product of three pieces of two-point spin vertices as is shown in Fig.(8.1.2), we construct the *three-point spin vertex*. This is an entangled state in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

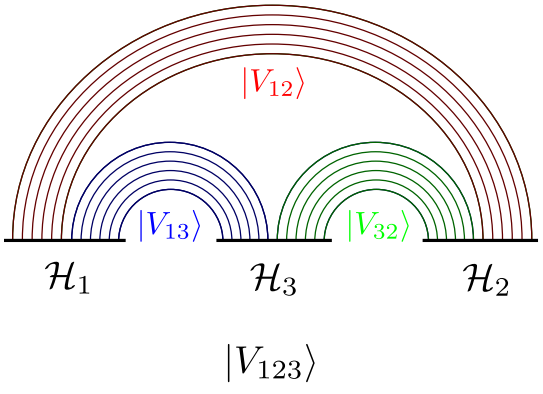

FIGURE 8.1.2: Construction of the three-point spin vertex by taking tensor products of two-point spin vertices.

8.1.2 Properties of spin vertex

We now discuss the two main properties of the spin vertex, which we shall call the *reflection property* and the *monodromy relation.*

Reflection property Let us first consider the one site spin vertex $\langle v_{12}|$. It is easy to show that it has the following properties under the action of $\mathfrak{su}(2)$ generators

$$\langle v_{12}|(S_a^{(1)} + S_a^{(2)}) = 0, \qquad a = \pm, z \tag{8.1.8}$$

where $S_a^{(i)}$ is the generator for the i-th spin chain. Since the symmetric generators for the spin chain is the sum of generators for each site, it is trivial to generalize (8.1.8) to the two-point vertex of length L

$$\langle V_{12}|(S_a^{(1)} + S_a^{(2)}) = 0, \qquad a = \pm, z. \tag{8.1.9}$$

This relation implies that the action of the generators on one of the spin chains can be 'reflected' to the action of other generators on the other spin chain. Using (8.1.9) we can find the reflection property of the two-point spin vertex for the generators of the elements of monodromy matrices. Recall that the Lax matrix at position-n can be written in terms of the symmetric generators as

$$L_{an}^{(1)}(u) = \begin{pmatrix} u + i\, S_z^{(1)} & i\, S_-^{(1)} \\ i\, S_+^{(1)} & u - i\, S_z^{(1)} \end{pmatrix}. \tag{8.1.10}$$

Acting on the two-point spin vertex, we have

$$\langle V_{12}|L_{an}^{(1)}(u) = \langle V_{12}| \begin{pmatrix} u + i\, S_z^{(1)} & i\, S_-^{(1)} \\ i\, S_+^{(1)} & u - i\, S_z^{(1)} \end{pmatrix} = \langle V_{12}| \begin{pmatrix} u - i\, S_z^{(2)} & -i\, S_-^{(2)} \\ -i\, S_+^{(2)} & u + i\, S_z^{(2)} \end{pmatrix} \tag{8.1.11}$$

which can be written compactly as

$$\langle V_{12}|L_{an}^{(1)}(u) = \langle V_{12}|\sigma_2 \left(L_{an}^{(2)}(u)\right)^t \sigma_2, \quad \text{or} \quad \langle V_{12}|L_{an}^{(1)}(u) = -\langle V_{12}|L_{an}^{(2)}(-u) \tag{8.1.12}$$

where "t" denotes the transposition in the auxiliary space and the multiplication of Pauli σ_2 matrices is also in the auxiliary space. Using the first relation of (8.1.12), it then follows

$$\langle V_{12}|T_a^{(1)}(u) = \langle V_{12}|\sigma_2 \left(T_a^{(2)}(u)\right)^t \sigma_2. \tag{8.1.13}$$

Notice that the order of the product of the Lax matrices is reversed in the other spin chain. In terms of components, we have the following relations

$$\begin{aligned} \langle V_{12}|A^{(1)}(u) &= \langle V_{12}|D^{(2)}(u), \qquad \langle V_{12}|B^{(1)}(u) = -\langle V_{12}|B^{(2)}(u), \\ \langle V_{12}|D^{(1)}(u) &= \langle V_{12}|A^{(2)}(u), \qquad \langle V_{12}|C^{(1)}(u) = -\langle V_{12}|C^{(2)}(u). \end{aligned} \tag{8.1.14}$$

The reflection properties (8.1.14) will be useful in the computation of three-point functions.

The monodromy relation The monodromy condition is the result of reflection property and the 'unitary relation' of the Lax operator, which simply states that

$$L_{an}(u+i/2)L_{an}(-u+i/2) = (u+i\,\mathrm{P}_{an})(-u+i\,\mathrm{P}_{an}) = -(u^2+1). \tag{8.1.15}$$

If we act two Lax operators on the two-point spin vertex and use the second equation of (8.1.12)

$$\begin{aligned} \langle V_{12}|L^{(1)}_{an'}(u-i/2)L^{(2)}_{an}(u+i/2) &= -\langle V_{12}|L^{(2)}_{an}(-u+i/2)L^{(2)}_{an}(u+i/2) \\ &= -(u^2+1)\langle V_{12}| \end{aligned} \tag{8.1.16}$$

here n' and n denote the position on the first and second spin chain which are connected by the arc in Fig.(8.1.1). The relation shows that the two-point spin vertex is the eigenstate of the operator $L^{(1)}_{an'}(u-i/2)L^{(2)}_{an}(u+i/2)$. Taking into account all the arcs that form the two-point spin vertex, we obtain

$$\langle V_{12}|T^{(1)}_a(u-i/2)T^{(2)}_a(u+i/2) = (-1)^L f(u)\langle V_{12}| \tag{8.1.17}$$

where the two monodromy matrix are defined as usual

$$\begin{aligned} T^{(1)}_a(u) &= \prod_{n=1}^{L} L^{(1)}(u-\theta_n+i/2), \\ T^{(2)}_a(u) &= \prod_{n=L}^{1} L^{(1)}(u-\theta_n+i/2) \end{aligned} \tag{8.1.18}$$

and the function $f(u)$ is given by

$$f(u) = \prod_{n=1}^{L}((u-\theta_n)^2+\frac{1}{4}). \tag{8.1.19}$$

Note that the impurities on the two ends of the same arc should be identified. For later convenience, we introduce the notation

$$T_{12}(u) \equiv (-1)^L f(u)^{-1} M^{(1)}(u - i/2) M^{(2)}(u + i/2), \tag{8.1.20}$$

so that $\langle V_{12}|M_{12}(u) = \langle V_{12}|$. Graphically this can be represented by Fig.(8.1.3) The mon-

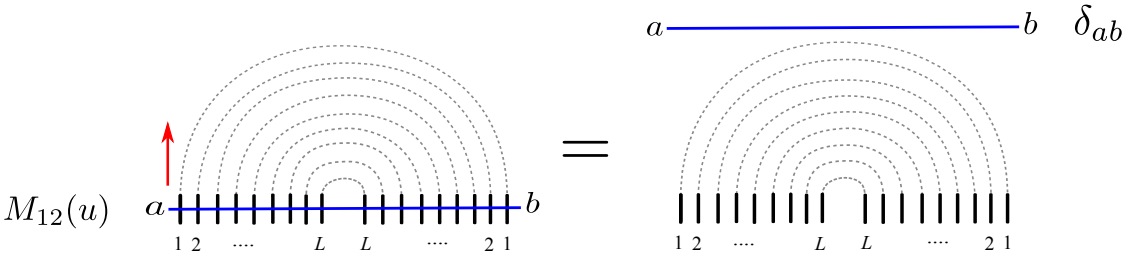

FIGURE 8.1.3: Graphical representation of the monodromy relation for two-point spin vertex.

odromy relation can be generalized to three-point spin vertex. We define the following monodromy like quantity for the three-point spin vertex

$$T_{123}(u) = f_{123}(u)^{-1}\, t_{12}^{-}(u) t_{13}^{+}(u) t_{31}^{-}(u) t_{32}^{+}(u) t_{23}^{-}(u) t_{21}^{+}(u) \tag{8.1.21}$$

where $t_{ij}^{\pm}(u) = t_{ij}(u \pm \frac{i}{2})$ and

$$\begin{aligned}
t_{12}(u) &= \prod_{k=1}^{L_{12}} L_{ak}^{(1)}(u - \theta_k^{(1)}), \quad t_{13}(u) = \prod_{k=L_{12}+1}^{L_1} L_{ak}^{(1)}(u - \theta_k^{(1)}), \\
t_{31}(u) &= \prod_{k=1}^{L_{13}} L_{ak}^{(3)}(u - \theta_k^{(3)}), \quad t_{32}(u) = \prod_{k=L_{13}+1}^{L_3} L_{ak}^{(1)}(u - \theta_k^{(3)}), \\
t_{23}(u) &= \prod_{k=1}^{L_{23}} L_{ak}^{(2)}(u - \theta_k^{(2)}), \quad t_{21}(u) = \prod_{k=L_{23}+1}^{L_2} L_{ak}^{(2)}(u - \theta_k^{(2)}).
\end{aligned} \tag{8.1.22}$$

and

$$f_{123}(u) = (-1)^{\frac{1}{2}(L_1+L_2+L_3)} \prod_{n=1}^{L_1}((u - \theta_n^{(1)})^2 + 1) \prod_{n=1}^{L_2}((u - \theta_n^{(2)})^2 + 1) \prod_{n=1}^{L_3}((u - \theta_n^{(3)})^2 + 1). \tag{8.1.23}$$

By similar analysis to the two-point spin vertex, we obtain

$$\langle V_{123}|T_{123}(u) = \langle V_{123}| \tag{8.1.24}$$

There are at least two reasons that the monodromy relation is interesting.

Firstly, in the semi-classical limit, when the shift of $\pm i/2$ in (8.1.21) is not important, $T_{123}(u)$ is approximately the product of the three monodromy matrices of the spin chains,

$$T_{123}(u) \propto T_1(u)T_2(u)T_3(u). \tag{8.1.25}$$

In fact at the strong coupling side, the relation $\Omega_1(u)\Omega_2(u)\Omega_3(u) = 1$ plays an essential role in computing the three-point function. What we obtained in (8.1.21) is the weak coupling counterpart of the monodromy relation at strong coupling. In fact, in the coherent state approximation, the spin chain can be described by the Landau-Ginzburg sigma model, then it is possible to use the same technique from strong coupling and obtain the semi-classical limit of the three-point function without computing any scalar products [151] ! This explains the fundamental importance of monodromy relation at least in the semi-classical limit.

For a spin chain, the states $|\Psi\rangle$ which satisfie $T(u)|\Psi\rangle \propto |\Psi\rangle$ where $T(u)$ is the monodromy matrix are called *Yangian invariants* since they are annihilated by all the Yangian generators. The Yangian invariant is studied in different contexts including the recent works of scattering amplitudes [152–155]. It is interesting to see whether there are common integrable structures for both on-shell and off-shell quantities.

8.1.3 Three-point functions and spin vertex

In this section, we explain how to use the spin vertex to compute the three-point functions. All the three operators under consideration are in $\mathfrak{su}(2)$ sector. Let us characterize the types of three-point functions. Using the double spin notation, each operator can be excited either in the left sector or in the right sector, but not both. Therefore, the possible types of three-point functions are LLL, LLR, LRR and RRR. It is obvious that there are only two independent types, namely the LLL and LLR because the rest two types are obtained by just renaming left and right.

We will show that the most general quantity we are going to compute is the following LLL type

$$C^L_{123} = \langle V_{123}|e^{z_1 S^{(1)}_-}|\mathbf{u}_1\rangle \otimes e^{z_2 S^{(2)}_-}|\mathbf{u}_2\rangle \otimes e^{z_3 S^{(3)}_-}|\mathbf{u}_3\rangle \tag{8.1.26}$$

where $|\mathbf{u}_i\rangle$, $(i = 1, 2, 3)$ are on-shell Bethe states constructed from the algebraic Bethe ansatz

$$|\mathbf{u}_i\rangle = B(u_{i,1}) \cdots B(u_{i,N_i})|\Omega\rangle. \tag{8.1.27}$$

The most general external state in the $\mathfrak{su}(2)$ sector is the on-shell Bethe state rotated by some global rotation $\mathcal{R}$. Any global rotation can be written in the following 'normal ordered' form

$$\mathcal{R} = e^{z\,S_-} e^{y\,S_z} e^{x\,S_+} \tag{8.1.28}$$

where x, y, z are three parameters. Acting such an rotation on the on-shell state, and using the fact that the on-shell Bethe state is the highest weight state $S_+|\mathbf{u}\rangle = 0$, we can eliminate the last term. The action of S_z on the Bethe state gives the eigenvalue which is a constant number for a given state, so the action of the second term amounts to multiplication of some numerical factor. The only non-trivial piece is the first term. We thus have

$$\mathcal{R}|\mathbf{u}\rangle \propto e^{z\,S_-}|\mathbf{u}\rangle. \tag{8.1.29}$$

For the LLR case, the structure constant is given by the product of result from two sectors. In each sector, the result is given as special cases of (8.1.26).

Let us first explain how to compute two-point functions using the spin vertex. Using the reflection properties (8.1.14) we have

$$\begin{aligned}\langle V_{12}|\mathbf{u}\rangle|\mathbf{v}\rangle =& \langle V_{12}|e^{x\,S^{(1)}_-} B^{(1)}(u_1)\cdots B^{(1)}(u_{M_1})|\uparrow^L\rangle \otimes e^{x\,S^{(2)}_-} B^{(2)}(v_1)\cdots B^{(2)}(v_{M_2})|\uparrow^L\rangle \\ =& \langle V_{12}|\uparrow^L\rangle \otimes e^{(y-x)S^{(2)}_-} B^{(2)}(u_1)\cdots B^{(2)}(u_{M_1}) B^{(2)}(v_1)\cdots B^{(1)}(v_{M_2})|\uparrow^L\rangle \\ =& \langle\downarrow^L| e^{(y-x)S^{(2)}_-} B^{(2)}(u_1)\cdots B^{(2)}(u_{M_1}) B^{(2)}(v_1)\cdots B^{(1)}(v_{M_2})|\uparrow^L\rangle.\end{aligned} \tag{8.1.30}$$

The matrix element which takes the form of the last line of (8.1.30) is nothing but the pDWPF defined in (3.3.85).

In a similar way, we compute the three-point functions. We first cut the spin chain state into two subchains, as described in Chapter 4. Then the computation is reduced to the the cases in (8.1.30) for three pairs of subchains. The final result of (8.1.26) is given in [150]

$$C^L_{123} = \sum_{\mathbf{u}'_k \cup \mathbf{u}''_k = \mathbf{u}_k} z_{21}^{L_{12}-|\mathbf{u}''_1|-|\mathbf{u}'_2|} z_{32}^{L_{23}-|\mathbf{u}''_2|-|\mathbf{u}'_3|} z_{13}^{L_{31}-|\mathbf{u}''_3|-|\mathbf{u}'_1|}\, \mathcal{C}(\mathbf{u}'_k, \mathbf{u}''_k) \tag{8.1.31}$$

The factor $\mathcal{C}(\mathbf{u}'_k, \mathbf{u}''_k)$ is independent of global rotation and is given by

$$\begin{aligned}\mathcal{C}(\mathbf{u}'_k, \mathbf{u}''_k) = (-1)^{|\mathbf{u}'_1|+|\mathbf{u}'_2|+|\mathbf{u}'_3|} \prod_{k=1}^{3} \mathrm{H}_{\mathbf{u}'_k,\mathbf{u}''_k} \\ \times Z_p\left(\mathbf{u}''_1 \cup \mathbf{u}'_2|\boldsymbol{\theta}^{(12)}\right) Z_p\left(\mathbf{u}''_2 \cup \mathbf{u}'_3|\boldsymbol{\theta}^{(23)}\right) Z_p\left(\mathbf{u}''_3 \cup \mathbf{u}'_1|\boldsymbol{\theta}^{(31)}\right).\end{aligned} \tag{8.1.32}$$

where the $\mathrm{H}_{\mathbf{u}',\mathbf{u}''}$ was the factor originates from cutting the Bethe states and is given in (4.2.13). Comparing to the result from tailoring (4.2.17), we find that the main advantage of (8.1.31) is that each term in the sum are the product of three determinants, instead of three off-shell/off-shell scalar products.

8.2 Oscillator representation of $\mathcal{N} = 4$ SYM theory

In what follows, we will generalize the spin vertex formalism to all sectors in $\mathcal{N} = 4$ SYM theory. To this end, we introduce the oscillator representation of the $\mathfrak{psu}(2,2|4)$ algebra in this section. We will give the explicit construction of the spin vertex in terms of the oscillators and discuss the main properties of the spin vertex in the next section.

8.2.1 Oscillator representation for $\mathfrak{psu}(2,2|4)$

Let us first discuss the oscillator representation of the compact version of $\mathfrak{psu}(2,2|4)$, namely $\mathfrak{psu}(4|4)$. It uses four copies of bosonic oscillators a_i, b_i, $(i = 1,2)$ and four copies of fermionic oscillators c_i, $(i = 1,2,3,4)$ satisfying the standard commutation and anti-commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [b_i, b_j^\dagger] = \delta_{ij}, \quad \{c_k, c_l^\dagger\} = \delta_{kl}, \qquad i,j = 1,2, \quad k,l = 1,\cdots,4. \tag{8.2.33}$$

we organize the oscillators in an eight-dimensional vector

$$\phi = \begin{pmatrix} a_i & b_i & c_k \end{pmatrix} \tag{8.2.34}$$

such that the generators of the $\mathfrak{su}(4|4)$ can be written as

$$\mathrm{E}^{AB}_{\mathrm{com}} = \phi^{A\dagger}\phi^{B}, \qquad \mathrm{E}^{AB\dagger}_{\mathrm{com}} = \mathrm{E}^{BA}_{\mathrm{com}} \tag{8.2.35}$$

It is straightforward to check that the generators satisfy the $\mathfrak{gl}(4|4)$ super algebra

$$[\mathrm{E}^{AB}, \mathrm{E}^{CD}] = \delta^{BC}\,\mathrm{E}^{AD} - (-1)^{(|A|+|B|)(|C|+|D|)}\delta^{AD}\,\mathrm{E}^{CB} \tag{8.2.36}$$

where $|A|$ denote the grading of the index. $|A| = 0, 1$ for bosonic and fermionic indices, respectively. The non-compact form $\mathfrak{su}(2,2|4)$ can be achieved by performing a particle-hole transformation on one set of the bosonic oscillators, say $b_i, b_i^\dagger$. The particle-hole transformation is defined by

$$b \to -b^\dagger, \qquad b^\dagger \to b. \tag{8.2.37}$$

The generators for the non-compact form are given by

$$\mathrm{E}^{AB} = \mathrm{E}^{AB}_{\mathrm{com}}(b \to -b^\dagger, b^\dagger \to b). \tag{8.2.38}$$

This transformation preserves the commutation relation (8.2.36) but changes the conjugation properties, which now become

$$\mathrm{E}^{AB\dagger} = \gamma\, \mathrm{E}^{BA}\, \gamma, \qquad \gamma = \mathrm{diag}(\mathbf{1}_2, -\mathbf{1}_2, \mathbf{1}_4). \tag{8.2.39}$$

For the sake of symmetry, sometimes it is also convenient to perform a fermionic particle-hole transformation

$$d_i = c_{i+2}^\dagger, \qquad d_i^\dagger = c_{i+2}, \qquad i = 1, 2. \tag{8.2.40}$$

In contrary to the bosonic case, the particle-hole transformation for the fermionic oscillators does not change the real form of the algebra. We can arrange the oscillators after the particle-hole transformation as the following

$$\psi = (a_i, -b_i^\dagger, c_i, d_i^\dagger), \qquad \bar{\psi} = \psi^\dagger \gamma = (a_i^\dagger, b_i, c_i^\dagger, d_i). \tag{8.2.41}$$

The generators can be written as

$$\mathrm{E}^{AB} = \bar{\psi}^A \psi^B. \tag{8.2.42}$$

In order to obtain the representation of $\mathfrak{psu}(2,2|4)$, we need to impose the projective condition that the identity generator, which is one of the central charges of the algebra, is zero. This is equivalent to

$$\sum_{A=1}^{8} \mathrm{E}^{AA} = \sum_{i=1}^{2} (N_{a_i} - N_{b_i} + N_{c_i} - N_{d_i}) = 0 \tag{8.2.43}$$

where N_a, N_b, N_c, N_d are the number of respective types of oscillators.

8.2.2 Oscillator representation and conformal symmetry

Let us now concentrate on the conformal algebra $\mathfrak{so}(4,2) \simeq \mathfrak{su}(2,2)$. In the above oscillator representation, there is a natural grading with respect to the maximal compact subalgebra $\mathfrak{u}(1) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)$. We denote the $\mathfrak{u}(1)$ generator by E, the grading is given by the value of E

$$[E, L_\mu^\pm] = \pm L_\mu^\pm, \qquad [E, L^0] = 0. \tag{8.2.44}$$

In terms of oscillators, they can be written explicitly as

$$\begin{aligned} E &= 1 + \frac{1}{2}(N_a + N_b) = 1 + \frac{1}{2}(a^\dagger a + b^\dagger b), \\ L_\mu^+ &= -a^\dagger \bar{\sigma}_\mu b^\dagger, \qquad L_\mu^- = a \sigma_\mu b \end{aligned} \tag{8.2.45}$$

with $\sigma_\mu = (-1, \vec{\sigma})$ and $\bar{\sigma}_\mu = (-1, -\vec{\sigma})$ and the summation over bosonic indices is understood.

We would like to identify the above generators with the standard presentation of the conformal group, which is the group of rotation in the 6d spacetime with the signature $\eta_{AB} = (-,+,+,+,+,-)$. Following the convention of [156], the 6d indices are $A, B = 0,1,2,3,4,5$ with the first four being the indices for the 4d Minkowski spacetime $\mu, \nu = 0,1,2,3$. The commutation relations are

$$[M_{PQ}, M_{RS}] = i(\eta_{QR} M_{PS} - \eta_{PR} M_{QS} - \eta_{QS} M_{PR} + \eta_{PS} M_{QR}). \tag{8.2.46}$$

The identification of the translations P_μ, special conformal transformations K_μ and the dilatation D is made as

$$P_\mu = M_{\mu 6} + M_{\mu 5}, \qquad K_\mu = M_{\mu 6} - M_{\mu 5}, \qquad D = -M_{56}. \tag{8.2.47}$$

On the other hand, the $\mathfrak{u}(1)$ generator in the oscillator representation E is given by

$$E = M_{06} = \frac{1}{2}(P_0 + K_0). \tag{8.2.48}$$

The authors of [156] suggests that the oscillator representation and the standard representation can be related by a rotation of an imaginary angle $-i\pi/2$ in the 05-plane. Since the rotation in 05-plane is generated by M_{05}, the rotation can be realized as

$$U = \exp \frac{\pi}{2} M_{05} = \exp \frac{\pi}{4}(P_0 - K_0). \tag{8.2.49}$$

The generators from the two representations are related by

$$U^{-1}\,K_\mu\,U = L^-_\mu, \quad U^{-1}\,P_\mu\,U = L^+_\mu, \quad U^{-1}\,D\,U = iE. \tag{8.2.50}$$

Note that the generator $L^+_0 - L^-_0 = U^{-1}(P_0 - K_0)U$, which helps to make contact between the rotated and non-rotated representations. The operator U^2 acts like a PT transformation which changes sign both for the 0 and 5 direction

$$U^{-2}\,D\,U^2 = -D, \qquad U^{-2}\,E\,U^2 = -E. \tag{8.2.51}$$

The relations in (8.2.51) plays an essential role in obtaining the correct form of the two-point functions in CFT, as is shown in [156]. The derivation is purely algebraic and holds at all loops. We refer to Appendix C for more details.

At tree level, the rotation U can be written in terms of oscillators

$$U = \exp -\frac{\pi}{4}\sum_{i=1}^{2}(a_i^\dagger b_i^\dagger + a_i b_i), \qquad U^\dagger = U. \tag{8.2.52}$$

By explicit calculation, we find that

$$\begin{aligned} &U^2 L_0^\pm U^{-2} = -L_0^\mp, \qquad U^2 L_m^\pm U^{-2} = L_m^\mp, \\ &U^2 P_0 U^{-2} = -K_0, \qquad U^2 P_m U^{-2} = K_m, \quad m = 1,2,3. \end{aligned} \tag{8.2.53}$$

These relations can be obtained from the action of U on the oscillators

$$U^2 a\,U^{-2} = b^\dagger, \quad U^2 a^\dagger U^{-2} = -b, \quad U^2 b\,U^{-2} = a^\dagger, \quad U^2 b^\dagger U^{-2} = -a. \tag{8.2.54}$$

We can also see that the operator U^2 sends the bosonic Fock vacuuum $|0\rangle_B$ $(a, b|0\rangle_B = 0)$ to the dual vacuum $|\bar{0}\rangle_B$ defined as

$$|\bar{0}\rangle_B = U^2|0\rangle_B, \qquad a^\dagger, b^\dagger|\bar{0}\rangle_B = 0. \tag{8.2.55}$$

We note that from (8.2.54), we have

$$U^{-4} x\,U^4 = -x, \qquad x = a_i, b_i, a_i^\dagger, b_i^\dagger. \tag{8.2.56}$$

8.2.3 Oscillator representation and R-symmetry

Now we consider the R-symmetry in a similar way. The R-symmetry algebra is $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$. The generators can be given in terms of fermionic oscillators as

$$R_{kl} = c_k^\dagger c_l - \frac{1}{4}\delta_{kl} c^\dagger c. \tag{8.2.57}$$

In analogy to the bosonic case, we can introduce a fermionic rotation defined as

$$U_F = \exp -\frac{\pi}{4}\sum_{i=1}^{2}(c_i^\dagger d_i^\dagger + c_i d_i), \qquad U_F^\dagger = U_F^{-1}. \tag{8.2.58}$$

The action of this rotation on the fermionic oscillators reads

$$U_F^2 c_i\, U_F^{-2} = d_i^\dagger, \quad U_F^2 c_i^\dagger U_F^{-2} = d_i, \quad U_F^2 d_i\, U_F^{-2} = -c_i^\dagger, \quad U_F^2 d_i^\dagger U_F^{-2} = -c_i, \tag{8.2.59}$$

which implies that

$$U_F^4 x\, U_F^{-4} = -x, \qquad x = c_i, c_i^\dagger, d_i, d_i^\dagger. \tag{8.2.60}$$

The U_F^2 operator also sends the fermionic Fock vacuum $|0\rangle_F$ into its dual Fock vacuum $|\bar{0}\rangle_F$

$$|\bar{0}\rangle_F = U_F^2|0\rangle_F, \qquad c_i^\dagger, d_i^\dagger|\bar{0}\rangle_F = 0. \tag{8.2.61}$$

Finally, we can combine the rotations from both sectors and define the rotation for the whole sector $\mathrm{U} = UU_F$.

8.2.4 States and correlation functions

In this subsection, we explain how to use oscillators to represent the gauge invariant operators and compute their correlation functions. The operators considered in this dissertation are gauge invariant single trace operators which are made of fundamental fields of the theory, or the string bits in view of gauge-string correspondence,

$$\mathcal{O} \sim \mathrm{Tr}\,(XXZZY\Psi_i\cdots)(x). \tag{8.2.62}$$

When the gauge coupling is zero, these string bits are independent and each of them is in a state corresponding to the $\mathfrak{psu}(2,2|4)$ representation described above. Gauge invariant operators can then be represented by elements in the tensor product of the individual string bits. In the spin chain language, string bits are the sites of the spin chain and we

will introduce a copy of oscillators at each site

$$\psi_s = \left(a_{i,s}, -b^\dagger_{i,s}, c_{i,s}, d^\dagger_{i,s}\right), \quad s = 1, \cdots, L. \tag{8.2.63}$$

In the non-interacting theory, the oscillator representation of the super-conformal algebra generators are

$$\mathrm{E}^{AB} = \sum_{s=1}^{L} \mathrm{E}^{AB}_s, \quad \mathrm{U} = \mathrm{U}_1 \otimes \cdots \otimes \mathrm{U}_L. \tag{8.2.64}$$

At higher loops, the radiative corrections will introduce interactions between the string bits. The space of *conformal primary operators* $\mathcal{O}(x)$ situated at $x = 0$ is selected by the highest weight condition

$$K_\mu \, \mathcal{O}(0) = 0. \tag{8.2.65}$$

On the other hand, we have the Fock vacuum $|0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_L$ such that

$$L^-_\mu |0\rangle = 0 \quad \Rightarrow \quad K_\mu \, U|0\rangle = 0. \tag{8.2.66}$$

Similarly, we can relate the space of primary operators with the space of Fock states $|\mathcal{O}\rangle$ annihilated by L^-_μ operator

$$L^-_\mu |\mathcal{O}\rangle = 0 \quad \Rightarrow \quad K_\mu \, U|\mathcal{O}\rangle. \tag{8.2.67}$$

Translating the operator to other spacetime points can be done with the help of momentum operator

$$\mathcal{O}(x) = e^{iPx} \mathcal{O}(0) e^{-iPx}, \tag{8.2.68}$$

with the corresponding Fock space representative

$$e^{iPx} U|\mathcal{O}\rangle. \tag{8.2.69}$$

Similar relations can be established for the operators and bra states. To sum up, we have the following *operator state correspondence*

$$\begin{aligned} \mathcal{O}(x) \quad &\Leftrightarrow \quad e^{iPx} U|\mathcal{O}\rangle, \\ \mathcal{O}^\dagger(x) \quad &\Leftrightarrow \quad \langle \mathcal{O}| U^\dagger e^{-iPx}. \end{aligned} \tag{8.2.70}$$

The operator state correspondence can be used to compute the two-point functions,

$$\langle \mathcal{O}_2^\dagger(y)\mathcal{O}_1(x)\rangle = \langle \mathcal{O}_2|U^\dagger e^{iP(x-y)}U|\mathcal{O}_1\rangle = \langle \mathcal{O}_2|U^2 e^{L^+(x-y)}|\mathcal{O}_1\rangle. \tag{8.2.71}$$

The authors in [156] check explicitly that the result reproduces correctly the Wick contractions of different kinds of fields.

8.3 Spin vertex for all sectors

In this section, we will construct the spin vertex for all sectors at tree level, which is a generalization of the spin vertex introduced in section 8.1. Before doing so, we first explain one subtlety when we consider all sectors. In the scalar sector, the operators under considerations are all scalar operators and their three-point functions are fixed up to one structure constant. When we consider all sectors, we need to take into account the operators with non-zero Lorentz spins such as operators involving fermions and covariant derivatives. As we discussed in Chapter 2, the three-point functions for operators with non-zero Loretnz spins are only fixed up to tensor structures, each one comes with its own structure constant. In the general case, it is not yet clear how to compute all these structure constants. It is easier to compute the three-point functions instead of structure constants in the generic case. In what follows, we will construct the spin vertex which will be used to compute two- and three-point functions. If we restrict ourselves to the scalar sector and factor out the trivial spacetime dependence, we will recover the formulation we have in section 8.1.

8.3.1 General discussion

We first introduce the two-point spin vertex. Let us first write the two-point correlation function (8.2.71) in a slightly different way, considering now both operators are represented by ket states. In order to do this, we need a mapping from a bra state $\langle\mathcal{O}|$ to the ket state $|\overline{\mathcal{O}}\rangle$, which will be done via a specially prepared state $\langle V_{12}$ which lives in the tensor product of two chains,

$${}^{(12)}\langle\mathcal{O}| = \langle V_{12}|U_F^{-2(1)}|\overline{\mathcal{O}}\rangle^{(2)}. \tag{8.3.72}$$

The factor $U_F^{-2(1)}$ was added for later convenience. In this language, the two-point function is given written as

$$\begin{aligned}\langle \mathcal{O}_2^\dagger(y)\mathcal{O}(x)\rangle =& \langle V_{12}|\mathrm{U}_1^{\dagger 2} e^{L^+_{(1)}(y-x)}|\mathcal{O}_1^{(1)}\rangle \otimes |\overline{\mathcal{O}}_2\rangle^{(2)} \\ =& \langle \mathcal{V}_{12}|e^{-iL^+_{(1)}(y-x)}|\mathcal{O}_1^{(1)}\rangle \otimes |\overline{\mathcal{O}}_2\rangle^{(2)} \\ =& \langle \mathcal{V}_{12}|e^{iL^+_{(1)}x+iL^+_{(2)}y}|\mathcal{O}_1^{(1)}\rangle \otimes |\overline{\mathcal{O}}_2\rangle^{(2)}\end{aligned} \tag{8.3.73}$$

where $\mathrm{U}^2 = U^2 U_F^2$, $\mathrm{U}^{\dagger 2} = U^2 U_F^{-2}$ and we have defined

$$\langle \mathcal{V}_{12}| = \langle V_{12}|\mathrm{U}_1^{\dagger 2} \tag{8.3.74}$$

We will show later that $\langle V_{12}|$ can be chosen such that

$$\langle V_{12}|\mathrm{U}_1^{2\dagger}(L^+_{(1)} + L^+_{(2)}) = \langle \mathcal{V}_{12}|(L^+_{(1)} + L^+_{(2)}) = 0. \tag{8.3.75}$$

The state $\langle \mathcal{V}_{12}|$, or its conjugate $|\mathcal{V}_{12}\rangle$ should play the role of vacuum state, in the sense that it carries the same quantum numbers as the vacuum. But it is clear that it can not be the tensor product of two Fock vacua. At tree level, $\langle V_{12}|$ should provide the correct Wick contractions between the elementary fields in $\mathcal{O}_1$ and $\mathcal{O}_2^\dagger$.

In a similar way, we can define the three-point spin vertex $\langle V_{123}|$ and $\langle \mathcal{V}_{123}|$ such that the three-point function is given by

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \langle \mathcal{V}_{123}|e^{i\left(L^+_{(1)}x+L^+_{(2)}y+L^+_{(3)}z\right)}|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle. \tag{8.3.76}$$

As before, at tree level, the three-point spin vertex can be built from the two-point vertices

$$\begin{aligned}|\mathcal{V}_{123}\rangle =& |\mathcal{V}_{12}\rangle \otimes |\mathcal{V}_{13}\rangle \otimes |\mathcal{V}_{23}\rangle \\ =& \mathrm{U}^2_{(12)}|V_{12}\rangle \otimes \mathrm{U}^2_{(13)}|V_{13}\rangle \otimes \mathrm{U}^2_{(23)}|V_{23}\rangle.\end{aligned} \tag{8.3.77}$$

8.3.2 Explicit construction by oscillators

The construction of the two-point spin vertex is given by

$$
\begin{aligned}
|\mathcal{V}_{12}\rangle &\equiv \mathrm{U}_1^2 |V_{12}\rangle \qquad (8.3.78)\\
&= \mathrm{U}_1^2 \exp\left[\sum_{s=1}^{L}\sum_{i=1}^{2}\left(b_{i,s}^{(1)\dagger}a_{i,s}^{(2)\dagger} - a_{i,s}^{(1)\dagger}b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)\dagger}c_{i,s}^{(2)\dagger} - c_{i,s}^{(1)\dagger}d_{i,s}^{(2)\dagger}\right)\right]|0\rangle^{(1)}\otimes|0\rangle^{(2)}\\
&= \exp\left[-\sum_{s=1}^{L}\sum_{i=1}^{2}\left(a_{i,s}^{(1)}a_{i,s}^{(2)\dagger} - b_{i,s}^{(1)}b_{i,s}^{(2)\dagger} + d_{i,s}^{(1)}c_{i,s}^{(2)\dagger} + c_{i,s}^{(1)}d_{i,s}^{(2)\dagger}\right)\right]|\bar{0}\rangle^{(1)}\otimes|0\rangle^{(2)}
\end{aligned}
$$

where $|\bar{0}\rangle = \mathrm{U}^2|0\rangle$. In order to mimic the planar Wick contraction we revert the order of the tensor product in the second spin chain

$$
|0\rangle^{(1)}\otimes|0\rangle^{(2)} = \left(|0\rangle_1^{(1)}\otimes\cdots\otimes|0\rangle_L^{(1)}\right)\otimes\left(|0\rangle_L^{(2)}\otimes\cdots\otimes|0\rangle_1^{(2)}\right). \qquad (8.3.79)
$$

The three-point spin vertex can be constructed by combining (8.3.77) and (8.3.78). The two-point spin vertex (8.3.78) can be expanded as the following

$$
|\mathcal{V}_{12}\rangle = \sum_{N_a,N_b,N_c,N_d} |\bar{N}_a,\bar{N}_b,\bar{N}_c,\bar{N}_d\rangle^{(1)}\otimes|N_a,N_b,N_c,N_d\rangle^{(2)}, \qquad (8.3.80)
$$

with

$$
\begin{aligned}
|\bar{N}_a,\bar{N}_b,\bar{N}_c,\bar{N}_d\rangle &= \frac{(-1)^{N_a+N_c-N_d}}{\sqrt{N_a!N_b!}}\prod_{k=1}^{2} a_k^{N_{a_k}} b_k^{N_{b_k}} c_k^{N_{c_k}} d_k^{N_{d_k}}|\bar{0}\rangle \qquad (8.3.81)\\
|N_a,N_b,N_c,N_d\rangle &= \frac{1}{\sqrt{N_a!N_b!}}\prod_{k=1}^{2}(a_k^\dagger)^{N_{a_k}}(b_k^\dagger)^{N_{b_k}}(c_k^\dagger)^{N_{c_k}}(d_k^\dagger)^{N_{d_k}}|0\rangle
\end{aligned}
$$

where $N_a! = N_{a_1}!N_{a_2}!$ and $N_b! = N_{b_1}!N_{b_2}!$. One can easily project the vertex (8.3.80) on the states satisfying zero charge condition $N_a - N_b + N_c - N_d = 0$.

8.3.3 Properties of the spin vertex

We now study the properties of the spin vertex. The most important properties are again the reflection property and the monodromy relation.

Reflection property From the explicit construction, one can see the reflection properties for the oscillators easily

$$\begin{aligned}
(a_{i,s}^{(1)\dagger} + b_{i,s}^{(2)})|V_{12}\rangle &= (b_{i,s}^{(1)\dagger} - a_{i,s}^{(2)})|V_{12}\rangle = 0, \\
(a_{i,s}^{(1)} + b_{i,s}^{(2)\dagger})|V_{12}\rangle &= (b_{i,s}^{(1)} - a_{i,s}^{(2)\dagger})|V_{12}\rangle = 0, \\
(c_{i,s}^{(1)} + d_{i,s}^{(2)\dagger})|V_{12}\rangle &= (d_{i,s}^{(1)} - c_{i,s}^{(2)\dagger})|V_{12}\rangle = 0, \\
(d_{i,s}^{(1)\dagger} + c_{i,s}^{(2)})|V_{12}\rangle &= (c_{i,s}^{(1)\dagger} - d_{i,s}^{(2)})|V_{12}\rangle = 0.
\end{aligned} \tag{8.3.82}$$

Using these relations, we can deduce the reflection properties for the generators of the algebra, which leads to

$$\left(\mathrm{E}_s^{AB(1)} + \tilde{\mathrm{E}}_s^{AB(2)} + (-1)^{|B|}\delta^{AB}\right)|V_{12}\rangle = 0, \quad s = 1, \cdots, L. \tag{8.3.83}$$

where

$$\tilde{\mathrm{E}}^{AB} = \mathrm{U}^2(\mathrm{E}^{AB} + (-1)^{|B|}\delta^{AB})\mathrm{U}^{-2}. \tag{8.3.84}$$

The reflection property can be written in $|\mathcal{V}_{12}\rangle$ simply as

$$\left(\mathrm{E}^{AB(1)} + \mathrm{E}^{AB(2)} + (-1)^{|B|}\delta^{AB}\right)|\mathcal{V}_{12}\rangle = 0, \quad s = 1, \cdots, L. \tag{8.3.85}$$

Using the reflection properties of the symmetric generators, we can derive the reflection properties of the Lax matrix and monodromy matrix as before. The definition of the Lax matrix depends on the representations of auxiliary space and quantum space. Here we choose both spaces in the fundamental representation and the one-loop Lax matrix is given by

$$L_{an}(u) = u - \frac{i}{2} - i(-1)^{|A|}\,\mathrm{E}_a^{AB}\,\mathrm{E}_n^{BA} \tag{8.3.86}$$

where a and n denotes the auxiliary space and the quantum space, respectively. Then it is straightforward to show that

$$L^{(1)}(u)|\mathcal{V}_{12}\rangle = -L^{(2)}(-u)|\mathcal{V}_{12}\rangle. \tag{8.3.87}$$

The reflection properties of the elements of monodromy relation follows from (8.3.87).

The monodromy relation The other important property, namely the monodromy relation can also be generalized to all sectors. Here we construct the monodromy matrix with the auxiliary space in the fundamental representation. For the monodromy matrix

with the auxiliary space in the physical representation, the construction for the compact $\mathfrak{so}(6)$ sector is relatively straightforward, but the construction for the non-compact sectors like $\mathfrak{sl}(2)$ sector is more subtle and will not be discussed here.

In the representations chosen by us, the R-matrix for the $\mathfrak{psu}(2,2|4)$ reads

$$R_{01}(u) = u - i\Pi_{01}, \qquad \Pi_{01} = (-1)^{|A|}\mathrm{E}_0^{AB}\,\mathrm{E}_1^{AB} \tag{8.3.88}$$

where Π_{01} is the super-permutation. For the definition in (8.3.88), the super permutation satisfies

$$\Pi_{01}^2 = -\Pi_{01}. \tag{8.3.89}$$

The derivation of the above relation uses the fact that in the physical representation $c = \mathrm{E}_1^{AA} = \psi^A\bar{\psi}^A = 0$ and in the auxiliary representation $\mathrm{E}_0^{AA} = 1$. Using (8.3.89), we can show that the R-matrix satisfies the following unitary representation

$$R_{01}(u)R_{01}(-u-i) = -u(u+i). \tag{8.3.90}$$

In general, the form of unitary condition depends on the central charge $c = \mathrm{E}_1^{AA}$ of the physical representation. For generic c, the unitary condition reads

$$R_{01}(u)R_{01}(i(c-1)-u) = -u(i(c-1)+u) - c. \tag{8.3.91}$$

Combing the reflection property (8.3.87) and unitary condition (8.3.90), we arrive at the following relation

$$R_{01}(u)R_{02}(u)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{01}(-i-u)|\mathcal{V}_{12}\rangle = u(u+i)|\mathcal{V}_{12}\rangle. \tag{8.3.92}$$

By taking products of the above relation, we obtain the monodromy relation for the two-point spin vertex

$$T_{12}(u)|\mathcal{V}_{12}\rangle = R_{01}^{(1)}(u)\cdots R_{0L}^{(1)}(u)R_{0L}^{(2)}(u)\cdots R_{01}^{(2)}(u)|\mathcal{V}_{12}\rangle = (u(u+i))^L|\mathcal{V}_{12}\rangle. \tag{8.3.93}$$

as well as for the three-point spin vertex with $T_{123}(u)$ defined by

$$T_{123}(u) = t_{12}(u)t_{13}(u)t_{31}(u)t_{32}(u)t_{23}(u)t_{21}(u) \tag{8.3.94}$$

where $t_{ij}(u)$ are defined as products of the corresponding R-matrices with the proper shifts.

The subsectors The $\mathfrak{psu}(2,2|4)$ R-matrix can be readily reduced to different subsectors, just by restricting the sum in the definition of the central charge (8.2.43) to the corresponding subsector. As a result, the central charge can take non-zero value $c = \mathrm{E}_1^{AA}$.

– In the $\mathfrak{su}(1|1)$, $\mathfrak{su}(2|3)$ and $\mathfrak{su}(2)$ sector, where the fields belong to the fundamental representation, $c = 1$, so the unitary condition is modified

$$\Pi_{01}^2 = 1, \qquad R_{01}(u)R_{01}(-u) = -(u^2+1) \tag{8.3.95}$$

and the corresponding monodromy relation reads

$$R_{01}(u)R_{02}(u-i)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{02}(-u)|\mathcal{V}_{12}\rangle = (u^2+1)|\mathcal{V}_{12}\rangle. \tag{8.3.96}$$

– In the $\mathfrak{sl}(2)$ sector, $c = 0$, so the unitary condition and monodromy condition are the same as for $\mathfrak{psu}(2,2|4)$.

– In the $\mathfrak{so}(6)$ sector, we have $c = 2$, so that

$$\Pi_{01}^2 = \Pi_{01} + 2, \quad R_{01}(u)R_{01}(i-u) = u(i-u) - 2. \tag{8.3.97}$$

The monodromy condition is then

$$R_{01}(u)R_{02}(u-2i)|\mathcal{V}_{12}\rangle = -R_{01}(u)R_{01}(i-u)|\mathcal{V}_{12}\rangle = (u(u-i)+2)|\mathcal{V}_{12}\rangle. \tag{8.3.98}$$

8.4 From spin vertex to string vertex

In this section, we take the BMN limit of the spin vertex and compare to the string vertex in the light-cone string field theory.

8.4.1 A brief review of light-cone string field theory

We review briefly the light-cone string field theory for strings on the pp-wave background [157–161] and refer the interested readers to [162, 163] and references therein for more detail.

The fundamental object in SFT is the string field operator Φ which creates and destroys strings. The Hilbert space on which the string field operator acts is a direct sum of n-string Hilbert spaces:

$$\mathcal{H} = |\mathrm{vac}\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ..., \quad \mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_1}_{n} \tag{8.4.99}$$

where $\mathcal{H}_1$ is the Hilbert space of a single string. The string interactions are described by the matrix elements of the Hamiltonian which has the following expansion in coupling constant g_s

$$H = H_2 + g_s H_3 + g_s^2 H_4 + ..., \tag{8.4.100}$$

where H_2 is the free part of the Hamiltonian and H_k describes interactions involving k strings. In this paper, we focus on the cubic interactions which are described by the following matrix elements

$$\lambda_{123} = \langle 2|\langle 3|H_3|1\rangle = \langle 1|\langle 2|\langle 3|H_3\rangle, \tag{8.4.101}$$

where $\langle 1|, \langle 2|, \langle 3|$ are three string states. In the second equality the matrix element is written in a more customary way by introducing the so-called cubic *string vertex* $|H_3\rangle$.

The principle to construct the string vertex is simple. For bosonic strings, the string vertex $|H_3\rangle$ is constructed by requiring worldsheet continuity at interacting point, which can be realized by imposing a delta functional $\Delta\Big(X_1(\sigma) - X_2(\sigma) - X_3(\sigma)\Big)$ in the functional integral over all possible configurations of three strings. The integral can be computed straightforwardly, leading to the following form of the bosonic string vertex [1]

$$|V\rangle = \exp\left(-\frac{1}{2} \sum_{m,n=-\infty}^{\infty} \sum_{r,s=1}^{3} \sum_{i=1}^{8} a_m^{(r)i\dagger} N_{mn}^{rs} a_n^{(s)i\dagger} \right) |0\rangle\rangle. \tag{8.4.102}$$

Here $|0\rangle\rangle$ denotes the vacuum of three-string Hilbert space $|0\rangle\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$. The indices r and s denote the number of strings, i denotes the polarization of the excitation and m, n are the mode numbers of the excitations. The quantities N_{mn}^{rs} are called Neumann coefficients and characterize the interactions between excitations of different strings.

For superstrings, in addition to worldsheet continuity, one also needs to require that supersymmetry is respected by the string vertex. This can be achieved by acting a new operator $\mathcal{P}$ on the exponential part (8.4.102). This operator can be written as polynomials of the creation operators $a_m^{(r)i}$ and is called the *prefactor*. The string vertex for superstring thus takes the following form

$$|H_3\rangle = \mathcal{P}|V\rangle. \tag{8.4.103}$$

However, it turns out that supersymmetry is not restrictive enough to fix the prefactor uniquely and there have been several proposals in the literature originating from different

1. Note that we use different notations from the ones in [164]. Our creation operator $a_m^{(r)i\dagger}$ is denoted by $\alpha_m^{(r)i\dagger}$ in [164] and our Neumann coefficient N_{mn}^{rs} is denoted by $\tilde{N}_{mn}^{rs}$ in [164].

motivations. Here we will use the prefactor proposal by Dobashi and Yoneya [165]. The reason is that their proposal has the virtue that works for both extremal and non-extremal [2] correlation functions [164]. Interestingly, the prefactor of Dobashi and Yoneya is the half sum of two prefactors $\mathcal{P}_1$ and $\mathcal{P}_2$ proposed in [157–160] and [166] respectively

$$\mathcal{P}_h = \frac{1}{2}\mathcal{P}_1 + \frac{1}{2}\mathcal{P}_2, \tag{8.4.104}$$

more explicitly

$$\mathcal{P}_h = \sum_{r=1}^{3}\left(\sum_{i=5}^{8}\sum_{m=0}^{\infty}\frac{\omega_m^{(r)}}{\alpha_{(r)}}a_m^{(r)i\dagger}a_m^{(r)i} + \sum_{i=1}^{4}\sum_{m=0}^{\infty}\frac{\omega_m^{(r)}}{\alpha_{(r)}}a_{-m}^{i(r)\dagger}a_{-m}^{(r)i}\right), \tag{8.4.105}$$

where $\omega_n^{(r)} = \sqrt{n^2+\mu^2\alpha_{(r)}^2}$ and $\alpha_{(r)} = \alpha' p_{(r)}^+$.

After one fixes the string vertex, the matrix elements of H_3 can be computed straightforwardly. According to [165], the holographic relation between matrix element of the H_3 and OPE coefficients in BMN limit is given by

$$C_{123} = \frac{\sqrt{J_1J_2J_3}}{N}\frac{G(\Delta_1,\Delta_2,\Delta_3)}{\mu(\Delta_2+\Delta_3-\Delta_1)}\lambda_{123}. \tag{8.4.106}$$

Here C_{123} is the structure constant of the three-point correlation function [3]

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \tag{8.4.107}$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$, Δ_i is the conformal dimension of the operator $\mathcal{O}_i$ and the function $G(\Delta_1,\Delta_2,\Delta_3)$ reads

$$G(\Delta_1,\Delta_2,\Delta_3) = \left(f\frac{J_2J_3}{J_1}\right)^{-(\Delta_2+\Delta_3-\Delta_1)/2}\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}+1\right). \tag{8.4.108}$$

Finally, we want to emphasis that the holographic relation between the matrix elements of H_3 and the OPE coefficient in $\mathcal{N}=4$ is not completely understood. The holographic relation (8.4.106) works well at the leading order [164, 167]. However, at higher loop order, the large μ expansion of the function $G(\Delta_1,\Delta_2,\Delta_3)$ give rises to non-perturbative terms such as $\log\mu$, the interpretation of which is still unclear.

2. The extremal and non-extremal correlation functions corresponds to the impurity preserving and impurity non-preserving processes in [164] respectively.

3. Note that here the definition for C_{123} is slightly different from the one we used before. We change the normalization here because (8.4.107) is more commonly used in the BMN limit.

8.4.2 Polynomial Representation of Spin Vertex

We consider the $\mathfrak{so}(6)$ sector of $\mathcal{N}=4$ SYM. The spin vertex for the $\mathfrak{so}(6)$ sector is constructed by *fermionic oscillators*. On the other hand, the cubic string vertex in SFT is constructed in terms of *bosonic oscillators*. In order to derive SFT from spin vertex, we introduce a polynomial representation for the spin vertex in the $\mathfrak{so}(6)$ sector. From the polynomial representation, it is then straightforward to construct the spin vertex using bosonic oscillators and make contact with SFT.

In what follows, as a convention, we choose operator $\mathcal{O}_1$ to be the "incoming" state and $\mathcal{O}_2$, $\mathcal{O}_3$ to be "outgoing" states, as is depicted in Fig. 8.4.4. The BMN vacuum for the

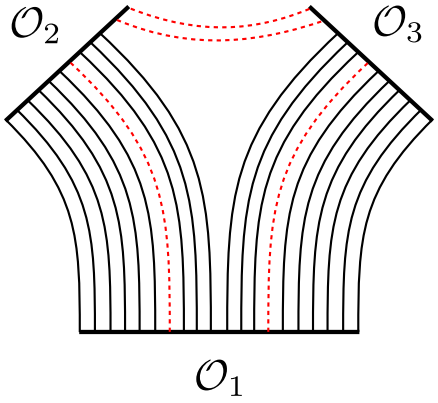

FIGURE 8.4.4: The configuration for three-point functions. The black lines correspond to the contractions of Z and $\bar{Z}$ and the red dashed lines correspond to the contractions of excitations ϕ^i, $i=1,2,3,4$.

incoming and outgoing states at each site are a scalar field Z and $\bar{Z}$ respectively. The "vacuum" can have different excitations, namely scalars excitations, vector excitations and fermionic excitations. As mentioned before, we consider here only scalar excitations. The three operators are thus made of the following fields

$$\mathcal{O}_1 : \{Z, \phi^i\}, \quad \mathcal{O}_2 : \{\bar{Z}, \phi^i\}, \quad \mathcal{O}_3 : \{\bar{Z}, \phi^i\}, \quad i = 1,2,3,4 \tag{8.4.109}$$

Following [164], we normalize the operators as

$$\langle \overline{\mathcal{O}}_i(x_1)\mathcal{O}_i(x_2)\rangle = \frac{1}{(x_1-x_2)^{2\Delta_i}}, \quad i=1,2,3 \tag{8.4.110}$$

where at leading order we have $\Delta_i = L_i$. The three-point function of three scalar operators is determined up to a constant called *structure constant* by conformal symmetry

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\sqrt{L_1L_2L_3}}{N_c}\frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{8.4.111}$$

At the leading order, we have

$$\Delta_i + \Delta_j - \Delta_k = L_i + L_j - L_k = 2L_{ij}, \quad i,j,k = 1,2,3 \tag{8.4.112}$$

where L_{ij} is the number of propagators between operators i and j. The structure constant c_{123} is related to the one in (8.4.107) by $C_{123} = \frac{\sqrt{L_1 L_2 L_3}}{N} c_{123}$. The computation of c_{123} can be formulated in the spin vertex formalism. In our case, we only need the scalar sector of the spin vertex, which can be obtained by a truncation from the full spin vertex. The scalar sector corresponds to the fermionic part of the spin vertex. Let us recall the fermionic part of the two-point spin vertex at each site from

$$|v_{12}\rangle = \exp \sum_{i=1,2} \left(d_i^{(1)\dagger} d_i^{(2)} + c_i^{(1)\dagger} c_i^{(2)} \right) |0\rangle^{(1)} \otimes |\bar{0}\rangle^{(2)} \tag{8.4.113}$$

The vacuum $|0\rangle$ and dual vacuum $|\bar{0}\rangle$ correspond to the scalar field Z and $\bar{Z}$ respectively and satisfy

$$c_i|0\rangle = d_i|0\rangle = 0, \quad c_i^\dagger|\bar{0}\rangle = d_i^\dagger|\bar{0}\rangle = 0, \quad i = 1,2. \tag{8.4.114}$$

More explicitly, we can write $|\bar{0}\rangle = c_1^\dagger c_2^\dagger d_1^\dagger d_2^\dagger |0\rangle$. We want to expand (8.4.113) and write the spin vertex in a more transparent way. The expansion leads to a sum of many terms, among which we keep only the terms with $N_c = N_d$, where N_c and N_d counts the number of operators $c_i, c_i^\dagger$ and $d_i, d_i^\dagger$, respectively. The terms which satisfy this condition in the expansion read

$$|v_{12}\rangle_{\text{scalar}} \equiv |v_{12}\rangle_{N_c=N_d} = |0\rangle \otimes |\bar{0}\rangle + \sum_{i,j=1}^{2} d_i^{(1)\dagger} c_j^{(1)\dagger} |0\rangle \otimes d_i^{(2)} c_j^{(2)} |\bar{0}\rangle + |\bar{0}\rangle \otimes |0\rangle \tag{8.4.115}$$

Let us define the following states corresponding to the scalar fields X, Y and their hermitian conjugates $\bar{X}$, $\bar{Y}$

$$\begin{aligned} |X\rangle &\equiv d_1^\dagger c_1^\dagger |0\rangle = d_2 c_2 |\bar{0}\rangle, \quad |\bar{X}\rangle \equiv d_2^\dagger c_2^\dagger |0\rangle = d_1 c_1 |\bar{0}\rangle \\ |Y\rangle &\equiv d_1^\dagger c_2^\dagger |0\rangle = c_1 d_2 |\bar{0}\rangle, \quad |\bar{Y}\rangle \equiv c_1^\dagger d_2^\dagger |0\rangle = d_1 c_2 |\bar{0}\rangle. \end{aligned} \tag{8.4.116}$$

Then we have

$$|v_{12}\rangle_{\text{scalar}} = \sum_{a=X,Y,Z} \left(|a\rangle \otimes |\bar{a}\rangle + |\bar{a}\rangle \otimes |a\rangle \right), \tag{8.4.117}$$

We can further define the states corresponding to the real scalar fields ϕ_i, $i = 1, 2, 3, 4$ as the following

$$\begin{aligned} |X\rangle &= \frac{1}{\sqrt{2}}(|\Phi_1\rangle + i|\Phi_2\rangle), \quad |\bar{X}\rangle = \frac{1}{\sqrt{2}}(|\Phi_1\rangle - i|\Phi_2\rangle) \\ |Y\rangle &= \frac{1}{\sqrt{2}}(|\Phi_3\rangle + i|\Phi_4\rangle), \quad |\bar{Y}\rangle = \frac{1}{\sqrt{2}}(|\Phi_3\rangle - i|\Phi_4\rangle) \end{aligned} \tag{8.4.118}$$

which gives

$$|v_{12}\rangle_{\text{scalar}} = |Z\rangle \otimes |\bar{Z}\rangle + |\bar{Z}\rangle \otimes |Z\rangle + \sum_{i=1}^{4} |\phi_i\rangle \otimes |\Phi_i\rangle \tag{8.4.119}$$

The spin vertex for three-point functions of our set-up is obtained by taking the appropriate tensor products of the vertex in (8.4.119), which reads

$$\begin{aligned} |V_3\rangle = &\prod_{k=1}^{L_{12}} \left(|Z\rangle^{(1)}_{L_1-k+1} \otimes |\bar{Z}\rangle^{(2)}_{k} + |\bar{Z}\rangle^{(1)}_{L_1-k+1} \otimes |Z\rangle^{(2)}_{k} + \sum_{i=1}^{4} |\Phi_i\rangle^{(1)}_{L_1-k+1} \otimes |\Phi_i\rangle^{(2)}_{k} \right) \otimes \\ &\prod_{k=1}^{L_{13}} \left(|Z\rangle^{(1)}_{k} \otimes |\bar{Z}\rangle^{(3)}_{L_3-k+1} + |\bar{Z}\rangle^{(1)}_{k} \otimes |Z\rangle^{(3)}_{L_3-k+1} + \sum_{i=1}^{4} |\Phi_i\rangle^{(1)}_{k} \otimes |\Phi_i\rangle^{(3)}_{L_3-k+1} \right) \otimes \\ &\prod_{k=1}^{L_{23}} \left(|Z\rangle^{(2)}_{L_2-k+1} \otimes |\bar{Z}\rangle^{(3)}_{k} + |\bar{Z}\rangle^{(2)}_{L_2-k+1} \otimes |Z\rangle^{(3)}_{k} + \sum_{i=1}^{4} |\Phi_i\rangle^{(2)}_{L_2-k+1} \otimes |\Phi_i\rangle^{(3)}_{k} \right). \end{aligned} \tag{8.4.120}$$

where the upper indices denote the spin chains and the lower indices denote the positions on the corresponding spin chains. In order to obtain a polynomial representation, let us define the following generating states for the three spin chains

$$\begin{aligned} |F_1\rangle &= \prod_{k=1}^{L_1} \left(|Z\rangle_k + \sum_{i=1}^{4} x^i_k |\Phi^i\rangle_k \right) \\ |F_2\rangle &= \prod_{k=1}^{L_2} \left(|\bar{Z}\rangle_k + \sum_{i=1}^{4} y^i_k |\Phi^i\rangle_k \right) \\ |F_3\rangle &= \prod_{k=1}^{L_3} \left(|\bar{Z}\rangle_k + \sum_{i=1}^{4} z^i_k |\Phi^i\rangle_k \right) \end{aligned} \tag{8.4.121}$$

where we couple each excitation with an auxiliary variable. The three states can be obtained in the following way

$$|1\rangle = \Psi_1(\boldsymbol{\partial_x})|F_1\rangle|_{\boldsymbol{x}^i=0}\,, \quad |2\rangle = \Psi_2(\boldsymbol{\partial_y})|F_2\rangle|_{\boldsymbol{y}^i=0}\,, \quad |3\rangle = \Psi_3(\boldsymbol{\partial_z})|F_3\rangle|_{\boldsymbol{z}^i=0} \tag{8.4.122}$$

where $\Psi_1(\boldsymbol{\partial_x}), \Psi_2(\boldsymbol{\partial_y})$ and $\Psi_3(\boldsymbol{\partial_z})$ are three differential operators of the following form

$$\Psi(\boldsymbol{\partial_x}) = \sum_{\boldsymbol{n},I} c^I_{\boldsymbol{n}} \prod_{k=1}^{L} (\partial_k^{i_k})^{n_k}, \qquad \partial_k^{i_k} \equiv \frac{\partial}{\partial x_k^{i_k}}, \quad n_k = 0, 1 \tag{8.4.123}$$

Here $\boldsymbol{n} = \{n_1, \cdots, n_L\}$ and I is a collective index indicating the polarizations of the excitations. Each differential operator specifies one spin chain state. Integrability of the spin chain usually provides systematic way to construct the differential operators. In our case, for example, the differential operator can be constructed systematically by the nested Bethe ansatz. According the the spin vertex formalism

$$\begin{aligned} c_{123} =& \langle 1|\langle 2|\langle 3|V_3\rangle \\ =& \Psi_1(\boldsymbol{\partial_x})\Psi_2(\boldsymbol{\partial_y})\Psi_3(\boldsymbol{\partial_z})\langle F_1|\langle F_2|\langle F_3|V_3\rangle|_{\boldsymbol{x}^i,\boldsymbol{y}^i,\boldsymbol{z}^i=0} \\ =& \Psi_1(\boldsymbol{\partial_x})\Psi_2(\boldsymbol{\partial_y})\Psi_3(\boldsymbol{\partial_z})V_3(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})|_{\boldsymbol{x}^i,\boldsymbol{y}^i,\boldsymbol{z}^i=0} \end{aligned} \tag{8.4.124}$$

where $V_3(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i)$ is the polynomial representation of the spin vertex which reads

$$V_3(\boldsymbol{x^i}, \boldsymbol{y^i}, \boldsymbol{z^i}) = \prod_{k=1}^{L_{12}} (1 + y_k^i x_{L_1-k+1}^i) \prod_{k=1}^{L_{13}} (1 + x_k^i z_{L_3-k+1}^i) \prod_{k=1}^{L_{23}} z_k^i y_{L_2-k+1}^i \tag{8.4.125}$$

In the polynomial representation, the spin vertex (8.4.125) is given by a polynomial in terms auxiliary variables $\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i$. The states are given by three differential operators, which can be constructed by Bethe ansatz. Note that the commutation relations of $\partial/\partial x$ and x are the same as commutation relations of bosonic creation and annihilation operators. Therefore, we can map the auxiliary variables and the corresponding derivatives into creation and annihilation operators

$$\begin{aligned} &x_k^i \to \alpha_k^{(1)i\dagger}, \quad y_k^i \to \alpha_k^{(2)i\dagger}, \quad z_k^i \to \alpha_k^{(3)i\dagger} \\ &\frac{\partial}{\partial x_k^i} \to \alpha_k^{(1)i}, \quad \frac{\partial}{\partial y_k^i} \to \alpha_k^{(2)i}, \quad \frac{\partial}{\partial z_k^i} \to \alpha_k^{(3)i} \end{aligned} \tag{8.4.126}$$

and the spin vertex can be written as

$$|V_3\rangle_B \equiv V_3\left(\alpha_k^{(1)i\dagger}, \alpha_k^{(2)i\dagger}, \alpha_k^{(3)i\dagger}\right)|0\rangle_B \tag{8.4.127}$$

here we use $|V_3\rangle_B$ to emphasis that this vertex is constructed by *bosonic* oscillators instead of *fermionic* ones as in (8.4.120). The new Fock vacuum is defined to be the state that is annihilated by all the bosonic annihilation operators

$$\alpha_k^{(r)i}|0\rangle_B = 0 \tag{8.4.128}$$

The corresponding states can be written as

$$\langle 1| \equiv \langle 0|\Psi_1\left(\alpha_k^{(1)i}\right), \quad \langle 2| \equiv \langle 0|\Psi_2\left(\alpha_k^{(2)i}\right), \quad \langle 3| \equiv \langle 0|\Psi_3\left(\alpha_k^{(3)i}\right) \tag{8.4.129}$$

and the structure constant is given by

$$c_{123} = \langle 1|\langle 2|\langle 3|V_3\rangle_B. \tag{8.4.130}$$

where both the spin vertex and the states are now constructed by bosonic oscillators. The formulation we described up to now is applicable to any three states in our set-up. In the next section, we will take the BMN limit of (8.4.127) and show that it reproduces the string vertex in SFT.

8.4.3 BMN Limit of the Spin Vertex

The BMN limit for $\mathcal{N} = 4$ SYM mainly contains two approximations. The first is a dilute gas approximation, which means the number of excitations is finite and are distributed sparsely. The second approximation is that the momenta of the excitations are small and scales like $\sim 1/L$ where L is the length of the spin chain, which is taken to be very large.

We start by recalling the spin vertex in polynomial representation in (8.4.125)

$$V_3(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i) = \prod_{k=1}^{L_{12}}(1 + x^i_{L_1-k+1}y^i_k)\prod_{k=1}^{L_{13}}(1 + z^i_{L_3-k+1}x^i_k)\prod_{k=1}^{L_{23}} y^i_{L_2-k+1}z^i_k \tag{8.4.131}$$

The length of each spin chain is equal to the sum of number of vacuum field Z or $\bar{Z}$ and number of excitations. We denote the number of vacuum fields by J_i and the number of excitations by N_i so that the length of the spin chain L_i is given by $L_i = J_i + N_i$. By dilute gas assumption, we have $J_i \gg N_i$ and hence $L_i \simeq J_i$. Due to charge conservation, we have $J_1 = J_2 + J_3$, hence we can write $J_1 = J$, $J_2 = rJ$, $J_3 = (1-r)J$, where $0 < r < 1$. The number of contractions between different operators are approximately

$$L_{12} \simeq J_2 = rJ, \quad L_{13} \simeq J_3 = (1-r)J, \quad L_{23} = M = \frac{1}{2}(N_2 + N_3 - N_1). \tag{8.4.132}$$

In what follows, we use M to denote the number of contractions between the two 'outgoing' operators. By BMN assumption, $M \ll J$. The three-point functions for $M = 0$ are called *impurity preserving*, or *extremal* while for $M \neq 0$ are called *impurity non-preserving* or *non-extremal*. For the extremal correlator, when diagonalizing anomalous dimension matrix one have to deal with the mixing between single trace and double trace operators. On the contrary, for the non-extremal cases, the contribution from

double trace operators are $1/N_c$-suppressed and can be neglected in the planar limit, which makes the computation much simpler from gauge theory aspect. On the other hand, the earlier proposals for string field theory and duality relations work only for the extremal cases. A string field theory applicable to non-extremal cases as well as extremal is the holographic string field theory proposed by Dobashi and Yoneya [164, 165]. Here we consider only the non-extremal cases so we always assume $M \neq 0$.

By (8.4.126), we map x_k^i, y_k^i and z_k^i to creation operators. In order to obtain creation operators in the momentum space, we perform the mode expansion of the bosonic oscillator

$$\alpha_k^{(r)i\dagger} = \frac{1}{\sqrt{J_r}} \sum_{n=-\infty}^{\infty} e^{\frac{2\pi nk}{J_r}} a_n^{(r)i\dagger}, \quad r = 1, 2, 3 \tag{8.4.133}$$

Let us investigate the part of spin vertex corresponding to the contractions between operators 1 and 2.

$$V_{12} = \prod_{k=1}^{L_{12}} (1 + \alpha_{L_1-k+1}^{(1)i} \alpha_k^{(2)i}) \approx \exp\left(\frac{1}{J\sqrt{r}} \sum_{n_i^{(1)}, n_i^{(2)}} \sum_{k=0}^{J_2} e^{\frac{2\pi i n_i^{(2)} k}{J_2} - \frac{2\pi i n_i^{(1)} k}{J_1}} a_{n_i^{(1)}}^{(1)i\dagger} \, a_{n_i^{(2)}}^{(2)i\dagger} \right) \tag{8.4.134}$$

$$= \exp\left(- \sum_{n_i^{(1)}, n_i^{(2)}} \mathcal{N}^{12}_{n_i^{(1)} n_i^{(2)}} \, a_{n_i^{(1)}}^{(1)i\dagger} \, a_{n_i^{(2)}}^{(2)i\dagger} \right).$$

In the first line, the summation over k gives

$$\sum_{k=0}^{J_2} e^{\frac{2\pi i n_i^{(2)} k}{J_2} - \frac{2\pi i n_i^{(1)} k}{J_1}} \simeq J e^{-\pi i r n_i^{(1)}} \frac{\sin \pi r n_i^{(1)}}{\pi (n_i^{(1)} - n_i^{(2)}/r)} = J\sqrt{r} (-1)^{n_i^{(1)} + n_i^{(2)}} \, e^{-\pi i r n_i^{(1)}} \, N^{12}_{n_i^{(1)} n_i^{(2)}} \tag{8.4.135}$$

Therefore our Neumann coefficient from spin vertex is related to the Neumann coefficient in SFT [168, 169] by a simple phase factor

$$\mathcal{N}^{12}_{n_i^{(1)} n_i^{(2)}} = (-1)^{n_i^{(1)} + n_i^{(2)}} \, e^{-\pi i r n_i^{(1)}} \, N^{12}_{n_i^{(1)} n_i^{(2)}}. \tag{8.4.136}$$

The explicit form of Neumann coefficient at the leading order of large μ expansion can be found in Appendix D. Similarly, for the contractions between operators 1 and 3, we

have

$$V_{13} = \prod_{k=1}^{L_{13}} (1 + \alpha_k^{(1)i} \alpha_{L_3-k+1}^{(3)i}) \simeq \exp\left(\frac{\sqrt{1-r}}{J} \sum_{n_i^{(1)}, n_i^{(3)}} \sum_{k=0}^{J_3} e^{\frac{2\pi i n_i^{(1)} k}{J_1} - \frac{2\pi i n_i^{(3)} k}{J_3}} \, a^{(1)i\dagger}_{n_i^{(1)}} \, a^{(3)i\dagger}_{n_i^{(3)}} \right) \tag{8.4.137}$$

$$= \exp\left(- \sum_{n_i^{(1)}, n_i^{(3)}} \mathcal{N}^{13}_{n_i^{(1)} n_i^{(3)}} \, a^{(1)i\dagger}_{n_i^{(1)}} \, a^{(3)i\dagger}_{n_i^{(3)}} \right)$$

where our Neumann coefficient is related to the SFT Neumann coefficient by

$$\mathcal{N}^{13}_{n_i^{(1)} n_i^{(3)}} = (-1)^{n_i^{(1)}} \, e^{-i\pi r n_i^{(1)}} \, N^{13}_{n_i^{(1)} n_i^{(3)}} \tag{8.4.138}$$

For the contractions between operators 2 and 3, we have

$$V_{23} = \prod_{k=1}^{M} \alpha_k^{(3)i} \alpha_{L_2-k+1}^{(2)i} = \prod_{k=1}^{M} \frac{1}{J\sqrt{r(1-r)}} \sum_{n_i^{(2)}, n_i^{(3)}} e^{\frac{2\pi i k n_i^{(3)}}{J_3} - \frac{2\pi i k n_i^{(2)}}{J_2}} \, a^{(2)i\dagger}_{n_i^{(2)}} \, a^{(3)i\dagger}_{n_i^{(3)}} \tag{8.4.139}$$

By dilute gas approximation $N \ll J_i$, so that in the above product $k \leq N \ll J_2, J_3$. Also we assume that the momentum of the excitations are small, meaning $n_i^{(2)} \ll J_2$ and $n_i^{(3)} \ll J_3$. Therefore the phase factor in (8.4.139) is trivial

$$e^{\frac{2\pi i k n_i^{(3)}}{J_3} - \frac{2\pi i k n_i^{(2)}}{J_2}} \approx 1 \tag{8.4.140}$$

and V_{23} simplifies to

$$V_{23} = \left(\frac{1}{J\sqrt{r(1-r)}} \sum_{n_i^{(2)}, n_i^{(3)}} a^{\dagger(2)i}_{n_i^{(2)}} \, a^{\dagger(3)i}_{n_i^{(3)}} \right)^M = \left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} \left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}^{23}_{n_i^{(2)} n_i^{(3)}} \, a^{\dagger(2)i}_{n_i^{(2)}} \, a^{\dagger(3)i}_{n_i^{(3)}} \right)^M . \tag{8.4.141}$$

where our Neumann coefficient is given by

$$\mathcal{N}^{23}_{n_i^{(2)} n_i^{(3)}} = (-1)^{n_i^{(2)}} N^{23}_{n_i^{(2)} n_i^{(3)}} . \tag{8.4.142}$$

Let us consider the following vertex

$$\tilde{V}_{23} = \exp\left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}^{23}_{n_i^{(2)} n_i^{(3)}} \, a^{\dagger(2)i}_{n_i^{(2)}} \, a^{\dagger(3)i}_{n_i^{(3)}} \right) . \tag{8.4.143}$$

which can be expanded as

$$\tilde{V}_{23} = \sum_{M=0}^{\infty} \frac{1}{M!} \left(- \sum_{n_i^{(2)}, n_i^{(3)}} \mathcal{N}^{23}_{n_i^{(2)} n_i^{(3)}} \, a^{\dagger(2)i}_{n_i^{(2)}} \, a^{\dagger(3)i}_{n_i^{(3)}} \right)^M \tag{8.4.144}$$

For a given configuration, the number M is fixed and the action of states will automatically pick out the term in $\tilde{V}_{23}$ with the corresponding M. Hence we can replace V_{23} by

$$V_{23} \longrightarrow \left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} M! \cdot \tilde{V}_{23} \tag{8.4.145}$$

Recalling that at leading order

$$M = \frac{1}{2}(L_2 + L_3 - L_1) = \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1), \tag{8.4.146}$$

we can write the factors in front of $\tilde{V}_{23}$ in (8.4.145) as

$$\left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} M! = \left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-(\Delta_2+\Delta_3-\Delta_1)/2} \Gamma\left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2} + 1 \right) \tag{8.4.147}$$

In SFT [164], the factor $J_1/4\pi\mu|\alpha_{(1)}|$ is the leading term of the large μ expansion of the following factor

$$f\frac{J_2 J_3}{J_1} = \frac{J}{4\pi\mu|\alpha_{(1)}|} + \mathcal{O}(\frac{1}{\mu^2}) \tag{8.4.148}$$

so that at the leading order, we have

$$\left(\frac{J}{4\pi\mu|\alpha_{(1)}|} \right)^{-M} M! \simeq \left(f\frac{J_2 J_3}{J_1} \right)^{-(\Delta_2+\Delta_3-\Delta_1)/2} \Gamma\left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2} + 1 \right) \tag{8.4.149}$$

The right hand side is nothing but the function $G(\Delta_1, \Delta_2, \Delta_3)$ in (8.4.106) and (8.4.108). From our derivation, it is clear that the function $G(\Delta_1, \Delta_2, \Delta_3)$ is intimately related to the interaction between the two outgoing states. Therefore it is crucial for the matching between SFT calculation and the non-extremal three-point functions.

We can define our spin vertex operator as

$$V_{\text{BMN}} = G(\Delta_1, \Delta_2, \Delta_3) V_{12} V_{13} \tilde{V}_{23}. \tag{8.4.150}$$

This is very close to the 3-point vertex from string theory including the correct G-factor, except that our Neumann coefficients seem to be different from those of SFT by some phase factors. We shall show that these phase factors are trivial if we consider the

physical states that satisfy the level matching conditions. When acting physical states on the spin vertex, we obtain the product of Neumann coefficients of the following type

$$\begin{aligned}&(-1)^{L_{12}+L_{23}+L_{13}}\prod_{12}\mathcal{N}^{12}_{p_i^{(1)}p_i^{(2)}}\prod_{13}\mathcal{N}^{13}_{q_i^{(2)}q_i^{(3)}}\prod_{23}\mathcal{N}^{23}_{r_i^{(2)}r_i^{(3)}} \\ =\,&\texttt{phase}\cdot(-1)^{L_{12}+L_{23}+L_{13}}\prod_{12}N^{12}_{p_i^{(1)}p_i^{(2)}}\prod_{13}N^{13}_{q_i^{(2)}q_i^{(3)}}\prod_{23}N^{23}_{r_i^{(2)}r_i^{(3)}}.\end{aligned} \tag{8.4.151}$$

From (8.4.136),(8.4.138) and (8.4.142), the phase factor is

$$\begin{aligned}\texttt{phase}=&\prod_{12}(-1)^{p_i^{(1)}+p_i^{(2)}}e^{-\pi i r p_i^{(1)}}\prod_{13}(-1)^{q_i^{(1)}}e^{-\pi i r q_i^{(1)}}\prod_{23}(-1)^{r_i^{(2)}} \\ =&(-1)^{\sum_i p_i^{(1)}+q_i^{(1)}}(-1)^{\sum_i p_i^{(2)}+r_i^{(2)}}e^{-\sum_i \pi i(p_i^{(1)}+q_i^{(1)})}=1\end{aligned} \tag{8.4.152}$$

where we have used the level matching conditions

$$\sum_i p_i^{(1)}+q_i^{(1)}=\sum_i p_i^{(2)}+r_i^{(2)}=0. \tag{8.4.153}$$

This means that if we consider the physical states, we can replace our Neumann coefficients $\mathcal{N}^{rs}_{m,n}$ by the Neumann coefficients of SFT $\tilde{N}^{rs}_{m,n}$ since the phase factors cancels out. To sum up, from the spin vertex in the BMN limit, we obtain at the leading order the following cubic vertex

$$|V_{\rm BMN}\rangle=G(\Delta_1,\Delta_2,\Delta_3)\exp\left[-\frac{1}{2}\sum_{\substack{r,s=1\\ r\neq s}}^{3}a_m^{(r)i\dagger}N^{rs}_{mn}a_n^{(s)i\dagger}\right]|0\rangle. \tag{8.4.154}$$

Notice that in the exponent we impose the condition $r\neq s$, while in SFT the Neumann coefficients $\tilde{N}^{rr}_{mn}$, which corresponds to interactions between the excitations of the same string, are non-zero. However these Neumann coefficients will appear only at higher orders in the large μ expansion. It is an interesting question whether we can obtain this kind of Neumann coefficients from weak coupling at higher loops, which we leave for future investigation. Therefore at the leading order, we have

$$|V_{\rm BMN}\rangle=G(\Delta_1,\Delta_2,\Delta_3)|V\rangle. \tag{8.4.155}$$

To complete our derivation, we also need to show that the spin chain states in the BMN limit also takes the same form as the ones in SFT. In the BMN limit, the scattering phases are zero which means there is no interaction between excitations. Therefore, the wave functions of the spin chain states are simply given by plane waves. In terms of

bosonic oscillators, a BMN state at the leading order can be represented by

$$\langle n_1, \cdots, n_N| = \frac{1}{\sqrt{L^N}} \sum_{x_1,\cdots,x_N=1}^{L} \langle 0|\alpha_{x_1}^{i_i} \cdots \alpha_{x_N}^{i_N} e^{-\frac{2\pi i}{L}(n_1 x_1 + \cdots + n_N x_N)} \tag{8.4.156}$$

where L is the length of the spin chain, $\alpha_{x_k}^{i_k}$ is the bosonic oscillator introduced in (8.4.126) and creates an excitation at position x_k with polarization i_k from the dual vacuum. Here $n_1, \cdots, n_N$ are the mode numbers of the excitations. Performing the mode expansion in (8.4.133), we have simply

$$\langle n_1, \cdots, n_N| = \langle 0|a_{n_1}^{i_1} \cdots a_{n_N}^{i_N} \tag{8.4.157}$$

which takes exactly the same form as the states in SFT.

There is another difference between the spin vertex and string vertex. In SFT, one has to take into account the non-trivial prefactor while we do not have similar prefactor in the spin vertex formalism. However, as argued by Dobashi and Yoneya [164], at the leading order, the action of prefactor on the states gives rise to a simple factor, which cancels neatly the holographic factor $1/\mu(\Delta_2 + \Delta_2 - \Delta_1)$. Let us briefly review how this works. The prefactor for scalar excitations at the leading order reads

$$\begin{aligned}\mathcal{P}_h =& \mu(a_0^{(2)i\dagger} a_0^{(2)i} + a_0^{(3)i\dagger} a_0^{(3)i} - a_0^{(1)i\dagger} a_0^{(1)i}) \\ &+ \frac{\mu}{2} \sum_{m=1}^{\infty} (a_m^{(2)i\dagger} a_m^{(2)i} + a_m^{(3)i\dagger} a_m^{(3)} - a_m^{(1)i\dagger} a_m^{(1)i} + [m \to -m]) \\ &+ \frac{\mu}{2} \sum_{m=1}^{\infty} (a_m^{(2)i\dagger} a_{-m}^{(2)i} + a_m^{(3)i\dagger} a_{-m}^{(3)i} - a_m^{(1)i\dagger} a_{-m}^{(1)i} + [m \to -m])\end{aligned} \tag{8.4.158}$$

The operators in the first two lines take the form of counting operators $a_m^\dagger a_m$, hence once acted on physical states, they just count the number of difference of excitations between the incoming state 1 and out-going states 2 and 3, which is $N_2 + N_3 - N_1 = 2M$. The contribution from the first two lines is simply μM. In order to calculate the contribution from the third line, we need to make use of the symmetry of the Neumann coefficients $N_{-n,m}^{rs} = N_{n,-m}^{rs}$, $N_{mn}^{rs} = N_{-m,-n}^{rs}$. The operators in the third line $a_m^\dagger a_{-m}$ change the sign of mode numbers. Consider first the contractions between operators 1 and 2. Changing the sign of mode numbers of either operator gives the same result due to the symmetry of the Neumann coefficient. It is the same for the contractions between operators 1 and 3. Therefore the contributions for the impurity preserving part cancel and we only need to consider the contractions between impurity non-preserving part 2↔3. We recall the the Neumann coefficient $N_{m,n}^{23}$ at the leading order does not depend on mode number at all, hence the sign changing operators have the same effect as counting operators. The

contribution is again μM. To sum up, the action of the prefactor on physical states gives rise to a multiplication of the simple factor $2\mu M$, which can be written in the following way at the leading order

$$2\mu M = \mu(\Delta_2 + \Delta_3 - \Delta_1). \tag{8.4.159}$$

Therefore, we have

$$\frac{1}{\mu(\Delta_2+\Delta_3-\Delta_1)}\langle 1|\langle 2|\langle 3|H_3\rangle = \frac{1}{\mu(\Delta_2+\Delta_3-\Delta_1)}\langle 1|\langle 2|\langle 3|\mathcal{P}_h|V\rangle = \langle 1|\langle 2|\langle 3|V\rangle \tag{8.4.160}$$

Finally, gathering all the pieces, we have

$$\begin{aligned} C_{123} &= \frac{\sqrt{J_1J_2J_3}}{N}\ \langle 1|\langle 2|\langle 3|V_{\text{BMN}}\rangle \\ &= \frac{\sqrt{J_1J_2J_3}}{N}\ G(\Delta_1,\Delta_2,\Delta_3)\ \langle 1|\langle 2|\langle 3|V\rangle \\ &= \frac{\sqrt{J_1J_2J_3}}{N}\ \frac{G(\Delta_1,\Delta_2,\Delta_3)}{\mu(\Delta_2+\Delta_3-\Delta_1)}\ \langle 1|\langle 2|\langle 3|\mathcal{P}_h|V\rangle \\ &= \frac{G(\Delta_1,\Delta_2,\Delta_3)}{\mu(\Delta_2+\Delta_3-\Delta_1)}\ \langle 1|\langle 2|\langle 3|\frac{\sqrt{J_1J_2J_3}}{N}|H_3\rangle \end{aligned} \tag{8.4.161}$$

where the first line is from the *spin vertex* while the last line is exactly the holographic relation (8.4.106) proposal by Dobashi and Yoneya. Therefore we have shown that gauge theory computation reproduces exactly SFT result at tree level.

We want to emphasis here that we not only obtain same results on both sides, but we are able to identify the common *structures*. This shows further that the spin vertex is indeed the weak coupling counterpart of the string vertex.

Chapter 9

Form Factor Methods

In this chapter, we discuss another direction towards an all-loop formulation of three-point functions in $\mathcal{N} = 4$ SYM theory. This direction is intimately related to the form factor bootstrap program in 2d integrable quantum field theories.

We first review the form factor bootstrap program in 2d integrable QFT in section 1. Then we discuss how to generalize and apply the bootstrap method to the context of AdS/CFT in section 2. The form factor bootstrap method provides us the form factors in *infinite volume*. However, the three-point functions are related to the form factors in *finite volume* where finite volume corrections are important. In general, computing finite volume corrections for form factors is a challenging problem. However, if one neglects wrapping corrections, then there is a systematic way to obtain finite volume corrections for the form factors, this is reviewed in section 3. It is conjectured by Bajnok, Janik and Wereszczynski [59] that the heavy-heavy-light (HHL) three-point functions with two heavy operators conjugated to each other are equivalent to diagonal form factors. Based on previous works in 2d integrable QFT, they proposed the volume dependence of the structure constant up to wrapping corrections and conjectured that it should hold at all loops. They checked the structure at strong coupling for some cases, it remains to be checked at weak coupling. We showed that their conjecture is indeed true at weak coupling using spin chain technique. This is discussed in section 4.

9.1 The form factor bootstrap program

In this section, we give a brief review of the form factor bootstrap program in 2d integrable QFT. For a detail review, we refer to [170]. Consider an integrable quantum field theory in 1+1 dimensions, which is defined completely by its S-matrix. For simplicity,

we restrict our discussions for the diagonally scattering theories with a self-conjugated particle.

In infinite volume, the states can be characterized by the set of momenta of the particles. In 2d, it is more convenient to parameterize the particles by rapidities u. The momentum $p(u)$ and energy $\varepsilon(u)$ of a particle are functions of the rapidity. In the remote past, we can define the *in* state as the state with ordered rapidities: the fastest one is the leftmost and the slowest one is the rightmost. Contrarily, we can define in the remote future the *out* state with the reversed order

$$|u_1, \cdots, u_N\rangle = \begin{cases} |u_1, \cdots, u_N\rangle^{\text{in}} & u_1 > \cdots > u_N \\ |u_1, \cdots, u_N\rangle^{\text{out}} & u_1 < \cdots < u_N \end{cases} \tag{9.1.1}$$

The infinite volume states that differ only in the order of rapidities are related by the two particle S-matrix

$$|u_1, \cdots, u_i, u_{i+1}, \cdots, u_N\rangle = S(u_i, u_{i+1})\, |u_1, \cdots, u_{i+1}, u_i, \cdots, u_N\rangle \tag{9.1.2}$$

The energy of a multiparticle state is the sum of the one particle energies

$$E(u_1, ..., u_N) = \sum_{i=1}^{N} \varepsilon(u_i). \tag{9.1.3}$$

In infinite volume we normalize the *in* states as

$${}^{\text{in}}\langle u'_1, \cdots, u'_M | u_1, \cdots, u_N\rangle^{\text{in}} = (2\pi)^N \delta_{NM}\, \delta(u_1 - u'_1) \cdots \delta(u_N - u'_N), \tag{9.1.4}$$

and the norm of a general state can be determined from (9.1.4) by (9.1.2). Let us consider the matrix elements of a *local* operator $\mathcal{O}(t, x)$ between asymptotic states. The space-time dependence can be easily factored out

$${}^{\text{out}}\left\langle u'_1, \cdots, u'_M \right| \mathcal{O}(x, t) \left| u_1, \cdots, u_N \right\rangle^{\text{in}} = e^{it\Delta E - ix\Delta P}\ {}^{\text{out}}\left\langle u'_1, \cdots, u'_M \right| \mathcal{O}(0, 0) \left| u_1, \cdots, u_N \right\rangle^{\text{in}},$$

where

$$\begin{aligned} \Delta E &= \sum_{j=1}^{M} \varepsilon(u'_j) - \sum_{k=1}^{N} \varepsilon(u_k), \\ \Delta P &= \sum_{j=1}^{M} p(u'_j) - \sum_{k=1}^{N} p(u_k), \end{aligned} \tag{9.1.5}$$

and we define the *form factor* of operator $\mathcal{O}$ as

$$F^{\mathcal{O}}_{M,N}\left(u'_1,\cdots,u'_M|u_1,\cdots,u_N\right) = {}^{\text{out}}\left\langle u'_1,\cdots,u'_M\right|\mathcal{O}(0,0)\left|u_1,\cdots,u_N\right\rangle^{\text{in}} \tag{9.1.6}$$

The form factors are *a priori* defined for ordered set of incoming and outgoing rapidities but can be analytically continued by (9.1.2). A form factor is a meromorphic function in all variables and each pole has a physical origin [170].

Suppose that the theory possesses a crossing symmetry, i.e. a transformation which maps an outgoing particle with rapidity u to an incoming anti-particle with rapidity $\bar{u}$, under which the physical quantities stay invariant. The crossing symmetry implies the crossing equation for the form factors which, in case of a single, self-conjugated particle, reads as

$$\begin{aligned} F^{\mathcal{O}}_{M,N}\left(u'_1,\cdots,u'_M|u_1,\cdots,u_N\right) &= F^{\mathcal{O}}_{M-1,N+1}\left(u'_1,\cdots,u'_{M-1}|\bar{u}_M,u_1,\cdots,u_N\right) \\ &+\sum_{k=1}^{N}\left\langle u'_M|u_k\right\rangle\prod_{l=1}^{k-1}S\left(u_l,u_k\right)F^{\mathcal{O}}_{M-1,N-1}\left(u'_1,\cdots,u'_{M-1}|u_1,\cdots,\hat{u}_k,\cdots,u_N\right) \end{aligned} \tag{9.1.7}$$

where the terms on the second line of (9.1.7) describe disconnected processes that occur if one of the incoming and outgoing particle has the same rapidity. The hat $\hat{u}_k$ denotes that u_k is missing from the list of rapidities. By using the crossing relation all form factors can be expressed in terms of *elementary form factors*

$$F^{\mathcal{O}}_N\left(u_1,...,u_N\right) = \langle 0|\mathcal{O}(0,0)|u_1,...,u_N\rangle. \tag{9.1.8}$$

The elementary form factors are fundamental building blocks of general form factors and correlation functions. The idea of bootstrap is that, instead of computing the elementary form factors directly from first principles, one proposes a set of functional equations called *bootstrap axioms* that they should satisfy. By solving these equations with appropriate analyticity properties from other physical considerations, one obtains the form factors. By construction, the result obtained in this way is non-perturbative since there is no perturbation expansion involved at all. We list the axioms in the following

1. **Exchange**

$$F^{\mathcal{O}}_N(u_1,\cdots,u_k,u_{k+1},\cdots,u_N) = S(u_k-u_{k+1})F^{\mathcal{O}}_N(u_1,\cdots,u_{k+1},u_k,\cdots,u_N) \tag{9.1.9}$$

2. **Cyclic permutation**

$$F^{\mathcal{O}}_N(u_1+2\pi i,u_2,\cdots,u_N) = F^{\mathcal{O}}_N(u_2,\cdots,u_N,u_1) \tag{9.1.10}$$

3. **Kinematical singularity**

$$-i\,\mathrm{Res}_{u=u'}\,F^{\mathcal{O}}_{N+2}(u+i\pi,u',u_1,\cdots,u_N)=\left(1-\prod_{k=1}^{N}S(u-u_k)\right)F^{\mathcal{O}}_{N}(u_1,\cdots,u_N) \tag{9.1.11}$$

4. **Dynamic singularity**

$$-i\,\mathrm{Res}_{u=u'}\,F^{\mathcal{O}}_{N+2}(u+i\bar{u}^i_{jk}/2,u'-i\bar{u}^j_{ik}/2,u_1,\cdots,u_N)=\Gamma^k_{ij}\,F^{\mathcal{O}}_{N+1}(u,u_1,\cdots,u_N) \tag{9.1.12}$$

whenever k is the bound state of the particles i and j, corresponding to the bound state pole of the S-matrix of the form

$$S(u-iu^k_{ij})\sim\frac{i\left(\Gamma^k_{ij}\right)}{u-iu^k_{ij}}. \tag{9.1.13}$$

Here Γ^k_{ij} is the on-shell three-particle coupling and u^k_{ij} is the so-called fusion angle, which satisfy the following relations

$$\begin{aligned} m_k^2 &= m_i^2+m_j^2+2m_im_j\cos u^k_{ij}\,, \\ 2\pi &= u^k_{ij}+u^j_{ik}+u^i_{jk}. \end{aligned} \tag{9.1.14}$$

and we use the notation that $\bar{u}^k_{ij}=\pi-u^k_{ij}$.

The above axioms 1-4 are supplemented by the maximal analyticity assumption which assumes that the form factor as a meromorphic function only have singularities that prescribed by the axioms. Namely, each singularity should has its physical origin. The solution of these axioms with maximal analyticity assumption gives the form factors of the theory non-perturbatively. Of course, these axioms allows many possible solutions, it is commonly believed that the space of solution has the same dimension as the space of local operators. In the form factor boostrap approach, it is non-trivial to relate the solution of form factor axioms to the corresponding operator.

The advantage of form factor approach is clear. Once we know the S-matrix of the theory, we can immediately write down the axioms. The solution of these axioms give the non-perturbative form factor. On the other hand, it is usually a challenging question to solve the form factor axioms. The solution of certain operators can be found in some simple models [ref], but is becomes very hard for more complicated models like the one in AdS/CFT.

9.2 Form factor approach in AdS/CFT

In this section, we discuss the ideas to generalize the form factor bootstrap method to the three-point function problem in the AdS/CFT context. The general idea is to relate the three-point functions to some form factors and then write down a set of axioms similar to the ones discussed in the last section. At present, there are several works in this direction. The first method, which is in direct analogy of the form factor bootstrap method discussed above was proposed by Klose and McLaughlin [74, 171]. They proposed a set of axioms for the worldsheet form factors of the $\mathrm{AdS}_5 \times S^5$ superstring thoery. The worldsheet theory in the light-cone gauge is not Lorentz invariant. Therefore, the form factors depend on individual rapidities of the external states, instead of their differences as is the case in relativistic quantum field theory. Solutions of these axioms give the worldsheet form factors. However, no solution has been found for the worldsheet form factor axioms up to now.

Another direction was pursued recently by Bajnok and Janik [75]. They consider three-point functions in the decompactification limit such that two of the operators are taken to be in the large volume limit where one can neglect the wrapping corrections. In this limit, the three-point function can be regarded as a generalized form factor. In the worldsheet formulation, the third operator is a non-local operator which 'eats up' part of the spacetime. This is different from the usual form factors in 2d integrable QFT where the operators are local. The authors of [75] proposed a set of axioms for the generalized form factors. Due to the non-locality of the operator, the form factor axioms contains additional phase factors. They checked some special cases of three-point functions both at strong and weak coupling in the decompactification limit, and find that they indeed satisfy the axioms. The solution of this set of axioms in general remains a hard problem.

In a related work, Bajnok, Janik and Wereszczyski [59] studied a special kind of HHL three-point function where the two heavy operators are conjugate to each other. They conjectured this kind of three-point function is related to the diagonal form factors in the worldsheet theory. The diagonal form factors are usually easier to study then the generic ones. Based on the work of [76, 77], the authors conjectured the structure of finite volume correction of the three-point functions up to wrapping corrections. They checked this conjecture at the strong coupling for some special case. The conjectured structure is believed to hold at any coupling and should also be valid at weak coupling. The description of the problem, however is different at weak coupling where one describe the operators in terms of spin chain states. We will discuss the finite volume corrections of the diagonal form factor in the next section and confirm their conjecture at tree level

and one loop level in the $\mathfrak{su}(2)$ sector using the spin chain technique introduced in section 4.

9.3 Finite volume corrections of diagonal form factor

In this section, we discuss the finite volume corrections of the diagonal form factor up to wrapping. The diagonal form factor of a local operator $\mathcal{O}$, defined as

$$^{\text{out}}\langle u_1, \cdots, u_N|\mathcal{O}(0,0)|u_N, \cdots, u_1\rangle^{\text{in}}, \tag{9.3.15}$$

is singular due to the disconnected terms in the crossing relation (9.1.7). To avoid the singularities we regularize it by slightly shifting the outgoing rapidities. After crossing we get

$$F_{2N}^{\mathcal{O}}\left(\bar{u}_1+\epsilon_1, \cdots, \bar{u}_N+\epsilon_N, u_N, \cdots, u_1\right) = \ ^{\text{out}}\langle 0|\mathcal{O}|\bar{u}_1+\epsilon_1, \cdots, \bar{u}_N+\epsilon_N, u_N, \cdots, u_1\rangle^{\text{in}} \tag{9.3.16}$$

The diagonal limit, $\epsilon_i \to 0$, of (9.3.16) is not well-defined. It was first noticed in [172] that the singular parts vanish in the limit when all $\epsilon_i \to 0$, but the result depends on the direction of the limit. Its general structure can be written as

$$\begin{aligned} &F_{2N}^{\mathcal{O}}\left(\bar{u}_1+\epsilon_1, \cdots, \bar{u}_N+\epsilon_N, u_N, \cdots, u_1\right) \\ &= \prod_{i=1}^{N}\frac{1}{\epsilon_i}\cdot\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_N=1}^{N} a_{i_1 i_2 \cdots i_N}(u_1, \cdots, u_N)\,\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_N} + \cdots \end{aligned} \tag{9.3.17}$$

where $a_{i_1 i_2 \ldots i_N}$ is a completely symmetric tensor of rank N. The ellipsis denote terms which vanish in the $\epsilon_i \to 0$ limit.

There are two generally used regularization scheme in the literature. The first is the so-called *symmetric* scheme, in which all ϵ_i are set to be the same

$$F_{2N}^{\mathcal{O},s}\left(u_1, \cdots, u_N\right) = \lim_{\epsilon\to 0} F_{2N}^{\mathcal{O}}\left(\bar{u}_1+\epsilon, \cdots, \bar{u}_N+\epsilon, u_N, \cdots, u_1\right). \tag{9.3.18}$$

The second scheme is called *connected*, in which the diagonal form factors are defined as the finite part of (9.3.17), i.e. the ϵ-independent term,

$$F_{2N}^{\mathcal{O},c}\left(u_1, \cdots, u_N\right) = N!\, a_{12\cdots N}\ . \tag{9.3.19}$$

Both the symmetric and the connected diagonal form factors are completely symmetric in the rapidity variables $u_1, \cdots, u_N$. Of course these two quantities are not independent and each can be expressed in terms of the other [77].

9.3.1 Diagonal form factors in finite volume

In this section we summarize the results about the volume dependence of the diagonal form factors in all polynomial orders in the inverse of the volume, neglecting the exponentially small wrapping corrections, following [76, 77].

In finite volume L, the rapidities are quantized and a generic multiparticle state can be labeled by the Bethe quantum numbers $|\{I_1, \cdots, I_N\}\rangle_L$. In finite volume we cannot order the particles by spatial separation in the remote past or future, as we did in the infinite volume case (9.1.2). In finite volume the states are completely symmetric under the exchange of particles and can be characterized by the *set* of quantum numbers. We adapt our notation to the conventions used in [76, 77] and order the quantum numbers in a monotonly decreasing sequence, $I_1 \geq \cdots \geq I_N$ [1]. The quantized rapidities with the quantum numbers $\{I_1, \cdots, I_N\}$ are solutions of the corresponding Bethe Ansatz equations. We define

$$\Phi_j(\{u_1, \cdots, u_N\}) = p(u_j)L - i \sum_{\substack{k=1 \\ k \neq j}} \log S(u_j, u_k)\,, \tag{9.3.20}$$

and the quantization conditions are

$$\Phi_j(\{u_1, \cdots, u_N\}) = 2\pi I_j, \qquad j = 1, \cdots, N. \tag{9.3.21}$$

These finite volume states are orthogonal to each other

$${}_L\langle\{J_1, \cdots, , J_M\}|\{I_1, \cdots, I_N\}\rangle_L \propto \delta_{NM}\, \delta_{I_1,J_1} \cdots \delta_{I_N,J_N} \tag{9.3.22}$$

and their normalization is a question of convention.

One can change from the quantum number representation of states to the rapidity representation which gives the direct connection between the finite and infinite volume states [76]. This change of variables involves the Jacobian, which is the density of N-particle

1. Apart from the free boson case all known S-matrix obey the property $S(u,u) = -1$ which is an effective Pauli exclusion. In this cases we have $I_1 > \cdots > I_N$.

states, defined as

$$\varrho_N(u_1,\cdots,u_N)_L = \det \mathcal{J}^{(N)}(u_1,\cdots,u_N)_L \qquad (9.3.23)$$
$$\mathcal{J}^{(N)}_{k,l}(u_1,\cdots,u_N)_L = \frac{\partial \Phi_k(u_1,\cdots,u_N)}{\partial u_l} \quad , \quad k,l=1,\cdots,N\,.$$

where we explicitly indicated the volume dependence of these quantities. The determinant (9.3.23) is closely related to the Gaudin norm of Bethe states. [2] Then the relation between the infinite and finite volume states reads

$$|\{I_1,\cdots,I_N\}\rangle_L = \frac{1}{\sqrt{\varrho_N(u_1,\cdots,u_N)_L \prod_{i<j} S(u_i,u_j)}}|u_1,\cdots,u_N\rangle \qquad (9.3.24)$$

where the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (9.3.20) corresponding to the quantum numbers $\{I_1,\cdots,I_N\}$. This identification holds up to wrapping corrections. The product of S-matrices in the denominator ensures that the finite volume state is indeed symmetric under the exchange of particles.

Defining the system in finite volume regularizes all the divergences appearing in the diagonal limit of form factors (9.3.17), thus the normalized finite volume diagonal matrix element

$$\frac{{}_L\langle\{I_1,\cdots,I_N\}|\mathcal{O}(0,0)|\{I_1,\cdots,I_N\}\rangle_L}{{}_L\langle\{I_1,\cdots,I_N\}|\{I_1,\cdots,I_N\}\rangle_L} \qquad (9.3.25)$$

is finite, completely well defined and does not depend on the normalization of states. However, it is a challenging task to relate them to the infinite volume ones in the general case [173, 174]. The problem become considerably simpler if we neglect the exponentially small wrapping corrections.

Up to wrapping, the finite volume N-particle diagonal form factor (9.3.25) can be expressed as a sum over the bipartite partitions of the full set $\{1,2,\cdots,N\}$, involving the infinite volume form factors and some kind of densities of states. As the diagonal form factors in infinite volume depend on the regularization scheme, this series is also scheme dependent. In case of the connected evaluation the relation reads as [77, 175]

$$\frac{{}_L\langle\{I_1,\cdots,I_N\}|\mathcal{O}(0,0)|\{I_1,\cdots,I_N\}\rangle_L}{{}_L\langle\{I_1,\cdots,I_N\}|\{I_1,\cdots,I_N\}\rangle_L} = \frac{1}{\rho_N(\{1,\cdots,N\})}\sum_{\alpha\subseteq\{1,\ldots,N\}} f^{\mathcal{O}}\left(\{u_k\}_{k\in\bar{\alpha}}\right)\rho_N\left(\alpha\right) \qquad (9.3.26)$$

2. The Gaudin norm itself is not physical as it depends on the conventions. However, in any convention, it is proportional to the Jacobian (9.3.23).

where $\bar{\alpha}$ denotes the complement of α in the full set. The functions appearing on the right hand side are exactly the connected diagonal form factors

$$f^{\mathcal{O}}(u_1,\cdots,u_l) = F_{2l}^{\mathcal{O},c}(u_1,\cdots,u_l) \tag{9.3.27}$$

The functions ρ_N are defined as the diagonal minor determinants of the N-particle Jacobian (9.3.23),

$$\rho_N(\alpha) = \det_{k,l\in\alpha} \mathcal{J}_{k,l}^{(N)}(u_1,\cdots,u_N)_L \quad , \qquad \alpha \subseteq \{1,\cdots,N\}. \tag{9.3.28}$$

They can also be referred to as partial Gaudin norms. As special cases we have

$$\rho_N(\{1,\cdots,N\}) = \varrho_N(u_1,\cdots,u_N)_L \quad ; \qquad \rho_N(\emptyset) = 1. \tag{9.3.29}$$

We want to emphasize that the function $\rho_N(\alpha)$ depend on *all* the N rapidities. The set of rapidities $\{u_i\}$ in the right hand side of (9.3.26) is the solution of the Bethe Ansatz equations (9.3.20) corresponding to the quantum numbers $\{I_1,\cdots,I_N\}$. Thus, the *explicit* volume dependence is encoded *only* into the factors ρ_N, the connected form factors $f^{\mathcal{O}}$ depend on the volume only *implicitly* via the Bethe Ansatz equations.

As the connected and symmetric diagonal form factors are not independent, we can express the finite volume matrix element in the symmetric regularization scheme. In this case the series take the form [76]

$$\frac{{}_L\langle\{I_1,\cdots,I_N\}|\mathcal{O}(0,0)|\{I_1,\cdots,I_N\}\rangle_L}{{}_L\langle\{I_1,\cdots,I_N\}|\{I_1,\cdots,I_N\}\rangle_L} = \frac{1}{\rho_N(\{1,\cdots,N\})} \sum_{\alpha\subseteq\{1,\ldots,n\}} F_{2|\bar{\alpha}|}^{s}\left(\{u_k\}_{k\in\bar{\alpha}}\right)\rho_{|\alpha|}\left(\alpha\right). \tag{9.3.30}$$

Here again, the rapidities $\{u_i\}$ are the solutions of the Bethe Ansatz equations (9.3.20) with the quantum numbers $\{I_1,\cdots,I_N\}$. The $\rho_{|\alpha|}$ functions in the sum are the $|\alpha|$-particle densities of state (9.3.29,9.3.23) evaluated at the rapidities $\{u_i\}_{i\in\alpha}$. Note that, contrary to the connected expansion, they depend *only* on the rapidities labeled by the set α. The explicit volume dependence is carried only by the ρ functions.

9.3.1.1 Form factor of densities of conserved charges

An important special case of local operators is the density of a conserved quantity,

$$Q = \int_0^L J(x,t)dx \tag{9.3.31}$$

where Q acts diagonally and additively on the multiparticle states. Its density therefore satisfies

$$\frac{{}_L\langle\{I_1,\cdots,I_N\}|J(0,0)|\{I_1,\cdots,I_N\}\rangle_L}{{}_L\langle\{I_1,\cdots,I_N\}|\{I_1,\cdots,I_N\}\rangle_L}=\frac{1}{L}\sum_{j=1}^{N}q(u_j), \tag{9.3.32}$$

where $\{u_i\}$ are the solutions of the Bethe Ansatz equations (9.3.20) corresponding to the quantum numbers $\{I_1,\cdots,I_N\}$, and $q(u)$ is the one-particle eigenvalue of the operator Q.

A compact expression for the connected diagonal form factors of these densities was presented in [173, 175], however the mathematically rigorous proof was found recently [176]. The connected form factors can be cast into the form

$$F_{2N}^{J,c}(u_1\cdots,u_N)=\sum_{\sigma\in S_N}\varepsilon(u_{\sigma(1)})\varphi(u_{\sigma(1)},u_{\sigma(2)})\varphi(u_{\sigma(2)},u_{\sigma(3)})\cdots\varphi(u_{\sigma(N-1)},u_{\sigma(N)})q(u_{\sigma(N)}) \tag{9.3.33}$$

where the summation runs over all the permutation of the set $\{1,\cdots,N\}$.

9.3.2 Conjecture for the symmetric structure constants

In [59], Bajnok, Janik and Wereszczynski considered the three-point function of operators $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ with $\mathcal{O}_3=\overline{\mathcal{O}}_1$ and $\mathcal{O}_2$ is a light operator with zero R-charge. They conjectured that in this case the large operators $\mathcal{O}_1$ and $\overline{\mathcal{O}}_1$ can be regarded as external states and $\mathcal{O}_2$ as an local operator so that the structure constant is given by the following diagonal matrix element

$$C_{\text{HHL}}={}_L\langle u_1,\cdots,u_N|\mathcal{O}|u_1,\cdots,u_N\rangle_L. \tag{9.3.34}$$

The finite volume dependence of the structure constant (9.3.34) is given by

$$C_{\text{HHL}}=\frac{1}{\rho_N(\{1,\cdots,N\})}\sum_{\alpha\subseteq\{1,\cdots,N\}}F^s_{2|\bar{\alpha}|}(\{u_k\}_{k\in\bar{\alpha}})\,\rho_{|\alpha|}(\alpha) \tag{9.3.35}$$

In fact, this conjecture contains two parts. The first part is that the finite volume corrections are encoded in the Jacobians $\rho_{|\alpha|}(\alpha)$; the second part is that the coefficients in front of the Jacobian $F^s_{2|\bar{\alpha}|}(\{u_k\}_{k\in\bar{\alpha}})$ are the form factors of the same operator in the infinite volume. At weak coupling, for the first few orders before the wrapping corrections have to be taken into account, the formula (9.3.35) gives the *exact* result, as will be shown below.

9.4 Diagonal matrix element and three-point functions

In this section, we study the symmetric HHL three-point functions at weak coupling with the heavy operators in the $\mathfrak{su}(2)$ sector. We prove the first part of the conjecture at tree level and one loop and give predictions for the infinite volume matrix elements for certain operators in the XXX spin chain.

9.4.1 Set-up

We will consider the three-point functions that are related to matrix elements. We take two "heavy" states, conjugate to each other, $\mathcal{O}_1$ and $\mathcal{O}_{\bar{1}}$. In the $\mathfrak{su}(2)$ sector. The two operators are constructed from the following scalar fields

$$\mathcal{O}_1 : \{Z, X\}, \qquad \mathcal{O}_{\bar{1}} : \{\bar{Z}, \bar{X}\}. \tag{9.4.36}$$

At tree-level, we consider operators with definite one-loop anomalous dimension [49]. Since we are considering diagonal matrix elements, the wave functions of the two operators are conjugate to each other. The third operator, denoted hereafter by $\mathcal{O}_\alpha$, are made of complex scalar fields Z, X and their Hermitian conjugates, such as

$$\mathrm{Tr}\, Z\bar{Z}, \quad \mathrm{Tr}\, X\bar{X}, \quad \mathrm{Tr}\, XZ\bar{Z}\bar{X}, \quad \cdots \tag{9.4.37}$$

This kind of operators are in the $\mathfrak{so}(4)$ sector of $\mathcal{N} = 4$ SYM theory. In order to have non-zero contribution, the R-charge of the operator should be zero. In addition, we require $\mathcal{O}_\alpha$ have definite anomalous dimension. The three-point function is fixed by conformal symmetry up to the structure constant C_α

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_\alpha(x_2)\mathcal{O}_{\bar{1}}(x_3)\rangle = \frac{L^2 L_\alpha}{N_c} \frac{\mathcal{N}_1\sqrt{\mathcal{N}_\alpha} C_\alpha}{|x_{12}|^{\Delta_{12}}|x_{13}|^{\Delta_{13}}|x_{23}|^{\Delta_{23}}} \tag{9.4.38}$$

where

$$x_{ij}^\mu = x_i^\mu - x_j^\mu, \qquad \Delta_{ij} = \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k), \tag{9.4.39}$$

and L is the length of $\mathcal{O}_1$ while L_α is the length of the operator $\mathcal{O}_\alpha$. The two-point functions are normalized as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_{\bar{1}}(x_2)\rangle = \frac{L\mathcal{N}_1}{|x_{12}|^{2\Delta_1}}, \quad \langle \mathcal{O}_\alpha(x_1)\overline{\mathcal{O}}_\alpha(x_2)\rangle = \frac{L_\alpha \mathcal{N}_\alpha}{|x_{12}|^{2\Delta_\alpha}} \tag{9.4.40}$$

The structure constant C_α can be expressed in terms of matrix elements of Heisenberg spin chain

$$C_\alpha = \frac{\langle \mathbf{u}|\hat{O}_\alpha|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle} \tag{9.4.41}$$

where $|\mathbf{u}\rangle \equiv |\{u_1, \cdots, u_N\}\rangle$ denotes an on-shell Bethe state and $\hat{O}_\alpha = \hat{O}_\alpha(\sigma_i^\pm, \sigma_i^z)$ is an operator made of local spin operators. In this way, the computation of three-point function in planar $\mathcal{N} = 4$ SYM theory is recast into the calculation of matrix elements of Heisenberg spin chain.

9.4.2 From field theory correlation functions to spin chain matrix elements

In this section, we show how to map the field theoretic operators $\mathcal{O}_\alpha$ to spin chain operators $\hat{O}_\alpha$. Let us introduce the following notations

$$Z \equiv \phi^0, \quad X \equiv \phi^1, \quad \bar{Z} \equiv \bar{\phi}^0, \quad \bar{X} \equiv \bar{\phi}^1. \tag{9.4.42}$$

The light operator is the linear combination of the single trace operators $\mathrm{Tr}\, \phi^{i_1}\bar{\phi}^{i_2}\bar{\phi}^{i_3}\cdots$. By planarity, only operators of the following form will contribute to the three-point function

$$\mathrm{Tr}\, \phi_1^{i_1} \cdots \phi_l^{i_l} \bar{\phi}_{l+1}^{j_l} \cdots \bar{\phi}_{2l}^{j_1}, \quad i_k, j_k = 0, 1. \tag{9.4.43}$$

where the indices $1, \cdots, 2l$ denotes the position on the third spin chain and $2l = L_\alpha$ is the length of the third operator. The zero R-charge condition is given by

$$\sum_{n=1}^{l} (i_n - j_n) = 0. \tag{9.4.44}$$

It is not hard to see that the operator (9.4.43) can be mapped to the following spin operator [48]

$$\mathrm{Tr}\, \phi_1^{i_1} \cdots \phi_l^{i_l} \bar{\phi}_{l+1}^{j_l} \cdots \bar{\phi}_{2l}^{j_1} \longrightarrow \mathrm{E}_{n+l-1}^{i_1+1, j_1+1} \cdots \mathrm{E}_n^{i_l+1, j_l+1}. \tag{9.4.45}$$

where the indices $n, \cdots, n-l+1$ denote the positions on the long spin chain and the operators E_n^{ab} are the basis 2×2 matrices $(\mathrm{E}_n^{ab})_{ij} = \delta_{ai}\delta_{bj}$ in the local quantum space $\mathcal{H}_n = \mathbb{C}^2$. The operators E_n^{ab} are related to the local spin operators as follows

$$\mathrm{E}_n^{11} \equiv \frac{1}{2}(\mathbb{I} + \sigma_n^z), \quad \mathrm{E}_n^{12} \equiv \sigma_n^+, \quad \mathrm{E}_n^{21} \equiv \sigma_n^-, \quad \mathrm{E}_n^{22} \equiv \frac{1}{2}(\mathbb{I} - \sigma_n^z). \tag{9.4.46}$$

Here $\sigma_n^{\pm}$, σ_n^z are the usual Pauli matrices. By the mapping (9.4.45), we can translate the field theory operators into the spin operators. As an example, we consider the Konishi operator

$$\mathcal{O}_K = \mathrm{Tr} X\bar{X} + \mathrm{Tr} Y\bar{Y} + \mathrm{Tr} Z\bar{Z}. \tag{9.4.47}$$

Since the heavy operators are in the $\mathfrak{su}(2)$ sector, the contraction with Y and $\bar{Y}$ are zero and can be neglected. The Konishi operator can be mapped to the following spin operator

$$\mathcal{O}_K \longrightarrow \hat{O}_K = \mathrm{E}_n^{11} + \mathrm{E}_n^{22} = \mathbb{I}. \tag{9.4.48}$$

Therefore we see that at tree level the structure constants, with the light operator being the Konishi operator, is trivial

$$C_K = \frac{\langle \mathbf{u}|\hat{O}_K|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle} = 1. \tag{9.4.49}$$

In order to obtain non-trivial structure constant, the light operator must have at least $L_\alpha = 4$. This corresponds to the insertion of two spin operators between the Bethe states. An example for length-4 operator is given in Fig.(9.4.1).

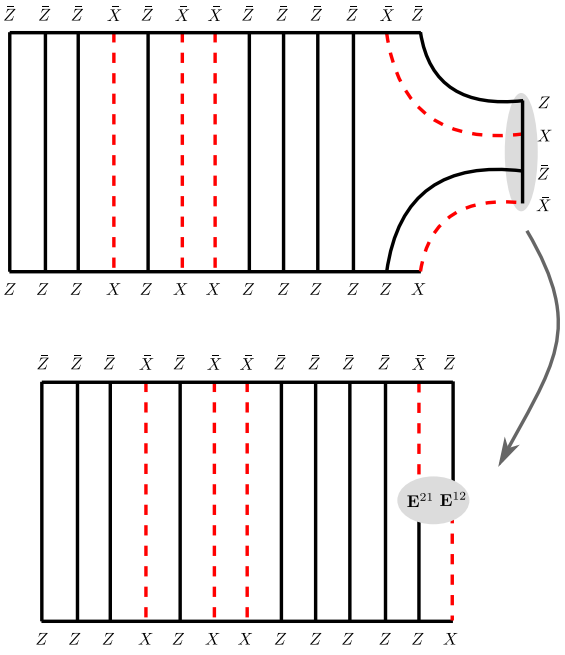

FIGURE 9.4.1: An example of the mapping between field operators to spin chain operators. The operator in the field theory $\mathcal{O} = \mathrm{Tr}\, ZX\bar{Z}\bar{X}$ is mapped to the spin chain operator $\hat{O} = \mathrm{E}_n^{21}\mathrm{E}_{n+1}^{12}$.

For simplicity, we focus on the length-4 field theory operators, which corresponds to operators in the spin chain that act on two neighboring sites. These are the simplest

non-trivial cases which can be studied thoroughly.

In order to have non-zero diagonal matrix element in (9.4.41), the spin chain operator should not alter the total spin S^z of the state it acts on. At $l_s = 1$, there are two independent operators, namely the identity and

$$o_1(n) = \mathrm{E}_n^{11}. \tag{9.4.50}$$

For $l_s = 2$, there are 6 independent operators: the identity, $o_1(n)$, $o_1(n+1)$ and the other three operators

$$o_2^1(n) = \mathrm{E}_n^{11}\mathrm{E}_{n+1}^{11}, \quad o_2^2(n) = \mathrm{E}_n^{12}\mathrm{E}_{n+1}^{21}, \quad o_2^3(n) = \mathrm{E}_n^{21}\mathrm{E}_{n+1}^{12}. \tag{9.4.51}$$

For later convenience, we also introduce the following operator

$$o_2^4(n) = \mathrm{E}_n^{22}\mathrm{E}_{n+1}^{22} = (\mathbb{I} - \mathrm{E}_n^{11})(\mathbb{I} - \mathrm{E}_{n+1}^{11}) = \mathbb{I} - o_1(n) - o_1(n+1) + o_2^1(n). \tag{9.4.52}$$

In what follows, we will study in detail the diagonal form factors of the spin operators o_1 and o_2^i $(i = 1, 2, 3, 4)$.

9.4.3 The Algebraic Bethe Ansatz approach

In this section, we review the algebraic Bethe Ansatz (ABA) approach to the correlation functions in the $\mathrm{XXX}_{1/2}$ spin chain, see [96, 177, 178] and references therein. This approach is based on two main elements: the Slavnov determinant formula which was introduced in Chapter 3 and the solution of the Quantum Inverse Scattering Problem (QISP), which will be introduced below. The Slavnov formula states that the scalar product of any Bethe state with an on-shell one can be written in terms of a determinant. The solution of the QISP enable us to express the local spin operators in terms of the elements of the monodromy matrix, which are non-local operators acting on the spin chain.

9.4.3.1 The solution of quantum inverse scattering problem

In the ABA approach, the idea to compute diagonal matrix element $\langle \mathbf{u}|\hat{O}_n|\mathbf{u}\rangle$ is to act the spin chain operator $\hat{O}_n$ on the bra state, so that the ket Bethe state $|\mathbf{u}\rangle$ is left on-shell. The operator $\hat{O}_n$ is typically a multilocal operator in the sense that it acts on a finite interval of the spin chain. In order to apply the Slavnov formula, the bra state also needs to be a Bethe state, although we do not require it to be on-shell. The solution of the quantum inverse scattering problem (QISP) relates the local spin operators to the

matrix elements of the monodromy matrix. In addition, from the RTT relation, it is clear that the action of A, B, C, D operators on a Bethe state always give a sum over Bethe states. Therefore, using the solution of QISP we can translate any local operator $\hat{O}_n$ into a sequence of ABA operators, and the state $\langle \mathbf{u}|\hat{O}_n$ can be written as a sum of dual Bethe states. This enable us to apply the Slavnov formula discussed in the last subsection. We present the main statement of the solution of QISP in this subsection and refer to [177] for the proof and details.

Let E_n^{ab} $(a,b=1,2)$ be the operators defined in (9.4.46) which act on the local quantum space $\mathcal{H}_n=\mathbb{C}^2$. They can be represented by the elements of monodromy matrix as

$$\mathrm{E}_n^{ab}=\left\{\prod_{k=1}^{n-1}\mathcal{T}(\theta_k+i/2)\right\}T_{ab}(\theta_n+i/2)\left\{\prod_{k=1}^{n}\mathcal{T}(\theta_k+i/2)\right\}^{-1} \tag{9.4.53}$$

where

$$T_{11}(u)=A(u),\quad T_{12}(u)=B(u),\quad T_{21}(u)=C(u),\quad T_{22}(u)=D(u). \tag{9.4.54}$$

Once inserted inside a correlator, the transfer matrices in (9.4.53) act on a Bethe state and can be replaced by their eigenvalues

$$\langle \mathbf{u}|\mathrm{E}_n^{ab}|\mathbf{u}\rangle=\frac{\langle \mathbf{u}|T_{ab}(\theta_n+i/2)|\mathbf{u}\rangle}{t_{\mathbf{u}}(\theta_n+i/2)} \tag{9.4.55}$$

where $t_{\mathbf{u}}(u)$ is the eigenvalue of the transfer matrix. At $u=\theta_n+i/2$, we have

$$t_{\mathbf{u}}(\theta_n+i/2)=Q_{\boldsymbol{\theta}}(\theta_n+i)\frac{Q^-_{\mathbf{u}}(\theta_n)}{Q^+_{\mathbf{u}}(\theta_n)}. \tag{9.4.56}$$

The generalization of (9.4.55) to a string of $l+1$ operators is straightforward

$$\langle \mathbf{u}|\mathrm{E}_n^{a_0b_0}\cdots\mathrm{E}_{n+l}^{a_lb_l}|\mathbf{u}\rangle=\left(\prod_{k=1}^{l}\frac{Q^+_{\mathbf{u}}(\theta_{n+k})}{Q_{\boldsymbol{\theta}}(\theta_{n+k}+i)Q^-_{\mathbf{u}}(\theta_{n+k})}\right)\langle \mathbf{u}|\prod_{k=0}^{l}T^{a_kb_k}_{n+k}(\theta_{n+k}+i/2)|\mathbf{u}\rangle \tag{9.4.57}$$

We can now compute the r.h.s. of (9.4.57) by using the RTT relation and the Slavnov formula.

9.4.4 Matrix elements of spin operators

In this section we study the diagonal matrix elements of spin operators of the Heisenberg spin chain using the ABA and the solution of QISP discussed in section 9.4.3. We show that, in general, the matrix elements can be written as linear combinations of a special

kind of scalar product. In section 9.4.5, we show that these scalar products has the structure conjectured in [59], namely it can be written as linear combinations of diagonal minors of Gaudin norms (9.3.26). We call the procedure of expanding quantities in terms of diagonal minors of Gaudin norms the *finite volume expansion*. After this expansion, all the finite volume dependence of the quantity are captured. We will discuss the case $l_s = 1, 2$ in detail and comment on the general $l_s > 2$ case.

9.4.4.1 Form factors of length-2 operators

We have shown in section 9.4.2 that all the length-2 diagonal matrix elements can be written as linear combinations of the following building blocks

$$\mathcal{F}^{o_1} = \langle \mathbf{u}|o_1(n)|\mathbf{u}\rangle, \qquad \mathcal{F}^{o_2^i} = \langle \mathbf{u}|o_2^i(n)|\mathbf{u}\rangle, \quad i = 1, ..., 3 \tag{9.4.58}$$

where the local operators $o_1(n)$ and $o_2^i(n)$ are given in (9.4.50) and (9.4.51). According to (9.4.57), these matrix elements are proportional to the following quantities

$$\mathcal{F}^{o_1} \propto F^A \equiv \langle \mathbf{u}|A(\theta_n + i/2)|\mathbf{u}\rangle. \tag{9.4.59}$$

$$\begin{aligned}
\mathcal{F}^{o_2^1} &\propto F^{AA} \equiv \langle \mathbf{u}|A(\theta_n + i/2)A(\theta_{n+1} + i/2)|\mathbf{u}\rangle, \\
\mathcal{F}^{o_2^2} &\propto F^{BC} \equiv \langle \mathbf{u}|B(\theta_n + i/2)C(\theta_{n+1} + i/2)|\mathbf{u}\rangle, \\
\mathcal{F}^{o_3^2} &\propto F^{CB} \equiv \langle \mathbf{u}|C(\theta_n + i/2)B(\theta_{n+1} + i/2)|\mathbf{u}\rangle, \\
\mathcal{F}^{o_2^4} &\propto F^{DD} \equiv \langle \mathbf{u}|D(\theta_n + i/2)D(\theta_{n+1} + i/2)|\mathbf{u}\rangle.
\end{aligned} \tag{9.4.60}$$

In order to compute the building blocks (9.4.59) and (9.4.59), we act all the operators on the ket state $|\mathbf{u}\rangle$. The action of A and D on a Bethe state is

$$\begin{aligned}
A(v)|\mathbf{u}\rangle &= a(v)\frac{Q_{\mathbf{u}}(v-i)}{Q_{\mathbf{u}}(v)}|\mathbf{u}\rangle + \sum_{n=1}^{N} \mathrm{M}_n(v)\,|\{\mathbf{u}, v\} \setminus \{u_n\}\rangle, \\
D(v)|\mathbf{u}\rangle &= d(v)\frac{Q_{\mathbf{u}}(v+i)}{Q_{\mathbf{u}}(v)}|\mathbf{u}\rangle + \sum_{n=1}^{N} \mathrm{N}_n(v)\,|\{\mathbf{u}, v\} \setminus \{u_n\}\rangle
\end{aligned} \tag{9.4.61}$$

where $\mathrm{M}_k(v)$, $\mathrm{N}_k(v)$ are given by

$$\begin{aligned}
\mathrm{M}_k(v) &= \frac{ia(u_n)}{v - u_n}\prod_{j\neq n}^{N}\frac{u_n - u_j - i}{u_n - u_j}, \\
\mathrm{N}_k(v) &= \frac{id(u_n)}{u_n - v}\prod_{j\neq n}^{N}\frac{u_n - u_j + i}{u_n - u_j}.
\end{aligned} \tag{9.4.62}$$

These relations can be derived from the Yangian algebra. From (9.4.61) we see that the action of the operators A and D on a Bethe state preserve the number of magnons. In addition to the original Bethe state $|\mathbf{u}\rangle$, there is a sum of Bethe states $|\{\mathbf{u}, v\} \setminus \{u_k\}\rangle$ where one of the rapidities u_k is replaced by the spectral parameter v of the operator. These are called the *unwanted terms* and are off-shell for generic v. On the other hand, they are not too far from the on-shell Bethe state $|\mathbf{u}\rangle$ since most of the rapidities remain unchanged.

The action of the C operator on the Bethe state is more involved

$$C(v)|\mathbf{u}\rangle = \sum_{n=1}^{N} \mathrm{K}_n \, |\{\mathbf{u}\} \setminus \{u_n\}\rangle + \sum_{k>n} \mathrm{K}_{kn} \, |\{\mathbf{u}, v\} \setminus \{u_k, u_n\}\rangle \tag{9.4.63}$$

where

$$\begin{aligned} \mathrm{K}_n = & \frac{ia(v)d(u_n)}{u_n - v} \prod_{j\neq n}^{N} \frac{u_j - u_n - i}{u_j - u_n} \cdot \frac{u_j - v + i}{u_j - v} + \\ & \frac{ia(u_n)d(v)}{v - u_n} \prod_{j\neq n}^{N} \frac{u_j - u_n + i}{u_j - u_n} \cdot \frac{u_j - v - i}{u_j - v} \\ \mathrm{K}_{kn} = & \frac{d(u_k)a(u_n)}{(u_k - v)(u_n - v)} \frac{u_k - u_n + i}{u_k - u_n} \prod_{j\neq k,n} \frac{u_j - u_k - i}{u_j - u_k} \cdot \frac{u_j - u_n + i}{u_j - u_n} + \\ & \frac{d(u_n)a(u_k)}{(u_n - v)(u_k - v)} \frac{u_k - u_n - i}{u_k - u_n} \prod_{j\neq k,n} \frac{u_j - u_k + i}{u_j - u_k} \cdot \frac{u_j - u_n - i}{u_j - u_n} \end{aligned} \tag{9.4.64}$$

The coefficients K_n and K_{kn} can be expressed in terms of M_n and N_n

$$\begin{aligned} \mathrm{K}_n(v) = & \mathrm{M}_n(v)\mathrm{N}_0(v) \frac{v - u_n}{v - u_n + i} + \mathrm{M}_0(v)\mathrm{N}_n(v) \frac{v - u_n}{v - u_n - i}, \\ \mathrm{K}_{nk}(v) = & \mathrm{M}_k(v)\mathrm{N}_n(v) \frac{u_n - u_k}{u_n - u_k + i} + \mathrm{M}_n(v)\mathrm{N}_k(v) \frac{u_n - u_k}{u_n - u_k - i} \end{aligned} \tag{9.4.65}$$

if we define

$$\mathrm{M}_0 \equiv a(v) \frac{Q_{\mathbf{u}}(v - i)}{Q_{\mathbf{u}}(v)}, \qquad \mathrm{N}_0 \equiv d(v) \frac{Q_{\mathbf{u}}(v + i)}{Q_{\mathbf{u}}(v)}. \tag{9.4.66}$$

From (9.4.63) it is clear that C reduces the number of magnons by one. For diagonal matrix elements, any C operator has to be accompanied by a B operator in order to preserve S^z, and obtain non-vanishing results. For length-2 operators the only possibilities are $B(u)C(v)$ and $C(u)B(v)$. Both combinations preserve the number of magnons but will lead to a sum of unwanted terms with one or two magnons replaced by the spectral parameters of the operators.

In general, the action of l_s ABA operators on a Bethe state generates unwanted terms

with at most l_s rapidities replaced by the spectral parameters of the operators. In particular, it is clear now that all the building blocks (9.4.59) and (9.4.60) can be written as the scalar products of the following three types

$$\langle \mathbf{u}|\mathbf{u}\rangle, \quad \langle \mathbf{u}|\{\mathbf{u},\theta_n^+\}\setminus\{u_k\}\rangle, \quad \langle \mathbf{u}|\{\mathbf{u},\theta_n^+,\theta_{n+1}^+\}\setminus\{u_j,u_k\}\rangle. \tag{9.4.67}$$

where we have used the notation $\theta_n^+ = \theta_n + i/2$. The first scalar product is the Gaudin norm. The rest two types of scalar products are the special kind of scalar products alluded before. We will study their finite volume dependence in section 9.4.5.

9.4.4.2 Form factors of length l_s operators

The discussion of the previous subsection can be generalized to the diagonal matrix elements of the operators with $l_s > 2$. As before, any such matrix element can be spanned by some building blocks such as $F^{A\cdots A}$, $F^{BCA\cdots A}$, $F^{D\cdots D}$. Of course, the number of the building blocks grows with the length of the operator.

We act all the ABA operators on the ket state which give rise to the unwanted terms

$$|\{\mathbf{u},\theta_n^+,\cdots,\theta_{n+M}^+\}\setminus\{u_{k_1},\cdots,u_{k_M}\}\rangle, \qquad M\le l_s. \tag{9.4.68}$$

The diagonal matrix element of any length-l_s operator can be written as a linear combination of the following scalar products

$$\langle \mathbf{u}|\{\mathbf{u},\theta_n^+,\cdots,\theta_{n+M}^+\}\setminus\{u_{k_1},\cdots,u_{k_M}\}\rangle, \quad M\le l_s. \tag{9.4.69}$$

The number of terms and the complexity of the coefficients will grow quickly with the increase of number of magnons and length of the operators, nevertheless the structure is robust.

9.4.5 Finite volume expansion of the modified Gaudin norm

In this section, we analyze the structure of the scalar products in (9.4.69) and show that any of them can be expanded in terms of diagonal minors of Gaudin norms. We call this procedure the *finite volume expansion*.

The scalar products under consideration are of on-shell/off-shell type and can be computed by the Slavnov determinant formula. In the Slavnov determinant (3.3.69), the denominator is a simple Cauchy determinant and can be computed readily. We therefore focus on the non-trivial numerator $\det_{jk}\Omega(u_j,v_k)$. Let us first consider the scalar

product for the length-2 operators, $\langle \mathbf{u}|\{\mathbf{u},\theta_n,\theta_{n+1}\}\setminus\{u_j,u_k\}\rangle$. The determinant takes the following form

$$\det\Omega=\begin{vmatrix} i\,\phi_{11} & \cdots & \Omega_{1j} & \cdots & \Omega_{1k} & \cdots & i\,\phi_{1N} \\ i\,\phi_{21} & \cdots & \Omega_{2j} & \cdots & \Omega_{2k} & \cdots & i\,\phi_{2N} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ i\,\phi_{N1} & \cdots & \Omega_{Nj} & \cdots & \Omega_{Nk} & \cdots & i\,\phi_{NN} \end{vmatrix}. \tag{9.4.70}$$

where we have defined $i\,\phi_{jk}=\Omega(u_j,u_k)$ and $\Omega_{ik}=\Omega_{ik}(u_i,\theta_n+i/2)$. The procedure is straightforward: *perform Laplace expansion with respect to the column or row that does not have any element of the form ϕ_{nn} repeatedly, until one can not do it further*. Note that after one Laplace expansion, we will obtain sub-determinants. We shall also perform the same procedure for all the sub-determinants until it terminates. This procedure will terminate when all the determinants in the expression take the form of diagonal minors (9.3.28) of Gaudin norm (9.3.23)

$$\rho_N(\{i_1,\cdots,i_m\})=(-1)^m\begin{vmatrix} \phi_{i_1i_1} & \cdots & \cdots & \cdots \\ \cdots & \phi_{i_2i_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \phi_{i_mi_m} \end{vmatrix} \tag{9.4.71}$$

Therefore, the scalar product allows the following expansion

$$\langle \mathbf{u}|\{\mathbf{u},\theta_n,\theta_{n+1}\}\setminus\{u_j,u_k\}\rangle=\sum_{\alpha\subseteq A}\mathcal{F}(\bar{\alpha})\,\rho_N(\alpha), \tag{9.4.72}$$

where $A=\{1,...,\hat{j},...,\hat{k},...,N\}$ [3] and the summation runs over all possible subsets α of A. Here $\bar{\alpha}$ is the complement of α in A. For an explicit and simple example, see Appendix B.

Finally we need to justify why we call this procedure "finite volume expansion". From a simple analysis below, it is clear that all the explicit L dependence are contained in the diagonal minors of the Gaudin norm. In the ABA approach, the diagonal matrix elements are given in terms of the following functions: the eigenvalue of the diagonal elements of the transfer matrix, $a(u)$ and $d(u)$, the products of functions $f(u-v)$ and $g(u-v)$ (9.4.61,9.4.63), and the matrix elements in the Slavnov determinant formula Ω_{jk} and ϕ_{jk}. Under proper normalization, the functions $a(u)$ and $d(u)$ always appear in the expression as the ratio $a(u)/d(u)=e^{ipL}$. In fact, this kind of phase factor is either canceled by the same factors from the norm, or be replaced by products of scattering matrices using the Bethe Ansatz equations and they do not appear in the final expression.

3. Here $\hat{j}$ and $\hat{k}$ mean these two indices are absent.

The products of $f(u-v)$ and $g(u-v)$ functions do not depend on L. The matrix element Ω_{jk} defined in (3.3.73) also has no dependence on L. Finally, ϕ_{jk} with $j \neq k$ reads

$$\phi_{jk} = \phi(u_j, u_k) = \frac{2}{(u_j - u_k)^2 + 1}, \quad j \neq k, \tag{9.4.73}$$

again, do not depend on L. The only dependence on L is hidden in the diagonal element ϕ_{nn}. Recall that we have

$$\phi_{nn} = \sum_{m=1}^{L} \frac{1}{(u_n - \theta_m)^2 + 1/4} - \sum_{\substack{l=1 \\ l\neq n}}^{N} \phi_{nl} \tag{9.4.74}$$

In the homogeneous limit, where $\theta_m = 0$ $(m = 1, \cdots, L)$, the first term of (9.4.74) becomes $L/(u_n^2 + 1/4)$ which depends linearly on L. When we perform the Laplace expansion for the the scalar product, we carefully avoid expansion with this kind of terms and they only appear in the diagonal minor $\rho_N(\alpha)$. Therefore, the finite volume corrections are all contained in $\rho_N(\alpha)$. This is one part of the conjecture in [59] [4].

We have shown in section 9.4.4 that any diagonal matrix element can be written as a linear combination of the special scalar product with coefficients that do not depend explicitly on L. Thus, we can perform the the finite volume expansion of any diagonal matrix element in the Heisenberg spin chain. As was shown in section 9.4.2, the diagonal matrix elements correspond to three-point functions of HHL type. Therefore we have shown that the structure of finite volume dependence of three-point functions conjectured in [59] is also valid at weak coupling at the leading order in the $\mathfrak{su}(2)$ sector. In section 9.4.7, we will show that the structure also holds at one-loop level.

This is only half of the story. In the conjecture [59], each coefficient $\mathcal{F}(\bar{\alpha})$ of $\rho(\alpha)$ is identified with the form factor of the same operator in *infinite volume*. In order to check this statement, it is desirable to have a formulation of the diagonal matrix elements of the Heisenberg spin chain directly in infinite volume. However, we are not aware of such a formulation, although it seems possible to do it in the framework of coordinate Bethe Ansatz. In principle, the infinite volume form factors for our case can also be obtained by first solving the Klose-McLoughlin axioms [73] and then take the weak coupling limit. However, no solution has been found up to now. Because of these reasons, we are not able to confirm that the coefficients we obtain from finite volume expansion are indeed the infinite volume form factors.

It is still of interest to know the explicit form of coefficients from our finite volume expansion. These will be our predictions for the diagonal form factors in the infinite

4. There the authors used an equivalent description of the diagonal matrix element in terms of the symmetric expansion (9.3.30), instead the connected one (9.3.26) that we used here.

volume theory. We perform the finite volume expansion for all the diagonal matrix elements of length-1 and length-2 operators and extract the coefficients. The results exhibit a nice structure and will be presented in section 9.4.6.

9.4.6 Infinite volume form factors

First, let us comment on the identity operator. As any multi-magnon diagonal matrix element of the identity operator equals to 1, matching it with the series (9.3.26), one can easily derive that all infinite volume connected form factors vanish except from the vacuum expectation value,

$$f^{\mathbb{I}}(\emptyset) = 1 \quad ; \qquad f^{\mathbb{I}}(u_1, ..., u_N) = 0 \quad , \quad N \geq 1. \tag{9.4.75}$$

We should also discuss separately the simple case of the vacuum expectation values of spin chain operators. In the series (9.3.26), the zero magnon diagonal matrix element only contains the vacuum expectation value of the given operator in the infinite volume theory. So that, for the length-1 and length-2 operators one can easily find

$$f^{o_1}(\emptyset) = f^{o_2^1}(\emptyset) = 1 \quad , \qquad f^{o_2^2}(\emptyset) = f^{o_2^3}(\emptyset) = f^{o_2^4}(\emptyset) = 0. \tag{9.4.76}$$

It holds also in the general case. Let us take the operators E_n^{ab}, $(a, b = 1, 2)$ as a basis on the local quantum space, and linearly extend it to l_s neighboring site. Then only one, among this 4^{l_s} basis element, has non-vanishing vacuum expectation value, namely the one containing E^{11} at each site.

In the rest of this section, we will perform the finite volume expansion for the diagonal matrix elements of length-1 and length-2 operators. We will discuss a simple example, namely the case of length-1 operator with 2 magnons in detail and present the results for more complicated form factors.

9.4.6.1 An example: length-1 operator with -2 magnons

We consider the finite volume diagonal matrix element for the operator $o_1(n) = \mathrm{E}_n^{11}$ with two magnons

$$\mathcal{F}_L^{o_1}(u_1, u_2) = \frac{\langle u_1, u_2 | o_1(n) | u_1, u_2 \rangle}{\langle u_1, u_2 | u_1, u_2 \rangle}. \tag{9.4.77}$$

It has the following structure in finite volume

$$\mathcal{F}_L^{o_1}(u_1,u_2)=\frac{1}{\rho_2(\{1,2\})}\left(\rho_2(\{1,2\})+f^{o_1}(u_2)\,\rho_2(\{1\})+f^{o_1}(u_1)\,\rho_2(\{2\})+f^{o_1}(u_1,u_2)\right) \tag{9.4.78}$$

where $f^{o_1}(\mathbf{u})$ is to be identified with the connected diagonal form factor of o_1 in the infinite volume theory (9.3.19).

We proceed as described in the previous sections. Using the solution of QISP, we have

$$\mathcal{F}_L^{o_1}(u_1,u_2)=\frac{1}{t_{\mathbf{u}}(\theta_n^+)}\frac{\langle u_1,u_2|A(\theta_n^+)|u_1,u_2\rangle}{\langle u_1,u_2|u_1,u_2\rangle}, \tag{9.4.79}$$

where the denominator is the Gaudin norm,

$$\langle u_1,u_2|u_1,u_2\rangle=\left(\prod_{j=1}^{2}a(u_j)d(u_j)\right)\frac{1+(u_1-u_2)^2}{(u_1-u_2)^2}\,\rho_2(\{1,2\}). \tag{9.4.80}$$

From (9.4.61),

$$\begin{aligned}\langle u_1,u_2|A(\theta_n^+)|u_1,u_2\rangle =&\,\mathrm{M}_0(\theta_n^+)\langle u_1,u_2|u_1,u_2\rangle \\ &+\mathrm{M}_1(\theta_n^+)\langle u_1,u_2|u_2,\theta_n^+\rangle+\mathrm{M}_2(\theta_n^+)\langle u_1,u_2|u_1,\theta_n^+\rangle.\end{aligned} \tag{9.4.81}$$

We introduce some notations in order to simplify the expressions. Let us define

$$\mathscr{C}_{\mathbf{u},\mathbf{v}}=\frac{\prod_{j=1}^{N}a(v_j)d(u_j)}{\det_{jk}\frac{1}{u_j-v_k+i}}, \tag{9.4.82}$$

so that

$$\langle\mathbf{v}|\mathbf{u}\rangle=\mathscr{C}_{\mathbf{u},\mathbf{v}}\det_{jk}\Omega(u_j,v_k). \tag{9.4.83}$$

By performing the finite volume expansion for the three scalar products, we obtain

$$\begin{aligned}\langle u_1,u_2|A(\theta_n^+)|u_1,u_2\rangle =&-\mathscr{C}_{\{u_1,u_2\},\{u_1,u_2\}}\mathrm{M}_0(\theta_n^+)\,\rho_2(\{1,2\}) \\ &-i\mathscr{C}_{\{u_1,u_2\},\{u_1,\theta_n^+\}}\mathrm{M}_2(\theta_n^+)\Omega(u_2,\theta_n^+)\,\rho_2(\{1\}) \\ &-i\mathscr{C}_{\{u_1,u_2\},\{u_2,\theta_n^+\}}\mathrm{M}_1(\theta_n^+)\Omega(u_1,\theta_n^+)\,\rho_2(\{2\}) \\ &-\phi_{12}\left(\mathscr{C}_{\{u_1,u_2\},\{u_2,\theta_n^+\}}\Omega(u_2,\theta_n^+)\mathrm{M}_1(\theta_n^+)+\mathscr{C}_{\{u_1,u_2\},\{u_1,\theta_n^+\}}\Omega(u_1,\theta_n^+)\mathrm{M}_2(\theta_n^+)\right)\end{aligned} \tag{9.4.84}$$

Plugging (9.4.84) into (9.4.79) and compare to the expansion (9.4.78), we obtain the expression for the various form factors in the infinite volume

$$\begin{aligned} f^{o_1}(u_1) &= i\frac{\mathscr{C}_{\{u_1,u_2\},\{u_2,\theta_n^+\}}}{\mathscr{C}_{\{u_1,u_2\},\{u_1,u_2\}}}\frac{\mathrm{M}_1(\theta_n^+)}{\mathrm{M}_0(\theta_n^+)}\,\Omega(u_1,\theta_n^+) \\ f^{o_1}(u_2) &= i\frac{\mathscr{C}_{\{u_1,u_2\},\{u_1,\theta_n^+\}}}{\mathscr{C}_{\{u_1,u_2\},\{u_1,u_2\}}}\frac{\mathrm{M}_2(\theta_n^+)}{\mathrm{M}_0(\theta_n^+)}\,\Omega(u_2,\theta_n^+) \\ f^{o_1}(u_1,u_2) &= \frac{\phi_{12}}{\mathscr{C}_{\{u_1,u_2\},\{u_1,u_2\}}\,\mathrm{M}_0(\theta_n^+)}\Big(\mathscr{C}_{\{u_1,u_2\},\{u_2,\theta_n^+\}}\Omega(u_2,\theta_n^+)\mathrm{M}_1(\theta_n^+) \\ &\quad +\mathscr{C}_{\{u_1,u_2\},\{u_1,\theta_n^+\}}\Omega(u_1,\theta_n^+)\mathrm{M}_2(\theta_n^+)\Big) \end{aligned} \tag{9.4.85}$$

Substituting the explicit expressions in (9.4.85) and, at the end, taking the homogeneous limit $\theta_n \to 0$, we obtain very compact results for the infinite volume connected form factors,

$$\begin{aligned} f^{o_1}(u_k) &= \frac{1}{u_k^2+1/4}, \qquad k=1,2 \\ f^{o_1}(u_1,u_2) &= \left(\frac{1}{u_1^2+1/4}+\frac{1}{u_2^2+1/4}\right)\frac{2}{1+(u_1-u_2)^2}. \end{aligned} \tag{9.4.86}$$

9.4.6.2 Length-1 operator with N magnons

We can perform the same calculation as in the previous subsection and extract the form factors with more magnons. The process becomes cumbersome for higher number of particles. However, from the first few magnon cases, we are able to observe a nice pattern of the connected form factors. The N-magnon connected diagonal form factor for $o_1(n)$ is given as

$$f^{o_1}(u_1,...,u_N) = \varepsilon_1\,\phi_{12}\,\phi_{23}...\phi_{N-1,N} + \text{permutations} \tag{9.4.87}$$

where ε_k is the energy of the magnon with rapidity u_k and ϕ_{jk} can be seen as some "propagator" defined as

$$\varepsilon_k = \varepsilon(u_k) = \frac{1}{u_k^2+1/4}, \qquad \phi_{jk} = \frac{2}{1+(u_j-u_k)^2} \quad , \quad j\neq k. \tag{9.4.88}$$

The expression (9.4.87) can be represented by the diagrams in Fig.(9.4.2). Each node is labeled by a number from 1 to N. The leftmost node is associated with the energy of its label. The lines between two neighboring nodes are associated with a propagator. Multiplying the factors we obtain the value of the diagram. Summing over all the permutations of the labeling gives the result for infinite volume form factor f^{o_1}. The result for an N magnon state is thus a sum over $N!$ terms.

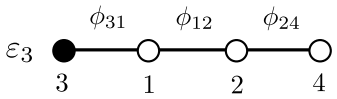

FIGURE 9.4.2: Diagrammatic representation of one term in (9.4.87) with 4 magnons.

The structure of f^{o_1} is exactly the structure of the connected form factors of conserved charge densities (9.3.33). This is not surprising, since $o_1(n) = \frac{1}{2}(1+\sigma_n^z)$ is indeed a length-1 conserved charge density of the Heisenberg spin chain. The nice feature is that once we know the one particle eigenvalue $q(u)$ of the charge, we can immediately write down the expression for the corresponding infinite volume form factors. We remark here that our result (9.4.87) is consistent with the determinant formula of [96].

9.4.6.3 Length-2 form factors of N magnons

The calculation of infinite volume matrix elements can be performed following the same line as in section 9.4.6.1. The process is more involved. Nevertheless, we again find some patterns for the various matrix elements which we present below. The structure for the length-2 operators can be encoded into diagrams similar to the one in Fig.(9.4.2). However, in this case we have two types of them, as are shown in Fig.(9.4.3).

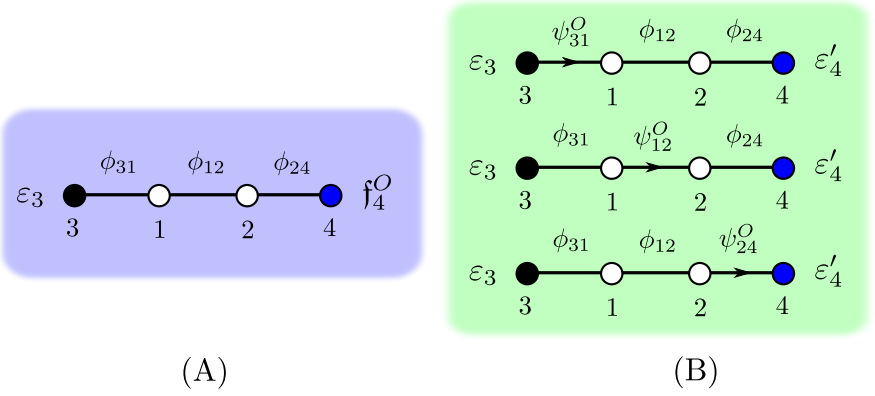

FIGURE 9.4.3: Different kinds of diagrams encoding length-2 form factors.

Each diagonal matrix element is given by two kinds of diagrams. The first kind is depicted in the blue region. We label the nodes by number from 1 to N. The leftmost node is associated with $\varepsilon(u_j)$ while the rightmost node is associated with a function denoted by $\mathfrak{f}^O(u_k)$, and it depends on the operator. The other kind, depicted in the green region, is more interesting. The leftmost and rightmost nodes are associated with ε_j and ε'_k where $\varepsilon'(u) = \frac{\partial}{\partial u}\varepsilon(u)$. In addition, for a given label of the nodes, one needs to sum over the diagrams which replaces one of the propagators by a "directed propagator", $\psi_{ij}^O = \psi^O(u_i, u_j)$. The directed propagator is antisymmetric with respect to its arguments $\psi_{ij}^O = -\psi_{ji}^O$ and its explicit form depends on the operator under consideration.

In sum, the infinite volume diagonal matrix element of a length-2 operator, $f^O(u_1, \cdots, u_N)$, is characterized by two functions $\mathfrak{f}^O(u)$ and $\psi^O(u,v)$. The result for N-magnon is given by

$$\begin{aligned} f^O(u_1,...,u_N) = &\left(\varepsilon_1\,\phi_{12}...\phi_{N-1,N}\,\mathfrak{f}^O_N + \text{permutations}\right) \\ &+ \left(\sum_{i=1}^{N-1} \varepsilon_1\,\phi_{12}...\psi^O_{i,i+1}...\phi_{N-1,N}\,\varepsilon'_N + \text{permutations}\right) \end{aligned} \tag{9.4.89}$$

We list the data for o_2^i (9.4.50,9.4.51) is the following:

$$\begin{aligned} &\mathfrak{f}^{o_2^1}(u) = 2 && \psi^{o_2^1}(u,v) = -(u-v)(uv-1/4)\phi(u,v) \\ &\mathfrak{f}^{o_2^2}(u) = -\frac{u+i/2}{u-i/2} && \psi^{o_2^2}(u,v) = (u-v)(u+i/2)(v+i/2)\phi(u,v) \\ &\mathfrak{f}^{o_2^3}(u) = -\frac{u-i/2}{u+i/2} && \psi^{o_2^3}(u,v) = (u-v)(u-i/2)(v-i/2)\phi(u,v) \\ &\mathfrak{f}^{o_2^4}(u) = 0 && \psi^{o_2^4}(u,v) = -(u-v)(uv-1/4)\phi(u,v) \end{aligned} \tag{9.4.90}$$

Let us comment on this results (9.4.90). These data for the operators can be read off simply from the computation of 2 magnon case. Therefore, one should simply compute the 2-magnon matrix elements and perform the finite volume expansion to extract the data. Once the functions in (9.4.90) are known, we can write down any diagonal form factor of length-2 operators in the infinite volume. Any length-2 operator is a linear combination of the identity operator, o_1 and o_2^i,

$$\mathcal{O} = b\,\mathbb{I} + c_0\,o_1(n) + \tilde{c}_0\,o_1(n+1) + \sum_{i=1}^{3} c_i\,o_2^i(n) \tag{9.4.91}$$

where b, c_0 and c_i $(i=1,2,3)$ are some numbers. Then the data of $\mathcal{O}$ is simply given by

$$\mathfrak{f}^{\mathcal{O}}(u) = c_0 + \tilde{c}_0 + \sum_{i=1}^{3} c_i\,\mathfrak{f}^{o_2^i}(u), \qquad \psi^{\mathcal{O}}(u,v) = \sum_{i=1}^{3} c_i\,\psi^{o_2^i}(u,v), \tag{9.4.92}$$

and its vacuum expectation value is

$$f^{\mathcal{O}}(\emptyset) = b + c_0 + \tilde{c}_0 + c_1. \tag{9.4.93}$$

For example, the operator o_2^4 is not independent

$$o_2^4(n) = \mathbb{I} - o_1(n) - o_1(n+1) + o_2^1(n).$$

Note that by translational invariance, $o_1(n)$ gives the same result as $o_1(n+1)$ when computing the form factors. It is easy to check that this resolution is consistent with

(9.4.76,9.4.90). The diagonal matrix elements of the operators o_2^2 and o_2^3 are related by complex conjugation which is also manifest in (9.4.90).

9.4.6.4 An examples of length-2 operator

As an example of the matrix elements of the length-2 we compute the permutation operator $\mathrm{P}_{k,k+1}$, or equivalently the Hamiltonian density $\mathrm{H}_{k,k+1} = \mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1}$. We will see that the data for the permutation operator simplifies and the final result takes exactly the form predicted in (9.3.33). This is a non-trivial check of our functions (9.4.90).

The permutation operator $\mathrm{P}_{k,k+1}$ is a length-2 operator of the Heisenberg spin chain. It can be written in terms of the operators o_2^i with equal weights

$$\mathrm{P}_{k,k+1} = \sum_{i,j=1}^{2} \mathrm{E}_k^{ij}\, \mathrm{E}_{k+1}^{ji} = \sum_{i=1}^{4} o_2^i. \tag{9.4.94}$$

According to (9.4.92), the data of permutation operator is given by

$$\mathfrak{f}^{\mathrm{P}}(u) = \frac{1}{u^2 + 1/4}, \qquad \psi^{\mathrm{P}}(u, v) = 0. \tag{9.4.95}$$

The infinite volume form factor (9.4.89) with the entires (9.4.95) has the structure as a conserved charge should have (9.3.33).

9.4.7 Matrix elements at one loop

In this section, we generalize the above considerations from tree level to one loop. We will show that the matrix elements at one loop can be again written in terms of a finite number of "building blocks". These building blocks are matrix elements of the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain with the inhomogeneities fixed to their BDS values. These building blocks can be again written in terms of the special scalar products of the inhomogeneous spin chain and one can perform the finite volume expansion, as at the tree level.

Let us recall from Chapter 6 and 7 that at one loop, we need to construct the eigenstates for the BDS spin chain and take into account the one-loop operator insertions. In what follows, we first discuss the effect of the operator insertions and show that the three-point function under consideration can be reduced into the calculation of matrix elements of the BDS spin chain. Then we consider the effect of the S operator on the spin operators and reduce the matrix elements of BDS spin chain into the correlation functions of the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain. As we discussed in the tree level case, the matrix

elements of the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain can be written in terms of the scalar products (9.4.69), and we can perform the finite volume expansion.

9.4.7.1 The effects of one-loop operator insertions

For each spin chain state, there are two operator insertions at the two splitting points. We first discuss the effects of insertions for the "light" operator $\mathcal{O}_\alpha$. The one-loop insertion takes the form of the Hamiltonian density

$$\mathrm{H}_l^{\mathfrak{so}(6)} = \mathrm{K}_{l,l+1} + 2\mathrm{I}_{l,l+1} - 2\mathrm{P}_{l,l+1} \tag{9.4.96}$$

where $\mathrm{I}_{l,l+1}$, $\mathrm{P}_{l,l+1}$ and $\mathrm{K}_{l,l+1}$ are the identity, permutation and trace operators. They act on the $\mathfrak{so}(6)$ spin chain states as

$$\begin{aligned}
&\mathrm{I}_{l,l+1}|\cdots\phi_l^i\phi_{l+1}^j\cdots\rangle = |\cdots\phi_l^i\phi_{l+1}^j\cdots\rangle, \\
&\mathrm{P}_{l,l+1}|\cdots\phi_l^i\phi_{l+1}^j\cdots\rangle = |\cdots\phi_l^j\phi_{l+1}^i\cdots\rangle, \\
&\mathrm{K}_{l,l+1}|\cdots\phi_l^i\phi_{l+1}^j\cdots\rangle = \delta_{ij}\sum_{k=1}^{6}|\cdots\phi_l^k\phi_{l+1}^k\cdots\rangle.
\end{aligned} \tag{9.4.97}$$

At one loop level, the light operator should diagonalize the two-loop dilatation operator and be of zero R-charge. The $\mathfrak{so}(6)$ sector is only closed at one-loop so in principle one needs fields outside the $\mathfrak{so}(6)$ sector, like fermionic fields, to construct the eigenstates of the two loop dilatation operator. However, when computing the three-point functions, the fields apart from $\{X, Z, \bar{X}, \bar{Z}\}$ will not contribute at one-loop order. Therefore the terms, which have non-zero contributions for the three-point function, are still made of scalar fields $\{X, Z, \bar{X}, \bar{Z}\}$ with zero R-charge, which are the cases that we already discussed at tree level. Under the action of the operator insertion (9.4.97), the terms which have non-zero contributions still have the same form, but with some different coefficients. To summarize, the one-loop structure constants can be recast to the calculation of correlation functions of the BDS spin chain ${}_{\mathrm{BDS}}\langle\mathbf{u}|\hat{O}(\sigma^{\pm},\sigma^z;g^2)|\mathbf{u}\rangle_{\mathrm{BDS}}$.

Now we consider the operator insertions for the heavy states. The effect of these insertions is increasing the length of the spin chain operator, as is shown in (9.4.4).

This can be seen easily by noticing that

$$\mathrm{P}_{k,k+1} = \sum_{i,j=1}^{2}\mathrm{E}_m^{ij}\otimes\mathrm{E}_{m+1}^{ji} \tag{9.4.98}$$

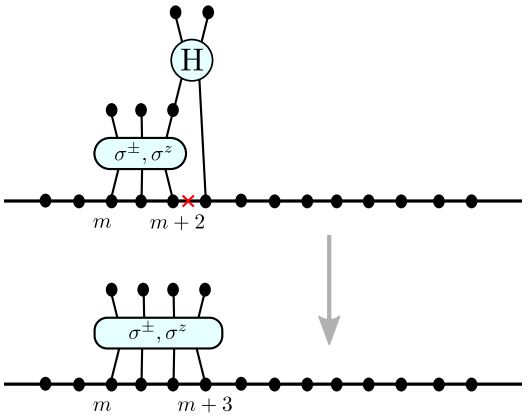

FIGURE 9.4.4: The effect of operator insertions for the heavy operators. They increase the length of the spin operator by 1. The red cross denotes the splitting point.

For example, we have the following

$$\star \mathrm{E}^{11}_{m} \mathrm{H}_{m} \star = (\star \mathrm{E}^{11}_{m} \star) - (\star \mathrm{E}^{11}_{m} \mathrm{E}^{11}_{m+1} \star) - (\star \mathrm{E}^{11}_{m} \mathrm{E}^{12}_{m+1} \star) \tag{9.4.99}$$

where the star stands for some strings of operators.

9.4.8 The effects of the unitary S operator

In this subsection, we discuss the action of unitary operator S on the spin operators. We are interested in the following quantity

$$\langle \mathbf{u}; \boldsymbol{\theta}^{\mathrm{BDS}} | \mathrm{S}^{-1} \hat{O}_{l+1}(\sigma^{\pm}, \sigma^{z}) \mathrm{S} | \mathbf{u}; \boldsymbol{\theta}^{\mathrm{BDS}} \rangle. \tag{9.4.100}$$

The S operator takes an exponential form $\mathrm{S} = \exp \hat{\mathrm{F}}$, thus we have

$$\mathrm{S}^{-1} \hat{O}_{l+1}(\sigma^{\pm}, \sigma^{z}) \mathrm{S} = \hat{O}_{l+1}(\sigma^{\pm}, \sigma^{z}) + [\hat{\mathrm{F}}, \hat{O}_{l+1}(\sigma^{\pm}, \sigma^{z})] + \frac{1}{2}[\hat{\mathrm{F}}^{2}, \hat{O}_{l+1}(\sigma^{\pm}, \sigma^{z})] + \mathcal{O}(g^{3}) \tag{9.4.101}$$

where we have truncated up to $\mathcal{O}(g^2)$ order. The action of S operator on the spin chain operator can be divided into two types. The first type is length preserving, it originates from the operators H_k and $[\mathrm{H}]_k$ that act within the range of the spin chain operator $\hat{O}_{l+1}$, which gives rise to an operator with the same length, this is depicted in Fig.(9.4.5).

The other type of the action increases the length of the operator by 1 or 2, which are generated from the operators at the boundary of the spin chain operator. There are two kinds of length changing processes at one loop. One process is generated by a single H_k or $[\mathrm{H}]_k$, which is given in Fig.(9.4.6).

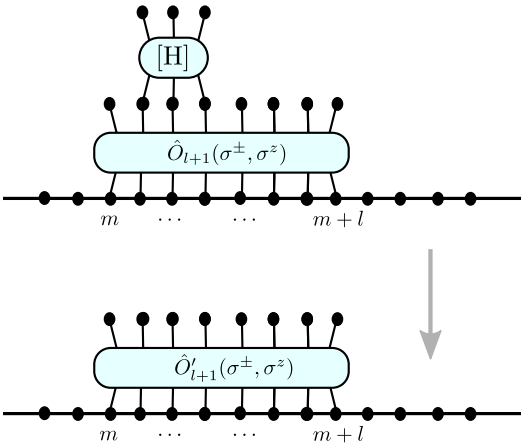

FIGURE 9.4.5: The length preserving action generated by $[\mathrm{H}]_k$ on the spin chain vertex.

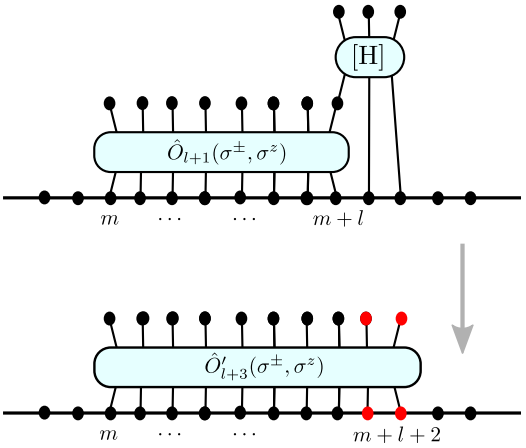

FIGURE 9.4.6: The length changing action generated by $[\mathrm{H}]_k$ on the spin chain vertex. In this example, it increases the length of the spin chain operator by 2.

The other process is generated by two H_k's, one example of which is given in Fig.(9.4.7).

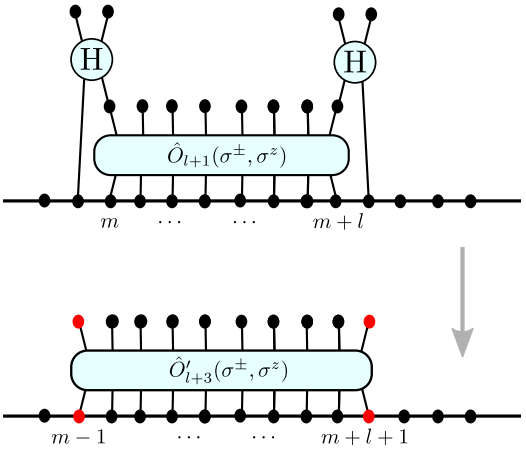

FIGURE 9.4.7: The length changing action generated by two H_k's on both ends of the spin chain vertex. The length of the spin operator also increases by 2 in this example.

From our analysis we see that the action of the S operator on the spin chain operators, in general, increases the length of the spin chain operator. Up to $\mathcal{O}(g^2)$ order, the length of the operator increases at most by 2.

$$\mathrm{S}^{-1}\,\hat{O}_l(\sigma^\pm,\sigma^z)\,\mathrm{S}=\hat{O}'_l(\sigma^\pm,\sigma^z)+\hat{O}'_{l+1}(\sigma^\pm,\sigma^z)+\hat{O}'_{l+2}(\sigma^\pm,\sigma^z)+\mathcal{O}(g^3) \tag{9.4.102}$$

This implies that in order to compute the form factor of length l operator for BDS spin chain, we need to compute the form factors of length $l+2$, $l+1$ and l operators in the inhomogeneous Heisenberg $\mathrm{XXX}_{1/2}$ spin chain.

Once we write the three-point function in terms of matrix elements of the inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain, we can perform the finite volume expansion and organize the results in the form conjectured in [59]. At one loop, the matrix element of Gaudin norm is modified. The Gaudin norm formula (3.3.76) is still valid, but the eigenvalues $a(u)$ and $d(u)$ are corrected

$$\begin{aligned} a(u)&=\prod_{k=1}^{L}(u-\theta_k^{\mathrm{BDS}}+i/2)=x(u+i/2)^L+\mathcal{O}(g^{2L}),\\ d(u)&=\prod_{k=1}^{L}(u-\theta_k^{\mathrm{BDS}}+i/2)=x(u-i/2)^L+\mathcal{O}(g^{2L}), \end{aligned} \tag{9.4.103}$$

where $x(u)$ is the Zhukowsky map (6.2.38). By replacing

$$p(u_k)=\frac{u_k+i/2}{u_k-i/2}\longrightarrow\frac{x(u_k+i/2)}{x(u_k-i/2)} \tag{9.4.104}$$

in (3.3.77) and expanding the result up to $\mathcal{O}(g^2)$ order, we obtain the Gaudin norm at one-loop, $\rho_n^{\text{1-loop}}$. In fact, the replacement (9.4.104) gives the correct Gaudin norm up to wrapping orders[55, 132]. Our conclusion is that the structure conjectured in [59] is also valid at one loop level with respect to the one-loop Gaudin norm.

Of course the coefficients or the infinite volume form factors at one-loop will be more complicated. It is an interesting question to see how the infinite volume form factors are deformed at one loop and whether it is possible to bootstrap to all loops.

Chapter 10

Conclusions and Outlook

In this thesis, we have studied the computation of three-point functions in $\mathcal{N} = 4$ SYM theory. The main idea is to use integrability of the theory to map operators with definite anomalous dimensions to on-shell Bethe states of the spin chains. By performing the cutting and gluing procedure, one can write the result in terms of scalar products between Bethe states. In the generic case, the result is given by a sum over partitions of the rapidities. However, for some special cases one can write the result in terms of one determinant. This is achieved by mapping the three-point functions to the partition functions of vertex models in some specific lattice.

The detail computations have been done mainly in the $\mathfrak{su}(2)$ sector and a special case in $\mathfrak{su}(3)$ sector where a determinant representation for the final result exist. By applying the mapping between long-range interacting spin chains and inhomogeneous spin chains, we managed to obtain a compact result for one-loop three-point function in the $\mathfrak{su}(2)$ sector, which can be written in terms of determinants. The determinant representation allows us to take the semi-classical limit of the result straightforwardly. The resultant expression is written in an elegant way just in terms of the three quasi-momenta of the corresponding Bethe states. Comparison with the computation from the strong coupling in the Frolov-Tseytlin limit shows a perfect match up to one-loop level.

In quest of a formalism which works for more general sectors and higher loops, we constructed the spin vertex at tree level, which is the weak coupling counterpart of the string vertex in light-cone string field theory. We constructed explicitly the spin vertex for all sectors and investigated the main properties of the spin vertex, namely the reflection properties and the monodromy relations. In the semi-classical limit, the monodromy relation plays an essential role in obtaining the final result without computing any scalar products. Since the string/spin vertex description appears both at strong and weak coupling, it is tempting to conjecture that it also exist at finite coupling. It is thus

interesting to see how does this formalism looks like at finite coupling and what should be the main principles to determine the spin vertex at higher loops. It is expected that as in the spectral problem, symmetry should play an essential role in this formalism.

Due to the intimate relation between three-point functions in $\mathcal{N} = 4$ SYM theory and the form factor bootstrap program, we also review some interesting proposals which try to formulate the three-point function in a non-perturbative manner by relating the three-point functions to form factor like objects and proposing boostrap axioms for these objects. The solution of these axioms determines the form factors in infinite volume. In order to take into account the finite volume dependence of the form factors, it is convenient to classify the finite volume correction into two kinds. The first kind takes the form of polynomials in $1/L$ and the second kind is the so-called the wrapping corrections. It is relatively easy to take into account the first kind of the finite volume correction, following the previous works in the form factors in 2d integrable QFT. However, it is a challenging problem to take into account wrapping corrections for form factors. Even after many years of hard work, there is no systematic way like TBA to take into account all finite volume corrections for the form factors. The form factors usually are very complicated, but it is expected to be simplified in the diagonal case where the incoming and outgoing states are conjugated to each other. We investigated also in this thesis the relation between diagonal form factors and symmetric HHL correlation functions in $\mathcal{N} = 4$ SYM from weak coupling where a spin chain description is available. By using the Slavnov determinant formula and the solution of QISP, we explore the finite volume dependence of the first kind for the diagonal matrix elements. Due to the absence of wrapping corrections at tree level and one-loop, the result we obtained is in fact exact.

There are many open questions in computing three-point functions of $\mathcal{N} = 4$ SYM theory. First of all, despite some proposals, the all loop formalism for three-point functions is still lacking. However, the very recent result by Basso, Komatsu and Vieira seems to improve this situation largely. These authors proposed a concrete framework for the all-loop three-point functions and obtain the result in the form of sum over partitions of rapidities. Their result is valid up to wrapping order. In a sense, this is similar to the BDS equation and BES dressing phase in the spectral problem where one can compute the quantity to all loops in the asymptotic regime. The next step, following the trajectory of development of the spectral problem, is taking into account the wrapping corrections. In the spectral problem, the methods for taking into account wrapping corrections such as Lüscher's method and TBA already exist and is rather systematic which can serve as guiding principle for the spectral problem in AdS/CFT. The generalization to the AdS/CFT, although far from being trivial can be achieved. For the three-point functions, however, the analogous method that takes into account the wrapping corrections

for form factors does not exist. Therefore the situation is more difficult for the three-point function case since the similar problem is not yet solved for the simpler models such as 2D integrable QFT's. Finite volume form factors are also related to the finite temperature correlation functions in 2d QFT's, which are very important observables for experiments. In this sense, finding a systematic way to take into account wrapping corrections for form factors both in 2d QFT and $\mathcal{N} = 4$ SYM theory are of fundamental importance.

In general, the three-point functions might be very complicated even at tree level. However, it is expected that in the semi-classical limit, the result will be much simplified and can be written in a more elegant way. This is supported by the existing results in the semi-classical limit both at weak and strong coupling, as we have shown in this dissertation. Therefore, it will be very interesting to develop semi-classical methods for the three-point functions, in the same way that the KMMZ algebraic curve description for the spectral problem. The weak coupling result for the moment is heavily based on the existence of determinant representation. This is not likely to be true for the general case and we need new insights and methods to tackle the problem. In this direction, we suspect that the monodromy relation will play an important role, based on the fact that it provide essential information both at strong and weak coupling when computing three-point functions.

Based on the great success of spectral problem, it will be interesting to build the relation between the three-point functions and two-point functions. In particular, is it possible to use the results from QSC as an input and compute the three-point functions non-perturbatively ? In this direction, Sklyanin's separation of variables (SoV) may prove to be important. It is observed by the authors of QSC that the Q-functions take the form of wave functions for the states in the SoV representation. Very recently, we show that at least at tree level, one can apply the SoV method to compute three-point functions. The final result is written in terms of a matrix model like integral for the separated variables. It is therefore an interesting question to see the relation between the QSC Q-functions and the Q-functions which are Bethe states in the SoV representation.

Finally, even if very optimistically one managed to determine all the OPE coefficients non-perturbatively, we need to *understand* its implication. Even for the spectral problem, we now have a very efficient method that allows us to determine the anomalous dimensions to all orders. It is not completely clear why such a nice formulation exists and can we obtain it from simpler methods ? What does the all-loop result teach us about the AdS/CFT correspondence, the strongly coupled gauge theory and the string theory on curved background ? With these questions in mind, we still have a long and exciting way to go.

Appendix A

Morphism and Theta-morphism

In this appendix we analyze the relation between the unitary S-transformation and the "theta-morphism" introduced by Gromov and Vieira [132]. The idea of Gromov and Vieira was to construct the states of the BDS long-range model by acting with a differential operator, that they called theta-morphism, on the states of the inhomogeneous model. We find that a purely differential operator cannot realize the morphism property, see below. The failure to fulfill the morphism property results in the cross-terms of [132]. Instead, we introduce the morphism associated to the S-operator via

$$T^{\mathrm{BDS}}(u) = \mathrm{S}\, T(u;\boldsymbol{\theta}^{\mathrm{BDS}})\, \mathrm{S}^{-1} \equiv \mathscr{D}_\theta T(u;\boldsymbol{\theta})|_{\theta=0}\,. \tag{A.0.1}$$

We find that, up to terms of order $\mathcal{O}(g^3)$, the action of $\mathscr{D}_\theta$ on (an arbitrary matrix element of) the monodromy matrix T amounts to

$$\mathscr{D}_\theta T_a(u;\boldsymbol{\theta})|_{\theta=0} \tag{A.0.2}$$

$$\equiv T_a - \frac{g^2}{2}\sum_{k=1}^{L}\mathrm{D}_k^2 T_a - g^2\sum_{k=1}^{L}\mathrm{D}_k T_a(\mathrm{P}_{k,k+1}+\delta_{k,L}\mathrm{Q}_2) + g^2\left[\frac{\mathrm{Q}_2^2}{2} + i\mathrm{Q}_3 + \mathrm{P}_{1L}\mathrm{Q}_2, T_a\right]\Bigg|_{\theta=0}\,. \tag{A.0.3}$$

The operator $\mathscr{D}_\theta$ differs from the theta-morphism of Gromov and Vieira [132] given by

$$\mathcal{D}_\theta^{\mathrm{GV}} = 1 - \frac{g^2}{2}\sum_{k=1}^{L}\mathrm{D}_k^2 + \mathcal{O}(g^4), \tag{A.0.4}$$

by the last two terms in the second line of equation (A.0.2). These two extra terms account for the cross-terms in [132] and they insure that the morphism property is exact

$$\mathscr{D}_\theta\left(T_a(u_1;\boldsymbol{\theta})T_a(u_2;\boldsymbol{\theta})\right)|_{\theta=0} = \mathscr{D}_\theta T_a(u_1;\boldsymbol{\theta})|_{\theta=0}\,\mathscr{D}_\theta T_a(u_2;\boldsymbol{\theta})|_{\theta=0}\,. \tag{A.0.5}$$

On the Bethe vectors, the action of the operator $\mathcal{D}_\theta$ reduces to

$$|\mathbf{u}\rangle_{\rm BDS} = \mathrm{S}|\mathbf{u};\boldsymbol{\theta}^{\rm BDS}\rangle = \mathscr{D}_\theta|\mathbf{u};\boldsymbol{\theta}\rangle|_{\theta=0} = \left(\mathcal{D}_\theta^{\rm GV} + \frac{g^2}{2}\left(\mathrm{Q}_2^2 + 2i\mathrm{Q}_3 + 2\mathrm{P}_{1L}\mathrm{Q}_2\right)\right)|\mathbf{u};\boldsymbol{\theta}\rangle|_{\theta=0}\,. \tag{A.0.6}$$

To obtain this expression, we use that $\sum_k \mathrm{D}_k = 0$ and that the vacuum eigenvalues of Q_2 and Q_3 are zero. If the Bethe vectors are on-shell, the charges Q_2 and Q_3 become numbers and we obtain

$$\begin{aligned}|\mathbf{u}\rangle_{\rm BDS} &= \left[\mathcal{D}_\theta^{\rm GV} + g^2\left(\frac{1}{2}\mathrm{E}_2^2 + \mathrm{E}_2 + i\mathrm{E}_3 - \mathrm{H}_{1L}\mathrm{Q}_2\right)\right]|\mathbf{u};\boldsymbol{\theta}\rangle|_{\theta=0}\\ &= \left[1 + g^2\left(\frac{1}{2}\mathrm{E}_2^2 + \mathrm{E}_2 + i\mathrm{E}_3\right)\right]|\mathbf{u}\rangle_{\rm BDS}^{\rm GV}\,.\end{aligned} \tag{A.0.7}$$

We thus see that our eigenvectors differ from those of Gromov and Vieira by a state-dependent factor. The imaginary contribution does not affect the norms, while the real part changes the normalization. The scalar products of two arbitrary Bethe states, on-shell or off-shell, is:

$${}_{\rm BDS}\langle\mathbf{u}|\mathbf{v}\rangle_{\rm BDS} = \mathscr{D}_\theta\langle\mathbf{u};\boldsymbol{\theta}|\mathbf{v};\boldsymbol{\theta}\rangle|_{\theta=0} = \mathcal{D}_\theta^{\rm GV}\langle\mathbf{u};\boldsymbol{\theta}|\mathbf{v};\boldsymbol{\theta}\,|_{\theta=0}\,. \tag{A.0.8}$$

Let us now sketch the derivation of the expression (A.0.2). First, we account for the shift in the inhomogeneities by

$$T_a(u;\boldsymbol{\theta}^{\rm BDS}) = \exp\left(\sum_{j=1}^{L}\theta_l^{\rm BDS}\,\partial_{\theta_l}\right)\,T_a(u;\boldsymbol{\theta})|_{\theta=0}\,. \tag{A.0.9}$$

Then we apply the PD relations that have been derived in section 7.2. The case of non-overlapping permutations is simple,

$$[\mathrm{H}_l,[\mathrm{H}_k,T_a]] = (\mathrm{Q}_2\,\delta_{l,L} - \mathrm{D}_l)(\mathrm{Q}_2\,\delta_{k,L} - \mathrm{D}_k)\,T_a\,,\quad |l-k|>1\,. \tag{A.0.10}$$

For overlapping permutations in the bulk, $k \neq L-1, L$, we obtain,

$$[[\mathrm{H}]_k, T_a] = \frac{1}{2}\left(\mathrm{D}_k^2 - \mathrm{D}_{k+1}^2\right)T_a + \mathrm{D}_k T_a \mathrm{P}_{k,k+1} - \mathrm{D}_{k+1}T_a\mathrm{P}_{k+1,k+2}\,, \tag{A.0.11}$$

$$\begin{aligned}[[\mathrm{H}]_k, T_a]] = {}& \frac{1}{2}\left(\mathrm{D}_k^2 + \mathrm{D}_{k+1}^2\right)T_a + \mathrm{D}_k T_a \mathrm{P}_{k,k+1} + \mathrm{D}_{k+1}T_a\mathrm{P}_{k+1,k+2}\\ &+2\mathrm{D}_k\mathrm{D}_{k+1}T_a + 2\mathrm{D}_k T_a\mathrm{P}_{k+1,k+2} + \mathrm{D}_{k+1}T_a\mathrm{P}_{k,k+1},\end{aligned} \tag{A.0.12}$$

where $[\mathrm{H}]_k = [\mathrm{H}_k, \mathrm{H}_{k+1}]$ and $[\mathrm{H}]_k = \{\mathrm{H}_k, \mathrm{H}_{k+1}\}$. When $k = L-1, L$ the action in (A.0.11) has to be supplemented with boundary terms,

$$\begin{aligned} [[\mathrm{H}]_{L-1}, T_a] &= [[\mathrm{H}]_{L-1}, T_a]_{\text{bulk}} + [\delta_{\text{bound}} - 2i\mathrm{Q}_3, T_a] \\ [[\mathrm{H}]_L, T_a] &= [[\mathrm{H}]_L, T_a]_{\text{bulk}} - [\delta_{\text{bound}}, T_a] \ , \qquad \delta_{\text{bound}} = \left(\frac{1}{2}\mathrm{Q}_2^2 + i\mathrm{Q}_3 + \mathrm{P}_{1L}\mathrm{Q}_2\right) \ . \end{aligned} \tag{A.0.13}$$

These expressions, together with the action of overlapping D_k and H_l that are derived in section 7.2, are all we need to obtain (A.0.2), provided that we choose $\nu_0 = \rho_0 = 0$. Let us notice that the expressions (A.0.11), (A.0.12),(A.0.13) obey the Leibniz rule,

$$[[\mathrm{H}]_k, T_1 T_2] = T_1[[\mathrm{H}]_k, T_2] + [[\mathrm{H}]_k, T_1]T_2 \ . \tag{A.0.14}$$

We can thus safely replace T_a by any product of monodromy matrices in all the commutators above. This feature is at the origin of the morphism property.

Appendix B

An example of finite Volume Expansion

In this appendix, we give an explicit example in order to illustrate how to perform the finite volume expansion of the special scalar products defined in Section 9.4.5. The scalar product under consideration is $\langle\{u_1,u_2,u_3,u_4\}|\{u_1,\theta,u_3,u_4\}\rangle$, where $\{u_1,u_2,u_3,u_4\}$ is a set of Bethe roots. Consider the numerator of the Slavnov determinant formula (**??**),

$$\langle\{u_1,u_2,u_3,u_4\}|\{u_1,\theta,u_3,u_4\}\rangle \propto \begin{vmatrix} \phi_{11} & \Omega_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \Omega_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \Omega_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \Omega_{42} & \phi_{43} & \phi_{44} \end{vmatrix} \tag{B.0.1}$$

We first perform the Laplace expansion for the second column, which gives

$$\begin{aligned} &-\Omega_{12}\begin{vmatrix} \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} + \Omega_{22}\begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{31} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} \\ &-\Omega_{32}\begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{41} & \phi_{43} & \phi_{44} \end{vmatrix} + \Omega_{42}\begin{vmatrix} \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{33} & \phi_{34} \end{vmatrix} \end{aligned} \tag{B.0.2}$$

The Laplace expansion gives rise to 4 terms, which we shall denote T_i, $i=1,\cdots,4$. For T_1, we do Laplace expansion by the first column

$$\mathrm{T}_1 = -\Omega_{12}\left\{\phi_{21}\rho_4(\{3,4\}) - \phi_{31}(\phi_{23}\rho_4(\{4\}) - \phi_{24}\phi_{43}) + \phi_{41}(\phi_{23}\phi_{34} - \phi_{24}\rho_4(\{3\}))\right\} \tag{B.0.3}$$

$$= -\Omega_{12}\phi_{21}\,\rho_4(\{3,4\}) + \Omega_{12}\phi_{41}\phi_{23}\,\rho_4(\{3\}) + \Omega_{12}\phi_{31}\phi_{23}\,\rho_4(\{4\}) - \Omega_{12}(\phi_{31}\phi_{24}\phi_{43} + \phi_{41}\phi_{24}\phi_{34})$$

The second term already takes the form of diagonal minor of the Gaudin norm

$$\mathrm{T}_2 = \Omega_{22}\ \rho_4(\{1,3,4\}) \tag{B.0.4}$$

For the third term, we perform Laplace expansion with respect to the second column

$$\mathrm{T}_3 = -\Omega_{32}\phi_{23}\,\rho_4(\{1,4\}) + \Omega_{32}\phi_{43}\phi_{24}\,\rho_4(\{1\}) + \Omega_{32}\phi_{13}\phi_{21}\,\rho_4(\{4\}) - \Omega_{32}(\phi_{13}\phi_{24}\phi_{41} + \phi_{43}\phi_{14}\phi_{21}) \tag{B.0.5}$$

For the last term, we perform Laplace expansion with respect to the last column

$$\mathrm{T}_4 = -\Omega_{42}\phi_{24}\,\rho_4(\{1,3\}) + \Omega_{42}\phi_{34}\phi_{23}\,\rho_4(\{1\}) + \Omega_{42}\phi_{14}\phi_{21}\,\rho_4(\{3\}) - \Omega_{42}(\phi_{14}\phi_{23}\phi_{31} + \phi_{34}\phi_{13}\phi_{21}) \tag{B.0.6}$$

Collecting terms from the above calculation, we obtain the finite volume expansion of the scalar product

$$\begin{aligned}
&\langle\{u_1,u_2,u_3,u_4\}|\{u_1,\theta,u_3,u_4\}\rangle \propto \\
&\Omega_{22}\,\rho_4(\{1,3,4\}) - \Omega_{42}\phi_{24}\,\rho_4(\{1,3\}) - \Omega_{32}\phi_{23}\,\rho_4(\{1,4\}) - \Omega_{12}\phi_{21}\,\rho_4(\{3,4\}) \\
&+ (\Omega_{32}\phi_{43}\phi_{24} + \Omega_{42}\phi_{34}\phi_{23})\,\rho_4(\{1\}) + (\Omega_{42}\phi_{14}\phi_{21} + \Omega_{12}\phi_{41}\phi_{24})\,\rho_4(\{3\}) \\
&+ (\Omega_{12}\phi_{31}\phi_{23} + \Omega_{32}\phi_{13}\phi_{21})\,\rho_4(\{4\}) - \Omega_{12}(\phi_{31}\phi_{24}\phi_{43} + \phi_{41}\phi_{23}\phi_{34}) \\
&- \Omega_{32}(\phi_{13}\phi_{24}\phi_{41} + \phi_{43}\phi_{14}\phi_{21}) - \Omega_{42}(\phi_{14}\phi_{23}\phi_{31} + \phi_{34}\phi_{13}\phi_{21})
\end{aligned} \tag{B.0.7}$$

In fact, it is not hard to convince ourselves that the similar expansion can be performed for general scalar products defined in section 9.4.5. For length-2 operators, we have the following finite volume expansion

$$\langle\mathbf{u}|\{\mathbf{u},\theta_n,\theta_{n+1}\}\setminus\{u_j,u_k\}\rangle = \sum_{\alpha\subseteq A} F_{\bar{\alpha}}\,\rho(\alpha), \quad A = \{1,2,...,\hat{j},...,\hat{k},...,N\} \tag{B.0.8}$$

In general, the terms of the expansion grows quickly with the number of excitations and the expansion coefficients $F_{\bar{\alpha}}$ might get quite involved. Therefore, in general we need a code to compute the finite volume expansion.

Appendix C

The operator U

In this Appendix we collect some formulas about the action of the operator $\mathrm{U} = UU_F$ which represents a finite super-conformal transformation. The operator is a product of a $\mathfrak{su}(2,2)$-rotation in imaginary angle

$$U = e^{-\frac{\pi}{4}(P_0 - K_0)} = e^{-\frac{\pi}{4}(L_0^+ - L_0^-)} = e^{-\frac{\pi}{4}(a_i^\dagger b_i^\dagger + b_i a_i)} \tag{C.0.1}$$

and a unitary $\mathfrak{su}(4)$-rotation

$$U_F = e^{-\frac{\pi}{4}(R_{13} - R_{31} + R_{24} - R_{42})} = e^{-\frac{\pi}{4}(c_i^\dagger d_i^\dagger - d_i c_i)} \ . \tag{C.0.2}$$

As it was suggested in [156], it is convenient to first to compute the action of a rotation in an arbitrary angle it

$$U_t = U_t^\dagger \equiv e^{t(a_i^\dagger b_i^\dagger + b_i a_i)}. \tag{C.0.3}$$

The action of U_t on the oscillators $a_i, a_i^\dagger, b_i, b_i^\dagger$ is

$$\begin{aligned} a_i(t) &\equiv U_t a_i U_t^{-1} = a_i \cos t - b_i^\dagger \sin t, \quad b_i(t) \equiv U_t b_i U_t^{-1} = b_i \cos t - a_i^\dagger \sin t, \\ a_i^\dagger(t) &\equiv U_t a_i^\dagger U_t^{-1} = a_i^\dagger \cos t + b_i \sin t, \quad b_i^\dagger(t) \equiv U_t b_i^\dagger U_t^{-1} = b_i^\dagger \cos t + a_i \sin t. \end{aligned} \tag{C.0.4}$$

From here one easily obtains the normal form of the operator U_t is [156]

$$U_t \equiv e^{t\left(a^\dagger b^\dagger + ba\right)} = \frac{1}{\cos^2 t}\ e^{\tan t\, a^\dagger b^\dagger} (\cos t)^{-a_i^\dagger a - b_i^\dagger b}\ e^{\tan t\, ba}, \tag{C.0.5}$$

or, in terms of the Lie-algebra generators,

$$U_t = e^{-t(L_0^+ - L_0^-)} = \frac{1}{\cos^2 t} \epsilon^{-L_0^+ \tan t} \cos(t)^{-2E} e^{L_0^- \tan t}. \tag{C.0.6}$$

Similarly one derives the normal form of the compact piece (C.0.2) by introducing the rotation at angle t,

$$U_t^F \equiv e^{t\left(c^\dagger d^\dagger + cd\right)} = \cos^2 t \; e^{\tan t \, c^\dagger d^\dagger} (\cos t)^{-\, c_i^\dagger c_i - d_i^\dagger d} \; e^{\tan t \, cd}. \tag{C.0.7}$$

In the normal form of the full operator, the $\cos t$ factors nicely cancel,

$$\begin{aligned} \mathrm{U}_t &\equiv e^{t\left(a^\dagger b^\dagger + ab + c^\dagger d^\dagger + cd\right)} \\ &= e^{\tan t \, (a^\dagger b^\dagger + c^\dagger d^\dagger)} e^{-\log\cos t (a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d} e^{\tan t \, (ab + cd)}. \end{aligned} \tag{C.0.8}$$

From (C.0.8) one obtains the regularized expression for the conjugate vacuum $|\bar{0}\rangle = |\bar{0}\rangle_B \otimes |\bar{0}\rangle_F$,

$$|\bar{0}\rangle \equiv \mathrm{U}^2 |0\rangle \approx e^{(a^\dagger b^\dagger + c^\dagger d^\dagger)/\epsilon} |0\rangle \tag{C.0.9}$$

$$\approx \frac{e^{a^\dagger b^\dagger/\epsilon}}{\epsilon^2} \; c_1^\dagger c_2^\dagger \; d_2^\dagger d_1^\dagger |0\rangle, \qquad \epsilon \to 0. \tag{C.0.10}$$

Appendix D

Neumann coefficients in large μ expansion

In this appendix, we list the leading order of Neumann coefficients in the large μ expansion. We take the same convention as in [164]. For $(m,n) \neq (0,0)$

$$
\begin{aligned}
N^{22}_{mn} &= \frac{(-1)^{m+n}}{4\pi\mu|\alpha_{(1)}|r}, \quad N^{23}_{mn} = \frac{(-1)^{m+1}}{4\pi\mu|\alpha_{(1)}|\sqrt{r(1-r)}} \\
N^{33}_{mn} &= \frac{1}{4\pi\mu|\alpha_{(1)}|(1-r)}, \quad N^{11}_{mn} = \frac{(-1)^{m+n+1}\sin(\pi mr)\sin(\pi nr)}{\pi\mu|\alpha_{(1)}|} \\
N^{21}_{mn} &= \frac{(-1)^{m+n+1}\sin\pi nr}{\pi\sqrt{r}(n-m/r)}, \quad N^{31}_{mn} = \frac{(-1)^{n}\sin(\pi nr)}{\pi\sqrt{1-r}(n-m/(1-r))}.
\end{aligned}
\tag{D.0.1}
$$

For $(m,n) = (0,0)$, we have

$$
\begin{aligned}
&N^{11}_{00} = 0, \quad N^{12}_{00} = -\sqrt{r}, \quad N^{13}_{00} = -\sqrt{1-r} \\
&N^{23}_{00} = -\frac{1}{4\pi\mu|\alpha_{(1)}|\sqrt{r(1-r)}}, \quad N^{22}_{00} = \frac{1}{4\pi\mu|\alpha_{(1)}r}, \quad N^{33}_{00} = \frac{1}{4\pi\mu|\alpha_{(1)}|(1-r)}
\end{aligned}
\tag{D.0.2}
$$

Bibliography

[1] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl.Phys.* **B72** (1974) 461.

[2] J. Maldacena, "The Large-N Limit of Superconformal Field Theories and Supergravity," *Inter. J. Theor. Phys.* **38** (1999) 1113–1133, `hep-th/9711200`.

[3] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, `arXiv:hep-th/9802109 [hep-th]`.

[4] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, `arXiv:hep-th/9802150 [hep-th]`.

[5] D. Berenstein, J. Maldacena, and H. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills," *JHEP* **4** (2002) 13, `hep-th/0202021`.

[6] J. Minahan and K. Zarembo, "The Bethe ansatz for N=4 superYang-Mills," *JHEP* **0303** (2003) 013, `arXiv:hep-th/0212208 [hep-th]`.

[7] I. Bena, J. Polchinski, and R. Roiban, "Hidden symmetries of the AdS(5) x S**5 superstring," *Phys.Rev.* **D69** (2004) 046002, `arXiv:hep-th/0305116 [hep-th]`.

[8] N. Beisert and *et al.*, "Review of AdS/CFT Integrability: An Overview," *Lett. Math. Phys.* **99** (2012) 3–32, `arXiv:1012.3982`.

[9] D. Serban, "Integrability and the AdS/CFT correspondence," *J.Phys.* **A44** (2011) 124001, `arXiv:1003.4214 [hep-th]`.

[10] N. Beisert, "The Dilatation operator of N=4 super Yang-Mills theory and integrability," *Phys.Rept.* **405** (2004) 1–202, `arXiv:hep-th/0407277 [hep-th]`.

[11] G. Arutyunov and S. Frolov, "Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring. Part I," *J.Phys.* **A42** (2009) 254003, `arXiv:0901.4937 [hep-th]`.

[12] N. Beisert, "The complete one loop dilatation operator of N=4 superYang-Mills theory," *Nucl.Phys.* **B676** (2004) 3–42, `arXiv:hep-th/0307015 [hep-th]`.

[13] N. Beisert and M. Staudacher, "The N=4 SYM integrable super spin chain," *Nucl.Phys.* **B670** (2003) 439–463, arXiv:hep-th/0307042 [hep-th].

[14] N. Beisert, C. Kristjansen, and M. Staudacher, "The Dilatation operator of conformal N=4 superYang-Mills theory," *Nucl.Phys.* **B664** (2003) 131–184, arXiv:hep-th/0303060 [hep-th].

[15] M. Staudacher, "The Factorized S-matrix of CFT/AdS," *JHEP* **0505** (2005) 054, arXiv:hep-th/0412188 [hep-th].

[16] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models," *Annals Phys.* **120** (1979) 253–291.

[17] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, "Classical/quantum integrability in AdS/CFT," *JHEP* **5** (2004) 24, hep-th/0402207.

[18] V. A. Kazakov and K. Zarembo, "Classical/quantum integrability in non-compact sector of AdS/CFT," *JHEP* **10** (2004) 60, hep-th/0410105.

[19] N. Beisert, V. A. Kazakov, and K. Sakai, "Algebraic Curve for the SO(6) Sector of AdS/CFT," *Comm. Math. Phys.* **263** (2006) 611–657, hep-th/0410253.

[20] N. Beisert, V. Kazakov, K. Sakai, and K. Zarembo, "Complete spectrum of long operators in N=4 SYM at one loop," *JHEP* **0507** (2005) 030, arXiv:hep-th/0503200 [hep-th].

[21] N. Beisert, V. Dippel, and M. Staudacher, "A Novel long range spin chain and planar N=4 super Yang-Mills," *JHEP* **0407** (2004) 075, arXiv:hep-th/0405001 [hep-th].

[22] N. Beisert and M. Staudacher, "Long-range psu(2,2—4) Bethe Ansatze for gauge theory and strings," *Nucl.Phys.* **B727** (2005) 1–62, arXiv:hep-th/0504190 [hep-th].

[23] N. Beisert, "The SU(2—2) dynamic S-matrix," *Adv.Theor.Math.Phys.* **12** (2008) 945–979, arXiv:hep-th/0511082 [hep-th].

[24] G. Arutyunov, S. Frolov, and M. Staudacher, "Bethe ansatz for quantum strings," *JHEP* **0410** (2004) 016, arXiv:hep-th/0406256 [hep-th].

[25] R. Hernandez and E. Lopez, "Quantum corrections to the string Bethe ansatz," *JHEP* **0607** (2006) 004, arXiv:hep-th/0603204 [hep-th].

[26] R. A. Janik, "The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry," *Phys.Rev.* **D73** (2006) 086006, arXiv:hep-th/0603038 [hep-th].

[27] N. Beisert, B. Eden, and M. Staudacher, "Transcendentality and Crossing," *J.Stat.Mech.* **0701** (2007) P01021, arXiv:hep-th/0610251 [hep-th].

[28] N. Dorey, D. M. Hofman, and J. M. Maldacena, "On the Singularities of the Magnon S-matrix," *Phys.Rev.* **D76** (2007) 025011, arXiv:hep-th/0703104 [HEP-TH].

[29] B. Basso, G. Korchemsky, and J. Kotanski, "Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling," *Phys.Rev.Lett.* **100** (2008) 091601, arXiv:0708.3933 [hep-th].

[30] I. Kostov, D. Serban, and D. Volin, "Functional BES equation," *JHEP* **0808** (2008) 101, arXiv:0801.2542 [hep-th].

[31] M. Luscher, "Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States," *Commun.Math.Phys.* **105** (1986) 153–188.

[32] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling: The Giant magnon," *Phys.Rev.* **D76** (2007) 126008, arXiv:0708.2208 [hep-th].

[33] Z. Bajnok and R. A. Janik, "Four-loop perturbative Konishi from strings and finite size effects for multiparticle states," *Nucl.Phys.* **B807** (2009) 625–650, arXiv:0807.0399 [hep-th].

[34] Z. Bajnok, A. Hegedus, R. A. Janik, and T. Lukowski, "Five loop Konishi from AdS/CFT," *Nucl.Phys.* **B827** (2010) 426–456, arXiv:0906.4062 [hep-th].

[35] A. B. Zamolodchikov, "On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories," *Phys.Lett.* **B253** (1991) 391–394.

[36] J. Ambjorn, R. A. Janik, and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain/string duality," *Nucl.Phys.* **B736** (2006) 288–301, arXiv:hep-th/0510171 [hep-th].

[37] D. Bombardelli, D. Fioravanti, and R. Tateo, "Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal," *J.Phys.* **A42** (2009) 375401, arXiv:0902.3930 [hep-th].

[38] N. Gromov, V. Kazakov, and P. Vieira, "Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory," *Phys.Rev.Lett.* **103** (2009) 131601, arXiv:0901.3753 [hep-th].

[39] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, "Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states," *Lett.Math.Phys.* **91** (2010) 265–287, arXiv:0902.4458 [hep-th].

[40] G. Arutyunov and S. Frolov, "Thermodynamic Bethe Ansatz for the AdS(5) x S(5) Mirror Model," *JHEP* **0905** (2009) 068, arXiv:0903.0141 [hep-th].

[41] G. Arutyunov, S. Frolov, and R. Suzuki, "Five-loop Konishi from the Mirror TBA," *JHEP* **1004** (2010) 069, arXiv:1002.1711 [hep-th].

[42] J. Balog and A. Hegedus, "5-loop Konishi from linearized TBA and the XXX magnet," *JHEP* **1006** (2010) 080, arXiv:1002.4142 [hep-th].

[43] A. Cavaglia, D. Fioravanti, and R. Tateo, "Extended Y-system for the AdS_5/CFT_4 correspondence," *Nucl.Phys.* **B843** (2011) 302–343, arXiv:1005.3016 [hep-th].

[44] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Solving the AdS/CFT Y-system," *JHEP* **1207** (2012) 023, arXiv:1110.0562 [hep-th].

[45] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Quantum Spectral Curve for Planar $\mathcal{N}$ = Super-Yang-Mills Theory," *Phys.Rev.Lett.* **112** no. 1, (2014) 011602, arXiv:1305.1939 [hep-th].

[46] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4," arXiv:1405.4857 [hep-th].

[47] K. Okuyama and L.-S. Tseng, "Three-Point Functions in N=4 SYM Theory at One-Loop," *JHEP* **8** (2004) 55, hep-th/0404190.

[48] R. Roiban and A. Volovich, "Yang-Mills Correlation Functions from Integrable Spin Chains," *JHEP* **9** (2004) 32, hep-th/0407140.

[49] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring three-point functions and integrability," *JHEP* **9** (2011) 28, arXiv:1012.2475.

[50] O. Foda, "$\mathcal{N} = 4$ SYM structure constants as determinants," *JHEP* **3** (2012) 96, arXiv:1111.4663.

[51] O. Foda, Y. Jiang, I. Kostov, and D. Serban, "A tree-level 3-point function in the su(3)-sector of planar SYM," *JHEP* **10** (2013) 138, arXiv:1302.3539.

[52] P. Vieira and T. Wang, "Tailoring Non-Compact Spin Chains," *JHEP* **1410** (2014) 35, arXiv:1311.6404 [hep-th].

[53] J. Caetano and T. Fleury, "Three-point functions and $\mathfrak{su}(1|1)$ spin chains," *JHEP* **1409** (2014) 173, arXiv:1404.4128 [hep-th].

[54] L. F. Alday, J. R. David, E. Gava, and K. Narain, "Structure constants of planar N = 4 Yang Mills at one loop," *JHEP* **0509** (2005) 070, arXiv:hep-th/0502186 [hep-th].

[55] Y. Jiang, I. Kostov, F. Loebbert, and D. Serban, "Fixing the Quantum Three-Point Function," *JHEP* **1404** (2014) 019, arXiv:1401.0384 [hep-th].

[56] K. Zarembo, "Holographic three-point functions of semiclassical states," *Journal of High Energy Physics* **9** (Sept., 2010) 30, arXiv:1008.1059 [hep-th].

[57] M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, "On three-point correlation functions in the gauge/gravity duality," *JHEP* **11** (2010) 141, arXiv:1008.1070.

[58] R. Roiban and A. Tseytlin, "On semiclassical computation of 3-point functions of closed string vertex operators in AdS5xS5,".

[59] Z. Bajnok, R. A. Janik, and A. Wereszczyski, "HHL correlators, orbit averaging and form factors," *JHEP* **1409** (2014) 050, arXiv:1404.4556 [hep-th].

[60] R. A. Janik, P. Surowka, and A. Wereszczynski, "On correlation functions of operators dual to classical spinning string states," *JHEP* **1005** (2010) 030, arXiv:1002.4613 [hep-th].

[61] R. A. Janik and A. Wereszczynski, "Correlation functions of three heavy operators: The AdS contribution," *JHEP* **1112** (2011) 095, arXiv:1109.6262 [hep-th].

[62] Y. Kazama and S. Komatsu, "On holographic three point functions for GKP strings from integrability," *JHEP* **1201** (2012) 110, arXiv:1110.3949 [hep-th].

[63] Y. Kazama and S. Komatsu, "Wave functions and correlation functions for GKP strings from integrability," *JHEP* **1209** (2012) 022, arXiv:1205.6060 [hep-th].

[64] Y. Kazama and S. Komatsu, "Three-point functions in the SU(2) sector at strong coupling," *JHEP* **1403** (2014) 052, arXiv:1312.3727 [hep-th].

[65] J. A. Minahan, "Holographic three-point functions for short operators," *JHEP* **1207** (2012) 187, arXiv:1206.3129 [hep-th].

[66] T. Bargheer, J. A. Minahan, and R. Pereira, "Computing Three-Point Functions for Short Operators," *JHEP* **1403** (2014) 096, arXiv:1311.7461 [hep-th].

[67] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability II. Weak/strong coupling match," arXiv:1104.5501 [hep-th].

[68] A. Bissi, T. Harmark, and M. Orselli, "Holographic 3-Point Function at One Loop," *JHEP* **1202** (2012) 133, arXiv:1112.5075 [hep-th].

[69] G. Georgiou, "SL(2) sector: weak/strong coupling agreement of three-point correlators," *JHEP* **1109** (2011) 132, arXiv:1107.1850 [hep-th].

[70] N. Gromov, A. Sever, and P. Vieira, "Tailoring three-point functions and integrability III. Classical tunneling," *JHEP* **7** (2012) 44, arXiv:1111.2349.

[71] I. Kostov, "Classical Limit of the Three-Point Function of N=4 Supersymmetric Yang-Mills Theory from Integrability," *Phys. Rev. Lett.* **108** no. 26, (2012) 261604, arXiv:1203.6180.

[72] I. Kostov, "Three-point function of semiclassical states at weak coupling," *J. Phys. A* **45** (2012) 4018, arXiv:1205.4412.

[73] T. Klose and T. McLoughlin, "Worldsheet Form Factors in AdS/CFT," *Phys.Rev.* **D87** no. 2, (2013) 026004, arXiv:1208.2020 [hep-th].

[74] T. Klose and T. McLoughlin, "Comments on World-Sheet Form Factors in AdS/CFT," *J.Phys.* **A47** no. 5, (2014) 055401, arXiv:1307.3506.

[75] Z. Bajnok and R. A. Janik, "String field theory vertex from integrability," arXiv:1501.04533 [hep-th].

[76] B. Pozsgay and G. Takacs, "Form-factors in finite volume I: Form-factor bootstrap and truncated conformal space," *Nucl.Phys.* **B788** (2008) 167–208, arXiv:0706.1445 [hep-th].

[77] B. Pozsgay and G. Takacs, "Form factors in finite volume. II. Disconnected terms and finite temperature correlators," *Nucl.Phys.* **B788** (2008) 209–251, arXiv:0706.3605 [hep-th].

[78] Y. Jiang, I. Kostov, A. Petrovskii, and D. Serban, "String Bits and the Spin Vertex," *Nucl.Phys.* **B897** (2015) 374–404, arXiv:1410.8860 [hep-th].

[79] B. Basso, S. Komatsu, and P. Vieira, "Structure Constants and Integrable Bootstrap in Planar N=4 SYM Theory," arXiv:1505.06745 [hep-th].

[80] M. F. Sohnius and P. C. West, "Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory," *Phys.Lett.* **B100** (1981) 245.

[81] S. Mandelstam, "Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model," *Nucl.Phys.* **B213** (1983) 149–168.

[82] L. Brink, O. Lindgren, and B. E. Nilsson, "The Ultraviolet Finiteness of the N=4 Yang-Mills Theory," *Phys.Lett.* **B123** (1983) 323.

[83] B. Basso, A. Sever, and P. Vieira, "Space-time S-matrix and Flux tube S-matrix II. Extracting and Matching Data," *JHEP* **1401** (2014) 008, `arXiv:1306.2058 [hep-th]`.

[84] B. Basso, A. Sever, and P. Vieira, "Spacetime and Flux Tube S-Matrices at Finite Coupling for N=4 Supersymmetric Yang-Mills Theory," *Phys.Rev.Lett.* **111** no. 9, (2013) 091602, `arXiv:1303.1396 [hep-th]`.

[85] B. Basso, A. Sever, and P. Vieira, "Space-time S-matrix and Flux-tube S-matrix III. The two-particle contributions," *JHEP* **1408** (2014) 085, `arXiv:1402.3307 [hep-th]`.

[86] B. Basso, A. Sever, and P. Vieira, "Space-time S-matrix and Flux-tube S-matrix IV. Gluons and Fusion," *JHEP* **1409** (2014) 149, `arXiv:1407.1736 [hep-th]`.

[87] D. Correa, J. Maldacena, and A. Sever, "The quark anti-quark potential and the cusp anomalous dimension from a TBA equation," *JHEP* **1208** (2012) 134, `arXiv:1203.1913 [hep-th]`.

[88] N. Gromov and A. Sever, "Analytic Solution of Bremsstrahlung TBA," *JHEP* **1211** (2012) 075, `arXiv:1207.5489 [hep-th]`.

[89] P. M. P. Di Francesco and D. Séneéchal, *Conformal Field Theory*. Springer-Verlag New York, inc., U.S.A., 1997.

[90] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, "Spinning Conformal Correlators," *JHEP* **1111** (2011) 071, `arXiv:1107.3554 [hep-th]`.

[91] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, "Bounding scalar operator dimensions in 4D CFT," *JHEP* **0812** (2008) 031, `arXiv:0807.0004 [hep-th]`.

[92] V. S. Rychkov and A. Vichi, "Universal Constraints on Conformal Operator Dimensions," *Phys.Rev.* **D80** (2009) 045006, `arXiv:0905.2211 [hep-th]`.

[93] H. Bethe, "Zur theorie der metalle," *Z. Phys.* **71** no. 3-4, (1931) 205–226.

[94] N. A. Slavnov, "Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz," *Theor. Math. Phys.* **79** (1989) 502–208.

[95] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge University Press, UK, 1993.

[96] N. Kitanine, J. M. Maillet, and V. Terras, "Form factors of the XXZ Heisenberg spin- 1/2 finite chain," *Nucl. Phys.* **B554** (1999) 647–678, `math-ph/9807020`.

[97] M. Gaudin, B. M. McCoy, and T. T. Wu, "Normalization sum for the bethe's hypothesis wave functions of the heisenberg-ising chain," *Phys. Rev. D* **23** (1981) 417–419.

[98] V. E. Korepin, "Calculation of norms of bethe wave functions," *Comm. Math. Phys.* **86** no. 3, (1982) 391–418.

[99] A. G. Izergin, "Partition function of the six-vertex model in a finite volume," in *Soviet Physics Doklady*, vol. 32, p. 878. 1987.

[100] A. G. Izergin, D. A. Coker, and V. E. Korepin, "Determinant formula for the six-vertex model," *J. Phys. A* **25** no. 16, (1992) 4315.

[101] I. Kostov and Y. Matsuo, "Inner products of Bethe states as partial domain wall partition functions," *JHEP* **10** (2012) 168, `arXiv:1207.2562 [hep-th]`.

[102] E. Bettelheim and I. Kostov, "Semi-classical analysis of the inner product of Bethe states," *J.Phys.* **A47** (2014) 245401, `arXiv:1403.0358 [hep-th]`.

[103] B. Sutherland, "Low-lying eigenstates of the one-dimensional heisenberg ferromagnet for any magnetization and momentum," *Phys. Rev. Lett.* **74** (1995) 816–819. `http://link.aps.org/doi/10.1103/PhysRevLett.74.816`.

[104] A. Dhar and B. Sriram Shastry, "Bloch walls and macroscopic string states in bethe's solution of the heisenberg ferromagnetic linear chain," *Phys. Rev. Lett.* **85** (2000) 2813–2816. `http://link.aps.org/doi/10.1103/PhysRevLett.85.2813`.

[105] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, "The Algebraic Curve of Classical Superstrings on $\mathrm{AdS}_5 \times S^5$," *Comm. Math. Phys.* **263** (2006) 659–710, `hep-th/0502226`.

[106] A. Izergin and V. Korepin, "$\mathcal{N} = 4$ SYM structure constants as determinants," *Comm. Math. Phys.* **94** (1984) 67–92.

[107] V. E. Korepin, "Calculation of norms of Bethe wave functions," *Comm. Math. Phys.* **86** (1982) 391–418.

[108] E. Sobko, "A new representation for two- and three-point correlators of operators from $sl(2)$ sector," *JHEP* **1412** (2014) 101, `arXiv:1311.6957 [hep-th]`.

[109] J. Escobedo, "Integrability in AdS/CFT: Exact Results for Correlation Functions,".

[110] P. P. Kulish and N. Y. Reshetikhin, "Diagonalisation of $gl(n)$ invariant transfer matrices and quantum n-wave system (lee model)," *J. Phys. A* **16** no. 16, (1983) L591.

[111] S. Belliard and E. Ragoucy, "The nested bethe ansatz for 'all' closed spin chains," *J. Phys. A* **41** (2008) C5202, `arXiv:0804.2822 [math-ph]`.

[112] M. Wheeler, "Scalar products in generalized models with SU(3)-symmetry," *Commun.Math.Phys.* **327** (2014) 737–777, `arXiv:1204.2089 [math-ph]`.

[113] N. Y. Reshetikhin, "Calculation of the norm of bethe vectors in models with $su(3)$-symmetry," *J. Sov. Math.* **46** no. 1, (1989) 1694–1706.

[114] N. Gromov and P. Vieira, "Complete 1-loop test of AdS/CFT," *JHEP* **0804** (2008) 046, `arXiv:0709.3487 [hep-th]`.

[115] S. Belliard, S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "The algebraic bethe ansatz for scalar products in su(3)-invariant integrable models," *J. Stat. Mech.* **10** (2012) 17, `arXiv:1207.0956 [math-ph]`.

[116] N. Beisert, J. A. Minahan, M. Staudacher, and K. Zarembo, "Stringing spins and spinning strings," *JHEP* **09** (2003) 010, `arXiv:hep-th/0306139`.

[117] M. Takahashi, "One-Dimensional Hubbard Model at Finite Temperature," *Prog. Theo. Phys.* **47** (1972) 69–82.

[118] T. Bargheer, N. Beisert, and F. Loebbert, "Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains," *J.Stat.Mech.* **0811** (2008) L11001, `arXiv:0807.5081 [hep-th]`.

[119] T. Bargheer, N. Beisert, and F. Loebbert, "Long-Range Deformations for Integrable Spin Chains," *J.Phys.* **A42** (2009) 285205, `arXiv:0902.0956 [hep-th]`.

[120] R. Baxter, "Corner transfer matrices of the eight-vertex model. 1. Low-temperature expansions and conjectured properties," *J.Statist.Phys.* **15** (1976) 485–503.

[121] H. B. Thacker, "Corner Transfer Matrices and Lorentz Invariance on a Lattice," *Physica* **18D** (1986) 348–359.

[122] H. Itoyama and H. Thacker, "Lattice Virasoro Algebra and Corner Transfer Matrices in the Baxter Eight Vertex Model," *Phys.Rev.Lett.* **58** (1987) 1395.

[123] M. Tetelman, "Lorentz group for two-dimensional integrable lattice systems," *Sov. Phys. JETP.* **55** (1982) 306.

[124] E. Sklyanin, "Quantum inverse scattering method. Selected topics," arXiv:hep-th/9211111 [hep-th].

[125] F. Loebbert, "Integrable Spin Chains in $\mathcal{N} = 4$ super Yang–Mills Theory,". PhD thesis.

[126] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, "Yang-baxter equation in long-range interacting systems," *J Phys. A* **26** no. 20, (1993) 5219.

[127] D. Serban and M. Staudacher, "Planar N=4 gauge theory and the Inozemtsev long range spin chain," *JHEP* **0406** (2004) 001, arXiv:hep-th/0401057 [hep-th].

[128] D. Serban, "A note on the eigenvectors of long-range spin chains and their scalar products," *JHEP* **1301** (2013) 012, arXiv:1203.5842 [hep-th].

[129] D. Serban, "Eigenvectors and scalar products for long range interacting spin chains II: the finite size effects," *JHEP* **1308** (2013) 128, arXiv:1302.3350 [hep-th].

[130] T. Bargheer, N. Beisert, and F. Loebbert, "Long-Range Deformations for Integrable Spin Chains," *J.Phys.* **A42** (2009) 285205, arXiv:0902.0956 [hep-th].

[131] N. Gromov and P. Vieira, "Quantum Integrability for Three-Point Functions of Maximally Supersymmetric Yang-Mills Theory," *Phys.Rev.Lett.* **111** no. 21, (2013) 211601, arXiv:1202.4103 [hep-th].

[132] N. Gromov and P. Vieira, "Tailoring Three-Point Functions and Integrability IV. Theta-morphism," arXiv:1205.5288.

[133] V. Inozemtsev, "Integrable Heisenberg-van Vleck chains with variable range exchange," *Phys.Part.Nucl.* **34** (2003) 166–193, arXiv:hep-th/0201001 [hep-th].

[134] F. D. M. Haldane, "Exact jastrow-gutzwiller resonating-valence-bond ground state of the spin-1/2 antiferromagnetic heisenberg chain with $1/r^2$ exchange," *Phys. Rev. Lett.* **60** (1988) 635–638.

[135] B. S. Shastry, "Exact solution of an $s = 1/2$ heisenberg antiferromagnetic chain with long-ranged interactions," *Phys. Rev. Lett.* **60** (1988) 639–642.

[136] A. Rej, D. Serban, and M. Staudacher, "Planar N=4 gauge theory and the Hubbard model," *JHEP* **0603** (2006) 018, arXiv:hep-th/0512077 [hep-th].

[137] F. Loebbert, "Recursion Relations for Long-Range Integrable Spin Chains with Open Boundary Conditions," *Phys.Rev.* **D85** (2012) 086008, arXiv:1201.0888 [hep-th].

[138] N. Beisert, L. Fievet, M. de Leeuw, and F. Loebbert, "Integrable Deformations of the XXZ Spin Chain," arXiv:1308.1584 [math-ph].

[139] N. Beisert, B. Eden, and M. Staudacher, "Transcendentality and crossing," *J. Stat. Mech.* **0701** (2007) P021, arXiv:hep-th/0610251.

[140] J. Caetano and T. Fleury, "Three-Point Functions and $\mathfrak{su}(1|1)$ Spin Chains," arXiv:1404.4128.

[141] N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability III. Classical Tunneling," *ArXiv e-prints* (Nov., 2011) .

[142] I. Kostov, "Three-point function of semiclassical states at weak coupling," *ArXiv e-prints* (May, 2012) , arXiv:1205.4412 [hep-th].

[143] S. Frolov and A. A. Tseytlin, "Semiclassical quantization of rotating superstring in AdS5xS5," *Journal of High Energy Physics* **6** (June, 2002) 7, arXiv:hep-th/0204226.

[144] J. Callan, Curtis G., H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson, *et al.*, "Quantizing string theory in AdS(5) x S**5: Beyond the pp wave," *Nucl.Phys.* **B673** (2003) 3–40, arXiv:hep-th/0307032 [hep-th].

[145] D. Serban and M. Staudacher, "Planar N=4 gauge theory and the Inozemtsev long range spin chain," *JHEP* **0406** (2004) 001, arXiv:hep-th/0401057 [hep-th].

[146] S. Komatsu, "Private discussions.,".

[147] J. A. Minahan, "The SU(2) sector in AdS/CFT," *Fortsch. Phys.* **53** (2005) 828–838, arXiv:hep-th/0503143.

[148] V. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, "Classical / quantum integrability in AdS/CFT," *JHEP* **05** (2004) 024, arXiv:hep-th/0402207.

[149] N. Beisert, V. Kazakov, and K. Sakai, "Algebraic curve for the SO(6) sector of AdS/CFT," *Commun.Math.Phys.* **263** (2006) 611–657, arXiv:hep-th/0410253 [hep-th].

[150] Y. Kazama, S. Komatsu, and T. Nishimura, "Novel construction and the monodromy relation for three-point functions at weak coupling," *JHEP* **1501** (2015) 095, arXiv:1410.8533 [hep-th].

[151] Y. Kazama, S. Komatsu, and T. Nishimura, "To appear,".

[152] L. Ferro, T. ?ukowski, C. Meneghelli, J. Plefka, and M. Staudacher, "Spectral Parameters for Scattering Amplitudes in N=4 Super Yang-Mills Theory," *JHEP* **1401** (2014) 094, arXiv:1308.3494 [hep-th].

[153] R. Frassek, N. Kanning, Y. Ko, and M. Staudacher, "Bethe Ansatz for Yangian Invariants: Towards Super Yang-Mills Scattering Amplitudes," *Nucl.Phys.* **B883** (2014) 373–424, arXiv:1312.1693 [math-ph].

[154] N. Kanning, T. Lukowski, and M. Staudacher, "A shortcut to general tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM via integrability," *Fortsch.Phys.* **62** (2014) 556–572, arXiv:1403.3382 [hep-th].

[155] L. Ferro, T. ?ukowski, and M. Staudacher, "$\mathcal{N} = 4$ scattering amplitudes and the deformed Gra?mannian," *Nucl.Phys.* **B889** (2014) 192–206, arXiv:1407.6736 [hep-th].

[156] L. F. Alday, J. R. David, E. Gava, and K. Narain, "Towards a string bit formulation of N=4 super Yang-Mills," *JHEP* **0604** (2006) 014, arXiv:hep-th/0510264 [hep-th].

[157] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background," *Phys. Rev. D* **66** no. 8, (2002) 086004, hep-th/0204146.

[158] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background II," *JHEP* **1** (2003) 36, hep-th/0206073.

[159] A. Pankiewicz, "More comments on superstring interactions in the pp wave background," *JHEP* **0209** (2002) 056, arXiv:hep-th/0208209 [hep-th].

[160] A. Pankiewicz and J. Stefanski, B., "PP wave light cone superstring field theory," *Nucl.Phys.* **B657** (2003) 79–106, arXiv:hep-th/0210246 [hep-th].

[161] A. Pankiewicz, "An Alternative formulation of light cone string field theory on the plane wave," *JHEP* **0306** (2003) 047, arXiv:hep-th/0304232 [hep-th].

[162] J. Plefka, "Lectures on Plane-Wave String/Gauge Theory Duality," *Fortsch.Phys.* **52** (2004) 264–301, arXiv:hep-th/0307101.

[163] A. Pankiewicz, "Strings in plane wave backgrounds," *Fortsch.Phys.* **51** (2003) 1139–1203, arXiv:hep-th/0307027 [hep-th].

[164] S. Dobashi and T. Yoneya, "Impurity non-preserving 3-point correlators of BMN operators from PP-wave holography I: bosonic excitations," *Nucl. Phys. B* **711** (2005) 54–82, hep-th/0409058.

[165] S. Dobashi and T. Yoneya, "Resolving the holography in the plane-wave limit of AdS/CFT correspondence," *Nucl. Phys. B* **711** (2005) 3–53, hep-th/0406225.

[166] P. Di Vecchia, J. L. Petersen, M. Petrini, R. Russo, and A. Tanzini, "The Three string vertex and the AdS / CFT duality in the PP wave limit," *Class.Quant.Grav.* **21** (2004) 2221–2240, arXiv:hep-th/0304025 [hep-th].

[167] W. Schulgin and A. Zayakin, "Three-BMN Correlation Functions: Integrability vs. String Field Theory. One-Loop Mismatch," *JHEP* **1310** (2013) 053, arXiv:1305.3198 [hep-th].

[168] Y.-H. He, J. H. Schwarz, M. Spradlin, and A. Volovich, "Explicit formulas for Neumann coefficients in the plane wave geometry," *Phys.Rev.* **D67** (2003) 086005, arXiv:hep-th/0211198 [hep-th].

[169] J. Lucietti, S. Schafer-Nameki, and A. Sinha, "On the plane wave cubic vertex," *Phys.Rev.* **D70** (2004) 026005, arXiv:hep-th/0402185 [hep-th].

[170] F. Smirnov, "Form-factors in completely integrable models of quantum field theory," *Adv.Ser.Math.Phys.* **14** (1992) 1–208.

[171] T. Klose and T. McLoughlin, "Worldsheet Form Factors in AdS/CFT," *Phys.Rev.* **D87** (2013) 026004, arXiv:1208.2020 [hep-th].

[172] G. Delfino, G. Mussardo, and P. Simonetti, "Nonintegrable quantum field theories as perturbations of certain integrable models," *Nucl.Phys.* **B473** (1996) 469–508, arXiv:hep-th/9603011 [hep-th].

[173] A. Leclair and G. Mussardo, "Finite temperature correlation functions in integrable QFT," *Nucl.Phys.* **B552** (1999) 624–642, arXiv:hep-th/9902075 [hep-th].

[174] B. Pozsgay, "Form factor approach to diagonal finite volume matrix elements in Integrable QFT," *JHEP* **1307** (2013) 157, arXiv:1305.3373 [hep-th].

[175] H. Saleur, "A Comment on finite temperature correlations in integrable QFT," *Nucl.Phys.* **B567** (2000) 602–610, arXiv:hep-th/9909019 [hep-th].

[176] J. Balog, "Private discussions.,".

[177] J. Maillet and V. Terras, "On the quantum inverse scattering problem," *Nucl.Phys.* **B575** (2000) 627–644, arXiv:hep-th/9911030 [hep-th].

[178] N. Kitanine, J. Maillet, N. Slavnov, and V. Terras, "On the algebraic Bethe ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain," `arXiv:hep-th/0505006 [hep-th]`.