EXISTENCE OF BOUTROUX CURVES, g-FUNCTIONS AND SPECTRAL NETWORKS FROM NEWTON'S POLYGON

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ABSTRACT. We prove the existence of an algebraic plane curve of equation $P(x,y)=0$, with prescribed asymptotic behaviors at punctures, and with the Boutroux property, namely, periods have vanishing real part, i.e, $\mathrm{Re}(\int_\gamma y\,dx)=0$ for every closed loop γ. This has applications in the Riemann-Hilbert problem, in random matrix theory, in spectral networks, in WKB analysis and Stokes phenomenon, in algebraic and enumerative geometry, and many applications in mathematical physics. From Newton's polygon we can define an affine space such that there exists always a Boutroux curve. This result is applied to random matrix and asymptotic theory, in which a key ingredient is called the g-function, the function $g(x)=\int_o^x Y\,dX$ is a g-function precisely if and only if the algebraic plane curve is a Boutroux curve.

1. INTRODUCTION

In all what follows, $P(x,y)\in\mathbb{C}[x,y]$ is a bivariate complex polynomial. $P'_x(x,y)$ and $P'_y(x,y)$ denotes its partial derivatives

$$P'_x(x,y)=\frac{\partial}{\partial x}P(x,y),\quad P'_y(x,y)=\frac{\partial}{\partial y}P(x,y). \tag{1.1}$$

1.1. **Purpose and results.**

Let $P(x,y)\in\mathbb{C}[x,y]$, we define the Newton's polygon as a convex polytope $\mathcal{N}\subset\mathbb{Z}_+\times\mathbb{Z}_+$

$$\mathcal{N}=\{(i,j)\mid \mathcal{P}_{i,j}\neq 0\}. \tag{1.2}$$

It is a vector space $\mathbb{C}[\mathcal{N}]\sim\mathbb{C}^{\#\mathcal{N}}$.

We define the interior of the Newton's polygon (but shifted by $(-1,-1)$):

$$\mathring{\mathcal{N}}=\{(i-1,j-1)\mid (i,j) \text{ strictly interior of convex envelope}\}. \tag{1.3}$$

Let $\mathcal{P}\in\mathbb{C}[\mathcal{N}]$ a once for all fixed bivariate polynomial. We want to study the set of bivariate polynomials that differ from $\mathcal{P}$ just from the interior, i.e. the affine space:

$$\mathcal{M}=\mathcal{P}+\mathbb{C}[\mathring{\mathcal{N}}]. \tag{1.4}$$

Fixing the exterior part (i.e. $\mathcal{P}$) is equivalent to fixing the asymptotic behaviors of solutions $y=Y(x)$ of $P(x,y)=0$ at points where x and/or y tend to ∞ (called punctures).

Our goal is to prove that there exists $P_{\text{Boutroux}}\in\mathcal{M}$, that has the Boutroux property:

$$\forall\,\gamma=\text{Jordan loop}\qquad \mathrm{Re}\oint_\gamma Y dx=0. \tag{1.5}$$

Boutroux curves have many applications:

- For example in asymptotic theory, the Riemann-Hilbert method of [DZ92; Dei+99] strongly relies on the existence of a so-called g-*function*, whose differential $dg=y\,dx$ has prescribed asymptotic behaviors and has the Boutroux property. In some sense this article provides a **theorem of existence of g-functions**.

- Also in geometry, cutting surfaces along some "horizontal trajectories" is a way to make combinatorial models of moduli spaces of surfaces. This was used by Strebel, Harrer-Zagier, Kontsevich, Penner, Thurston, and many others [Str84; W T02; Kon92; HZ86; Pen03a; Pen03b].

- A seminal work of Gaiotto, Moore, and Neitzke relates WKB asymptotic expansion to "spectral networks" [GMN13], and is again closely related to Boutroux curves.

All these authors have considered foliations of surfaces by cutting along "horizontal trajectories". Horizontal trajectories of the differential form $\frac{1}{2\pi i} y dx$ give a good foliation, typically when it has the Boutroux property. The existence of this differential, and thus the existence of this foliation, is what gives the bijection between the combinatorial moduli space and the geometric moduli space. During an IHES seminar, M. Kontsevich was quoted saying that if proved, "this theorem of existence would be the most useful tool possible".

Our method is to obtain the Boutroux curve by minimizing some real function called "Energy" $F : \mathcal{M} \to \mathbb{R}$, which can be interpreted as the "area" of the curve. In other words, the Boutroux curve will be the "minimal surface".

The plan of the article is:

Section 1 is a brief introduction to the property of Boutroux curves, and also to different applications of this property ranging from the existence of g-functions to foliations of surfaces by the so-called spectral networks.

Section 2: we recall basic notions about Newton's polygon, algebraic Riemann surfaces and plane curves. We shall in particular introduce "times" and "periods".

Section 3: we define the energy as a regularized area of the surface, by removing some small discs around punctures and adding appropriate correction terms. Then proving that the energy is bounded from below, continuous and with tight compact level sets. This will imply the existence of a minimum (the intersection of all decreasing compact level sets is a non-empty compact). In addition we rewrite the energy as a function of times and periods (this requires choosing a basis of cycles on the curve, called a "marking").

Section 4: we can finally prove that the minimum of the energy is a Boutroux curve.

Section 5: we associate a spectral network to a Boutroux curve. This is in fact done in two ways. The first kind of the spectral network is similar to the notion of "Strebel graph", and provides a canonical atlas of the curve, made of rectangular pieces.

Section 6: the second kind of spectral network associated to a Boutroux curve, is the one useful for random matrices, and spectral networks as in [GMN13].

Section 7: we study examples of applications, like Strebel graphs, and random matrices.

Section 8: we gather a number of concluding remarks.

2. Newton's polygon and algebraic curves

2.1. Newton's polygon.

Newton's polygon. From now on we choose $\mathcal{P} = \sum_{i,j} \mathcal{P}_{i,j} x^i y^j \in \mathbb{C}[x, y]$ a bivariate polynomial, fixed once for all. Let $\mathcal{N} = \{(i, j) \mid \mathcal{P}_{i,j} \neq 0\}$.

We require that $\mathcal{N}$ has at least three points non aligned.

We define $\mathcal{P}_d(x)$ the coefficient of y^d the highest degree term in y of $\mathcal{P}$.

Our goal is to study the space of bivariate polynomials that differ from $\mathcal{P}$ only by "interior" coefficients.

Definition 2.1 (Newton's polygon)**.** *The Newton's polytope*

$$\mathcal{N} := \{(i,j) \in \mathbb{Z}^2 \mid \mathcal{P}_{i,j} \neq 0\}. \tag{2.1}$$

is a set of points in $\mathbb{Z}_+ \times \mathbb{Z}_+$. We define its completion with all integer points enclosed within its convex envelope:

$$\bar{\mathcal{N}} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, (i,j) \in \textit{ inside or on the boundary of the convex envelope of } \mathcal{N}\}. \tag{2.2}$$

*We define its **interior** $\overset{\circ}{\mathcal{N}} \subset \mathbb{Z} \times \mathbb{Z}$, shifted by $(-1,-1)$:*

$$\overset{\circ}{\mathcal{N}} := \{(i,j) \in \bar{\mathcal{N}} \,|\, (i+1, j+1) \in \textit{ strictly interior of the convex envelope of } \mathcal{N}\}. \tag{2.3}$$

*and its **boundary** (the integer points of the convex envelope)*

$$\partial\mathcal{N} := \bar{\mathcal{N}} \setminus (\overset{\circ}{\mathcal{N}} + (1,1)), \tag{2.4}$$

and we define

$$\mathcal{N}''' := \{(i,j) \in \bar{\mathcal{N}} \,|\, (i+1, j+1) \in \partial\mathcal{N}\} = \textit{"3rd kind points"} \tag{2.5}$$

$$\mathcal{N}'' := \{(i,j) \in \bar{\mathcal{N}} \,|\, (i+1, j+1) \ \notin \bar{\mathcal{N}}\} = \textit{"2nd kind points"}. \tag{2.6}$$

The points of $\overset{\circ}{\mathcal{N}}$ are also called "1st kind".

- *1st kind $\overset{\circ}{\mathcal{N}}$:= interior : $(i+1, j+1) \in$ strict interior*
- *3rd kind $\mathcal{N}'''$:= boundary : $(i+1, j+1) \in$ boundary*
- *2nd kind $\mathcal{N}''$:= exterior : $(i+1, j+1) \in$ exterior*

To recall why they are called 1st, 2nd or 3rd kind, cf lectures notes [Eyn18].

We want now to study the moduli space of polynomials sharing the same exterior and boundary as $\mathcal{P}$, i.e. differ only by interior points.

Definition 2.2 (Moduli space of a Newton polygon)**.** *If $\deg \mathcal{P}_d(x) = 0$, we let*

$$\mathbb{C}[\overset{\circ}{\mathcal{N}}] := \left\{ Q(x,y) \in \mathbb{C}[x,y] \,|\, Q = \sum_{(i,j) \in \overset{\circ}{\mathcal{N}}} Q_{i,j} x^i y^j \right\}. \tag{2.7}$$

It is a complex vector space of dimension $\#\overset{\circ}{\mathcal{N}}$, or a real vector space of dimension $2\#\overset{\circ}{\mathcal{N}}$.

In the general case, if $\deg \mathcal{P}_d(x) > 0$, let

$$\begin{aligned} \mathbb{C}[\overset{\circ}{\mathcal{N}}] \quad := \quad & \left\{ Q(x,y) \in \mathbb{C}[x,y] \,|\, Q = \sum_{(i,j) \in \overset{\circ}{\mathcal{N}}} Q_{i,j} x^i y^j \right\} \\ & \bigcap_{x_0 = \textit{zero of } \mathcal{P}_d} \left\{ Q(x,y) = \sum_{(i,j) \in \overset{\circ}{\mathcal{N}}(\mathcal{P}(x+x_0, y))} \tilde{Q}_{i,j} (x - x_0)^i y^j \right\}. \end{aligned} \tag{2.8}$$

In both cases we define the moduli-space

$$\mathcal{M}(\mathcal{P}) := \mathcal{P} + \mathbb{C}[\overset{\circ}{\mathcal{N}}], \tag{2.9}$$

which is a complex affine space. It is equipped with the canonical topology of $\mathbb{C}^{\dim \mathcal{M}}$.

Since we work with a once for all fixed $\mathcal{P}$, for easier readability we shall drop $\mathcal{P}$ from the notations and write

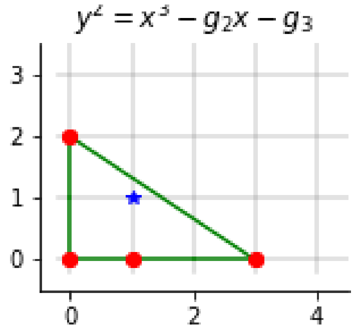
$$\mathcal{M} = \mathcal{M}(\mathcal{P}). \tag{2.10}$$

Remark 2.1. It may seem an "overkill" to call $\mathcal{M}$ a "moduli-space", because it is just an affine space. However, we shall later decompose it into strata by genus $\mathcal{M} = \cup_g \mathcal{M}^{(g)}$, which correspond to usual notions of moduli spaces.

Remark 2.2. **[Hypothesis 1**: $\mathbb{C}[\mathring{\mathcal{N}}] \neq 0]$ From now on, we shall only consider the case with $\mathbb{C}[\mathring{\mathcal{N}}] \neq 0$. Proving the Boutroux curve when $\mathbb{C}[\mathring{\mathcal{N}}] = 0$ is trivial.

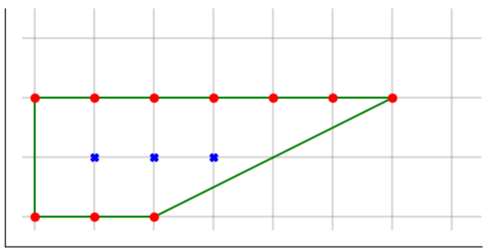
We shall often illustrate our proposal with the following examples:

Example 2.1 (Weierstrass curve). $\mathcal{P}(x,y) = y^2 - x^3 + g_2 x + g_3$ *has the following Newton's polygon*



where red points represents non-zero coefficients $\mathcal{P}_{i,j}$, the dots . represent zero coefficients, and $*$ in position $(1,1)$ is the only interior point to the polygon. Therefore $\mathring{\mathcal{N}} = \{(0,0)\}$, $\partial\mathcal{N} = \{(0,0),(0,1),(0,2),(1,0),(2,0),(3,0)\}$, $\mathcal{N}''' = \emptyset$, $\mathcal{N}'' = \{(0,2),(0,1),(1,0),(2,0),(3,0),(1,1)\}$. $\mathcal{M} \sim \mathbb{C}$ is the 1-dimensional affine space $\mathcal{M} = \{P_{0,0}\}$. In other words $\mathcal{M}$ corresponds to choices of $P_{0,0} = g_3$.

Example 2.2 (Strebel-3). $\mathcal{P}(x,y) = y^2(x-z_1)^2(x-z_2)^2(x-z_3)^2 - Ax^2 - Bx - C$ *has the following Newton's polygon.*

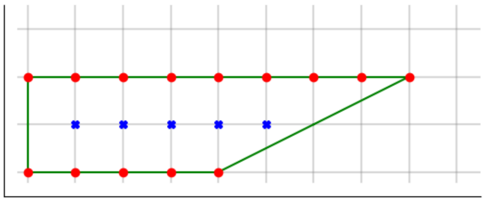


There are 3 interior points $\mathring{\mathcal{N}} = \{(0,0),(1,0),(2,0)\}$. However since $\mathcal{P}_d = \prod_{i=1}^{3}(x-z_i)^2$ is of degree 6, with 3 double zeros, Definition 2.2 gives

$$\mathbb{C}[\mathring{\mathcal{N}}] = \{0\}, \tag{2.11}$$

which has dimension 0, and $\mathcal{M} = \{\mathcal{P}\}$.

Example 2.3 (Strebel-4). *$\mathcal{P}(x,y) = y^2(x-z_1)^2(x-z_2)^2(x-z_3)^2(x-z_4)^2 - Ax^4 - Bx^3 - Cx^2 - Dx - E$ has the following Newton's polygon.*



There are 5 interior points $\overset{\circ}{\mathcal{N}} = \{(0,0),(1,0),(2,0),(3,0),(4,0)\}$. However since $\mathcal{P}_d = \prod_{i=1}^4 (x-z_i)^2$ is of degree 8, Definition 2.2 gives

$$\mathbb{C}[\overset{\circ}{\mathcal{N}}] = (x-z_1)(x-z_2)(x-z_3)(x-z_4)\mathbb{C}, \tag{2.12}$$

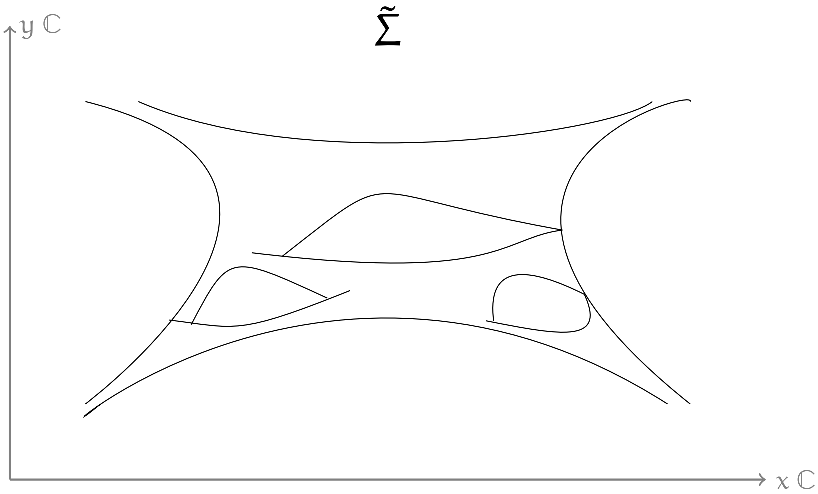
which has dimension 1.

$$\mathcal{M} = \mathcal{P} + (x-z_1)(x-z_2)(x-z_3)(x-z_4)\mathbb{C} \quad , \quad \dim \mathcal{M} = 1. \tag{2.13}$$

2.2. **Riemann surface.** For $P \in \mathcal{M}$, the zero-locus of P defines a subset of $\mathbb{C} \times \mathbb{C}$, which is locally a Riemann surface

$$\tilde{\Sigma} = \tilde{\Sigma}_P := \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid P(x,y) = 0\}, \tag{2.14}$$

(most of the time, we shall drop the P index when no confusion is possible). This surface might be not connected (if P is factorizable), it is not compact (there are punctures, where x and/or y tend to ∞), and in fact it is not even a surface, as it may have self intersections points with neighborhoods not homeomorphic to a Euclidean disc (rather union of discs), called nodal points, viewed as "pinchings" in the figure below.



Let Σ the normalization of $\tilde{\Sigma}$ (possibly disconnected), a compact Riemann surface, equipped with two meromorphic functions, $X : \Sigma \to \mathbb{C}$ and $Y : \Sigma \to \mathbb{C}$, such that

$$\tilde{\Sigma} = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid P(x,y) = 0\} = \{(X(p), Y(p)) \mid p \in \Sigma \setminus \{\text{punctures}\}\}. \tag{2.15}$$

• The map

$$\begin{array}{rcl} i:\Sigma & \hookrightarrow & \mathbb{C}\times\mathbb{C} \\ p & \mapsto & (X(p),Y(p)) \end{array} \tag{2.16}$$

is a meromorphic immersion, whose image is $i(\Sigma)=\tilde{\Sigma}$.

• The punctures are the locus where either x or y tends to ∞, i.e. the poles of X and/or Y. It is well known (See appendix A or literature [Eyn18]) that there is a 1-1 correspondence between punctures and boundaries of the convex envelope of the Newton's polygon. A pole α of X and Y of degree $a_\alpha=\deg_\alpha X$, $b_\alpha=\deg_\alpha Y$, is associated to a boundary of $\partial\mathcal{N}$ of slope $-a_\alpha/b_\alpha$.

• At all points $(X,Y)\in\tilde{\Sigma}$ where the vector $\nabla P=(P'_x(X,Y),P'_y(X,Y))\neq(0,0)$, the surface $\tilde{\Sigma}$ is smooth, it has a well defined tangent plane $T_{(X,Y)}\tilde{\Sigma}=(P'_y(X,Y),-P'_x(X,Y))\mathbb{C}$.

• The meromorphic map

$$\begin{array}{rcl} X:\Sigma & \to & \mathbb{CP}^1 \\ p & \mapsto & X(p) \end{array} \tag{2.17}$$

is a holomorphic ramified covering of $\mathbb{CP}^1$ by Σ. Its ramification points occur when two (or more) branches meet, and thus at $p=$ zeros of $P'_y(X(p),Y(p))$, and/or possibly at punctures. The degree of the covering is

$$d=\deg X=\deg_y \mathcal{P}(x,y)=\text{height of the Newton's polygon.} \tag{2.18}$$

• Zeros of $P'_y(X,Y)$ can be either regular ramification points, or they can also be nodal points i.e. self-intersection points, and they can be higher ramified.

• For generic $P\in\mathcal{M}$, the zeros of $P'_y(X,Y)$ and $P'_x(X,Y)$ are distinct on Σ, $\tilde{\Sigma}$ has everywhere a tangent and is smooth. However for non-generic points these may coincide, and the surface is not smooth. We have a degenerate curve with nodal points of possibly higher degeneracy order.

Example 2.4 (Weierstrass curve). $P(x,y)=y^2-x^3+g_2x+g_3$.

• *For generic g_2,g_3, the curve Σ is a torus. Every torus is conformally isomorphic to a parallelogram $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ whose modulus τ satisfies $\text{Im}\tau>0$. The map $i:\Sigma\to\mathbb{C}\times\mathbb{C}$ is then worth*

$$i\ :\quad \begin{array}{rl} X(z)= & -\nu^2\wp(z,\tau), \\ Y(z)= & \frac{i}{2}\nu^3\wp'(z,\tau), \end{array} \tag{2.19}$$

where $\wp$ is the Weierstrass function (the unique ellitptic function biperiodic $\wp(z+1)=\wp(z+\tau)=\wp(z)$ and with a double pole $\wp(z)\sim z^{-2}+O(z^2)$ at $z=0$). The parameters (ν,τ) are functions of (g_2,g_3), whose inverse map $(\nu,\tau)\mapsto(g_2,g_3)$ is:

$$g_2=15\nu^4G_4(\tau),\quad g_3=-35\nu^6G_6(\tau), \tag{2.20}$$

with G_4 and G_6 the modular Eisenstein G-series.

There are three ramification points, at $z=\frac{1}{2}$, $z=\frac{\tau}{2}$, $z=\frac{1}{2}(1+\tau)$, corresponding to branch points $X(\frac{1}{2}),X(\frac{\tau}{2}),X(\frac{1}{2}(1+\tau))$. There is one puncture (pole of X and Y) at $z=0$, with $a_0=\deg_0 X=2$ and $b_0=\deg_0 Y=3$, at which $Y\sim X^{\frac{3}{2}}$, and notice that the boundary of the Newton's polygon has indeed a slope $-a_0/b_0=-\frac{2}{3}$.

• *If $4g_2^3-27g_3^2=0$, then the torus is degenerate, $\Sigma=\mathbb{CP}^1$ is then a sphere, and $\tilde{\Sigma}$ has a nodal point. We parametrize the sphere $\Sigma=\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}$ with a complex variable z, and up to an automorphism of the sphere, we can write the immersion map $i:\Sigma\to\mathbb{C}\times\mathbb{C}$ as*

$$\begin{aligned} X(z)&=z^2-2u \\ Y(z)&=z^3-3uz, \end{aligned} \tag{2.21}$$

with $u = -\frac{3}{2} g_3/g_2$, *i.e.*

$$g_2 = 3u^2, \quad g_3 = -2u^3. \tag{2.22}$$

The nodal point is $\beta = i(\sqrt{3u}) = i(-\sqrt{3u})$, *with* $x_\beta = X(\beta) = u$ *and* $y_\beta = Y(\beta) = 0$.

There is one branch point, at $z = 0$. *There is one puncture (pole of* X *and* Y*) at* $z = \infty$, *with* $a_\infty = \deg_\infty X = 2$ *and* $b_\infty = \deg_\infty Y = 3$, *at which* $Y \sim X^{\frac{3}{2}}$, *related to the unique boundary of the Newton's polygon, which has slope* $-a_\infty/b_\infty = -\frac{2}{3}$.

Definition 2.3 (Nodal points)**.** *A nodal or branch point* $\beta = (x_\beta, y_\beta) \in \tilde{\Sigma}$, *is a point at which* $P'_y(x_\beta, y_\beta) = 0$. *Let*

$$(\beta^{(1)}, \dots, \beta^{(\ell_\beta)}) = i^{-1}(\beta), \tag{2.23}$$

its pre-images (the labeling doesn't matter) on Σ. *These are smooth points on* Σ. *If* $\ell_\beta = 1$, *we say that it is a pure ramification point* $\beta^{(1)}$ *corresponding to a branch point* $x_\beta = X(\beta^{(1)})$, *and if* $\ell_\beta \geqslant 2$, *we say that it is a nodal point.*

Lemma 2.1. *In* $\mathcal{M}$, *there exists some* P *that have no nodal points, and have only simple ramification points. More precisely, the set of* P *that have no nodal points, and have only simple ramification points, is open dense in* $\mathcal{M}$.

Proof. The subset of $\mathcal{M}$ that have degenerate ramification points or nodal points, is a union of algebraic submanifolds, given by the vanishing of the discriminant, i.e. the condition that P, P'_y, P''_{yy}, or P, P'_y, P'_x have a common zero. It is thus the complement of an algebraic set, it is open dense.

2.3. **Canonical local coordinates.**

Definition 2.4. *Let* $p \in \Sigma$.

- *If* $X(p) = \infty$, *we define* $a_p := \deg_p X = -\operatorname{ord}_p X$, *and* $X_p = 0$. *We have* $a_p > 0$.
- *If* $X(p) \neq \infty$, *we define* $a_p := -\operatorname{ord}_p(X - X(p))$, *and* $X_p = X(p)$. *We have* $a_p < 0$.

We define the canonical local coordinate at p*:*

$$\zeta_p(z) := (X(z) - X_p)^{\frac{-1}{a_p}}. \tag{2.24}$$

$\zeta_p(z)$ *vanishes linearly (order 1) at* $z = p$.

The canonical local coordinate is defined modulo a root of unity. Let

$$\rho_k = e^{\frac{2\pi i}{k}}. \tag{2.25}$$

Other local coordinates are

$$\zeta_p(z)\rho^j_{a_p} \quad j = 1, \dots, |a_p|. \tag{2.26}$$

Choosing a canonical local coordinate is equivalent to choosing one of the rays (there are $|a_p|$ *of them) starting from* p *in the direction* $X(z) - X_p \in \mathbb{R}_-$.

2.4. **Genus and cycles.** The compact Riemann surface Σ is possibly disconnected $\Sigma = \Sigma_1 \cup \dots \Sigma_m$, and each connected component has some genus $\mathfrak{g}_i$. Let us define the total genus

$$\mathfrak{g} := \sum_{i=1}^{m} \mathfrak{g}_i. \tag{2.27}$$

It is well known (and we shall recover it below) that the genus is at most the number of interior points to the Newton's polygon:

$$0 \leqslant \mathfrak{g} \leqslant \dim \mathbb{C}[\mathring{\mathcal{N}}] \leqslant \#\mathring{\mathcal{N}}. \tag{2.28}$$

The Homology space $H_1(\Sigma, \mathbb{Z})$ has dimension

$$\dim H_1(\Sigma, \mathbb{Z}) = 2\mathfrak{g}, \tag{2.29}$$

which means that there exists $2\mathfrak{g}$ independent non-contractible cycles, and it is possible (but not uniquely) to choose a symplectic basis:

$$\mathcal{A}_1, \dots, \mathcal{A}_\mathfrak{g}, \ \mathcal{B}_1, \dots, \mathcal{B}_\mathfrak{g}, \tag{2.30}$$

such that

$$\mathcal{A}_i \cap \mathcal{A}_j = 0 \ , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0 \ , \quad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}. \tag{2.31}$$

Such a choice of symplectic basis of cycles is called a Torelli marking of Σ.

Cycles are defined as linear combinations of homotopy classes of Jordan loops. However, here so far we are considering cycles on Σ, and we are going to integrate 1-forms (for example YdX) that have poles, and one should consider the homotopy classes of Jordan loops on $\Sigma \setminus$ poles. We could also consider removing nodal and ramification points. A way to avoid this, is just to choose Jordan loops rather than cycles.

So here we need the following notion of marking:

Definition 2.5 (Marking). *We call a marking of Σ, a choice of $2\mathfrak{g}$ **Jordan loops** on $\Sigma \setminus \{$punctures, ramification points, nodal points$\}$, satisfying*

$$\mathcal{A}_i \cap \mathcal{A}_j = 0, \quad \mathcal{B}_i \cap \mathcal{B}_j = 0, \quad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}. \tag{2.32}$$

Their projection to $H_1(\Sigma, \mathbb{Z})$ is a Torelli marking of Σ.

Remark 2.3. We insist that $\mathcal{A}_i$ and $\mathcal{B}_i$ are not cycles, they are Jordan loops.

Lemma 2.2 (Continuous Jordan cycles). *Let $P \in \mathcal{M}$ such that Σ_P has total genus $\mathfrak{g}$, and let $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1,\dots,\mathfrak{g}}$ a marking of symplectic Jordan loops on $\Sigma_P \setminus \{$punctures, branch points, nodal points$\}$, whose projection to $H_1(\Sigma, \mathbb{Z})$ forms a symplectic basis.*

There exists some neighborhood U of P in $\mathcal{M}$, such that for each $Q \in U$, there exists a unique family of symplectic Jordan loops on $\Sigma_Q \setminus \{$punctures, branch points, nodal points$\}$, whose projection by i are continuous on U, and whose projection by X is constant over U. We shall call it a "continuous choice of Jordan cycles" in U.

Proof. Away from punctures or ramification or nodal points, X is locally a homeomorphism, and the restriction of i is locally continuous on U. Use X to push the Jordan loops to the base and pull them back on any $Q \in U$.

Remark 2.4. Notice that in a neighborhood of P, there can be some Q with higher genus, and thus the family of symplectic Jordan loops is not a basis, it is only an independent family. One can obtain a basis by completing it with other cycles. For example one could add new cycles corresponding to unpinching the nodal points. However, this will not be needed in this article.

2.4.1. *Holomorphic forms.* Let $\Omega^1(\Sigma)$ the space of holomorphic differential 1-forms on Σ.

The following is a classical theorem going back to Riemann

Theorem 2.1 (Riemann). *We have*

$$\dim \Omega^1(\Sigma) = \mathfrak{g}. \tag{2.33}$$

Having made a choice of Torelli marking of Σ, there exists a unique basis $\omega_1, \dots, \omega_\mathfrak{g}$ of $\Omega^1(\Sigma)$, such that

$$\oint_{\mathcal{A}_i} \omega_j = \delta_{i,j}. \tag{2.34}$$

This is used to define the Riemann matrix of periods

$$\tau_{i,j} = \oint_{\mathcal{B}_i} \omega_j. \tag{2.35}$$

τ is a $\mathfrak{g} \times \mathfrak{g}$ Siegel matrix, i.e. a complex symmetric matrix, whose imaginary part is positive definite:

$$\tau^t = \tau, \quad \operatorname{Im} \tau > 0. \tag{2.36}$$

We can also obtain $\Omega^1(\Sigma)$ algebraically from the Newton's polygon, the following is a classical theorem

Theorem 2.2. *For any $Q \in \mathbb{C}[\overset{\circ}{\mathcal{N}}]$, the following differential form*

$$\frac{Q(X,Y)\, dX}{P'_y(X,Y)} \tag{2.37}$$

is holomorphic at all the punctures. Its only poles could be at the zeros of $P'_y(X,Y)$ if these are not compensated by zeros of dX, i.e. these can be only nodal points. We define

$$H'^1(\Sigma) = \frac{dX}{P'_y(X,Y)} \mathbb{C}[\overset{\circ}{\mathcal{N}}]. \tag{2.38}$$

In the generic case, all zeros of $P'_y(X,Y)$ are simple and are zeros of dX, so that this 1-form has no pole at all, it is holomorphic.

• *In the generic case we have*

$$\Omega^1(\Sigma) = H'^1(\Sigma), \qquad \mathfrak{g} = \dim \Omega^1(\Sigma) = \dim \mathbb{C}[\overset{\circ}{\mathcal{N}}]. \tag{2.39}$$

• *In the non-generic case we only have*

$$\Omega^1(\Sigma) \subset H'^1(\Sigma), \qquad \mathfrak{g} = \dim \Omega^1(\Sigma) \leqslant \dim \mathbb{C}[\overset{\circ}{\mathcal{N}}]. \tag{2.40}$$

In all cases there exists a rectangular matrix $\mathcal{K}_{k;(i,j)}$ of size $\mathfrak{g} \times \dim \mathbb{C}[\overset{\circ}{\mathcal{N}}]$, such that the normalized holomorphic differentials can be written

$$\forall\, k = 1, \dots, \mathfrak{g}, \qquad \omega_k = \sum_{(i,j) \in \overset{\circ}{\mathcal{N}}} \mathcal{K}_{k;(i,j)} \frac{X^i Y^j\, dX}{P'_y(X,Y)}. \tag{2.41}$$

Let the $\dim \mathbb{C}[\overset{\circ}{\mathcal{N}}] \times \mathfrak{g}$ rectangular matrix

$$\hat{\mathcal{K}}_{(i,j);k} = \oint_{\mathcal{A}_k} \frac{X^i Y^j\, dX}{P'_y(X,Y)}. \tag{2.42}$$

By definition we have $\mathcal{K}\hat{\mathcal{K}} = Id_{\mathfrak{g}}$, i.e.

$$\sum_{(i,j) \in \overset{\circ}{\mathcal{N}}} \mathcal{K}_{k;(i,j)}\, \hat{\mathcal{K}}_{(i,j);k} = \delta_{k,l}. \tag{2.43}$$

This shows that when $\mathfrak{g} = \dim \mathbb{C}[\overset{\circ}{\mathcal{N}}]$, $\mathcal{K}$ is invertible.

In all cases we have

$$\Omega^1(\Sigma) = \frac{dX}{P'_y(X,Y)} \mathcal{K}(\mathbb{C}[\overset{\circ}{\mathcal{N}}]). \tag{2.44}$$

Proof. Classical theorem, see [FK12; Fay73; Mum07; Eyn18].

Example 2.5 (Weierstrass curve). $P(x,y) = y^2 - x^3 + g_2 x + g_3$, *for generic* g_2, g_3. $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ *is a torus, with the immersion map* $i : \Sigma \to \mathbb{C} \times \mathbb{C}$ *given by*

$$\begin{aligned} X(z) &= -\nu^2 \wp(z,\tau) \\ Y(z) &= \frac{i}{2}\nu^3 \wp'(z,\tau). \end{aligned} \tag{2.45}$$

We have

$$\frac{dX}{P'_y(X,Y)} = \frac{dX}{2Y} = \frac{i\nu^2 \wp'(z,\tau)dz}{\nu^3 \wp'(z,\tau)} = \frac{i}{\nu} dz. \tag{2.46}$$

dz *is indeed a holomorphic form, it has no pole in the parallelogram* $(0, 1, 1+\tau, \tau)$, *and it is biperiodic* $dz = d(z+1) = d(z+\tau)$. *We choose the Jordan loops* $\mathcal{A} = [p, p+1]$ *and* $\mathcal{B} = [p, p+\tau]$ *with* p *a generic point. The matrix* $\hat{\mathcal{K}}$ *is a* 1×1 *matrix, worth*

$$\hat{\mathcal{K}} = \oint_{\mathcal{A}} \frac{dX}{P'_y(X,Y)} = \frac{i}{\nu} \int_p^{p+1} dz = \frac{i}{\nu}. \tag{2.47}$$

Its inverse is

$$\mathcal{K} = -i\nu. \tag{2.48}$$

The normalized holomorphic differential is

$$\omega = dz = -i\nu \frac{dX}{P'_y(X,Y)}. \tag{2.49}$$

Its $\mathcal{B}$*-cycle integral is*

$$\oint_{\mathcal{B}} \omega = \int_p^{p+\tau} dz = \tau. \tag{2.50}$$

Riemann's theorem ensures that $\mathrm{Im}\tau > 0$.

Definition 2.6 (Cells of fixed genus). *We define*

$$\mathcal{M}^{(\mathfrak{g})} := \{P \in \mathcal{M} \mid \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m, \ \ \mathfrak{g} = \sum_{i=1}^m \mathfrak{g}_i\}. \tag{2.51}$$

Proposition 2.1. *Alternatively*

$$\mathcal{M}^{(\mathfrak{g})} = \left\{ P \in \mathcal{M} \mid \mathfrak{g} = \dim \left\{ Q \in \mathbb{C}[\mathring{\mathcal{N}}] \,\Big|\, \frac{Q(X,Y)dX}{P'_y(X,Y)} \text{ has no pole} \right\} \right\}. \tag{2.52}$$

- $\mathcal{M}^{(\mathfrak{g})}$ *is an algebraic subset of* $\mathcal{M}$.
- *Each* $\mathcal{M}^{(\mathfrak{g})}$ *has a finite number of connected components.*

Proof. Indeed if the form $Q dX/P'_y$ has no pole, then it belongs to $\Omega^1(\Sigma)$, and vice-versa, i.e.

$$\Omega^1(\Sigma) = \left\{ Q \in \mathbb{C}[\mathring{\mathcal{N}}] \,\Big|\, \frac{Q(X,Y)dX}{P'_y(X,Y)} \text{ has no pole at nodal points} \right\}. \tag{2.53}$$

It has thus dimension $\mathfrak{g}$. The fact that $\mathcal{M}^{(\mathfrak{g})}$ is an algebraic subset of $\mathcal{M}$, comes from the fact that requiring that $dX/P'_y(X,Y)$ having zeros of a certain order, can be formulated with resultants of P, P'_y, P'_x and higher order derivatives having to vanish, i.e. some polynomials relations of the $P_{i,j}$'s. Each algebraic equations has a finite number of solutions.

2.4.2. *Non-generic case.*

Definition 2.7. *Let* β *a nodal point, with preimages* $i^{-1}(\beta) = \{\beta^{(1)}, \dots, \beta^{(\ell_\beta)}\}$*, Let*

$$\begin{aligned} H'^1_\beta(\Sigma) \quad := \quad & \Big\{ Q(X,Y) \frac{dX}{P'_y(X,Y)} \, | Q \in \mathbb{C}[x,y] \text{ having pole(s) at most} \\ & \text{at preimage(s) of } \beta \text{ and no other pole} \Big\} \, / \, \Omega^1(\Sigma). \end{aligned} \tag{2.54}$$

We define the algebraic genus of the nodal point β *as*

$$\mathfrak{g}_\beta := \dim H'^1_\beta(\Sigma). \tag{2.55}$$

Theorem 2.3. *We have*

$$H'^1(\Sigma) = \Omega^1(\Sigma) \bigoplus_{\beta = \text{nodal points}} H'^1_\beta(\Sigma), \tag{2.56}$$

whose dimensions are

$$\dim \mathbb{C}[\overset{\circ}{\mathcal{N}}] = \mathfrak{g} + \sum_{\beta = \text{nodal points}} \mathfrak{g}_\beta. \tag{2.57}$$

Proof. $H'^1(\Sigma)$ is the space of forms having no poles at punctures. The only places where an element of $H'^1(\Sigma)$ could have poles is where $P'_y(X,Y)$ vanishes at an order higher than that of dX, i.e. nodal points. Either this form has no poles at all, and is in $\Omega^1(\Sigma)$, or must be in some $H'^1_\beta(\Sigma)$. By definition all the $H'^1_\beta(\Sigma)$ are disjoint for different β, so we have a direct sum.

Example 2.6 (Degenerate Weierstrass curve)**.** $P(x,y) = y^2 - x^3 + g_2 x + g_3$, *with* $4g_2^3 - 27g_3^2 = 0$*. The immersion map* $i : \Sigma \to \mathbb{C} \times \mathbb{C}$ *is*

$$\begin{aligned} X(z) &= z^2 - 2u \\ Y(z) &= z^3 - 3uz \end{aligned} \tag{2.58}$$

with $g_2 = 3u^2$, $g_3 = -2u^3$*. The nodal point is* $\beta = (u,0)$ *and* $i^{-1}(\beta) = \{\sqrt{3u}, -\sqrt{3u}\}$*, so that* $\ell_\beta = 2$*. Since* $\Sigma = \mathbb{C}P^1$*, we have* $\dim \Omega^1(\Sigma) = \mathfrak{g} = 0$*, and we have* $H'^1(\Sigma) = \frac{dX}{P'_y(X,Y)} \mathbb{C}$ *a one dimensional space. We have*

$$\frac{dX}{P'_y(X,Y)} = \frac{2zdz}{2Y} = \frac{dz}{z^2 - 3u}. \tag{2.59}$$

It has simple poles at $z = \pm\sqrt{3u}$*, each with degree one. The algebraic genus of* β *is thus*

$$\mathfrak{g}_\beta = 1. \tag{2.60}$$

The Newton's polygon has one interior point, we have

$$1 = \#\overset{\circ}{\mathcal{N}} = \dim \mathbb{C}[\overset{\circ}{\mathcal{N}}] = \mathfrak{g} + \mathfrak{g}_\beta = 0 + 1. \tag{2.61}$$

2.5. **Period coordinates.**

Definition 2.8 (Period coordinates)**.** *Let* $P \in \mathcal{M}^{(\mathfrak{g})}$*. Having chosen a symplectic marking of Jordan loops in a neighborhood of* P *(lemma 2.2) we define:*

$$\mathcal{A} - \textit{periods :} \qquad \forall\, i = 1, \dots, \mathfrak{g}, \quad \eta_i := \frac{1}{2\pi i} \oint_{\mathcal{A}_i} Y dX \tag{2.62}$$

$$\mathcal{B} - \textit{periods :} \qquad \forall\, i = 1, \dots, \mathfrak{g}, \quad \tilde{\eta}_i = \eta_{i+\mathfrak{g}} := \frac{1}{2\pi i} \oint_{\mathcal{B}_i} Y dX \tag{2.63}$$

$$\textit{Real periods :} \qquad \forall\, i = 1, \dots, 2\mathfrak{g}, \quad \epsilon_i := \operatorname{Re} \eta_i \ , \ \zeta_i := \operatorname{Im} \eta_i. \tag{2.64}$$

$$\forall\, i = 1, \dots, \mathfrak{g}, \quad \tilde{\epsilon}_i := \operatorname{Re} \tilde{\eta}_i = \epsilon_{\mathfrak{g}+i} \ , \ \tilde{\zeta}_i := \operatorname{Im} \tilde{\eta}_i = \zeta_{\mathfrak{g}+i}. \tag{2.65}$$

Theorem 2.4 (periods = local coordinates)**.** *We have the following:*

- *The periods $\eta_1,\dots,\eta_{\mathfrak{g}}$ are local complex coordinates on $\mathcal{M}^{(\mathfrak{g})}$.*
- *The periods $\epsilon_1,\dots,\epsilon_{2\mathfrak{g}}$ are local real coordinates on $\mathcal{M}^{(\mathfrak{g})}$.*

Proof. This is a well known theorem, however, since it plays an important role in this article, lets give a proof. First from lemma 2.2 a marking with Jordan loops can be chosen continuous in some neighborhood of $P\in\mathcal{M}^{(\mathfrak{g})}$.

The tangent space is generated by tangent vectors

$$\partial_k = -\sum_{(i,j)\in\overset{\circ}{\mathcal{N}}} \mathcal{K}_{k;(i,j)}\frac{\partial}{\partial P_{i,j}}. \tag{2.66}$$

We have

$$\begin{aligned}\partial_k\eta_l &= -\frac{1}{2\pi i}\oint_{\mathcal{A}_l}\sum_{(i,j)\in\overset{\circ}{\mathcal{N}}}\mathcal{K}_{k;(i,j)}\frac{\partial Y}{\partial P_{i,j}}\,dX\\ &= \frac{1}{2\pi i}\oint_{\mathcal{A}_l}\sum_{(i,j)\in\overset{\circ}{\mathcal{N}}}\mathcal{K}_{k;(i,j)}\frac{X^iY^j\,dX}{P'_y(X,Y)}\\ &= \sum_{(i,j)\in\overset{\circ}{\mathcal{N}}}\mathcal{K}_{k;(i,j)}\hat{K}_{(i,j);l}\\ &= \delta_{k,l}.\end{aligned} \tag{2.67}$$

This implies that $\eta_1,\dots,\eta_{\mathfrak{g}}$ are coordinates because the Jacobian matrix is invertible.

Then compute the differential

$$\begin{aligned}-2\pi i\, d\tilde{\eta}_k &= \sum_{(i,j)\in\overset{\circ}{\mathcal{N}}} dP_{i,j}\oint_{\mathcal{B}_k}\frac{X^iY^j}{P'_y(X,Y)}\,dx\\ &= \sum_{(i,j)\in\overset{\circ}{\mathcal{N}}} dP_{i,j}\oint_{\mathcal{B}_k}\sum_{l=1}^{\mathfrak{g}}\mathcal{K}^{-1}_{(i,j);l}\omega_l\\ &= \sum_{(i,j)\in\overset{\circ}{\mathcal{N}}} dP_{i,j}\sum_{l=1}^{\mathfrak{g}}\mathcal{K}^{-1}_{(i,j);l}\tau_{l,k}\\ &= -2\pi i\sum_l \tau_{k,l}d\eta_l.\end{aligned} \tag{2.68}$$

Let us decompose τ in its real and imaginary part

$$\tau = R + iI, \tag{2.69}$$

and recall that $I>0$ so that in particular I is invertible. We have

$$\begin{aligned}d\eta &= d\epsilon + id\zeta\\ d\tilde{\eta} &= \tau d\eta = (R+iI)(d\epsilon+id\zeta)\\ &= (Rd\epsilon - Id\zeta) + i(Id\epsilon + Rd\zeta),\end{aligned} \tag{2.70}$$

i.e.

$$d\tilde{\epsilon} = R\,d\epsilon - I\,d\zeta, \tag{2.71}$$

and thus

$$Id\zeta = -d\tilde{\epsilon} + Rd\epsilon, \tag{2.72}$$

$$d\eta = (1+iI^{-1}R)d\epsilon - i\,I^{-1}d\tilde{\epsilon}. \tag{2.73}$$

$$d\epsilon = \mathrm{Re}\, d\eta, \qquad d\tilde{\epsilon} = \mathrm{Re}\,\tau d\eta. \tag{2.74}$$

In other words the Jacobian of the change of variable $\eta \to (\epsilon, \tilde{\epsilon})$ is invertible. This implies that $(\epsilon, \tilde{\epsilon})$ is also a set of local coordinates.

2.5.1. *Nodal point coordinates.* We should think of nodal points, as cycles that have been pinched (collapsed). It is useful to associate also period coordinates to them.

Definition 2.9. *Let β a nodal point, with preimages $i^{-1}(\beta) = \{\beta^{(1)}, \dots, \beta^{(\ell_\beta)}\}$, let $\zeta_i = \zeta_{\beta^{(i)}}$ the canonical local coordinate at $\beta^{(i)}$. Let*

$$\eta_{\beta^{(i)},k} = \operatorname*{Res}_{\beta^{(i)}} \zeta_i^k Y dX = 0, \qquad \tilde{\eta}_{\beta^{(i)},k} = \frac{1}{k} \operatorname*{Res}_{\beta^{(i)}} \zeta_i^{-k} Y dX. \tag{2.75}$$

($\eta_{\beta^{(i)},k} = 0$ because YdX has no pole at nodal points).

This allows to consider the period-vector $(\eta_1, \dots, \eta_g)$ of dimension $\mathfrak{g} = \dim \mathcal{M}^{(\mathfrak{g})}$ as a period-vector $(\eta_1, \dots, \eta_\mathfrak{g}, 0, \dots, 0)$ of dimension $\mathfrak{g} + \sum_\beta \mathfrak{g}_\beta = \dim \mathbb{C}[\mathring{\mathcal{N}}] = \dim \mathcal{M}$. With this definition the period-vector is continuous in a neighborhood in $\mathcal{M}$ of any $P \in \mathcal{M}^{(\mathfrak{g})}$ (with the topology of $\mathcal{M}$).

2.6. **Punctures coordinates.** We use the canonical local coordinates of Definition 2.4.

Definition 2.10 (Times at punctures)**.** *Let α a puncture.*

The 1-form YdX has a local Laurent series expansion near α:

$$YdX \sim \sum_{k=0}^{r_\alpha} t_{\alpha,k}\, \zeta_\alpha^{-k-1} d\zeta_\alpha + \textit{analytic at } \alpha. \tag{2.76}$$

We have

$$t_{\alpha,k} = \operatorname*{Res}_{\alpha} \zeta_\alpha^k Y dX. \tag{2.77}$$

The coefficients $t_{\alpha,k}$ are called the "times" of P.

We have

$$Y \sim \frac{-t_{\alpha,r_\alpha}}{a_\alpha} \zeta_\alpha^{-b_\alpha}, \quad b_\alpha = r_\alpha - a_\alpha. \tag{2.78}$$

In other words

$$Y \sim \frac{-t_{\alpha,r_\alpha}}{a_\alpha} (X - X_\alpha)^{b_\alpha/a_\alpha}. \tag{2.79}$$

There is only a finite number of non-vanishing times:

$$\{t_{\alpha,k} \,| \alpha = \textit{punctures}\,,\ k = 0, \dots, r_\alpha\}. \tag{2.80}$$

Proposition 2.2 (Times and exterior coefficients)**.** *The times $t_{\alpha,k}$ are algebraic functions of the coefficients $P_{i,j}$ of P, and it is well known (see proposition A.2 of appendix A) that they are algebraic functions of only the exterior coefficients of P. In other words they are algebraic functions of the coefficients $\mathcal{P}_{i,j} = P_{i,j}$ of $\mathcal{P}$ and are the same for all $P \in \mathcal{M}$.*

Vice–versa, the exterior coefficients of $\mathcal{P}$ are polynomials of the times.

Proof. Done in appendix A.2.

It is well known that the exponent $-b_\alpha/a_\alpha$ is a slope of the convex envelope of the Newton's polygon, and there exists a line of equation

$$D_\alpha = \{(i,j) \mid a_\alpha i + b_\alpha j = m_\alpha\} \tag{2.81}$$

tangent to the Newton's polygon. See appendix A.

Remark 2.5. **[Hypothesis: real residues]** From now on, we shall assume that $\mathcal{P}$ has been chosen so that:

$$\forall\alpha\,,\quad t_{\alpha,0}=\operatorname*{Res}_{\alpha} YdX\in\mathbb{R}. \tag{2.82}$$

In fact this is necessary for having a chance to satisfy Boutroux property. Indeed if γ is a small circle around α we have

$$\mathrm{Re}\oint_{\gamma} YdX=\mathrm{Re}\left(2\pi i\operatorname*{Res}_{\alpha} YdX\right)=-2\pi\,\mathrm{Im}\,t_{\alpha,0}=0\quad\text{if Boutroux.} \tag{2.83}$$

Example 2.7 (Weierstrass curve). $P(x,y)=y^2-x^3+g_2x+g_3$. *There is one puncture $\alpha=\infty$, at which both x and y become infinite with the asymptotic behavior:*

$$y\sim x^{\frac{3}{2}}\,\left(1-g_2x^{-2}-g_3x^{-3}\right)^{\frac{1}{2}}. \tag{2.84}$$

Using the canonical local coordinate $\zeta=\zeta_\infty=x^{-\frac{1}{2}}$, i.e. $a_\alpha=2$, we have

$$x=\zeta^{-2},\qquad dx=-2\zeta^{-3}d\zeta, \tag{2.85}$$

and

$$y=\zeta^{-3}\left(1-\frac{g_2}{2}\zeta^4-\frac{g_3}{2}\zeta^6+O(\zeta^8)\right). \tag{2.86}$$

The exponents $a_\alpha=2$ and $b_\alpha=3$ (degrees of poles of x and y in function of ζ), are related to the boundary of the Newton's polygon with normal vector $(2,3)$. We have

$$YdX=-2\zeta^{-6}\left(1-\frac{g_2}{2}\zeta^4+O(\zeta^6)\right)d\zeta=-2\zeta^{-5-1}d\zeta+g_2\zeta^{-1-1}d\zeta+O(1)d\zeta. \tag{2.87}$$

which gives $r_\infty=5$ and the times

$$t_{\infty,5}=-2,\quad t_{\infty,1}=g_2, \tag{2.88}$$

and all the other times are vanishing. The times are independent of g_3, and thus are the same for all $P\in\mathcal{M}$.

For the non-degenerate case we have

$$\eta=3iv^5G_4'(\tau),\quad \tilde{\eta}=\tau\eta+12iv^5G_4(\tau). \tag{2.89}$$

Definition 2.11 (Conjugate times). *For $k>0$, let*

$$\tilde{t}_{\alpha,k}=\frac{1}{k}\operatorname*{Res}_{\alpha}\zeta_\alpha^{-k}YdX. \tag{2.90}$$

We have the Laurent series expansion:

$$YdX\underset{\alpha}{\sim}\sum_{k=0}^{r_\alpha}t_{\alpha,k}\zeta_\alpha^{-k-1}d\zeta_\alpha+\sum_{k=1}^{\infty}k\tilde{t}_{\alpha,k}\zeta_\alpha^{k-1}d\zeta_\alpha. \tag{2.91}$$

The coordinates $\tilde{t}_{\alpha,0}$ conjugated to $t_{\alpha,0}$ are slightly more tricky to define. We first need:

Definition 2.12 (Fundamental domain). *Let p_i a generic point in the i^{th} connected component Σ_i of Σ. For each connected component Σ_i, let us consider a set of disjoints smooth Jordan arcs e_α from p_i to all punctures α that are in Σ_i.*

$\Sigma\setminus\cup_\alpha e_\alpha$ is typically a finite union of disjoint surfaces of total genus $\mathfrak{g}$. On $\Sigma\setminus\cup_\alpha e_\alpha$ it is possible to choose a set of $2\mathfrak{g}$ smooth closed Jordan loops $\mathcal{A}_1,\dots,\mathcal{A}_\mathfrak{g},\mathcal{B}_1,\dots,\mathcal{B}_\mathfrak{g}$ (and we denote $\mathcal{A}_{\mathfrak{g}+i}=\mathcal{B}_i$) starting and ending at p_i, such that $\Sigma\setminus(\cup_\alpha e_\alpha\cup_{i=1}^{2\mathfrak{g}}\mathcal{A}_i)$ is simply connected, and we can choose them so that they form a symplectic marking of cycles.

Let Υ the graph of all these edges. It is a graph whose vertices are at p_i and at the punctures. Each edge has at least one boundary being p_i. Let

$$D:=\Sigma\setminus\Upsilon. \tag{2.92}$$

D is a finite union of simply connected open domains of Σ: D may be disconnected (if Σ was) and is a finite union of topological discs (as many as the connected components). The boundary of D is made of edges of Υ, and each edge e of Υ appears twice as a boundary of D, with two opposite orientations. We call e_+ (resp. e_-)

the edge of ∂D *corresponding to the edge* e *of* Υ *whose orientation with respect to* D *(having* D *on its left) is the same (resp. opposite) as* e *in* Υ*. We have*

$$\partial D = \sum_{e\in\Upsilon} e_+ - e_-. \tag{2.93}$$

Let o_i *a generic point inside the* i^{th} *connected component of* D*. We define for* z *is in the* i^{th} *connected component of* D

$$g(z) := \int_{o_i}^{z} YdX, \tag{2.94}$$

where the integration path is the (unique up to homotopy) path from o_i *to* z *in the fundamental domain* D*.*

Definition 2.13 (Conjugate times case $k=0$)**.** *Let* D *a fundamental domain, and let* α *a puncture. We may assume that in a neighborhood of* α*, the edge* e_α *is such that* $\zeta_\alpha\in\mathbb{R}_-$*, so that if* z *is a point close to* α *of coordinate* $\zeta_\alpha = re^{i\theta}$*, we shall define the logarithm with cut on* $\mathbb{R}_-$*:*

$$\ln\zeta_\alpha(z) := \ln r + i\theta \qquad \theta\in]-\pi,\pi]. \tag{2.95}$$

In a neighborhood of α *in* D *we define the local "potential"*

$$V_\alpha := -\sum_{k=1}^{r_\alpha}\frac{1}{k}t_{\alpha,k}\zeta_\alpha^{-k}. \tag{2.96}$$

It is such that $YdX - dV_\alpha$ *can have at most a simple pole at* α*:*

$$YdX - dV_\alpha = t_{\alpha,0}\zeta_\alpha^{-1}d\zeta_\alpha + \textit{holomorphic at } \alpha. \tag{2.97}$$

We define:

$$\begin{aligned} g_\alpha(z) &:= \int_\alpha^z (YdX - dV_\alpha - t_{\alpha,0}\frac{d\zeta_\alpha}{\zeta_\alpha}) + V_\alpha(z) + t_{\alpha,0}\ln\zeta_\alpha(z) \\ &= -\sum_{k=1}^{r_\alpha}\frac{t_{\alpha,k}}{k}\zeta_\alpha^{-k} + t_{\alpha,0}\ln\zeta_\alpha + \sum_{k=1}^{\infty}\tilde{t}_{\alpha,k}\zeta_\alpha^{k}. \end{aligned} \tag{2.98}$$

We define:

$$\tilde{t}_{\alpha,o} := g(z) - g_\alpha(z) \tag{2.99}$$

which is independent of $z\in D$*.*

Moreover, since the sum of residues of a meromorphic 1-form has to vanish, then $\sum_\alpha t_{\alpha,0} = 0$*, and this implies that*

$$\sum_\alpha t_{\alpha,0}\tilde{t}_{\alpha,o} \tag{2.100}$$

is independent of the point o_i *used to define the function* g*.*

3. Energy as regularized area

The Boutroux curve will be obtained by a variational principle: minimizing an "energy". The energy will be the "area" and Boutroux curves will then be "minimal surfaces".

3.1. **Regularized area.** Let us give a first definition of our energy here, and it will be shown later that it is equivalent to another definition.

Recall that on $\mathbb{C}$ the Euclidean metric is related to the symplectic metric
$$|dx|^2 = \overline{dx} \wedge dx = 2i\ d^2x = 2i\ d\mathrm{Re}x \wedge d\mathrm{Im}x. \tag{3.1}$$
In $\mathbb{C}\times\mathbb{C}$, we have the canonical symplectic form $\overline{dx}\wedge\overline{dy}\wedge dy\wedge dx$. Its reduction to $\tilde{\Sigma}$, is the canonical metric on $\tilde{\Sigma}$
$$|y\,dx|^2 = 2i|y|^2 d^2x, \tag{3.2}$$
and its pullback by i is the canonical metric $|YdX|^2$ on Σ. Because of punctures, the total area $\int_\Sigma |YdX|^2$ is infinite. We need to "regularize" it.

Definition 3.1 (Energy = regularized area)**.** *For each puncture α, let us choose a small radius $R_\alpha > 0$, and consider the disc D_α : $|\zeta_\alpha| < R_\alpha$ in Σ and its boundary $\mathcal{C}_\alpha$ the circle $|\zeta_\alpha| = R_\alpha$, i.e. $\mathcal{C}_\alpha = \{R_\alpha e^{i\theta}|\theta \in]-\pi,\pi]\}$. We choose R_α small enough so that all D_α are topological discs and are all disjoint. In particular, each of them encloses only one puncture, and doesn't enclose any nodal or branch point.*

*Then we define the "**regularized area**":*
$$\begin{aligned} 4F \;:=\; & \frac{1}{2\pi i}\int_{\Sigma\setminus\cup_\alpha D_\alpha} |YdX|^2 \\ & -\sum_\alpha\sum_{k=1}^{r_\alpha}\frac{1}{k}|t_{\alpha,k}|^2R_\alpha^{-2k} + 2\sum_\alpha |t_{\alpha,0}|^2\ln R_\alpha \\ & +\sum_\alpha\sum_{k=1}^{\infty} k|\tilde{t}_{\alpha,k}|^2R_\alpha^{2k} - 2\mathrm{Re}\sum_\alpha\sum_{k=1}^{r_\alpha} t_{\alpha,k}\tilde{t}_{\alpha,k}. \end{aligned} \tag{3.3}$$

Lemma 3.1. *F is independent of the choice of radius R_α.*

Proof. Let D a fundamental domain. Let $\tilde{R}_\alpha < R_\alpha$. The proof uses Stokes theorem, we compute the integral on the annulus $\tilde{R}_\alpha < |\zeta_\alpha| < R_\alpha$ in the fundamental domain:
$$\begin{aligned} \frac{1}{2\pi i}\int_{D\cap D_\alpha\setminus\tilde{D}_\alpha}\overline{YdX}\wedge YdX &= \frac{1}{2\pi i}\int_{\partial(D\cap D_\alpha\setminus\tilde{D}_\alpha)}\overline{g_\alpha}\,YdX \\ &= \frac{1}{2\pi i}\int_{|\zeta_\alpha|=R_\alpha}\overline{g_\alpha}\,YdX - \frac{1}{2\pi i}\int_{|\zeta_\alpha|=\tilde{R}_\alpha}\overline{g_\alpha}\,YdX \\ &\quad+\frac{1}{2\pi i}\int_{[\tilde{p}_\alpha,p_\alpha]_{\mathrm{left}}}\overline{g_\alpha}\,YdX - \frac{1}{2\pi i}\int_{[\tilde{p}_\alpha,p_\alpha]_{\mathrm{right}}}\overline{g_\alpha}\,YdX \end{aligned} \tag{3.4}$$

where p_α (resp. $\tilde{p}_\alpha$) is the point of coordinate $\zeta_\alpha = R_\alpha e^{i\pi}$ (resp. $\zeta_\alpha = \tilde{R}_\alpha e^{i\pi}$).

For the last two terms, remark that $g_\alpha(z_{\mathrm{right}}) - g_\alpha(z_{\mathrm{left}}) = 2\pi i t_{\alpha,0}$ therefore
$$\begin{aligned} \frac{1}{2\pi i}\int_{[\tilde{p}_\alpha,p_\alpha]_{\mathrm{left}}}\overline{g_\alpha}\,YdX - \frac{1}{2\pi i}\int_{[\tilde{p}_\alpha,p_\alpha]_{\mathrm{right}}}\overline{g_\alpha}\,YdX &= t_{\alpha,0}\int_{\tilde{p}_\alpha}^{p_\alpha} YdX \\ &= t_{\alpha,0}(g_\alpha(p_\alpha) - g_\alpha(\tilde{p}_\alpha)). \end{aligned} \tag{3.5}$$

The first two terms, we use lemma B.1 of appendix B, and we get
$$\begin{aligned} & \frac{1}{2\pi i}\int_{D\cap D_\alpha\setminus\tilde{D}_\alpha}\overline{YdX}\wedge YdX \\ = \;& t_{\alpha,0}(g_\alpha(p_\alpha) - g_\alpha(\tilde{p}_\alpha)) \\ & -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k} + \sum_{k=1}^{\infty}k|\tilde{t}_{\alpha,k}|^2R_\alpha^{2k} + 2t_{\alpha,0}^2\ln R_\alpha - t_{\alpha,0}g_\alpha(p_\alpha) \\ & +\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}\tilde{R}_\alpha^{-2k} - \sum_{k=1}^{\infty}k|\tilde{t}_{\alpha,k}|^2\tilde{R}_\alpha^{2k} - 2t_{\alpha,0}^2\ln\tilde{R}_\alpha + t_{\alpha,0}g_\alpha(\tilde{p}_\alpha) \end{aligned}$$

$$
\begin{aligned}
= \quad & -\sum_{k=1}^{r_\alpha} \frac{|t_{\alpha,k}|^2}{k} R_\alpha^{-2k} + \sum_{k=1}^{\infty} k|\tilde{t}_{\alpha,k}|^2 R_\alpha^{2k} + 2t_{\alpha,0}^2 \ln R_\alpha \\
& + \sum_{k=1}^{r_\alpha} \frac{|t_{\alpha,k}|^2}{k} \tilde{R}_\alpha^{-2k} - \sum_{k=1}^{\infty} k|\tilde{t}_{\alpha,k}|^2 \tilde{R}_\alpha^{2k} - 2t_{\alpha,0}^2 \ln \tilde{R}_\alpha.
\end{aligned} \tag{3.6}
$$

This proves the Lemma.

Theorem 3.1 (Continuity)**.** *The energy is continuous on $\mathcal{M}$.*

$$
F \in C^0(\mathcal{M}, \mathbb{R}). \tag{3.7}
$$

Proof. The immersion $\tilde{\Sigma} \setminus \cup_\alpha i(D_\alpha)$ in $\mathbb{C} \times \mathbb{C}$, being the locus of solutions of $P(x,y) = 0$, is continuous on $\mathcal{M}$. The integral over $\Sigma \setminus \cup_\alpha D_\alpha$ is the area of $\tilde{\Sigma} \setminus \cup_\alpha i(D_\alpha)$ with the metric $|y|^2 d^2x$ of $\mathbb{C} \times \mathbb{C}$, therefore it is continuous. The times $t_{\alpha,k}$ are constant on $\mathcal{M}$, the conjugate times $\tilde{t}_{\alpha,k}$ are continuous and the radius R_α are taken locally constant. Thus, F is continuous.

3.2. **Minimum.**

Theorem 3.2 (Bounded from below)**.** *F is bounded from below on $\mathcal{M}$.*

Proof. We shall compare $F(P)$ to $F(\mathcal{P})$. Recall that P and $\mathcal{P}$ have the same times $t_{\alpha,k}$, but their conjugate times $\tilde{t}_{\alpha,k}$ can be different. Choose the radius R_α small enough so that they can be used for both P and $\mathcal{P}$.

We let $\mathcal{X}, \mathcal{Y}$ denote the functions X, Y when $P = \mathcal{P}$. We have

$$
\begin{aligned}
4F(P) - 4F(\mathcal{P}) \quad = \quad & \frac{1}{2\pi i} \int_{\Sigma \setminus \cup_\alpha D_\alpha} |Y dX|^2 - \frac{1}{2\pi i} \int_{\Sigma_{\mathcal{P}} \setminus \cup_\alpha D_\alpha} |\mathcal{Y} d\mathcal{X}|^2 \\
& + \sum_\alpha \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{k}} \bar{t}_{\alpha,k} R_\alpha^{-k} - \sqrt{k} \tilde{t}_{\alpha,k} R_\alpha^{k} \right|^2 \\
& - \sum_\alpha \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{k}} \bar{t}_{\alpha,k} R_\alpha^{-k} - \sqrt{k} \tilde{t}_{\alpha,k}(\mathcal{P}) R_\alpha^{k} \right|^2 .
\end{aligned} \tag{3.8}
$$

Therefore:

$$
4F(P) \geqslant 4F(\mathcal{P}) - \frac{1}{2\pi i} \int_{\Sigma_{\mathcal{P}} \setminus \cup_\alpha D_\alpha} |\mathcal{Y} d\mathcal{X}|^2 - \sum_\alpha \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{k}} \bar{t}_{\alpha,k} R_\alpha^{-k} - \sqrt{k} \tilde{t}_{\alpha,k}(\mathcal{P}) R_\alpha^{k} \right|^2 . \tag{3.9}
$$

Since the rhs is independent of P this shows that F is bounded from below on $\mathcal{M}$.

Theorem 3.3. *The level sets of F are compact in the canonical topology of $\mathbb{C}[\overset{\circ}{\mathcal{N}}]$. (we recall that the level set of level L is the set $\{Q \in \mathbb{C}[\overset{\circ}{\mathcal{N}}] \mid F(\mathcal{P} + Q) \leqslant L\}$.)*

Proof. Since F is continuous, its level sets are closed.

It remains to prove that they are bounded. Let $L > \inf F$, so that the level set is not empty.

If $F(P) \leqslant L$, this implies:

$$
\begin{aligned}
& \frac{1}{2\pi i} \int_{\Sigma \setminus \cup_\alpha D_\alpha} |Y dX|^2 + \sum_\alpha \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{k}} \bar{t}_{\alpha,k} R_\alpha^{-k} - \sqrt{k} \tilde{t}_{\alpha,k} R_\alpha^{k} \right|^2 \\
\leqslant \quad & 4L + \frac{1}{2\pi i} \int_{\Sigma \setminus \cup_\alpha D_\alpha} |\mathcal{Y} d\mathcal{X}|^2 - 4F(\mathcal{P}) + \sum_\alpha \sum_{k=1}^{\infty} \left| \frac{1}{\sqrt{k}} \bar{t}_{\alpha,k} R_\alpha^{-k} - \sqrt{k} \tilde{t}_{\alpha,k}(\mathcal{P}) R_\alpha^{k} \right|^2 = \tilde{L}.
\end{aligned} \tag{3.10}
$$

In particular

$$\int_{\Sigma\setminus\cup_\alpha D_\alpha} |y|^2\, d^2x \leqslant \pi\tilde{L}. \tag{3.11}$$

Let U an open subset of $\Sigma_{\mathcal{P}} \setminus \cup_\alpha D_\alpha$, that excludes some small disks around all ramification points of $\mathcal{P}$. There exists some $K > 0$ such that

$$\min_{x\in U,\ i\neq j} |\mathcal{Y}_i(x) - \mathcal{Y}_j(x)| \geqslant K > 0, \tag{3.12}$$

where $\mathcal{Y}_i(x)$ are the roots of $\mathcal{P}(x,y) = 0$.

Let also $r > 0$ small enough so that there is V an open subset of U, such that for all $x_0 \in X(V)$, the ball $X^*D(x_0, r)$ is contained in U. Let r small enough, so that there are at least $2\#\mathcal{N} + 1$ disjoint discs of radius r in V, denoted $D_1, \ldots, D_{2\#\mathcal{N}+1}$, of respective centers $q_1, \ldots, q_{2\#\mathcal{N}+1}$.

Let

$$||Q|| \quad = \quad \left(\int_U d^2x\, \frac{|Q(x,\mathcal{Y})|^{2/d}}{|\mathcal{P}'_y(x,\mathcal{Y})|^{2/d}}\right)^{d/2}, \tag{3.13}$$

where $i^{-1}(x,\mathcal{Y}) \in U$ designates a point on the curve $\mathcal{P}(x,\mathcal{Y}) = 0$, i.e. $\mathcal{Y} = \mathcal{Y}_i(x)$ for some i.

Remark that $Q(x,\mathcal{Y}) = \mathcal{P}(x,\mathcal{Y}) + Q(x,\mathcal{Y}) = P(x,\mathcal{Y}) = \mathcal{P}_d(x)\prod_{i=1}^{d}(\mathcal{Y} - Y_i(x))$, and $\mathcal{P}'_y(x,\mathcal{Y}) = \mathcal{P}_d(x)\prod_{i=2}^{d-1}(\mathcal{Y} - \mathcal{Y}_i(x))$, where we labeled $\mathcal{Y}_1 = \mathcal{Y}$. This gives

$$\begin{aligned}
||Q||^{2/d} &= \int_U d^2x\, \frac{|Q(x,\mathcal{Y})|^{2/d}}{|\mathcal{P}'_y(x,\mathcal{Y})|^{2/d}} \\
&\leqslant \frac{1}{K^{2\frac{d-1}{d}}} \int_U d^2x\, |\prod_{i=1}^{d}(\mathcal{Y} - Y_i(x))|^{2/d} \\
&\leqslant \frac{1}{d\, K^{2\frac{d-1}{d}}} \int_U d^2x\, \sum_{i=1}^{d} |\mathcal{Y} - Y_i(x)|^2 \qquad \leftarrow \text{AM-GM inequality} \\
&\leqslant \frac{2}{d\, K^{2\frac{d-1}{d}}} \int_U d^2x\, \sum_{i=1}^{d} (|Y_i(x)|^2 + |\mathcal{Y}|^2) \\
&\leqslant \frac{2}{d\, K^{2\frac{d-1}{d}}} \left(\pi\tilde{L} + d\int_U d^2x\, |\mathcal{Y}|^2\right).
\end{aligned} \tag{3.14}$$

This implies that $||Q||$ is bounded on the level sets of F.

However, $||Q||$ is not a norm (it would be the Hölder norm if $d/2 \leqslant 1$ but here we have $d/2 \geqslant 1$), so we can not yet conclude.

Let us show the following lemma:

Lemma 3.2. *For all $x_0 \in X(V)$ and $i^{-1}(x_0, \mathcal{Y}(x_0)) \in V$, there exists $x \in D(x_0, r)$ such that $\left|\frac{Q(x,\mathcal{Y}(x))}{\mathcal{P}'_y(x,\mathcal{Y}(x))}\right| \leqslant \left(\pi r^2\right)^{-d/2} ||Q||$. There exist at least $\#\overset{\circ}{\mathcal{N}}$ points among $q_1, \ldots, q_{2\#\mathcal{N}+1}$, for which $\left|\frac{Q(q_i,\mathcal{Y}(q_i))}{\mathcal{P}'_y(q_i,\mathcal{Y}(q_i))}\right| \leqslant \left(\pi r^2\right)^{-d/2} ||Q||$.*

Proof. For all $i^{-1}(x_0, \mathcal{Y}(x_0)) \in V$, we have either:

- $\left|\frac{Q(x_0,\mathcal{Y}(x_0))}{\mathcal{P}'_y(x_0,\mathcal{Y}(x_0))}\right| \leqslant \left(\pi r^2\right)^{-d/2} ||Q||$
- or $\left|\frac{Q(x_0,\mathcal{Y}(x_0))}{\mathcal{P}'_y(x_0,\mathcal{Y}(x_0))}\right| > \left(\pi r^2\right)^{-d/2} ||Q||$

In this second case, $Q(x_0, \mathcal{Y}(x_0)) \neq 0$. Assume that there is no x in $D(x_0, r)$ such that $Q(x, \mathcal{Y}(x)) = 0$, we then have

$$\begin{aligned}
\left(\frac{Q(x_0,\mathcal{Y}(x_0))}{\mathcal{P}'_y(x_0,\mathcal{Y}(x_0))}\right)^{2/d} &= \operatorname*{Res}_{x\to x_0} \frac{dx}{x-x_0}\left(\frac{Q(x,\mathcal{Y}(x))}{\mathcal{P}'_y(x,\mathcal{Y}(x))}\right)^{2/d} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\frac{Q(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}{\mathcal{P}'_y(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}\right)^{2/d} \qquad \forall \tilde{r} \leqslant r \\
&= \frac{1}{\pi r^2}\int_0^{r} \tilde{r}d\tilde{r}\int_0^{2\pi} d\theta \left(\frac{Q(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}{\mathcal{P}'_y(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}\right)^{2/d}.
\end{aligned} \tag{3.15}$$

This implies that

$$\begin{aligned}
\left|\frac{Q(x_0,\mathcal{Y}(x_0))}{\mathcal{P}'_y(x_0,\mathcal{Y}(x_0))}\right|^{2/d} &\leqslant \frac{1}{\pi r^2}\int_{U\cap X^{-1}(D(x_0,r))} d^2x \left|\frac{Q(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}{\mathcal{P}'_y(x_0+\tilde{r}e^{i\theta},\mathcal{Y}(x_0+\tilde{r}e^{i\theta}))}\right|^{2/d} \\
&\leqslant \frac{1}{\pi r^2}\,||Q||^{2/d}.
\end{aligned} \tag{3.16}$$

This contradicts our hypothesis. Therefore there exists $x \in D(x_0, r)$ such that $Q(x, \mathcal{Y}(x)) = 0$.

Then, notice that $Q(x, \mathcal{Y}(x))$ can have at most $\#\mathcal{N}$ zeros on Σ, therefore, among the discs $D_1, \dots, D_{2\#\mathcal{N}+1}$, $Q(x, \mathcal{Y}(x))$ can have a zero in at most half of them, and therefore has no zero in the others, and thus is bounded by $(\pi r^2)^{-d/2}\,||Q||$ in at least $\#\mathring{\mathcal{N}}$ of them. This proves the lemma.

Then, consider the $\#\mathring{\mathcal{N}}$ points $u_l = (q_l, \mathcal{Y}(q_l))$ for $l = 1, \dots, \#\mathring{\mathcal{N}}$. By definition we have

$$A\vec{Q} = \vec{B} \tag{3.17}$$

with $\vec{Q} = (Q_{i,j})_{(i,j)\in\mathring{\mathcal{N}}}$ the $\#\mathring{\mathcal{N}}$ dimensional vector of coefficients of Q, $\vec{B} = (Q(u_l)/\mathcal{P}'_y(u_l))_{l=1,\#\mathring{\mathcal{N}}}$ the $\#\mathring{\mathcal{N}}$ dimensional vector of evaluations, and A the $\#\mathring{\mathcal{N}} \times \#\mathring{\mathcal{N}}$ square matrix $A_{l;(i,j)} = q_l^i \mathcal{Y}(q_l)^j$. The matrix A is invertible, and is independent of Q. This gives

$$\vec{Q} = A^{-1}\vec{B}. \tag{3.18}$$

With the sup-norm this gives

$$||Q||_{sup} \leqslant ||A^{-1}||\;||B||_{sup}, \tag{3.19}$$

which shows that all coefficients $Q_{i,j}$ are bounded.

Therefore the level sets are compact.

Theorem 3.4 (Minimum)**.** F *admits a minimum*

Proof. F is continuous, it is bounded from below, and its level sets are compact. The intersection of all level sets

$$\cap_{L>\inf F}\{Q \mid F(\mathcal{P}+Q) \leqslant L\} \tag{3.20}$$

is a decreasing intersection of non-empty compacts, therefore it is a non-empty compact. Let Q an element of this compact. We have

$$F(\mathcal{P}+Q) = \inf F, \tag{3.21}$$

so it is a minimum.

3.3. **Derivatives.** As a corollary of proposition 2.1, we have

Lemma 3.3 (Cotangent space)**.** *The cotangent space of an affine space is isomorphic to the underlying vector space*

$$T^*\mathcal{M} \sim \mathbb{C}[\mathring{\mathcal{N}}]. \tag{3.22}$$

It is isomorphic to $H'^1(\Sigma)$*:*

$$\begin{array}{rcl} T_P^*\mathcal{M} & \to & H'^1(\Sigma_P) \\ \delta P_{k,l} & \mapsto & \omega_{k,l} = -\dfrac{x^k y^l dx}{P'_y(X,Y)}. \end{array} \tag{3.23}$$

Proof. Since $P(X,Y)=0$, we have

$$P'_y(X,Y)\delta Y + \sum_{k,l} \delta P_{k,l} X^k Y^l = 0, \tag{3.24}$$

and thus

$$\delta Y dx = -\sum_{k,l} \delta P_{k,l} \frac{x^k y^l dx}{P'_y(X,Y)}. \tag{3.25}$$

If $(k,l)\in\mathring{\mathcal{N}}$, then $\omega\in H'^1(\Sigma)$ has no poles at punctures, it means that $\delta t_{\alpha,k}=0$ for all 2nd kind and 3rd kind times.

Proposition 3.1 (Derivative)**.** *Let us consider a deformation* $\delta\in T_P^*\mathcal{M}$*. Let*

$$\omega = \sum_{k,l} \omega_{k,l}\delta P_{k,l} = \delta Y dX, \qquad \textit{where} \qquad \omega_{k,l} = -\frac{x^k y^l dx}{P'_y(X,Y)}. \tag{3.26}$$

We have

$$4\delta F = \frac{1}{\pi}\mathrm{Im}\int_{\Sigma\setminus\cup_\alpha D_\alpha} \overline{YdX}\wedge\omega + \sum_\alpha \frac{1}{\pi}\mathrm{Im}\int_{D_\alpha} (\overline{YdX}-\overline{dV_\alpha})\wedge\omega - 2\sum_\alpha \mathrm{Re}\,\mathop{\mathrm{Res}}_\alpha V_\alpha\omega. \tag{3.27}$$

This is independent of the radius R_α*. Therefore it has a limit as* $R_\alpha\to 0$*. Since* $YdX - dV_\alpha$ *can have at most a simple pole at* α *and* ω *is holomorphic at* α*, the last integral tends to 0 as* $R_\alpha\to 0$*, and therefore the first integral has a limit. We write:*

$$\frac{1}{4\pi}\mathrm{Im}\int_\Sigma \overline{YdX}\wedge\omega = \lim_{R_\alpha\to 0}\frac{1}{4\pi}\mathrm{Im}\int_{\Sigma\setminus\cup_\alpha D_\alpha}\overline{YdX}\wedge\omega. \tag{3.28}$$

Proof. Stokes theorem.

Proposition 3.2 (Second Derivative)**.** *Let* $\delta\in T^*\mathcal{M}$*. Define*

$$\Omega = \sum_{(k,l)} \delta P_{k,l} \sum_{(i,j)} \delta P_{i,j} \frac{\omega_{k,l}\omega_{i,j}}{YdX}\left(\frac{yP''_{y,y}(x,y)}{P'_y(X,Y)} - l - j\right). \tag{3.29}$$

The Hessian, is the Hermitian quadratic form $H(\omega,\omega)$*:*

$$H(\omega,\omega) = \frac{1}{4\pi}\mathrm{Im}\int_\Sigma \overline{\omega}\wedge\omega + \frac{1}{4\pi}\mathrm{Im}\int_\Sigma \overline{YdX}\wedge\Omega - \frac{1}{2}\mathrm{Re}\sum_\alpha \mathop{\mathrm{Res}}_\alpha V_\alpha\Omega. \tag{3.30}$$

Proof. Simple computation.

3.4. **Minimization with constrained prescribed Integrals.** Let $\mathcal{L}$ an algebraic 3-dimensional submanifold of $\mathbb{C}\times\mathbb{C}$, with boundaries at most above the punctures. Generically, $\mathcal{L}\cap\tilde{\Sigma}$ will be a 1-dimensional algebraic submanifold of $\tilde{\Sigma}$, that we can write as a finite union of smooth Jordan arcs. These arcs may end at the punctures or not. Up to homotopic deformations, they can be moved to integer linear combinations of cycles $\mathcal{A}_i,\ i=1,\dots,2\mathfrak{g}$, or small circles around the punctures, or arcs that end at the punctures. Therefore

$$\begin{aligned}\int_{i^{-1}(\mathcal{L}\cap\tilde{\Sigma})} YdX &= \sum_{i=1}^{2\mathfrak{g}} c_i(\mathcal{L})\oint_{\mathcal{A}_i} YdX+\sum_\alpha c_\alpha(\mathcal{L})\tilde{t}_{\alpha,0}+\sum_\alpha \tilde{c}_\alpha(\mathcal{L})2\pi i \operatorname*{Res}_\alpha YdX \\ &= \sum_{i=1}^{2\mathfrak{g}} c_i(\mathcal{L})2\pi i\eta_i+\sum_\alpha c_\alpha(\mathcal{L})\tilde{t}_{\alpha,0}+\sum_\alpha \tilde{c}_\alpha(\mathcal{L})2\pi i t_{\alpha,0},\end{aligned} \tag{3.31}$$

with $c_i(\mathcal{L}),c_\alpha(\mathcal{L}),\tilde{c}_\alpha(\mathcal{L})$ integers. If we take the real part, due to hypothesis of 2.5, we have

$$\operatorname{Re}\int_{i^{-1}(\mathcal{L}\cap\tilde{\Sigma})} YdX = \sum_{i=1}^{2\mathfrak{g}} c_i(\mathcal{L})2\pi\epsilon_i+\sum_\alpha c_\alpha(\mathcal{L})\tilde{t}_{\alpha,0}. \tag{3.32}$$

Remark that the integers $c_i(\mathcal{L}),c_\alpha(\mathcal{L}),\tilde{c}_\alpha(\mathcal{L})$ are locally constant, but they can be discontinuous over $\mathcal{M}$.

Definition 3.2 (Moduli space with prescribed integrals)**.** *Let $\mathcal{L}_1,\dots,\mathcal{L}_N$ be given. Let $\ell_1,\dots,\ell_N$ be* N *real numbers. Let*

$$\mathcal{M}(\mathcal{P};\mathcal{L}_i,\ell_i)=\left\{P=\mathcal{P}+Q\in\mathcal{M}\mid \forall i=1,\dots,N\quad \operatorname{Re}\int_{i^{-1}(\mathcal{L}_i\cap\tilde{\Sigma})} YdX=2\pi\ell_i\right\}. \tag{3.33}$$

Theorem 3.5. *If $\mathcal{M}(\mathcal{P};\mathcal{L}_i,\ell_i)$ is a non-empty closed subset of $\mathcal{M}$, then the restriction F is continuous on it, and it has an infimum*

$$\inf_{\mathcal{M}(\mathcal{P};\mathcal{L}_i,\ell_i)} F\geqslant\inf_{\mathcal{M}} F>-\infty. \tag{3.34}$$

It has a minimum.

Proof. The maps $P\mapsto\ell_i$ are continuous, so $\mathcal{M}(\mathcal{P};\mathcal{L}_i,\ell_i)$ is closed.

The continuity of F on it, and the infimum are trivial.

Since F is continuous on $\mathcal{M}(\mathcal{P};\mathcal{L}_i,\ell_i)$, the level sets are closed. We have already seen that level sets are bounded, so they are compact. The intersection of all level sets, is the intersection of a decreasing sequence of non-empty compacts, so is a non-empty compact. A point in the intersection is a minimum.

3.5. **Energy from Prepotential.** Here we shall see another definition of the energy F. We shall define a function $\check{F}$ on $\mathcal{M}$, from the prepotential F_0, and we shall then prove that $\check{F}$ and F are equals. The advantage is that this expression of F will be expressed in local period coordinates, and will allow to see how a minimum is related to the Boutroux condition.

Let us choose some genus $\mathfrak{g}\leqslant\dim\mathbb{C}[\mathring{\mathcal{N}}]$. Let $U\subset\mathcal{M}^{(\mathfrak{g})}$ a simply connected open domain of $\mathcal{M}^{(\mathfrak{g})}$, in which we choose a continuous symplectic Jordan cycles marking (lemma 2.2), and we choose a fundamental domain D as in Definition 2.12, continuous on U.

This allows to have period coordinates well defined over U, as well as puncture-times:

$$\begin{aligned} t_{\alpha,k} &= \operatorname*{Res}_\alpha \zeta_\alpha^k YdX, &&\alpha=\text{punctures}\ \ k=0,\dots,r_\alpha \\ \eta_i &= \frac{1}{2\pi i}\oint_{\mathcal{A}_i} YdX, && i=1,\dots\mathfrak{g}\end{aligned} \tag{3.35}$$

as well as their conjugate times

$$\begin{aligned} \tilde{t}_{\alpha,k} &= \frac{1}{k}\,\mathop{\mathrm{Res}}_{\alpha}\, \zeta_\alpha^{-k}\, Y dX, && \alpha = \text{punctures} \;\; k = 1, \dots, r_\alpha \\ \tilde{t}_{\alpha,o} &= g(z) - g_\alpha(z), && \text{independent of } z \in D \\ \tilde{\eta}_i &= \frac{1}{2\pi i}\oint_{\mathcal{B}_i} Y dX, && i = 1, \dots \mathfrak{g} \end{aligned} \tag{3.36}$$

Definition 3.3 (Free energy). *We define the prepotential F_0 as (see [EO07; Eyn17])*

$$F_0 \quad := \quad \frac{1}{2}\left(\sum_\alpha \sum_{k=1}^{r_\alpha} t_{\alpha,k}\tilde{t}_{\alpha,k} + \sum_\alpha t_{\alpha,0}\tilde{t}_{\alpha,o} + 2\pi i \sum_{i=1}^{\mathfrak{g}} \eta_i \tilde{\eta}_i\right). \tag{3.37}$$

We also define

$$\hat{F} := \frac{1}{2}\left(\sum_\alpha \sum_{k=1}^{r_\alpha} t_{\alpha,k}\tilde{t}_{\alpha,k} + \sum_\alpha t_{\alpha,0}\tilde{t}_{\alpha,o}\right) = F_0 - \pi i \sum_{i=1}^{\mathfrak{g}} \eta_i \tilde{\eta}_i. \tag{3.38}$$

In addition, define

$$\check{F} := -\mathrm{Re}\hat{F} + \pi(\tilde{\zeta}^t \epsilon - \zeta^t \tilde{\epsilon}) = -\mathrm{Re}\hat{F} + \pi \zeta^t E^{-1} \epsilon, \tag{3.39}$$

where E is the symplectic matrix of size $2\mathfrak{g}$

$$E = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}. \tag{3.40}$$

Remark 3.1. Notice that F_0 depends on the choice of fundamental domain D, and of the symplectic Jordan loops marking. It is not a function of P alone. In other words F_0 is not defined as a function on $\mathcal{M}$.

Proposition 3.3. *$\check{F}$ and $\hat{F}$ are independent of a choice of marking of Jordan loops. Moreover, we have in the cotangent space T^*U, the following differentials*

$$dF_0 = 2\pi i \sum_{i=1}^{\mathfrak{g}} \tilde{\eta}_i d\eta_i, \tag{3.41}$$

$$d\hat{F} = \pi i \left(\sum_i \tilde{\eta}_i d\eta_i - \eta_i d\tilde{\eta}_i\right), \tag{3.42}$$

and

$$d\check{F} = 2\pi \left(\sum_{i=1}^{\mathfrak{g}} \tilde{\zeta}_i d\epsilon_i - \zeta_i d\tilde{\epsilon}_i\right). \tag{3.43}$$

If we choose an arbitrary basis of cycles, not necessarily symplectic, then we have

$$d\hat{F} = \pi i\, \eta^t E^{-1} d\eta, \tag{3.44}$$

$$d\check{F} = 2\pi\, \zeta^t E^{-1} d\epsilon, \tag{3.45}$$

where E is the $2\mathfrak{g} \times 2\mathfrak{g}$ intersection matrix $E_{i,j} = \mathcal{A}_i \cap \mathcal{A}_j = -E_{j,i}$.

Proof. The fact that $\hat{F}$ is independent of a choice of marking is obvious, because we subtracted from F_0 the part that depends on it. Then, consider a change of Jordan loop marking, by taking linear combinations of them. This implies that η changes to $C\eta$ where C is an invertible matrix with integer coefficients, so that ϵ changes to $C\epsilon$ and $\zeta \to C\zeta$, and E changes to CEC^t. This shows that the $\zeta^t E^{-1}\epsilon$ is invariant under a change of basis, and therefore $\check{F}$ is independent of a choice of marking.

The relation

$$dF_0 = 2\pi i \sum_i \tilde{\eta}_i d\eta_i \tag{3.46}$$

comes from the fact that F_0 is the Seiberg-Witten prepotential, this was proved for instance in [EO07; Ber07].

The expression for $d\hat{F}$ follows immediately, and since we are in a symplectic basis with E of the form eq (3.40), this takes the form

$$d\hat{F} = \pi i\, \eta^t E^{-1} d\eta, \tag{3.47}$$

which is clearly invariant under any change of basis of cycles.

Then, taking the real part we have

$$\begin{aligned} d\mathrm{Re}\hat{F} = & \quad -\pi \left(\zeta^t E^{-1} d\epsilon + \epsilon^t E^{-1} d\zeta \right) \\ = & \quad -\pi \left(\zeta^t E^{-1} d\epsilon - d\zeta^t E^{-1} \epsilon \right), \end{aligned} \tag{3.48}$$

and thus

$$d\check{F} = -d\mathrm{Re}\hat{F} + \pi d(\zeta^t E^{-1} \epsilon) = 2\pi \zeta^t E^{-1} d\epsilon. \tag{3.49}$$

Proposition 3.4. *The map* $\hat{F}$*:*

$$\begin{aligned} \mathcal{M} & \to \mathbb{C} \\ P & \mapsto \hat{F} \end{aligned} \tag{3.50}$$

is well defined, and is holomorphic in each $\mathcal{M}^{(\mathfrak{g})}$ *(with respect to the complex structure of period coordinates* η_i*).*

The map $\mathrm{Re}\hat{F}$ *:*

$$\begin{aligned} \mathcal{M} & \to \mathbb{R} \\ P & \mapsto \mathrm{Re}\hat{F} \end{aligned} \tag{3.51}$$

is well defined, and is locally harmonic in each $\mathcal{M}^{(\mathfrak{g})}$*.*

The map $\check{F}$ *:*

$$\begin{aligned} \mathcal{M} & \to \mathbb{R} \\ P & \mapsto \check{F} \end{aligned} \tag{3.52}$$

is well defined (it is not harmonic).

Proof. There is no continuous section of Jordan loops marking over the full $\mathcal{M}$, and this is why we used some open set $U \subset \mathcal{M}^{(\mathfrak{g})}$ to define $\check{F}$. However, we have seen that $\check{F}$ and $\hat{F}$ are in fact independent of the choice of marking, so they are well defined over the full $\mathcal{M}^{\mathfrak{g}}$ and also other their disjoint union $\mathcal{M}$.

Theorem 3.6 (Hessian of $\check{F}$)**.** *Let* $\mathfrak{g} \leqslant \#\mathring{\mathcal{N}}$*, and let* $U \subset \mathcal{M}^{(\mathfrak{g})}$ *an open domain, in which we choose a continuous marking of Jordan cycles (lemma 2.2). In* U *we use the period coordinates* $\epsilon_1, \dots, \epsilon_{2\mathfrak{g}}$*. We consider the real and imaginary parts of the Riemann matrix of periods*

$$\tau = R + i\, I, \quad I > 0, \quad R = R^t, \quad I = I^t. \tag{3.53}$$

The $2\mathfrak{g} \times 2\mathfrak{g}$ *Hessian matrix of* $\check{F}$ *is*

$$\begin{aligned} \frac{1}{2\pi} \frac{\partial^2 \check{F}}{\partial \epsilon_i \partial \epsilon_j} & = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix} \\ & = \begin{pmatrix} \mathbf{1} & -R \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -R & \mathbf{1} \end{pmatrix}, \end{aligned} \tag{3.54}$$

which is symmetric (as any Hessian matrix) and positive definite. $\check{F}$ *is strictly convex in any convex subdomain of* U *(convex in period coordinates).*

Proof. This is a simple computation. Moreover, since $I > 0$ this matrix is clearly positive definite, and thus invertible. It is also clearly symmetric.

Corollary 3.1. *$\check{F}$ is strictly convex in U (in the real coordinates ϵ). If $\check{F}$ has a minimum in a convex U, then it is unique and is a Boutroux curve:*

$$\forall\, \gamma \in H_1(\Sigma \setminus \textit{punctures}, \mathbb{Z}), \quad \mathrm{Re} \oint_\gamma Y dX = 0. \tag{3.55}$$

Proof. Strict convexity comes from the fact that the Hessian is positive definite. From strict convexity, it is clear that the minimum if it exists is unique. (A minimum doesn't necessarily exist in U, it could be at the boundary of U and we shall discuss that issue later below).

Consider that a minimum is reached in U, this implies that $d\check{F} = 0$, i.e.

$$\forall\, i = 1, \ldots, 2\mathfrak{g}\,, \qquad \zeta_i = 0 = \frac{1}{2\pi} \mathrm{Re} \oint_{\mathcal{A}_i} Y dX, \tag{3.56}$$

where we used the convention that $\mathcal{B}_i = \mathcal{A}_{\mathfrak{g}+i}$.

Also, if γ is a small circle around a puncture α we have

$$\mathrm{Re} \int_\gamma Y dX = \mathrm{Re} 2\pi i \operatorname*{Res}_\alpha Y dX = -2\pi \mathrm{Im} t_{\alpha,0} = 0, \tag{3.57}$$

by our assumption 2.5.

Since any closed Jordan loop γ on $\Sigma \setminus$ puncture is homotopic to an integer linear combination of cycles $\mathcal{A}_i$s and circles around punctures, this implies the Boutroux condition.

3.6. **Uniqueness of the energy.**

Theorem 3.7. *The two definitions of F coincide*

$$F = \check{F}. \tag{3.58}$$

Proof. We recall that we chose a fundamental domain D, with its symplectic marking of cycles bordering D, and we have defined $g(z) = \int_{o_i}^z Y dX$ analytic in D.

On D we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\Sigma \setminus \cup_\alpha D_\alpha} \overline{Y dX} \wedge Y dX &= \frac{1}{2\pi i} \int_{\partial(\Sigma \setminus \cup_\alpha D_\alpha)} \overline{g}\, Y dX \\
&= \frac{-1}{2\pi i} \sum_\alpha \int_{\mathcal{C}_\alpha} \overline{g}\, Y dX \\
&\quad + \frac{1}{2\pi i} \sum_\alpha \int_{o_i}^{p_\alpha} \overline{2\pi i t_{\alpha,0}}\, Y dX \\
&\quad + \frac{1}{2\pi i} \sum_{i=1}^{\mathfrak{g}} \int_{\mathcal{A}_i} Y dX \left(\overline{- \int_{\mathcal{B}_i} Y dX} \right) \\
&\quad + \frac{1}{2\pi i} \sum_{i=1}^{\mathfrak{g}} \int_{\mathcal{B}_i} Y dX \left(\overline{\int_{\mathcal{A}_i} Y dX} \right) \\
&= \frac{-1}{2\pi i} \sum_\alpha \int_{\mathcal{C}_\alpha} \overline{(\tilde{t}_{\alpha,0} + g_\alpha)}\, Y dX \\
&\quad - \sum_\alpha t_{\alpha,0} (g(p_\alpha) - g(o_i)) \\
&\quad + \frac{1}{2\pi i} \sum_{i=1}^{\mathfrak{g}} 2\pi i \eta_i \left(\overline{-2\pi i \tilde{\eta}_i} \right) \\
&\quad + \frac{1}{2\pi i} \sum_{i=1}^{\mathfrak{g}} 2\pi i \tilde{\eta}_i \left(\overline{2\pi i \eta_i} \right)
\end{aligned}$$

$$
\begin{aligned}
&= && -\sum_\alpha \overline{\tilde t_{\alpha,0}}\, t_{\alpha,0} - \frac{1}{2\pi i}\sum_\alpha \int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX \\
& && -\sum_\alpha t_{\alpha,0}\, g(p_\alpha) \\
& && +2\pi i \sum_{i=1}^{\mathfrak{g}} \eta_i \overline{\tilde\eta_i} - \tilde\eta_i\, \overline{\eta_i} \\
&= && -\sum_\alpha \overline{\tilde t_{\alpha,0}}\, t_{\alpha,0} - \frac{1}{2\pi i}\sum_\alpha \int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX \\
& && -\sum_\alpha t_{\alpha,0}\, g(p_\alpha) + 4\pi \mathrm{Im} \sum_{i=1}^{\mathfrak{g}} \tilde\eta_i\, \overline{\eta_i} \\
&= && -\sum_\alpha \overline{\tilde t_{\alpha,0}}\, t_{\alpha,0} - \frac{1}{2\pi i}\sum_\alpha \int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX \\
& && -\sum_\alpha t_{\alpha,0}\, g(p_\alpha) + 4\pi \sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i \epsilon_i - \zeta_i \tilde\epsilon_i
\end{aligned}
\tag{3.59}
$$

Then we use lemma B.1 in appendix B:

$$
\frac{1}{2\pi i}\int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX = -\sum_{k=1}^{r_\alpha} \frac{|t_{\alpha,k}|^2}{k} R_\alpha^{-2k} + \sum_{k=1}^{\infty} k|\tilde t_{\alpha,k}|^2 R_\alpha^{2k} + 2t_{\alpha,0}^2 \ln R_\alpha - t_{\alpha,0} g_\alpha(p_\alpha) + \pi i t_{\alpha,0}^2. \tag{3.60}
$$

This implies

$$
\begin{aligned}
& && \frac{1}{2\pi i}\int_{\Sigma\setminus\cup_\alpha D_\alpha} \overline{YdX}\wedge YdX - \sum_\alpha \sum_{k=1}^{r_\alpha} \frac{1}{k}|t_{\alpha,k}|^2 R_\alpha^{-2k} + \sum_\alpha\sum_{k=1}^{\infty} k|\tilde t_{\alpha,k}|^2 R_\alpha^{2k} + 2\sum_\alpha |t_{\alpha,0}|^2 \ln R_\alpha \\
&= && -\sum_\alpha \overline{\tilde t_{\alpha,0}}\, t_{\alpha,0} - \sum_\alpha t_{\alpha,0}\, g(p_\alpha) + 4\pi\sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i\epsilon_i - \zeta_i\tilde\epsilon_i + \sum_\alpha t_{\alpha,0} g_\alpha(p_\alpha) \\
&= && -\sum_\alpha \overline{\tilde t_{\alpha,0}}\, t_{\alpha,0} - \sum_\alpha t_{\alpha,0}\, \tilde t_{\alpha,0} + 4\pi\sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i\epsilon_i - \zeta_i\tilde\epsilon_i \\
&= && -2\mathrm{Re}\left(\sum_\alpha t_{\alpha,0}\, \tilde t_{\alpha,0} - 2\pi \sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i\epsilon_i - \zeta_i\tilde\epsilon_i\right),
\end{aligned}
\tag{3.61}
$$

and thus

$$
\begin{aligned}
4F &= \frac{1}{2\pi i}\int_{\Sigma\setminus\cup_\alpha D_\alpha} \overline{YdX}\wedge YdX - \sum_\alpha \sum_{k=1}^{r_\alpha} \frac{1}{k}|t_{\alpha,k}|^2 R_\alpha^{-2k} + \sum_\alpha\sum_{k=1}^{\infty} k|\tilde t_{\alpha,k}|^2 R_\alpha^{2k} + 2\sum_\alpha |t_{\alpha,0}|^2 \ln R_\alpha \\
& \quad -2\mathrm{Re}\sum_\alpha\sum_{k=1}^{r_\alpha} t_{\alpha,k}\tilde t_{\alpha,k} \\
&= -2\mathrm{Re}\left(\sum_\alpha t_{\alpha,0}\, \tilde t_{\alpha,0} + \sum_{k=1}^{r_\alpha} t_{\alpha,k}\tilde t_{\alpha,k} - 2\pi\sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i\epsilon_i - \zeta_i\tilde\epsilon_i\right) \\
&= -4\mathrm{Re}\hat F + 4\pi\left(\sum_{i=1}^{\mathfrak{g}} \tilde\zeta_i\epsilon_i - \zeta_i\tilde\epsilon_i\right) \\
&= 4\check F.
\end{aligned}
\tag{3.62}
$$

4. Boutroux Curves

This is the main theorem

Theorem 4.1 (Boutroux Curve)**.** *There exists at least one Boutroux curve in $\mathcal{M}$. Boutroux curves are isolated in $\mathcal{M}$.*

Proof. Because of theorem 3.4, F admits at least one minimum on $\mathcal{M}$. Let $P = \mathcal{P} + Q$ a minimum of F. It belongs to some $\mathcal{M}_{\mathfrak{g}}$, with $\mathfrak{g} \leqslant \dim \mathbb{C}[\mathring{\mathcal{N}}]$.

It may happen that $\mathfrak{g} = 0$, in which case, the Boutroux condition is trivially satisfied (all cycles are contractible or reduce to small circles around punctures and we have $\mathrm{Re} \oint_{\mathcal{C}_\alpha} YdX = \mathrm{Re}(2\pi i \operatorname{Res}_\alpha YdX) = \mathrm{Re}(2\pi i t_{\alpha,0}) = 0$).

Otherwise, eq (3.43) i.e. corollary 3.1 implies that $dF = 0$ in $\mathcal{M}_{\mathfrak{g}}$, which implies $\zeta_i = 0$ for all $i = 1, \ldots, 2\mathfrak{g}$, and therefore we get Boutroux condition.

Boutroux curves are isolated, because in the period coordinates, F is locally strictly convex.

Theorem 4.2. *If $\mathcal{P} + Q$ is a Boutroux curve we have*

$$F(\mathcal{P} + Q) = -\mathrm{Re} F_0(\mathcal{P} + Q). \tag{4.1}$$

Proof. We have for any P

$$F = -\mathrm{Re} F_0 - 2\pi \zeta^t \tilde{\epsilon}, \tag{4.2}$$

and ζ vanishes for Boutroux curves.

5. Spectral network of first kind

A Boutroux curve has canonically some graphs associated to it, often called **spectral networks**. However, there is two versions used in many applications. For hyperelliptic curves (degree two in y) the two versions almost coincide as we shall see in subsection 6.1.

Let's denote $\Sigma \setminus$ punctures by Σ^*.

Theorem 5.1 (Harmonic function)**.** *If $P \in \mathcal{M}$ is a Boutroux curve, let $o_i \in \Sigma_i$ a generic point in each connected component Σ_i of Σ. The following function:*

$$\begin{aligned} \phi : \quad \Sigma \to & \quad \mathbb{R} \\ p \mapsto & \quad \phi(p) = \mathrm{Re} \int_{o_i}^{p} YdX \end{aligned} \tag{5.1}$$

is well defined and harmonic on Σ^*.

Remark 5.1. For $x \leftrightarrow y$ symmetry, we have the following function:

$$\begin{aligned} \tilde{\phi} : \quad \Sigma \to & \quad \mathbb{R} \\ p \mapsto & \quad \phi(p) = \mathrm{Re} \int_{o_i}^{p} XdY \end{aligned} \tag{5.2}$$

which is also well defined and harmonic on Σ^*.

Proof. The integration path from o_i to p is not unique, but two different paths differ by a closed Jordan loop γ, and $\mathrm{Re} \int_\gamma YdX = 0$, so $\phi(p)$ is independent of the chosen path. This makes it a well defined function on Σ^*. It is the real part of a locally analytic function, so it is harmonic.

For $\tilde{\phi}$, notice that by integration by parts, on any Jordan loop one has $\int_\gamma XdY = -\int_\gamma YdX$, and also $\operatorname{Res}_\alpha XdY = -\operatorname{Res}_\alpha YdX = -t_{\alpha,0}$.

Definition 5.1 (Spectral Network)**.** *For each* $a \in \Sigma$ *that is a ramification point or a zero of* YdX *(in particular this includes ramification points, and all zeros of* Y*), let*

$$\check{\Gamma}_a := \text{connected component of } \{p \mid \phi(p) = \phi(a)\} \text{ that contains } a, \tag{5.3}$$

and

$$\check{\Gamma} := \cup_a \check{\Gamma}_a. \tag{5.4}$$

$\check{\Gamma}_a$ *is called the "vertical trajectory" passing through* a*.*

Theorem 5.2. *Each* $\check{\Gamma}_a$ *is a finite union of smooth Jordan arcs.*

Except at zeros or poles of YdX*, these arcs have in the* x*-chart, a tangent in the direction* $e^{-i\arg Y(x)}$*.*

These arcs can cross only at points where $YdX = 0$ *or at punctures. Let the arcs that end at punctures be called "non-compact", and arcs that don't end at punctures be called "compact".*

If a *is a zero of* YdX*, possibly a ramification point with canonical local coordinate* $\zeta_a = (x - X_a)^{-1/a_a}$*, and where* $y \sim \eta_a \zeta_a^{-1/b_a}$*, so that* $YdX \sim -a_a \eta_a \zeta_a^{-a_a - b_a - 1} d\zeta_a$*. With* $a_a + b_a + 1 < 0$*, the arcs of* $\check{\Gamma}_a$ *start from* a *at angles*

$$e^{i\frac{-\arg\eta_a + \frac{\pi}{2} + k\pi}{-a_a - b_a}} \qquad k = 1, \dots, 2|a_a + b_a|. \tag{5.5}$$

Proof. There is a well defined tangent YdX at each point, in a direction given by $\arg(YdX) \in \frac{\pi}{2} + \pi\mathbb{Z}$, which implies $\arg dx = \frac{\pi}{2} - \arg Y + \pi\mathbb{Z}$. The only points where this is not the case is when $YdX = 0$ or YdX has a pole, and at these points we use the local coordinate ξ.

Remark 5.2. Since the spectral networks are described by algebraic equations the number of these trajectories is always finite.

Definition 5.2 (Cellular decomposition)**.** *The complement*

$$\Sigma \setminus \check{\Gamma} = \cup_{i=1}^m \check{\mathcal{D}}_i \tag{5.6}$$

is a finite union of disjoint connected open sets $\check{\mathcal{D}}_i \subset \Sigma$*, not containing any zero nor pole of* YdX*.* ϕ *is a harmonic function on each of them. The boundaries of* $\check{\mathcal{D}}_i$ *are arcs of* $\check{\Gamma}$*, and must contain at least a zero* a *of* YdX*. Let* a *a zero of* YdX *at the boundary of* $\check{\mathcal{D}}_i$*. Let*

$$\begin{aligned} g_a : \quad \check{\mathcal{D}}_i &\to \quad \mathbb{C} \\ p &\mapsto \quad g_a(p) = \int_a^p YdX. \end{aligned} \tag{5.7}$$

The map g_a *is well defined in a neighborhood of* a*. The real part* $\mathrm{Re} g_a(p) = \phi(p) - \phi(a)$ *is globally well defined.*

Theorem 5.3 (Elementary pieces)**.** *The image* $g_a(\check{\mathcal{D}}_i) \in \mathbb{C}$*, is a domain of* $\mathbb{C}$*, or of* $\mathbb{C}/i\tilde{c}\mathbb{Z}$ *for some* $\tilde{c} \in \mathbb{R}^*$*, whose boundaries are (if several) vertical lines. One of the boundaries is the imaginary axis. Since there are only two types of domains bounded by vertical lines in* $\mathbb{C}$ *and only two types of domains bounded by vertical lines in* $\mathbb{C}/i\tilde{c}\mathbb{Z}$*, only four possibilities can occur:*

- $g_a(\check{\mathcal{D}}_i)$ *is a half plane in* $\mathbb{C}$*, either* $\mathrm{Re} z > 0$ *or* $\mathrm{Re} z < 0$*.*
- $g_a(\check{\mathcal{D}}_i)$ *is a vertical strip in* $\mathbb{C}$ *bounded by two lines* $\mathrm{Re} z = 0$ *and* $\mathrm{Re} z = c$ *where* $c \neq 0$ *is some real constant.* c *must be of the form* $c = \phi(b) - \phi(a)$ *for some* b *another zero of* YdX*.*
- $g_a(\check{\mathcal{D}}_i)$ *is a half-cylinder, i.e. a half-plane quotiented by* $z \to z + i\tilde{c}$ *for some* $\tilde{c} \in \mathbb{R}^*$*.*
- $g_a(\check{\mathcal{D}}_i)$ *is a cylinder (or annulus) a vertical strip bounded by two lines* $\mathrm{Re} z = 0$ *and* $\mathrm{Re} z = c$ *where* $c = \phi(b) - \phi(a)$*, and quotiented by* $z \to z + i\tilde{c}$ *for some* $\tilde{c} \in \mathbb{R}^*$*.*

In all cases, the map $p \mapsto g_a(p)$ *(resp.* $p \mapsto g_a(p) \mod i\tilde{c}$ *if cylinder or half-cylinder), is a conformal isomorphism between* $\check{\mathcal{D}}_i$ *and its image.*

In the first two cases (strip or half-plane), $\check{\mathcal{D}}_i$ is simply connected, and in the last two cases (cylinder or half-cylinder), $\check{\mathcal{D}}_i$ is not simply connected.

In all cases except cylinder, $\check{\mathcal{D}}_i$ has a puncture on its boundary.

Together, the $g_a(\check{\mathcal{D}}_i)$ form charts of an atlas of Σ, with transition maps that are translations $g_a(p) = g_b(p) + c + i\tilde{c}$. The charts are either half-planes, strips, cylinders or half-cylinders.

*For strips or cylinders, $c = \phi(b) - \phi(a)$ is called the **width**.*

*For cylinders and half-cylinders, $\tilde{c}$ is called the **perimeter** of the cylinder.*

Proof. The image of local patches of $\check{\mathcal{D}}_i$ must be patches of $\mathbb{C}$ bounded by vertical lines. Since $dg_a = YdX$ never vanishes nor has poles in $\check{\mathcal{D}}_i$, g_a is locally a holomorphic isomorphism. g_a is not globally defined, it is defined only up to additive constants, which means that transition maps must be translations. Since the transition maps must match the boundary $\mathrm{Re} z = 0$, they must be vertical translations $g \mapsto g + i\tilde{c}$ with $\tilde{c} \in \mathbb{R}$.

The only connected domains of $\mathbb{C}$ that have only vertical lines as boundaries, can only be a half-plane or a strip. If $\tilde{c} \neq 0$, the transition maps being $g \mapsto g + i\tilde{c}$ imply that the image can be a half-plane or a strip quotiented by a vertical translation, i.e. a cylinder or a half-cylinder. These are the only possibilities.

If $\tilde{c} = 0$ then g_a is an isomorphism, and if $\tilde{c} \neq 0$ then $g_a \mod i\tilde{c}$ is an isomorphism.

Theorem 5.4. *In each $\check{\mathcal{D}}_i$, the map $X : \check{\mathcal{D}}_i \to \mathbb{C}$ is a conformal isomorphism to its image.*

Proof. If $\check{\mathcal{D}}_i$ is a cylinder or half-cylinder and contains a non-contractible loop γ, the projection $X(\gamma)$ in $\mathbb{C}$ is contractible (because $\mathbb{C}$ is simply connected). Hence, $\int_{X(\gamma)} dX = 0$ and, if $\check{\mathcal{D}}_i$ is a half-plane or strip, it is simply connected.

By definition $\check{\mathcal{D}}_i$ contains no ramification point, so X is a conformal isomorphism.

Theorem 5.5 (Metric and geodesics)**.** *The restriction of the metric $|YdX|^2$ of $\mathbb{C} \times \mathbb{C}$, to Σ is equal to $\frac{1}{2i}\overline{dg_a} \wedge dg_a$. It is thus the canonical Euclidean metric of $g_a(\check{\mathcal{D}}_i)$. The geodesics are fixed angles lines $\arg dg_a =$ constant, i.e. Eulcidian straight lines in the charts $g_a(\check{\mathcal{D}}_i)$. As a consequence, the vertical trajectories $\phi =$ constant, and therefore the edges of $\check{\Gamma}$ are geodesic.*

Theorem 5.6 (Half-cylinder=Fuchsian)**.** *Half-cylinders have some puncture α at their boundary, and their perimeter is $\tilde{c} = 2\pi t_{\alpha,0}$. The puncture α is then necessarily a simple pole of YdX. We call it a "Fuchsian" puncture. Therefore, half-cylinders are Fuchsian punctures.*

Proof. Near α we have $YdX \sim t_{\alpha,r_\alpha} \zeta_\alpha^{-r_\alpha - 1} d\zeta_\alpha$, and thus:

- If $r_\alpha = 0$ we have $\phi \sim t_{\alpha,0} \ln|\zeta_\alpha|$ whose vertical trajectories are circles around α.

- If $r_\alpha > 0$ we have $\phi \sim -\mathrm{Re}\left(\frac{t_{\alpha,r_\alpha}}{r_\alpha} \zeta_\alpha^{-r_\alpha}\right)$ whose vertical trajectories can't be circles around α.

- If we have a half-cylinder, we see that there is a foliation of circles as vertical trajectories surrounding α, and this can be compatible only with $r_\alpha = 0$, i.e. a simple pole.

Theorem 5.7 (No cylinders)**.** *There is no cylinders on the graph $\check{\Gamma}$ of a Boutroux curve.*

Proof. The proof uses combinatorics of graphs to compute the Euler characteristics. The Euler characteristics of Σ can be computed from the number of vertices, edges and faces of $\check{\Gamma}$. Let:

- ν = number of zeros of ydx. Each zero of ydx is of some degree ν_i.
- N = number of poles of ydx. Each pole of ydx is of some degree d_α.
- $c_{1/2}$ =number of Fuchsian poles, i.e. with $d_\alpha = 1$.
- $f = h+s+c+c_{1/2}$ =the number of faces, where h = number of half-planes, s =number of strips, c =number of cylinders and $c_{1/2}$ = number of half-cylinders.
- e_c = number of compact edges of $\check{\Gamma}$, i.e. going from a zero of ydx to a zero of ydx.
- e_{nc} = number of non-compact edges of $\check{\Gamma}$, i.e. going from a zero of ydx to a pole of ydx.
- $e = e_c + e_{nc}$ = the total number of edges.

We have the following relations:

- Since non-compact edges can end only on half-planes and on strips, and each half plane has 2 non-compact edges and each strip has 4, and all are doubly counted:
$$2e_{nc} = 2h + 4s. \tag{5.8}$$
- Since from a zero of degree ν_i of ydx we have $2(\nu_i + 1)$ half edges, and half edges can be either compact or non-compact we have
$$2e_c + e_{nc} = \sum_i 2(\nu_i + 1) = 2\nu + 2\deg_{zeros} ydx. \tag{5.9}$$
- Since from a pole of degree d_α of ydx we have $2(d_\alpha - 1)$ half-planes, we have
$$h = \sum_\alpha 2(d_\alpha - 1) = -2N + 2\deg_{poles} ydx. \tag{5.10}$$
- Together these relations imply that the total number of edges is
$$\begin{aligned} e &= e_c + e_{nc} \\ &= \nu + \deg_{zeros} ydx + s - N + \deg_{poles} ydx. \end{aligned} \tag{5.11}$$
- The Euler characteristic is thus:
$$\begin{aligned} 2-2g &= f - e + (\nu + N - c_{1/2}) \\ &= f - s - c_{1/2} - \deg_{zeros} ydx - \deg_{poles} ydx + 2N \\ &= f - s - c_{1/2} - h - \deg_{zeros} ydx + \deg_{poles} ydx \\ &= c - \deg_{zeros} ydx + \deg_{poles} ydx. \end{aligned} \tag{5.12}$$

  Every meromorphic 1-form on a Riemann surface of genus g satisfies
$$\deg_{poles} ydx - \deg_{zeros} ydx = 2 - 2g, \tag{5.13}$$
  this implies that
$$c = 0. \tag{5.14}$$
  There is no cylinders on a Boutroux curve.

Definition 5.3 (Tiles). *We can further subdivide each*
$$\check{\mathfrak{D}}_i = \cup_j \check{\mathfrak{D}}_{i,j} \tag{5.15}$$
by cutting along horizontal trajectories emanating from the vertices of $\check{\Gamma}$ that are on the boundary of $\check{\mathfrak{D}}_i$.

Each $\check{\mathfrak{D}}_{i,j}$ can have two, three or four sides, that cross at right angles:

- *If it has two sides, we call it a "corner tile" or "L tile", it has infinite width and height.*
- *If it has three sides, we call it a "U tile", it has either finite width, infinite height (vertical "U") or infinite width, finite height (horizontal "U").*
- *If it has four sides, we call it a "rectangle tile" or "R tile", it has finite width and finite height. In particular it has a finite area = width $\times$ height.*

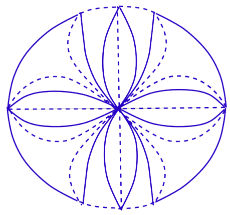


FIGURE 1. Example of a domain $\check{\mathcal{D}}_\alpha$, it is a gluing of L and U tiles. Vertical edges are continuous and horizontal edges are dashed.

Each tile is simply connected.

Theorem 5.8. *Let α a puncture. Consider the union of all tiles that have α at their boundary. Let $\check{\mathcal{D}}_\alpha$ the union of these tiles, all horizontal and vertical trajectories (their interior) ending at α, and α itself. $\check{\mathcal{D}}_\alpha$ is topologically a disc.*

The discs $\check{\mathcal{D}}_\alpha$ are disjoint.

The complement

$$\Sigma \setminus \cup_\alpha \check{\mathcal{D}}_\alpha \tag{5.16}$$

is the union of a graph (all compact horizontal and vertical lines) and all rectangle tiles.

See Fig.1.

Proof. For each puncture α, let D_α a disc of radius R_α small enough around α, such that D_α contains no zero of YdX nor ramification or nodal point nor other punctures. Consider all tiles that intersect D_α. They must be L or U tiles. Moreover, no tile can touch α without intersecting D_α, therefore $\check{\mathcal{D}}_\alpha$ is precisely the union of all tiles that intersect D_α, and they must be U or L tiles.

Since each U and L tiles have exactly two edges going to α, the gluing of U and L tiles around α has necessarily the topology of a disc around α.

Every L or U tile is connected, and touches at most one puncture. This implies that $\check{\mathcal{D}}_\alpha$ are disjoint.

The complement is the set of edges that don't touch punctures, i.e. all compact vertical and horizontal edges, and also all rectangle tiles.

6. Spectral network of second kind

There is another way of defining the spectral network that is very useful in applications, in particular in WKB analysis.

We mention that for hyperelliptic curves (of the form $P(x,y) = y^2 - R(x)$ with $R(x) \in \mathbb{C}(x)$), the first and second kinds are closely related as we shall see in subsection 6.1.

So here, we define the spectral network as:

Definition 6.1 (Spectral network)**.** *We define $\Gamma \subset \mathbb{C}$:*

$$\Gamma := \{x \in \mathbb{C} \,|\, \exists\, p \neq p',\ X(p) = X(p') = x \text{ and } \phi(p) = \phi(p')\} \tag{6.1}$$

(this set is independent of the choice of basepoint "o" in the definition of ϕ). Γ is a graph embedded in $\mathbb{C}$. We complete Γ by adding the vertices, i.e. we take the closure of Γ.

Definition 6.2 (Spectral network on Σ). *The set*

$$\tilde{\Gamma} := \{p \in \Sigma \,|\, \exists\, p' \neq p,\ X(p) = X(p') \text{ and } \phi(p) = \phi(p')\} \tag{6.2}$$

forms a graph $\tilde{\Gamma}$ on Σ. We complete $\tilde{\Gamma}$ by adding the vertices and the punctures, i.e. we take the closure of $\tilde{\Gamma}$. We have $X(\tilde{\Gamma}) = \Gamma$, and $\tilde{\Gamma} \subset X^{-1}(\Gamma)$.

Remark 6.1. For an arbitrary generic $x \in \mathbb{C}$, $X^{-1}(\{x\}) = \{p_1(x), \ldots, p_d(x)\}$ contains d points where $d = \deg_y P$. In addition, $\{\phi(p_1(x)), \phi(p_2(x)), \ldots, \phi(p_d(x))\}$ are all distinct.

For a point $x \in \mathbb{C} \setminus \Gamma$, let us order the p_i's by the ordering of the $\phi(p_i(x))$.

$$\forall\, x \text{ generic}, \qquad \phi(p_1(x)) < \phi(p_2(x)) < \cdots < \phi(p_d(x)) \tag{6.3}$$

Definition 6.3. *For a point $p \in \Sigma \setminus X^{-1}(\Gamma)$, we call the index of p the integer $i(p)$, such that:*

$$p_{i(p)}(x) = p. \tag{6.4}$$

For a point p that belongs to $X^{-1}(\Gamma)$ and doesn't belong to $\tilde{\Gamma}$, we define the index by continuity from its neighborhood in Σ. Thus, the index is defined on $\Sigma \setminus \tilde{\Gamma}$. We have

$$1 \leqslant i(p) \leqslant d = \deg X. \tag{6.5}$$

Definition 6.4 (Domains of given index). *Let:*

$$\tilde{\mathcal{D}}_i := \{p \in \Sigma \setminus \tilde{\Gamma} \,|\, i(p) = i\} = \cup_j \tilde{\mathcal{D}}_{i,j} \quad , \ \tilde{C}_{i,j} = X(\tilde{\mathcal{D}}_{i,j}) \tag{6.6}$$

i.e. $\tilde{\mathcal{D}}_i \subset \Sigma$ is the open set of points of index i, and $\tilde{\mathcal{D}}_{i,j}$ are its connected components. Let

$$\tilde{\Gamma}_i = \partial \tilde{\mathcal{D}}_i, \qquad \Gamma_i = X(\tilde{\Gamma}_i). \tag{6.7}$$

Proposition 6.1. *For any (i, j), the map $X : \tilde{\mathcal{D}}_{i,j} \to \tilde{C}_{i,j}$ is an analytic bijection, whose inverse is analytic.*

Proof. the map $X : \tilde{\mathcal{D}}_{i,j} \to \tilde{C}_{i,j}$ is surjective by definition. It is a bijection whose inverse is $x \mapsto p_i(x)$. The only point where X or its inverse would be non-analytic can only be punctures or ramification points, which are vertices of the graph, and are at the boundaries of $\tilde{\mathcal{D}}_{i,j}$s, they are outside.

Proposition 6.2. *For any fixed index i, the disjoint union of all $\tilde{C}_{i,j}$ with index i is the complex plane itself (except the graph Γ_i).*

$$\mathbb{C} \setminus \Gamma_i = \sqcup_j \tilde{C}_{i,j}, \tag{6.8}$$

and thus

$$\overline{\cup_j \tilde{C}_{i,j}} = \mathbb{C}. \tag{6.9}$$

In other words, we have d copies of the complex plane, cut along the spectral network graph.

Proof. for each $x \in \mathbb{C} \setminus \Gamma$, $p_i(x)$ belongs to $\tilde{\mathcal{D}}_i$, and thus is necessarily in some $\tilde{\mathcal{D}}_{i,j}$, and thus $x \in \tilde{C}_{i,j}$.

Moreover, the $\tilde{C}_{i,j}$ are disjoints, indeed, imagine that there exists some $x \in \tilde{C}_{i,j} \cap \tilde{C}_{i,j'}$, that means that $p_i(x) \in \tilde{\mathcal{D}}_{i,j} \cap \tilde{\mathcal{D}}_{i,j'} = \emptyset$, so this is impossible. We thus have

$$\sqcup_j \tilde{C}_{i,j} = \mathbb{C} \setminus \Gamma_i. \tag{6.10}$$

Those d copies of $\mathbb{C}$ with cuts, provide an atlas of Σ, whose charts are the $\tilde{C}_{i,j}$s. The transition maps are obtained by gluing the charts along edges and at vertices of the graph, with transition function $x \mapsto x$.

Lemma 6.1. *For every domain of given index $\mathcal{D}_i$, there is a finite number of connected components $\mathcal{D}_{i,j}$.*

Proof. The edges of the graph $\tilde{\Gamma}_i$ are algebraic lines, therefore the number of connected components is finite.

Edges:

FIGURE 2. two sheets meet along an edge e. They have adjacent index $i, i+1$. σ_e permutes the domains that analytically continue each other across e (it permutes the half planes of the same color). τ_e permutes the two domains on top of each other along e (it permutes the colors).

Definition 6.5 (Edges permutations)**.** *Each edge e of $\tilde{\Gamma}$ is at the intersection of two domains, whose index differ by one, $\tilde{\mathcal{D}}_{i,j}$ and $\tilde{\mathcal{D}}_{i+1,j'}$. The edge $X(e)$ of Γ is also at the intersection of domains $\tilde{C}_{i,j}$ and $\tilde{C}_{i,j''}$, with the same index i. The domain $\tilde{\mathcal{D}}_{i,j''}$ has an edge e' such that $X(e') = X(e)$, and on the other side on Σ, there is a domain $\tilde{\mathcal{D}}_{i+1,j'''}$.*

We define the permutations (these are products of two transpositions):

$$\sigma_e : ((i,j) \leftrightarrow (i+1,j'))\,((i,j'') \leftrightarrow (i+1,j''')) \tag{6.11}$$

$$\tau_e : ((i,j) \leftrightarrow (i+1,j'''))\,((i,j'') \leftrightarrow (i+1,j')) \tag{6.12}$$

σ_e permutes the domains that analytically continue each other across e.

τ_e permutes the two domains on top of each other along e.

See Fig.2.

Lemma 6.2. *Along each edge e we have*

$$\sigma_e^2 = Id, \quad \tau_e^2 = Id, \quad \tau_e\sigma_e = \sigma_e\tau_e. \tag{6.13}$$

Proof. simple computation.

Vertices:

• Branch points.

At a regular branch point we have $p_i(x) \to p_{i+1}(x)$. Let $z = \sqrt{x-a}$. We have $y \sim y(a) + y'(a)z + \frac{1}{2}y''(a)z^2 + \dots$, and thus $\phi(p_i) - \phi(p_{i+1}) \sim \frac{4}{3}\,\mathrm{Re}(y'(a)(x-a)^{3/2}) + \dots$, and thus a is a trivalent vertex.

• Higher Branch points.

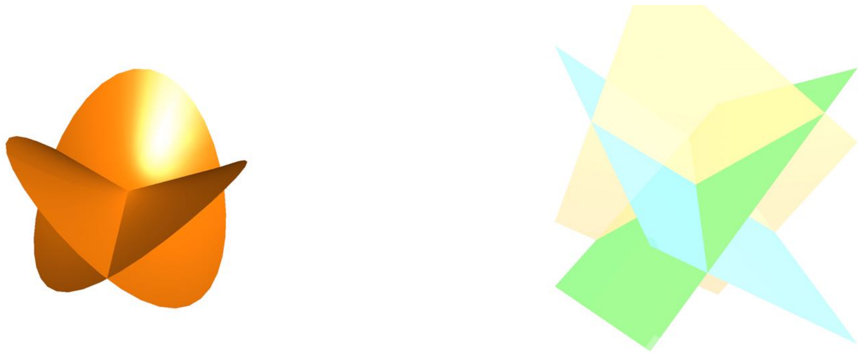


FIGURE 3. On the left a regular ramification point, it is a trivalent vertex, two sheets meet. On the right, a virtual vertex. three sheets meet and all have different normal vectors.

At a branch point of order r we have $p_i(x) = p_{i+1}(x) = \cdots = p_{i+r-1}(x)$. Let $z = (x-a)^{1/r}$. We have $y \sim y(a) + y'(a)z + \frac{1}{2}y''(a)z^2 + \ldots$, and thus $\phi(p_i) - \phi(p_{i+1}) \sim \frac{r}{r+1}\,\mathrm{Re}((1-e^{2i\pi/r})\,y'(a)(x-a)^{(r+1)/r}) + \ldots$, and thus a is a $r+1$ valent vertex.

• Nodal points.

It may happen that vertices can be nodal points (but not all nodal points are vertices). These are points where $y(p_i) = y(p_{i+1})$, with $p_i \neq p_{i+1}$, and in fact where $(y(p_i) - y(p_{i+1}))$ vanishes at an order k. At those points we have

$$\phi(p_i) - \phi(p_{i+1}) \sim \mathrm{Re}(y'(d)\,(x-d)^{k+1}), \tag{6.14}$$

and thus they give $2(k+1)$ valent vertex.

• Virtual vertices.

They are points where $\phi(p_i) = \phi(p_{i+1}) = \phi(p_{i-1})$, but $p_i \neq p_{i-1} \neq p_{i+1}$, and the differentials $d(\phi(p_i) - \phi(p_j))$ do not vanish. In other words, the boundaries of the $\tilde{\mathcal{D}}_{i,j}$ have smooth tangents there.

See Fig.3.

Lemma 6.3. *For any domain $\tilde{\mathcal{C}}_{i,j}$, there is no edge at the boundary of $\tilde{\mathcal{C}}_{i,j}$, such that both sides are in $\tilde{\mathcal{C}}_{i,j}$. We say that the boundary of $\tilde{\mathcal{C}}_{i,j}$ has no self-edge.*

Equivalently, for every $e \in \tilde{\Gamma}$, $\sigma_e \circ \tau_e$ has no fixed point.

Proof. Assume that there would exist an edge $e \subset \partial\tilde{\mathcal{C}}_{i,j}$, such that both sides are in $\tilde{\mathcal{C}}_{i,j}$. Let $x \in e$, and x_+ and x_- some points very close to x on each side. By definition of $\tilde{\mathcal{C}}_{i,j}$, we have $p_i(x_+) \in \tilde{\mathcal{D}}_{i,j}$ and $p_i(x_-) \in \tilde{\mathcal{D}}_{i,j}$, and thus

$$\lim_{x_+ \to x} p_i(x_+) = \lim_{x_- \to x} p_i(x_-) = p_x. \tag{6.15}$$

Let $e' = \{p_x \mid x \in e\}$, such that $X(e') = e$. e' is a boundary edge of $\tilde{\mathcal{D}}_{i,j}$, in fact it is a self-edge of $\tilde{\mathcal{D}}_{i,j}$. This implies that $\tilde{\mathcal{D}}'_{i,j} = \tilde{\mathcal{D}}_{i,j} \cup e'$ is an open connected domain of Σ. This implies that YdX is holomorphic on a neighborhood of e'. Thus, $\sigma_{e'}((i,j)) = (i,j)$, which is impossible because σ_e must always move the index by one, i.e $\sigma_{e'}((i,j)) = (i \pm 1, j')$. This is a contradiction, so the assumption that $\partial\tilde{\mathcal{C}}_{i,j}$ has a self edge was impossible.

Definition 6.6 (Admissible and maximal domains)**.** *Let $\mathcal{D}$ a union of domains $\tilde{\mathcal{D}}_{i,j}$s, and edges of $\tilde{\Gamma}$.*

$\mathcal{D}$ is called admissible iff:

- *$\mathcal{D}$ is open,*
- *ϕ is harmonic on $\mathcal{D}$,*
- *X is injective on $\mathcal{D}$.*

Let

$$\mathcal{C} = X(\mathcal{D}) = \bigcup_{\tilde{\mathcal{D}}_{i,j} \subset \mathcal{D}} \tilde{\mathcal{C}}_{i,j} \bigcup_{e \subset \mathcal{D}} X(e). \tag{6.16}$$

We say that $\mathcal{D}$ is maximal iff there is no $\mathcal{D}'$ admissible such that $\mathcal{D} \subset \mathcal{D}'$ and $\mathcal{D} \neq \mathcal{D}'$.

The following lemmas are immediate:

Lemma 6.4. *Every single domain $\tilde{\mathcal{D}}_{i,j}$ is admissible. For every admissible domain, there exists a maximal admissible domain that contains it.*

Lemma 6.5. *If $\mathcal{D}$ is admissible, then $\mathcal{C} = X(\mathcal{D})$ is an open domain of $\mathbb{C}$. Its boundary is a graph (possibly empty). Its complement is a graph and possibly a finite union of open sets of $\mathbb{C}$. The map $X : \mathcal{D} \to \mathcal{C}$ is a conformal bijection.*

Theorem 6.1. *Let $\mathcal{D}$ a maximal admissible domain, and $\mathcal{C} = X(\mathcal{D})$.*

Then the complement of $\mathcal{C}$ can contain no open set, it must be a graph $\hat{\Gamma}$.

$X : \mathcal{D} \to \mathcal{C}$ is a conformal bijection. $\Phi = \phi \circ X^{-1}$ is harmonic on $\mathcal{C} = \mathbb{C} \setminus \hat{\Gamma}$. Φ can be extended by continuity to $\mathbb{C}$. Φ is then continuous on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus \hat{\Gamma}$, and the places where it is not harmonic is exactly on $\hat{\Gamma}$.

Proof. Assume that the complement of $\mathcal{C}$ contains an open domain. Let us choose x in this open domain. x is not in Γ, so it has d distinct preimages $p_1(x), \dots p_d(x)$. Each of them is in some $\tilde{\mathcal{D}}_{i,j}$.

If we assume that there exists some (i,j) such that $\tilde{\mathcal{C}}_{i,j} \cap \mathcal{C} = \emptyset$, then we would have $\tilde{\mathcal{D}}_{i,j} \cap \mathcal{D} = \emptyset$, and we could add $\tilde{\mathcal{D}}_{i,j}$ to $\mathcal{D}$ to obtain an admissible domain. This would contradict the maximality of $\mathcal{D}$.

Therefore, for every (i,j) we have $\tilde{\mathcal{C}}_{i,j} \cap \mathcal{C} \neq \emptyset$, which implies that there is some (i,j) such that $\tilde{\mathcal{D}}_{i,j} \cap \mathcal{D} \neq \emptyset$. This means that $\tilde{\mathcal{D}}_{i,j} \subset \mathcal{D}$, which contradicts our hypothesis that $x \notin X(\mathcal{D})$.

This implies that the complement of $\mathcal{C}$ can not contain any open domain, it can contain only edges.

By definition ϕ is harmonic on $\mathcal{D}$, and $\Phi = \phi \circ X^{-1}$ is harmonic on $\mathcal{C} = \mathbb{C} \setminus \hat{\Gamma}$.

The only places where it could be non-harmonic could be a subgraph of $\hat{\Gamma}$.

Let e and edge of $\hat{\Gamma}$. It is a boundary of $\mathcal{C}$, and thus there is an e' boundary of $\mathcal{D}$ for which $X(e') = e$. If we assume that Φ is harmonic on e, this would imply that ϕ is harmonic on e'. This implies that $\mathcal{D} \cup e'$ would be admissible. This would contradict the maximality of $\mathcal{D}$. Therefore Φ must be non-harmonic on the edges of $\hat{\Gamma}$.

Definition 6.7. *Each edge e of $\hat{\Gamma}$ has two sides e_+, e_- on $\partial\mathcal{D}$, oriented such that $\mathcal{D}$ is on their left. They border two domains, e_+ borders $\mathcal{D}_{e_+}$, and e_- borders $\mathcal{D}_{e_-}$. We must have*

$$\mathcal{D}_{e_-} = \sigma_e \circ \tau_e(\mathcal{D}_{e_+}). \tag{6.17}$$

On e_+ (resp. e_-), let

$$d\rho_e := \frac{1}{2\pi} \mathrm{Im}(dg - \tau_e^* dg). \tag{6.18}$$

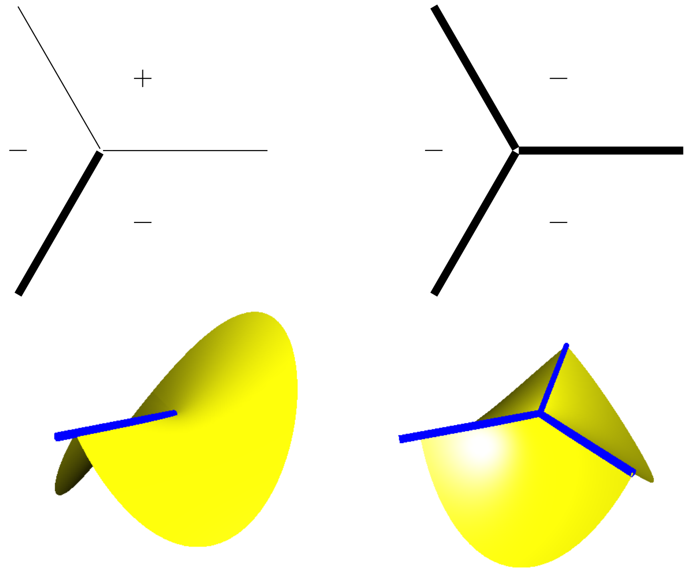


FIGURE 4. Generic branch point. An edge belongs to $\hat{\Gamma}$ only if the sign of $Y_i - Y_j$ is the same on both sides. If $--$ then $d\rho_e$ is positive and if $++$ then $d\rho_e$ is negative. Left: only one edge belongs to $\hat{\Gamma}$. Right: three edges belong to $\hat{\Gamma}$, and $d\rho_e$ is of the same sign on the three edges.

Adding the measure of all edges:

$$d\rho := \sum_{e = \text{ edges of } \hat{\Gamma}} \chi_e \, d\rho_e, \tag{6.19}$$

where χ_e is the characteristic function of e.

$d\rho$ *is a real measure on* $\hat{\Gamma}$.

Proposition 6.3. *The only places where $d\rho$ can vanish, are ramification or nodal points.*

Proof. By definition, on any edge e of $\hat{\Gamma}$, which is at the intersection of sheets $Y_i(x)$ and $Y_j(x)$, we have that $\mathrm{Re}(Y_i(x) - Y_j(x))dx$ vanishes. $d\rho_e = 0$ implies that the imaginary part is also vanishing, i.e. that $(Y_i(x) - Y_j(x))dx = 0$. This implies that either $dx = 0$ (ramification point) or $Y_i(x) - Y_j(x) = 0$ (nodal point).

Proposition 6.4. *At generic ramification points, there are either one or three edges of $\hat{\Gamma}$. If there are three edges, the sign of $d\rho$ is the same on all three edges.*

Proof. At generic ramification points we have $(Y_i(x) - Y_j(x))dx \sim C(x-a)^{\frac{1}{2}}dx$ and thus $g_i(x) - g_j(x) \sim \frac{2}{3}C(x-a)^{\frac{3}{2}}$. There are three lines where $\mathrm{Re}C(x-a)^{\frac{3}{2}} = 0$ (See Fig.4). If we consider the case where $\hat{\Gamma}$ has three lines meeting, then we choose the branches of the square root discontinuous accross each of them, and it is easy to see that the sign of $\mathrm{Im}(g_i - g_j)$ is the same on the three edges.

Proposition 6.5. *At generic nodal points, there are either 0, 2 or 4 edges of $\hat{\Gamma}$. If there are four edges, the sign of $d\rho$ is the same on all four edges.*

When dealing with two edges, if they are adjacent, $d\rho$ has the same sign along both edges, but if they are aligned, the sign of $d\rho$ becomes opposite.

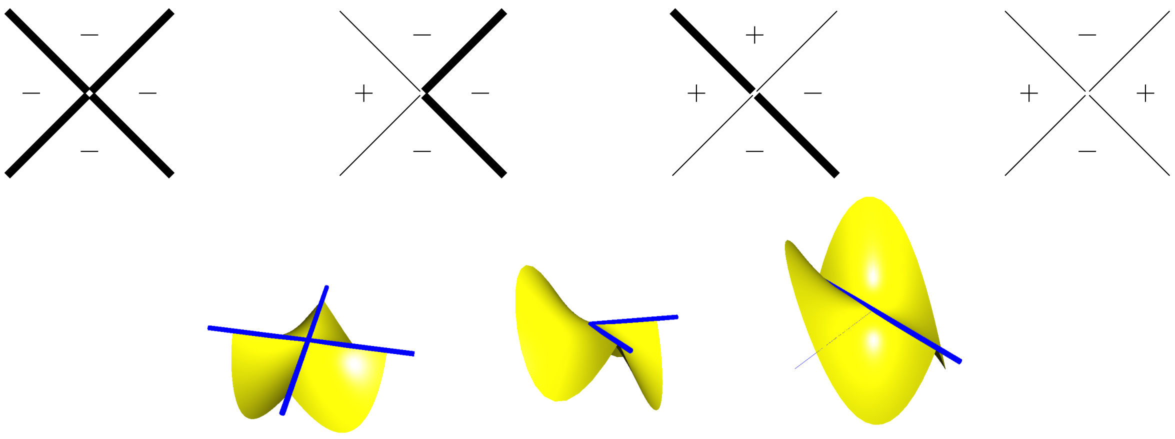


FIGURE 5. Generic nodal point. An edge belongs to $\hat{\Gamma}$ only if the sign of $Y_i - Y_j$ is the same on both sides. If $--$ then $d\rho_e$ is positive and if $++$ then $d\rho_e$ is negative. Left: four edges, the sign is the same on the four edges. Middle: two adjacent edges on $\hat{\Gamma}$, same sign. Right: two aligned edges on $\hat{\Gamma}$, opposite sign. Last case, no edge on $\hat{\Gamma}$.

Proof. At generic nodal points we have $(Y_i(x) - Y_j(x))dx \sim C(x-a)dx$ and thus $g_i(x) - g_j(x) \sim \frac{1}{2}C(x-a)^2$. There are four lines where $\mathrm{Re}C(x-a)^2 = 0$ (See Fig.5). The signs are easily computed in each case.

Theorem 6.2. *The measure* $d\rho$ *on the edges of* $\hat{\Gamma}$ *coincides with the measure* $\frac{1}{2\pi i}\Delta\Phi$ *on* $\mathbb{C}$. *Since* $\Delta\Phi = 0$ *on* $\mathbb{C}\setminus\hat{\Gamma}$, *the measure is localized on* $\hat{\Gamma}$.

Proof. This follows from Stokes theorem. Let $f : \mathbb{C} \to \mathbb{R}$ a real valued C^∞ bounded function. We have

$$\begin{aligned}
\int_{\mathbb{C}} f\Delta\Phi &= \int_{\mathbb{C}\setminus\hat{\Gamma}} f\Delta\Phi \\
&= \int_{\mathbb{C}\setminus\hat{\Gamma}} f\bar{\partial}d\Phi \\
&= \int_{\hat{\Gamma}} f\, d\Phi \\
&= \sum_{e=\text{edges of }\hat{\Gamma}} \int_e f\,(Y_{\text{left}} - Y_{\text{right}})dx \\
&= \int_{\hat{\Gamma}} f\, 2\pi i d\rho. \qquad (6.20)
\end{aligned}$$

Lemma 6.6. *On* Σ *we have*

$$\Delta\tilde{\phi} = -\Delta\phi. \qquad (6.21)$$

Proof. Remember that

$$\tilde{\phi} = -\phi + \mathrm{Re}(XY), \qquad (6.22)$$

and XY is an analytic function on Σ, whose Real part is then harmonic, and therefore

$$\Delta\tilde{\phi} = -\Delta\phi. \qquad (6.23)$$

Moreover, we mention the following theorem that can be very useful:

Theorem 6.3 (Change of functions that don't affect the spectral network)**.** *The spectral network* Γ *is unchanged if we change* $Y \to Y + V'(x)$ *where* $V(x) \in \mathbb{C}(x)$. *In particular the measure is unchanged*

$$\Delta(\Phi(x) + V(x)) = \Delta\Phi(x). \qquad (6.24)$$

In other words, we change

$$P(x,y) \to P(x, y + V'(x)). \tag{6.25}$$

This changes the Newton's polygon, and the moduli space, but to an isomorphic one.

Proof. This is obvious since $V(X(z^i(x))) = V(X(z^j(x)))$, so this change of function doesn't change the spectral network, the indices and the domains.

6.1. **Hyperelliptical case, comparison of the two kinds of spectral networks.** An hyperelliptical plane curve is a plane curve with a Newton's polygon of the form:

$$\mathcal{P}(x,y) = y^2 D(x) - U(x), \tag{6.26}$$

where D and U are polynomials of x. The moduli space $\mathcal{M} = \mathcal{P} + \mathbb{C}[\overset{\circ}{\mathcal{N}}]$ is an affine vector space of polynomials of x:

$$\mathcal{M} \subset \mathcal{P} + \mathbb{C}[x], \qquad \dim \mathcal{M} \leqslant \frac{1}{2}(\deg U + \deg D) - 1. \tag{6.27}$$

But notice that if D has multiple zeros, the dimension may be smaller, here we give only an upper bound.

Definition 6.8 (Hyperelliptic involution)**.** *There exists an involution $\sigma : \Sigma \to \Sigma$, such that $X \circ \sigma = X$ and $Y \circ \sigma = -Y$.*

The fixed points must have $y = 0$, i.e. $U(x) = 0$, and $x = \infty$ if $\deg U - \deg D$ is odd.

The fixed points of σ are the ramification points and odd punctures.

Nodal points are pairs $(p, \sigma(p))$, they are invariant as a pair, but p itself is not invariant.

6.1.1. *Geometry of hyperelliptic curves.*

- Punctures are zeros of $D(x)$, and $X^{-1}(\infty)$ if $\deg U > \deg D - 4$.
- Ramification points are the odd zeros of $U(x)$.
- Nodal points are the even zeros of $U(x)$.
- The genus of Σ is
$$\mathfrak{g} = -1 + \left\lfloor \frac{1}{2}\#\text{ramification points} \right\rfloor. \tag{6.28}$$
- Let
$$U(x) = U_-(x) U_+(x)^2, \tag{6.29}$$
where $U_-(x)$ has only odd zeros, and $U_+(x) = \sqrt{U(x)/U_-(x)}$ contains all the even zeros, chosen so that U_- and U_+ have no common zeros.
- Let r the number of ramification points, let n the number of nodal points.
$$\begin{aligned} r &= \#\text{zeros of } U_- \\ n &= \#\text{zeros of } U_+. \end{aligned} \tag{6.30}$$

From now on, we choose a Boutroux curve in $\mathcal{M}$.

We choose the origin for defining ϕ, to be a ramification point, i.e. a point invariant under the hyperelliptic involution.

Lemma 6.7. *ϕ is odd under the involution:*

$$\phi \circ \sigma = -\phi. \tag{6.31}$$

In particular, all ramification points have $\phi = 0$.

Proof. We have $\sigma^* YdX = -YdX$, and therefore $\sigma^* d\phi = -d\phi$. This implies that $\phi + \sigma^*\phi$ must be a constant, and since it vanishes at one point, it must be zero.

6.1.2. *Spectral Networks of 1st kind.* The spectral network is the graph whose edges are horizontal trajectories starting from every branch points or nodal points.

Lemma 6.8. *The spectral network graph $\tilde{\Gamma}$ of Definition 5.1 is invariant under the involution:*

$$\sigma(\tilde{\Gamma}) = \tilde{\Gamma}. \tag{6.32}$$

Moreover edges emanating from ramification points must have $\phi = 0$.

The graph cuts the curve into domains that are either half-planes, strips and half-cylinders (if Fuchsian punctures).

Proof. If a line is a vertical line, i.e. $d\phi = 0$, then its image by the involution is $-d\phi = 0$ and is also a vertical line.

6.1.3. *Spectral Networks of 2nd kind.*

Definition 6.9. *Let*

$$\tilde{\Gamma}_0 := \phi^{-1}(\{0\}), \quad \Gamma_0 := X(\tilde{\Gamma}_0). \tag{6.33}$$

Let

$$\mathcal{D}_+ := \{p \mid \phi(p) > 0\}, \quad \mathcal{D}_- := \{p \mid \phi(p) < 0\}, \tag{6.34}$$

and $\mathcal{C}_\pm = X(\mathcal{D}_\pm)$. We have $\partial\mathcal{D}_+ = \partial\mathcal{D}_- = \tilde{\Gamma}_0$.

Lemma 6.9. *Let $\mathcal{D}_{\pm,j}$ the connected components of $\mathcal{D}_+$. Each domain $\mathcal{D}_{\pm,j}$ is a finite union of strips, half-planes and half-cylinders with vertical boundaries, which are the connected components of $\mathcal{D}_{\pm,j} \setminus \check{\Gamma}$ where $\check{\Gamma}$ is the spectral network of 1st kind.*

Proof. Each connected components of $\mathcal{D}_{\pm,j} \cap \check{\Gamma}$ is obtained by cutting the connected components of $\Sigma \setminus \check{\Gamma}$ by the vertical line $\phi = 0$. In all cases cutting a vertical half-plane, a vertical strip or a vertical half-cylinder by the line $\phi = 0$ gives 1 or 2 half-plane, strip or half-cylinder.

Proposition 6.6. *$\mathcal{D}_-$ is a maximal admissible domain.*

Proof. It is admissible because ϕ is harmonic in each connected component, and X is $1:1$ on $\mathcal{D}_-$. It is maximal, because $X(\mathcal{D}_-) = \mathbb{C}P^1 \setminus \Gamma_0$, its complement is a graph. Therefore, no other open domain can be added to $\mathcal{D}_-$. Moreover, since the sign of ϕ is the same on both sides of each edge, ϕ is not harmonic on edges. Adding an edge to $\mathcal{D}_-$ would make it not admissible.

The following proposition is an immediate consequence

Proposition 6.7. *Let $\mathcal{D}$ a maximal admissible domain. It is a finite union of vertical half-plane, vertical strip or vertical half-cylinder (if Fuchsian puncture).*

Its boundary $\hat{\Gamma} = \partial X(\mathcal{D})$ is a subgraph of $X(\phi^{-1}(\{0\}))$, all edges have $\Phi = 0$. The measure $\frac{1}{2\pi i}\Delta\Phi$ is a real measure, localized on the boundary $\hat{\Gamma}$.

7. Applications and examples

There are many applications of Boutroux curves and their spectral networks. Most famous examples are vertical trajectory foliations of the moduli space of Riemann surfaces by Strebel graphs, and eigenvalues equilibrium density for random matrices.

7.1. **Example: Weierstrass curve.** Let us exemplify all the method for the Weierstrass curve. Let

$$\mathcal{P}(x,y) = y^2 - x^3 + g_2 x + g_3 \tag{7.1}$$

whose moduli space is

$$\mathcal{M} = \{g_3\} \sim \mathbb{C}, \qquad \dim \mathcal{M} = 1. \tag{7.2}$$

In other words we shall keep g_2 fixed and take g_3 as a coordinate of $\mathcal{M}$.

For $4g_2^3 - 27g_3^2 \neq 0$, the curve has genus $\mathfrak{g} = 1$, we have $g_2 = 15\nu^4 G_4(\tau)$, $g_3 = -35\nu^6 G_6(\tau)$, in other words we can view g_3 as a function of τ

$$g_3 = -35 g_2^{\frac{3}{2}} \frac{G_6(\tau)}{(15G_4(\tau))^{\frac{3}{2}}}, \tag{7.3}$$

and we can moreover view g_3 as a function of $q = e^{2\pi i\tau}$. In other words we can parametrize $\mathcal{M}$ by the coordinate q.

The degenerate curve $4g_2^3 - 27g_3^2 = 0$, will be considered to correspond to $\tau = i\infty$, i.e. $q = 0$.

For $q \neq 0$ we have

$$t_{\infty,5} = -2, \quad t_{\infty,1} = g_2, \quad \tilde{t}_{\infty,5} = \frac{1}{10} g_2 g_3, \quad \tilde{t}_{\infty,1} = g_3. \tag{7.4}$$

$$\eta = 3i\nu^5 G_4'(\tau), \quad \tilde{\eta} = \tau\eta + 12i\nu^5 G_4(\tau). \tag{7.5}$$

This gives

$$\hat{F} = \frac{2}{5} g_2 g_3 = -210\nu^{10} G_4(\tau) G_6(\tau), \tag{7.6}$$

and thus

$$\begin{aligned} F &= -\frac{2}{5}\mathrm{Re} g_2 g_3 + \pi(\tilde{\zeta}\epsilon - \zeta\tilde{\epsilon}) = -\frac{2}{5}\mathrm{Re} g_2 g_3 + \pi \mathrm{Im}\, \tilde{\eta}\bar{\eta} \\ &= 210\ \mathrm{Re}\left(\nu^{10} G_4(\tau) G_6(\tau)\right) + 9\pi|\nu|^{10}\ \mathrm{Im}\Big(\tau|G_4'(\tau)|^2 + 4\overline{G_4'(\tau)} G_4(\tau)\Big). \end{aligned} \tag{7.7}$$

When $4g_2^3 - 27g_3^2 = 0$, we have $F = -\frac{2}{5}\mathrm{Re} g_2 g_3$, which is the limit when $q \to 0$, i.e. F is continuous at $q = 0$.

The Boutroux curve, i.e. the minimum of F is reached at $\zeta = \tilde{\zeta} = 0$, i.e. when $F = -\frac{2}{5}\mathrm{Re} g_2 g_3$.

F as a function of $q = e^{2\pi i\tau}$ is plotted in Fig.6. Let us admit that if $g_2 \in \mathbb{R}$, the minimum is reached at the degenerate curve $q = 0$ (this is obvious on Fig.6).

We parametrize the degenerate curve as

$$\begin{cases} g_2 = -3u^2, \quad g_3 = 2u^3 \\ X(z) = z^2 - 2u \\ Y(z) = z^3 - 3uz \end{cases} \tag{7.8}$$

We have

$$g(z) = \frac{2}{5} z^5 - 2uz^3. \tag{7.9}$$

The 1st kind of spectral network has 10 half-planes and 2 strips. See Fig.7.

7.2. **Strebel graphs.** Another major example is the following.

Let $z_1, \dots, z_N$ fixed points in $\mathbb{C}^N$ with $N \geqslant 3$, and let $L_1, \dots, L_N$ fixed positive real numbers. Let

$$D(x) = \prod_{\alpha=1}^{N} (x - z_\alpha) \tag{7.10}$$

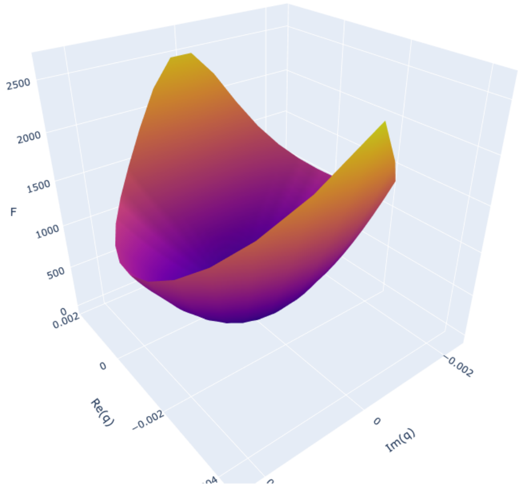


FIGURE 6. $F(q)$ for the Weierstrass curve. We plot F as a function of $q = e^{2\pi i\tau}$ rather than a function of $g_3 = -35 g_2^{\frac{3}{2}}\, G_6(\tau)\,(15 G_4(\tau))^{-\frac{3}{2}}$. We plot for q in the fundamental domain, i.e. $-\frac{1}{2} < \mathrm{Re}\tau \leqslant \frac{1}{2}$ and $|\tau| \geqslant 1$. The minimum is reached at $q = 0$.

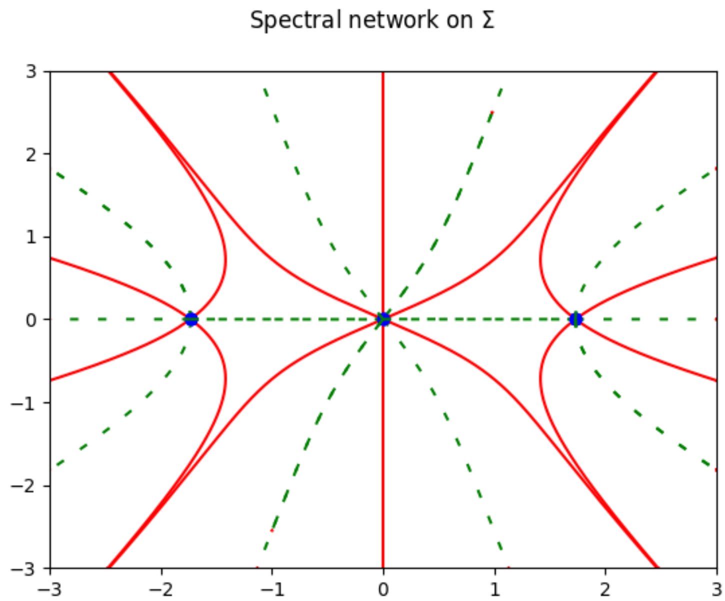


FIGURE 7. Spectral network in the z plane, for the Weierstrass curve. Continuous lines are vertical trajectories, Dashed lines are horizontal trajectories. Vertical trajectories make 10 half-planes and 2 strips.

7.2.1. *Newton's polygon.* Let

$$\mathcal{P}(x, y) = y^2 D(x)^2 - \sum_{\alpha=1}^{N} L_\alpha^2 D'(z_\alpha) \frac{D(x)}{x - z_\alpha}. \tag{7.11}$$

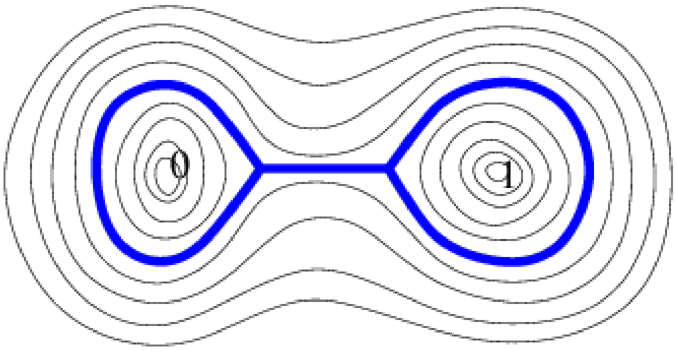


FIGURE 8. Strebel graph for $N = 3$. There are 3 faces of prescribed perimeters.

We have

$$\mathcal{M} = \mathbb{C}[\mathring{\mathcal{N}}] = D(x)\{Q \mid \deg Q \leqslant N-4\}, \qquad \dim \mathcal{M} = N-3. \tag{7.12}$$

7.2.2. *Boutroux curve and Strebel graph.* Theorem 4.1 implies that one can find $Q \in \mathcal{M}$ such that this is a Boutroux curve.

Thus, let the Boutroux curve

$$P(x,y) = y^2 D(x)^2 - \sum_{\alpha=1}^{N} L_\alpha^2 D'(z_\alpha) \frac{D(x)}{x - z_\alpha} + D(x)Q(x). \tag{7.13}$$

We choose the origin o for computing ϕ to be a point invariant under the involution $Y(o) = -Y(o)$, i.e. $Y(o) = 0$.

There are 2N punctures, which are simple poles, let us denote them $z_{\alpha,\pm}$, and at which we have

$$y \underset{z_{\alpha,\pm}}{\sim} \frac{\pm L_\alpha}{x - z_\alpha} + \text{hol}. \tag{7.14}$$

Near the puncture $z_{\alpha,\pm}$ we have

$$\phi \sim \pm L_\alpha \ln |x - z_\alpha|, \tag{7.15}$$

the vertical trajectories near the punctures are circles surrounding the punctures.

7.2.3. *First Kind.* Let $\check{\Gamma}$ the graph of Definition 5.1 of all vertical trajectories starting from all zeros of y (this includes ramification points and possibly nodal points). They cut Σ into connected domains. From theorem 5.3, connected domains can be only half-planes, strips, cylinders or half-cylinders, and from theorem 5.7 there is no cylinders. Moreover, since all punctures are Fuchsian, there are no edges ending at the punctures, and thus there is no half-planes neither strips. The only faces are half-cylinders ending at the punctures.

Moreover, since there is no strip, this implies that all vertical edges must have the same value of ϕ. Since we chose ϕ such that ϕ vanishes at a branch point, then ϕ must be zero on all the graph $\check{\Gamma}$:

$$\check{\Gamma} = \phi^{-1}(\{0\}). \tag{7.16}$$

$\check{\Gamma}$ is a cellular graph on Σ, whose faces are discs around the punctures, and its projection $\Gamma = X(\check{\Gamma})$ is a cellular graph on $\mathbb{CP}^1$ whose edges are vertical trajectories, and whose faces are discs around the points z_α, of perimeter $2\pi L_\alpha$.

This is the **Strebel graph**.

7.2.4. *Second Kind.* Let $\mathcal{D}_- = \{p \in \Sigma \mid \phi(p) < 0\}$ (resp. $\mathcal{D}_+ = \{p \in \Sigma \mid \phi(p) > 0\}$).

$\mathcal{D}_-$ (resp. $\mathcal{D}_+$) is a union of connected components, and X is 1:1 on each connected component. From proposition 6.6, we know that $\mathcal{D}_-$ is a maximal admissible domain. Each connected component contains exactly one puncture. There are N connected components. The two kinds of spectral networks coincide.

7.2.5. *Strebel differential and Strebel Graph.* We have thus found, that if we have chosen a Boutroux curve

$$y^2 = \frac{R(x)}{\prod_{\alpha=1}^N (x-z_\alpha)^2}, \tag{7.17}$$

with $\deg R \leqslant 2N-4$ and $R(z_\alpha) = L_\alpha^2 D'(z_\alpha)^2$, then the following quadratic differential

$$\Omega = (YdX)^2 = \left(\sum_{\alpha=1}^N \frac{L_\alpha^2}{(x-z_\alpha)^2} - \frac{R(x)}{D(x)}\right) dx^2 \tag{7.18}$$

is such that the vertical trajectories of $\sqrt{\Omega}$ form a cellular graph whose faces are discs surrounding the z_αs, and with perimeter (in the metric $\sqrt{\Omega}$) $2\pi L_\alpha$.

Ω is called a **Strebel differential**, and the cellular graph $\Gamma = \Gamma_0$ of its vertical trajectories is called the **Strebel graph**.

One can verify that the Strebel graph is left invariant by Möbius transformations $x \to (ax+b)/(cx+d)$ and $y \to (cx+d)^2 y$, with $ad - bc = 1$. In other words we have a map:

$$\begin{array}{lll} ({\mathbb{C}P^1}^N / \mathrm{Aut}(\mathbb{C}P^1)) \times \mathbb{R}_+^N & \to \text{Quadratic differentials} & \to \text{Graphs} \\ (z_i, L_i)_{i=1,\dots,N} \quad \text{mod Möbius} & \mapsto \Omega = (YdX)^2_{\text{Boutroux}} & \mapsto \text{Strebel Graph} \end{array} \tag{7.19}$$

and we notice that

$$({\mathbb{C}P^1}^N / \mathrm{Aut}(\mathbb{C}P^1)) = \mathcal{M}_{0,N} \tag{7.20}$$

is the moduli space of Riemann surfaces of genus 0 and N marked points.

Strebel's theorem extends this to $\mathcal{M}_{g,n}$ for every g, and our method above can be extended to that case.

7.3. **1 Matrix model.** Let $V(x) \in \mathbb{C}[x]$ a polynomial of degree $d \geqslant 2$, written as

$$V(x) = \sum_{k=1}^d \frac{t_k}{k} x^k. \tag{7.21}$$

7.3.1. *Newton's polygon.* Let

$$\mathcal{P}(x,y) = y^2 - \frac{1}{4} V'(x)^2 + t_d x^{d-2}. \tag{7.22}$$

It is an hyperelliptic curve.

There are exactly two punctures, that we denote ∞_+ and ∞_-. We have at $\infty_\pm$, $a_\pm = a_{\infty_\pm} = 1$, $b_\pm = b_{\infty_\pm} = \deg V' = d-1$, $r_\pm = r_{\infty_\pm} = \deg V = d$. We have

$$\zeta_{\infty_+} = \zeta_{\infty_-} = \zeta = x^{-1}, \quad Y \sim_{\infty_\pm} \pm(\frac{1}{2} V'(X) - \frac{1}{X}) + O(X^{-2}). \tag{7.23}$$

The times at $\infty_\pm$ are

$$t_{\infty_\pm, k} = \operatorname*{Res}_{\infty_\pm} \zeta^k YdX = \operatorname*{Res}_{\infty_\pm} X^{-k} YdX = \mp \frac{1}{2} t_k, \tag{7.24}$$

and

$$t_{\infty_\pm, 0} = \pm 1. \tag{7.25}$$

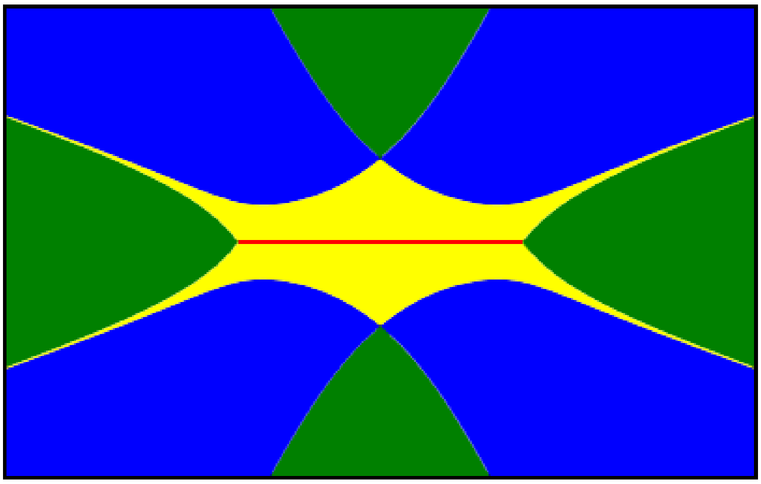


FIGURE 9. Example of spectral network for 1-Matrix model. One branch cut, two strips and eight half-planes. We have thirteen vertical trajectories in total.

We have

$$\mathring{\mathcal{N}} = \{(i,0),\ i = 0,\dots,d-3\},\ \ \#\mathring{\mathcal{N}} = d-2, \tag{7.26}$$

and

$$\mathcal{M} = \mathbb{C}[\mathring{\mathcal{N}}] = \{y^0 Q(x) \mid Q(x) \in \mathbb{C}[x],\ \deg Q \leqslant d-3\}, \qquad \dim \mathcal{M} = d-2. \tag{7.27}$$

7.3.2. *Boutroux curve.* Theorem 4.1 implies that there exists $Q \in \mathcal{M}$, such that the curve is Boutroux:

$$P(x,y) = y^2 - \frac{1}{4}V'(x)^2 + t_d x^{d-2} + Q(x), \tag{7.28}$$

with $\deg Q \leqslant d-3$.

7.3.3. *First Kind.* Let $\check{\Gamma}$ the graph of Definition 5.1.

Since punctures are not Fuchsian, there is no half-cylinder, and it follows from theorem 5.3, theorem 5.7 and theorem 5.6 that

Proposition 7.1. *The faces of* $\mathbb{C} \setminus \Gamma$*, where* $\Gamma = X(\check{\Gamma})$*, are half-planes and strips. All ramifications points are on* $\check{\Gamma} \cap \phi^{-1}(\{0\})$.

Let us further subdivide Σ by cutting along $\phi^{-1}(\{0\})$.

Proposition 7.2. *The faces of* $\mathbb{C} \setminus X(\phi^{-1}(\{0\})$ *are half-planes and strips. All branch points are on* $\Gamma \cap X(\phi^{-1}(\{0\}))$.

Proof. The faces of $\mathbb{C} \setminus X(\phi^{-1}(\{0\})$ are obtained by further cutting the faces of $\mathbb{C} \setminus \Gamma$, along the trajectories $\phi = 0$. Since all faces of $\mathbb{C} \setminus \Gamma$ are half-planes and strips, cutting them along $\phi = 0$ can only produce also half-planes and strips.

7.3.4. *Second Kind.* Let us consider the second kind of the spectral network. We cut Σ into domains of index 1 and domains of index 2, that we rename index $-$ and index $+$. Thanks to the hyperelliptical involution $y \to -y$, we always have that $\phi(p_+(x)) = -\phi(p_-(x))$, and therefore the 2 domains are:

$$\tilde{\mathcal{D}}_+ = \{p \mid \phi(p) > 0\} = \cup_j \tilde{\mathcal{D}}_{+,j}, \quad \tilde{\mathcal{D}}_- = \{p \mid \phi(p) < 0\} = \cup_j \tilde{\mathcal{D}}_{-,j}. \tag{7.29}$$

Their common boundary is the graph $\phi^{-1}(\{0\})$.

Lemma 7.1. *Every domain $\tilde{\mathcal{D}}_{\pm,j}$ is a finite union of half-planes and strips (half-planes and strips of proposition 7.2).*

As a consequence, every $\tilde{\mathcal{D}}_{\pm,j}$ must have $\infty_\pm$ at its boundary. Every $X(\tilde{\mathcal{D}}_{\pm,j})$ is non-compact.

Proposition 7.3. *Let $\mathcal{D}_{\infty_+}$ the interior of the union of all closed domains $\overline{\tilde{\mathcal{D}}_{\pm,j}}$ that have ∞_+ at their boundary.*

- *$\mathcal{D}_{\infty_+}$ is a maximal admissible domain.*
- *$\mathcal{D}_{\infty_+}$ is a finite union of half-plane and strips bounded by vertical trajectories.*

Proof. X has a simple pole at ∞_+, it is $1:1$ in a neighborhood of ∞_+, which implies that X is injective in $\mathcal{D}_{\infty_+}$. Moreover ϕ is harmonic in a neighborhood of ∞_+, so it is harmonic in $\mathcal{D}_{\infty_+}$. This implies that $\mathcal{D}_{\infty_+}$ is admissible.

If it were not maximal, it would be possible to add another domain to it. Since all domains either touch ∞_+ or ∞_-, and we have already taken all the domains that touch ∞_+, the only possibility would be to add a domain that contains ∞_-. But this is impossible because X would then not be injective in a neighborhood of ∞.

Let

$$\hat{\Gamma} = X(\partial\mathcal{D}_{\infty_+}). \tag{7.30}$$

Proposition 7.4. *$\hat{\Gamma}$ is a graph, whose edges are vertical trajectories $\phi = 0$*

$$\hat{\Gamma} \subset X(\phi^{-1}(\{0\})). \tag{7.31}$$

All branch points are on $\hat{\Gamma}$.

Proof. Since we included in $\mathcal{D}_{\infty_+}$ all the edges that end at ∞_+ and none of the edges that end at ∞_-, then all the compact edges of $\partial\mathcal{D}_{\infty_+}$ must be on $\phi^{-1}(\{0\})$.

Then, remark that a branch point, is a vertex of $X(\phi^{-1}(\{0\}))$ of odd valency, therefore it is impossible that Φ is analytic around a branch point. This implies that all branch points must be on $\hat{\Gamma}$.

Definition 7.1. *Let us define, for $n \in \mathbb{Z}/2d\mathbb{Z}$:*

$$\theta_n := -\frac{1}{d}\arg t_d + \frac{\pi}{2d} + \frac{n\pi}{d}. \tag{7.32}$$

Lemma 7.2. *All vertical trajectories $\Phi = 0$ arrive to ∞ at one of these angles, and there is exactly one non-compact half-edge of the graph $\Phi = 0$ ending at angle θ_n.*

For r large enough $\Phi(re^{i\theta})$ is an increasing function of θ when θ is close to θ_n with n odd and decreasing if n even.

Proof. Consider a disc neighborhood of ∞, on which Φ is harmonic. Consider a small circle inside the disc, parametrized by an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. We have $\Phi(x) \underset{\infty}{\sim} \frac{1}{2}\mathrm{Re}V(x) \underset{\infty}{\sim} \mathrm{Re}\frac{t_d}{2d}x^d$. This implies that the lines $\Phi = 0$ approach ∞ in directions θ_n for all $n \in \mathbb{Z}/2d\mathbb{Z}$.

Moreover $\Phi(re^{i\theta}) \underset{\infty}{\sim} (-1)^{n+1}\frac{|t_d|r^d}{2d}\sin(d(\theta-\theta_n))$ is an increasing function of θ when θ is close to θ_n with n odd and decreasing if n even.

Proposition 7.5. *We have the following properties:*

- *$\mathbb{C} \setminus \hat{\Phi}^{-1}(\{0\})$ is a finite union of connected domains bounded by vertical trajectories $\Phi = 0$. We write*

$$\mathbb{C} \setminus \hat{\Phi}^{-1}(\{0\}) = \bigcup_{j=1}^{m} C_j. \tag{7.33}$$

- *Each connected domain C_j is a finite union of strips and half-planes of the 1st kind spectral network, and the vertical trajectories with $\phi \neq 0$ are strictly inside the connected components.*
- *Every half plane reaches ∞ in a sector of argument $\theta \in]\theta_n, \theta_{n+1}[$.*
  - *If n is odd, this is a half-plane $\phi \to +\infty$.*
  - *If n is even, this is a half-plane $\phi \to -\infty$.*
- *Every strip reaches ∞ at two angles $\theta = \theta_{n_1}$ and θ_{n_2}, such that $n_1 - n_2$ is odd.*
- *Let us orient the boundary ∂C_j such that C_j sits on the left of its boundary. ∂C_j can have several connected components, let us say m_j, each of them is a vertical trajectory $\Phi = 0$:*

$$\partial C_j = \bigcup_{i=1}^{m_j} \partial C_{j,i}, \quad \partial C_{j,i} \subset \Phi^{-1}(\{0\}). \tag{7.34}$$

  C_j can reach ∞ in m_j distinct angular sectors of some angles of argument $\theta \in]\theta_n, \theta_{n+1}[$, with parity ϵ depends on the angular sectors in which $C_{j,i}$ goes to ∞.
- *If $\epsilon = 1$, the component $C_{j,i}$ reaches ∞ in domain $\Phi > 0$, and if $\epsilon = -1$, the component $C_{j,i}$ reaches ∞ in domain $\Phi < 0$.*

Proof. From proposition 7.2, C_j is a finite union of strips and half-planes. Lemma 7.2 says that half planes reach ∞ in sector of argument $\theta \in (\theta_n, \theta_{n+1})$, each strip ($\Phi = 0$ and $\Phi = c$) reaches ∞ in two sectors with different arguments, either in domains $\Phi > 0$ or $\Phi < 0$, which implies two angles such that $n_1 - n_2$ is odd. In addition, each angular sector has different parity which depend on the choices of connected components.

Proposition 7.6. *Consider the connected components of the graph $\Upsilon = \Phi^{-1}(\{0\})$. It is a tree.*

- *Each connected component Υ_i of $\Phi^{-1}(\{0\})$ is a tree.*
- *Each connected component Υ_i is the union of boundaries of connected domains of $C_{i,j}$ of $\mathbb{C} \setminus \Phi^{-1}(\{0\})$ adjacent to Υ_i.*

$$\Upsilon_{i,j} = \Upsilon_i \cap \partial C_{i,j}, \ j = 1, \dots, k_i. \tag{7.35}$$

  We order them cyclically around the tree Υ_i (in the trigonometric order) so that $j + k_i \equiv j$ and $C_{i,j+1}$ is adjacent and follows $C_{i,j}$.
- *Each $\Upsilon_{i,j}$ corresponds to a pair $(\theta_{n_{i,j}}, \theta_{n_{i,j+1}})$ with $n_{i,j}$ and $n_{i,j+1}$ of different parities. This implies that k_i must be even*

$$k_i \in 2\mathbb{Z}_+, \tag{7.36}$$

  and the signs alternate

$$(-1)^{n_{i,j+1}} = -(-1)^{n_{i,j}}. \tag{7.37}$$

  Proof. From proposition 7.5, the graph is made up of strips and half planes, since there is no cycles, the graph is a tree. Otherwise, the graph will have faces with finite boundaries, this is not possible since the only poles of the potential is at ∞.

7.3.5. *Measure.*

Definition 7.2 (Measure). *Let the real measure on Borel subsets of $\mathbb{C}$:*

$$\mu(E) = \frac{1}{2\pi} \int_E \Delta\Phi. \tag{7.38}$$

From theorem 6.2 we have

Lemma 7.3. *The measure is supported on $\hat{\Gamma}$. Along edges of $\hat{\Gamma}$, the measure has density*

$$d\mu = \frac{1}{\pi i} Y dX = \frac{1}{\pi} \mathrm{Im}\, Y dX, \tag{7.39}$$

and is real.

Theorem 7.1 (Stieltjes transform)**.** *The Stieltjes transform of* μ

$$W(x) = \int_{\hat{\Gamma}} \frac{d\mu(x')}{x - x'} \tag{7.40}$$

is analytic in $\mathbb{C} \setminus \hat{\Gamma}$*, and is worth*

$$W(x) = \frac{1}{2} V'(x) - Y(x). \tag{7.41}$$

Proof. Let $\tilde{W}(x) = \frac{1}{2}\left(V'(x) - 2Y(x)\right)$, we have

$$\tilde{W}(x_{\text{left}}) - \tilde{W}(x_{\text{right}}) = -(Y(x_{\text{left}}) - Y(x_{\text{right}})) = -2\pi i d\mu(x) \tag{7.42}$$

Moreover,

$$\tilde{W}(x) = x^{-1} + O(x^{-2}). \tag{7.43}$$

These two properties characterize the Stieltjes transform, and imply that $W = \tilde{W}$.

Theorem 7.2 (Energy)**.** *We have*

$$F = \int_{\hat{\Gamma}} \operatorname{Re} V(x) d\mu(x) - \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \ln|x - x'| d\mu(x') d\mu(x). \tag{7.44}$$

Proof. Remark that in $\mathbb{C} \setminus \hat{\Gamma}$ we have

$$\begin{aligned} \int_{\hat{\Gamma}} \ln(x - x') d\mu(x') &= \ln x + \int_{\infty}^{x} (W(x') - 1/x') dx' \\ &= \frac{1}{2} V(x) - g_{\infty_+}(x) \\ &= \frac{1}{2} V(x) - g(x) + \tilde{t}_{\infty_+,0}. \end{aligned} \tag{7.45}$$

Indeed both the left and right hand side behave as $\ln x + O(1/x)$ at large x, and both have the same derivative $\int_{\hat{\Gamma}} \frac{1}{x - x'} d\mu(x') = W(x)$. Taking the real part this gives

$$\int_{\hat{\Gamma}} \ln|x - x'| d\mu(x') = \frac{1}{2} \operatorname{Re} V(x) - \Phi(x) + \operatorname{Re} \tilde{t}_{\infty_+,0}, \tag{7.46}$$

and since $\Phi = 0$ on $\hat{\Gamma}$:

$$\int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \ln|x - x'| d\mu(x') d\mu(x) = \frac{1}{2} \int_{\hat{\Gamma}} \operatorname{Re} V(x) d\mu(x) + \operatorname{Re} \tilde{t}_{\infty_+,0} \tag{7.47}$$

and

$$\operatorname{Re} \tilde{t}_{\infty_+,0} = \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \ln|x - x'| d\mu(x') d\mu(x) - \frac{1}{2} \int_{\hat{\Gamma}} \operatorname{Re} V(x) d\mu(x). \tag{7.48}$$

Beside we have

$$t_{\infty_+,k} = -t_{\infty_-,k} = -\frac{1}{2} t_k, \tag{7.49}$$

and

$$\tilde{t}_{\infty_+,k} = -\tilde{t}_{\infty_-,k} = \frac{1}{k} \operatorname*{Res}_{\infty_+} x^k y dx = -\frac{1}{k} \operatorname*{Res}_{\infty_+} x^k W(x) dx = \frac{1}{k} \int_{\hat{\Gamma}} x^k \, d\mu(x). \tag{7.50}$$

This implies that

$$\sum_{k=1}^{\deg V} t_{\infty_+,k} \tilde{t}_{\infty_+,k} = \sum_{k=1}^{\deg V} t_{\infty_-,k} \tilde{t}_{\infty_-,k} = -\frac{1}{2} \int_{\hat{\Gamma}} V(x) \, d\mu(x). \tag{7.51}$$

$$\begin{aligned} 2F_0 &= \sum_{k=1}^{\deg V} t_{\infty_+,k} \tilde{t}_{\infty_+,k} + \sum_{k=1}^{\deg V} t_{\infty_-,k} \tilde{t}_{\infty_-,k} + t_{\infty_+,0} \tilde{t}_{\infty_+,0} + t_{\infty_-,0} \tilde{t}_{\infty_-,0} \\ &= -\int_{\hat{\Gamma}} V(x) d\mu(x) + 2 t_{\infty_+,0} \tilde{t}_{\infty_+,0}, \end{aligned} \tag{7.52}$$

and thus, and using eq (7.48):

$$\begin{aligned} F = -\mathrm{Re} F_0 &= \frac{1}{2}\int_{\hat\Gamma} \mathrm{Re} V(x) d\mu(x) - t_{\infty_+,0} \mathrm{Re}\, \tilde t_{\infty_+,0} \\ &= \frac{1}{2}\int_{\hat\Gamma} \mathrm{Re} V(x) d\mu(x) - \mathrm{Re}\, \tilde t_{\infty_+,0} \\ &= \int_{\hat\Gamma} \mathrm{Re} V(x) d\mu(x) - \int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d\mu(x') d\mu(x). \end{aligned} \tag{7.53}$$

Theorem 7.3 (Energy). *It is possible to choose the Boutroux curve such that μ is a probability measure (positive and total mass 1) on $\hat\Gamma$, and is the extremal measure of the following functional*

$$F = \inf_{\nu \in \text{probability measures on } \hat\Gamma} \left(\int_{\hat\Gamma} \mathrm{Re} V(x) d\nu(x) - \int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d\nu(x') d\nu(x) \right). \tag{7.54}$$

Proof. This is a classical theorem in potential theory. In the context of random matrices this is done for example in [AG97]. We refer to random matrix literature and potential theory literature.

Let us just sketch the main ideas: Let the functional defined on the space $\mathrm{Pr}(\hat\Gamma)$ of probability measures on $\hat\Gamma$, equipped with the weak topology:

$$\mathcal{F}(\nu) = \int_{\hat\Gamma} \mathrm{Re} V(x) d\nu(x) - \int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d\nu(x') d\nu(x). \tag{7.55}$$

• $\mathcal{F}$ is bounded from below: let ν_0 any given probability measure on $\hat\Gamma$, and $U_0(x) = \int_{\hat\Gamma} \ln|x-x'| d\nu_0(x')$. We have

$$\begin{aligned} \mathcal{F}(\nu) &= -\int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d(\nu(x')-\nu_0(x'))\, d(\nu(x)-\nu_0(x)) \\ &\quad + \int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d\nu_0(x')\, d\nu_0(x) \\ &\quad + \int_{\hat\Gamma} (\mathrm{Re} V(x) - 2U_0(x)) d\nu(x) \\ &\geqslant \int_{\hat\Gamma}\int_{\hat\Gamma} \ln|x-x'| d\nu_0(x')\, d\nu_0(x) + \inf_{\hat\Gamma}(\mathrm{Re} V(x) - 2U_0(x)), \end{aligned} \tag{7.56}$$

which shows that $\mathcal{F}$ is bounded from below.

• $\mathcal{F}$ is lower semi-continuous: let $\mathcal{F}_M(\nu) = \int_{\hat\Gamma} \mathrm{Re} V(x) d\nu(x) - \int_{\hat\Gamma}\int_{\hat\Gamma} \ln_M|x-x'| d\nu(x') d\nu(x)$ where $\ln_M(x) = \max(M, \ln|x|)$. One can verify that $\mathcal{F}_M$ is Lipschitzien, and thus continuous (with the weak topology). $\mathcal{F} = \limsup_{M\to-\infty} \mathcal{F}_M$ is a limsup of continuous functions, therefore is lower semi-continuous.

• Since $\hat\Gamma$ is compact and is a finite union of Jordan arcs, the space $\mathrm{Pr}(\hat\Gamma)$ is a Susslin space (image by a continuous function = the Jordan arc, of a Polish space = here an interval of $\mathbb{R}$), which implies that it is complete, and every probability measure on $\hat\Gamma$ is tight. The level sets of $\mathcal{F}$ are closed (because $\mathcal{F}$ is lower semi-continuous) and compact by Prokhorov's theorem.

• The intersection of a decreasing sequence of non-empty compacts is a non empty compact, and any element in this intersection is a minimum of $\mathcal{F}$. Therefore $\mathcal{F}$ admits at least one minimum.

• $\mathcal{F}$ is strictly convex, so the minimum is unique.

• The minimum must satisfy Euler-Lagrange equations, and this can be written in the following way:

a probability measure $\hat\mu$ on $\hat\Gamma$ is said to satisfy Euler-Lagrange equations if and only if

$$\exists \ell \in \mathbb{R}\,, \quad \Phi(x) = \int_{\mathrm{supp}(\hat\mu)} \left(\frac{1}{2}\mathrm{Re} V(x') - \ln|x-x'| \right) d\hat\mu(x') \quad \begin{cases} = \ell & \text{if } x \in \mathrm{supp}(\hat\mu) \\ \geqslant \ell & \text{if } x \in \hat\Gamma \setminus \mathrm{supp}(\hat\mu) \end{cases} . \tag{7.57}$$

Let $\hat{W}(x) = \int_{x' \in \hat{\Gamma}} \frac{1}{x-x'} d\hat{\mu}(x')$ the Stieltjes transform of $\hat{\mu}$. It is analytic outside $\mathrm{supp}(\hat{\mu}) \subset \hat{\Gamma}$. It may be discontinuous across $\mathrm{supp}(\hat{\mu}) \subset \hat{\Gamma}$, with discontinuity

$$x \in \mathrm{supp}(\hat{\mu}) \qquad \Longrightarrow \qquad \hat{W}(x_{\mathrm{left}})dx - \hat{W}(x_{\mathrm{right}})dx = 2\pi i d\hat{\mu}(x). \tag{7.58}$$

On the other hand, the Euler-Lagrange equations in the support imply that

$$x \in \mathrm{supp}(\hat{\mu}) \qquad \Longrightarrow \qquad \hat{W}(x_{\mathrm{left}}) + \hat{W}(x_{\mathrm{right}}) - V'(x) = 0. \tag{7.59}$$

Let us define $R(x) := V'(x)\hat{W}(x) - \hat{W}(x)^2$. $R(x)$ is analytic outside of $\mathrm{supp}(\hat{\mu})$, and on $\mathrm{supp}(\hat{\mu})$ it satisfies

$$\begin{aligned} R(x_{\mathrm{left}}) - R(x_{\mathrm{right}}) &= V'(x)(\hat{W}(x_{\mathrm{left}}) - \hat{W}(x_{\mathrm{right}})) \\ &\quad -(\hat{W}(x_{\mathrm{left}}) + \hat{W}(x_{\mathrm{right}}))(\hat{W}(x_{\mathrm{left}}) - \hat{W}(x_{\mathrm{right}})) \\ &= 0. \end{aligned} \tag{7.60}$$

Thanks to Cauchy-Riemann equations, this shows that $R(x)$ is in fact analytic in the whole complex plane. Moreover it behaves at ∞ as $O(|V(x)/x|)$, so $R(x)$ is an entire function bounded by a polynomial, it must be a polynomial. We have

$$\hat{W}(x)^2 = V'(x)\hat{W}(x) - R(x). \tag{7.61}$$

If we write $\hat{y} = \frac{1}{2}V'(x) - \hat{W}(x)$, we see that $\hat{y}$ is an algebraic function of x satisfying an equation

$$\hat{y}^2 = (\frac{1}{2}V'(x) - \hat{W}(x))^2 = \frac{1}{4}V'(x)^2 - R(x). \tag{7.62}$$

Remark that $\hat{P}(x,y) = y^2 - \frac{1}{4}V'(x)^2 + R(x)$ is a plane curve that belongs to our moduli space $\mathcal{M}$.

The Euler-Lagrange equations imply that $\hat{\Phi}(x) = \mathrm{Re}\int \hat{y}dx$ is constant on the support, which implies that $\hat{P}$ is a Boutroux curve. Moreover, it is a Boutroux curve with a positive probability measure $\hat{\mu}$.

This implies that it is possible to choose a Boutroux curve in $\mathcal{M}$ such that the Boutroux curve's measure $\mu = \frac{1}{2\pi}\Delta\Phi$ is a positive probability measure.

7.3.6. *g-function in the Riemann-Hilbert method.* The ingredients of the Steepest-descent method of [DZ92] are a graph, a set of "jump matrices" associated to each edge of the graph Υ, and a function g defined in the complex plane, and that has certain properties near the edges of the graph.

We claim that the following data is the data needed for the Steepest-descent method of [DZ92]:

- Υ contains all edges and vertices of $\hat{\Gamma}$.
- for each connected component of $\hat{\Gamma}$ (made of vertical edges that border of some half-plane $\phi < 0$), let a its highest vertex (highest value of $\mathrm{Im} g$). From a follow a horizontal trajectory where ϕ is non-decreasing. Each time you meet a vertex, choose the uppermost horizontal trajectory. Add this horizontal trajectory to Υ.
- add to Υ some "lenses" around edges of $\hat{\Gamma}$. Since edges of $\hat{\Gamma}$ border domains where $\Phi < 0$, we choose the lenses small enough to be entirely in domains $\Phi < 0$.
- add to Υ some small circle around vertices of $\hat{\Gamma}$,
- the g-function is the function g. It's real part is constant and vanishing on the edges of $\hat{\Gamma}$. It is growing on horizontal trajectories. $\mathrm{Re} g$ is < 0 on the lenses. g behaves like $C_a(x - x_a)^{r_a/a_a}$ near a vertex a.

We then refer to [DZ92; Dei+99; Ber07].

7.3.7. *Interpretation as matrix models.* Let N a positive integer. Consider the following measure on H_N the space of Hermitian matrices of size N:

$$\frac{1}{Z_N} e^{-N \operatorname{tr} V(M)} dM, \tag{7.63}$$

where dM is the canonical Lebesgue measure on H_N. (all this can be extended to $H_N(\gamma)$ the set of normal matrices with eigenvalues on γ, i.e. $H_N(\mathbb{R}) = H_N$ if $\gamma = \mathbb{R}$). The normalization constant is

$$Z_N = \int_{H_N(\gamma)} e^{-N \operatorname{tr} V(M)} dM. \tag{7.64}$$

In the large N limit, the empirical density of eigenvalue of M will tend to a limit, called the "equilibrium density" $d\mu(x)$. Equivalently in the large N limit, the Stieltjes transform of the empirical density of eigenvalue of M will tend to a limit $W(x)$.

The conjecture is that

$$W(x) = \frac{1}{2} V'(x) - Y(x), \quad d\mu(x) = \frac{1}{2\pi i} \left(Y(x_{\mathrm{left}}) - Y(x_{\mathrm{right}}) \right), \tag{7.65}$$

where $P(x,y)$ is a Boutroux curve, $\hat{\Gamma}$ is the cellular graph of a maximal domain containing all tiles adjacent to the puncture ∞_+, and $Y(x)$ is the solution of $P(x,y) = 0$ in $\mathbb{C} \setminus \hat{\Gamma}$, and $d\mu$ is the measure supported on $\hat{\Gamma}$ given by the discontinuity of Y.

In principle this conjecture can be proved by the Riemann–Hilbert method and the Steepest-descent method of [DZ92].

7.4. **2 Matrix model.** Let $V(x) \in \mathbb{C}[x]$, $\tilde{V}(y) \in \mathbb{C}[y]$ be two polynomials, of degree at least two. Denote their leading coefficients:

$$V(x) = \frac{t_{\tilde{d}}}{\tilde{d}} x^{\tilde{d}} + O(x^{\tilde{d}-1}), \quad \tilde{V}(y) = \frac{\tilde{t}_d}{d} y^d + O(y^{d-1}). \tag{7.66}$$

7.4.1. *Newton's polygon.* Let

$$\mathcal{P}(x,y) = (y - V'(x))(x - \tilde{V}'(y)) - t_{\tilde{d}} \tilde{t}_d x^{\tilde{d}-2} y^{d-2}. \tag{7.67}$$

There are exactly two punctures, that we denote ∞_+ and ∞_-. We have

- at ∞_+, $a_+ = a_{\infty_+} = 1$, $b_+ = b_{\infty_+} = \deg V' = \tilde{d} - 1$, $r_+ = r_{\infty_+} = \deg V = \tilde{d}$.

$$x = \zeta_+^{-1}, \qquad y \sim \eta_+ \zeta_+^{1-\tilde{d}}. \tag{7.68}$$

  The times at ∞_+

$$t_{\infty_+,k} = \operatorname*{Res}_{\infty_+} \zeta_+^{-k} Y dX, \tag{7.69}$$

  are such that

$$V'(x) = -\sum_{k=1}^{\tilde{d}} \frac{t_{\infty_+,k}}{k} x^k, \tag{7.70}$$

  in particular

$$t_{\infty_+,k} = -t_{\tilde{d}}, \qquad t_{\infty_+,0} = 1. \tag{7.71}$$

- at ∞_-, $a_- = a_{\infty_-} = \deg \tilde{V}' = d - 1$, $b_- = b_{\infty_-} = 1$, $r_- = r_{\infty_-} = \deg \tilde{V} = d$.

$$x = \zeta_-^{1-d}, \qquad y \sim \eta_- \zeta_-^{-1}. \tag{7.72}$$

7.4.2. *Boutroux curve and spectral network.* We have

$$\mathcal{M} = \{\mathcal{P}(x,y) + Q(x,y) \mid Q(x,y) = \sum_{i \leqslant d-2,\ j \leqslant \tilde{d}-2,\ i+j < d+\tilde{d}-4} Q_{i,j} x^i y^j\}. \tag{7.73}$$

$$\dim \mathcal{M} = (d-1)(\tilde{d}-1) - 1. \tag{7.74}$$

Consider now a Boutroux curve in $\mathcal{M}$:

$$P(x,y) = \mathcal{P}(x,y) + Q(x,y) \qquad = \text{Boutroux}. \tag{7.75}$$

The functions $\phi(p) = \operatorname{Re}\int_o^p YdX$ and $\tilde{\phi}(p) = \operatorname{Re}\int_o^p XdY$ are harmonic on $\Sigma \setminus \{\infty_+, \infty_-\}$. We use them to define the index $i(p)$ (resp. $\tilde{i}(p)$), and the spectral network graphs Γ (resp. $\tilde{\Gamma}$) of Section 6.

We use the second version of spectral networks.

Definition 7.3. *For ϕ (resp. $\tilde{\phi}$), let $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) a maximal admissible domain that contains the union of all domains $\tilde{\mathcal{D}}_{i,j}$ (resp. $\tilde{\mathcal{D}}_{i,j}$) that have ∞_+ (resp. ∞_-) at their boundary. Let $\mathcal{C}_+ = X(\mathcal{D}_+)$ (resp. $\mathcal{C}_- = Y(\mathcal{D}_-)$), and let $\hat{\Gamma}_+$ (resp. $\hat{\Gamma}_-$) its boundary.*

As an immediate consequence of theorem 6.1, we have:

Theorem 7.4. *ϕ (resp. $\tilde{\phi}$) is harmonic on $\mathcal{D}_+$ (resp. $\mathcal{D}_-$). $\Phi = \phi \circ X^{-1}$ (resp. $\tilde{\Phi} = \tilde{\phi} \circ Y^{-1}$) is harmonic on $\mathcal{C}_+$ (resp. $\mathcal{C}_-$).*

The complement $\hat{\Gamma}_+ = \mathbb{C} \setminus \mathcal{C}_+$ (resp. $\hat{\Gamma}_- = \mathbb{C} \setminus \mathcal{C}_-$) is a cellular graph.

The locus where Φ (resp. $\tilde{\Phi}$) is not harmonic is exactly on $\hat{\Gamma}_+$ (resp. $\hat{\Gamma}_-$).

7.4.3. *Measures.*

Definition 7.4 (Measures). *Let us define the following measures on $\mathbb{C}$, supported on $\hat{\Gamma}_+ = \mathbb{C} \setminus \mathcal{C}_+$ (resp. $\hat{\Gamma}_- = \mathbb{C} \setminus \mathcal{C}_-$)*

$$\begin{cases} d\mu = \frac{1}{2\pi i}(Y(p^i(x)) - Y(p^j(x)))dx & \textit{along an edge separating index } (i,j) \\ d\mu = 0 & \textit{inside open domains} \end{cases} \tag{7.76}$$

$$\begin{cases} d\tilde{\mu} = \frac{1}{2\pi i}(X(\tilde{p}^{\tilde{i}}(y)) - X(\tilde{p}^{\tilde{j}}(y)))dy & \textit{along an edge separating index } (\tilde{i},\tilde{j}) \\ d\tilde{\mu} = 0 & \textit{inside open domains} \end{cases} \tag{7.77}$$

They are such that

$$\mu(E) = \frac{1}{2\pi}\int_E \Delta\Phi \qquad \left(\textit{resp.}\ \ \tilde{\mu}(E) = \frac{1}{2\pi}\int_E \Delta\tilde{\Phi}\right). \tag{7.78}$$

Theorem 7.5 (Stieltjes transform). *The Stieltjes transform*

$$W(x) = \int_{\operatorname{supp} d\mu} \frac{d\mu(x')}{x - x'}, \tag{7.79}$$

$$\tilde{W}(y) = \int_{\operatorname{supp} d\tilde{\mu}} \frac{d\tilde{\mu}(y')}{y - y'}. \tag{7.80}$$

defined on the complement $\mathbb{C} \setminus \mathcal{C}_+$ (resp. $\mathbb{C} \setminus \mathcal{C}_-$), is worth

$$W(x) = V'(x) - Y(x), \tag{7.81}$$

$$\tilde{W}(y) = \tilde{V}'(y) - X(y). \tag{7.82}$$

Proof. The discontinuity of eq (7.81) across the support of μ is equal to eq (7.76) times $2\pi i$, and it behaves as $1/x + O(1/x^2)$ at large x, this characterizes the Stieltjes transform eq (7.79). Idem for $\tilde{W}$.

7.4.4. *Interpretation as matrix models.* Let N a positive integer. Consider the following measure on $H_N \times H_N$:

$$\frac{1}{Z_N} e^{-N \operatorname{tr}\left(V(M_1)+\tilde{V}(M_2)-M_1M_2\right)} dM_1 dM_2. \tag{7.83}$$

where dM is the canonical Lebesgue measure on H_N. (all this can be extended to $H_N(\gamma) \times H_N(\tilde{\gamma})$ the set of normal matrices with eigenvalues on γ and $\tilde{\gamma}$). In the large N limit, the empirical density of eigenvalue of M_1 (resp. M_2) will tend to a limit, called the "equilibrium density" $d\mu(x)$ (resp. $d\tilde{\mu}(y)$). Equivalently in the large N limit, the Stieltjes transform of the empirical density of eigenvalue of M_1 (resp. M_2) will tend to a limit $W(x)$ (resp. $\tilde{W}(y)$).

The conjecture is that

$$W(x) = V'(x) - Y(x), \quad (\text{resp. } \tilde{W}(y) = \tilde{V}'(y) - X(y)), \tag{7.84}$$

$$d\mu(x) = \frac{1}{2\pi i}\left(Y(x_{\text{left}}) - Y(x_{\text{right}})\right), \quad (\text{resp. } d\tilde{\mu}(y) = \frac{1}{2\pi i}\left(X(y_{\text{left}}) - X(y_{\text{right}})\right)), \tag{7.85}$$

where $P(x,y)$ is a Boutroux curve, $\hat{\Gamma}$ (resp. $\tilde{\hat{\Gamma}}$) is the cellular graph of a maximal domain containing all tiles adjacent to the puncture ∞_+ (resp. tiles of $\tilde{\phi}$ adjacent to ∞_-), and $Y(x)$ (resp. $X(y)$) is the solution of $P(x,y)=0$ in $\mathbb{C}\setminus\hat{\Gamma}$ (resp. $\mathbb{C}\setminus\tilde{\hat{\Gamma}}$), and $d\mu$ (resp. $d\tilde{\mu}$) is the measure supported on $\hat{\Gamma}$ (resp. $\tilde{\hat{\Gamma}}$) given by the discontinuity of Y (resp. X).

We believe that this should be provable by Deift-Zhou's steepest descent method [DZ92; Dei+99]. This proof was achieved so far in very few examples of low degree.

7.4.5. *Matytsin property.* In the random 2-matrix model, it is conjectured (and proved in some cases [GZ02]) that the partition function Z_N, or more precisely $\frac{1}{N^2}\ln Z_N$ has a limit at large N.

$$\exists \lim_{N\to\infty} \frac{-1}{N^2} \ln Z_N = \mathcal{F}. \tag{7.86}$$

Since the equilibrium measures μ and $\tilde{\mu}$ are functions of the potentials V and $\tilde{V}$, we can locally describe $\mathcal{F}$ as a functional of two measures:

$$\mathcal{F} = \mathcal{F}(\mu, \tilde{\mu}). \tag{7.87}$$

We also define

$$\begin{aligned} \mathcal{I}(\mu,\tilde{\mu}) \quad = \quad & -\mathcal{F}(\mu,\tilde{\mu}) + \int_{\text{supp}(\mu)} \operatorname{Re}V(x) d\mu(x) + \int_{\text{supp}(\tilde{\mu})} \operatorname{Re}\tilde{V}(y) d\tilde{\mu}(y) \\ & - \int_{\text{supp}(\mu)\times\text{supp}(\mu)} \ln|x-x'| d\mu(x) d\mu(x') \\ & - \int_{\text{supp}(\tilde{\mu})\times\text{supp}(\tilde{\mu})} \ln|y-y'| d\tilde{\mu}(y) d\tilde{\mu}(y'). \end{aligned} \tag{7.88}$$

The interpretation of $\mathcal{I}(\mu,\tilde{\mu})$ is that

$$\begin{aligned} e^{N^2\mathcal{I}} \quad & \sim_{N\to\infty} \quad \mathbb{E}\left(e^{N^2 \operatorname{Tr} M_1 M_2} \mid \text{ knowing that } \ \operatorname{sp}(M_1)\to\mu, \operatorname{sp}(M_2)\to\tilde{\mu}\right) \\ & \sim_{N\to\infty} \quad \int_{U\in U(N)} e^{N^2 \operatorname{Tr}\Lambda U\tilde{\Lambda}U^\dagger} dU \mid \text{ with } \ \operatorname{sp}(\Lambda)\to\mu, \operatorname{sp}(\tilde{\Lambda})\to\tilde{\mu}. \end{aligned} \tag{7.89}$$

i.e. the expectation value of $e^{N^2 \operatorname{Tr} M_1 M_2}$ knowing that the spectrum Λ of M_1 (resp. $\tilde{\Lambda}$ of M_2), empirical spectral measure, tends to the measure μ (resp. $\tilde{\mu}$) at large N.

For fixed diagonal matrices Λ and $\tilde{\Lambda}$ of size N, the following integral

$$I_N(\Lambda,\tilde{\Lambda}) = \int_{U\in U(N)} e^{N^2 \operatorname{Tr}\Lambda U\tilde{\Lambda}U^\dagger} dU, \tag{7.90}$$

with dU the Haar measure on $U(N)$, is known as the Itzykson-Zuber (case $\Lambda, \tilde{\Lambda}$ real) or Harish-Chandra integral (case $\Lambda, \tilde{\Lambda}$ purely imaginary) and is worth

$$I_N(\Lambda, \tilde{\Lambda}) = \frac{\det e^{N^2 \Lambda_i \tilde{\Lambda}_j}}{\prod_{i<j}(\Lambda_i - \Lambda_j)(\tilde{\Lambda}_i - \tilde{\Lambda}_j)}. \tag{7.91}$$

In [Mat94], Matytsin derived heuristically from this exact formula, that in the large N limit, the functional $\mathcal{I}(\mu, \tilde{\mu})$ should satisfy some functional equation:

Let

$$\hat{Y}(x) = \frac{d}{dx} \frac{\partial}{\partial d\mu(x)} \left(\mathcal{I}(\mu, \tilde{\mu}) + \frac{1}{2} \int_{\mathrm{supp}(\mu) \times \mathrm{supp}(\mu)} \ln|x - x'| d\mu(x) d\mu(x') \right), \tag{7.92}$$

$$\hat{X}(y) = \frac{d}{dy} \frac{\partial}{\partial d\tilde{\mu}(y)} \left(\mathcal{I}(\mu, \tilde{\mu}) + \frac{1}{2} \int_{\mathrm{supp}(\tilde{\mu}) \times \mathrm{supp}(\tilde{\mu})} \ln|y - y'| d\tilde{\mu}(y) d\tilde{\mu}(y') \right). \tag{7.93}$$

Matytsin claimed that they must be functional inverse of one-another:

$$\hat{X} \circ \hat{Y} = \mathrm{Id}. \tag{7.94}$$

Let us verify that this is satisfied by the measures we have obtained from the Boutroux curve and its spectral network:

Theorem 7.6. *The measures μ, $\tilde{\mu}$ satisfy the Matytsin property*

Proof. Let here $\hat{\mu}$ and $\hat{\tilde{\mu}}$ be the measures that minimize the energy $\mathcal{F}$. This implies that

$$\frac{\partial}{\partial d\hat{\mu}(x)} \mathcal{F} = 0 = \frac{\partial}{\partial d\hat{\tilde{\mu}}(y)} \mathcal{F}. \tag{7.95}$$

Therefore this implies

$$\hat{Y}(x) = \frac{d}{dx} \frac{\partial}{\partial d\hat{\mu}(x)} \mathcal{I} = V'(x) - \int_{\mathrm{supp}(\hat{\mu})} \frac{d\hat{\mu}(x')}{x - x'} = V'(x) - \hat{W}(x), \tag{7.96}$$

$$\hat{X}(y) = \frac{d}{dy} \frac{\partial}{\partial d\hat{\tilde{\mu}}(y)} \mathcal{I} = \tilde{V}'(y) - \int_{\mathrm{supp}(\hat{\tilde{\mu}})} \frac{d\hat{\tilde{\mu}}(y')}{y - y'} = \tilde{V}'(y) - \hat{\tilde{W}}(y), \tag{7.97}$$

where $\hat{W}(x)$ (resp. $\hat{\tilde{W}}(y)$) designates the Stieltjes transform of the measure $\hat{\mu}$ (resp. $\hat{\tilde{\mu}}$).

The Matytsin property is thus formulated as

$$\hat{X} \circ \hat{Y} = \mathrm{Id}. \tag{7.98}$$

Here, the measures μ and $\tilde{\mu}$ constructed from the Boutroux curve satisfy:

$$Y(x) = V'(x) - W(x), \quad X(y) = \tilde{V}'(y) - \tilde{W}(y), \tag{7.99}$$

where $W(x)$ (resp. $\tilde{W}(y)$) designates the Stieltjes transform of the measure μ (resp. $\tilde{\mu}$). And the function $Y(x)$ is the solution of $P(x, Y(x)) = 0$, while the function $X(y)$ is the solution of the $P(X(y), y) = 0$ for the same Boutroux curve $P(x, y) \in \mathcal{M}$. Therefore they satisfy on Σ:

$$X \circ Y = \mathrm{Id}_\Sigma. \tag{7.100}$$

Remark 7.1. Remark that the Matytsin property is a consequence that the Boutroux property is invariant under the exchange $x \leftrightarrow y$. In other words, if $\mathrm{Re} \oint_\gamma y dx = 0$ for all closed γ, then $\mathrm{Re} \oint_\gamma x dy = 0$ as well.

7.5. **Matrix model with external field.** Let $V(x) \in \mathbb{C}[x]$ be a polynomial of degree $d \geqslant 2$

$$V(x) = \sum_{k=1}^{d} \frac{t_k}{k} x^k. \tag{7.101}$$

Let r a positive integer $r \geqslant 1$, and let r distinct complex numbers $A_1, \dots, A_r \in \mathbb{C}^r$, and let $\nu_1, \dots, \nu_r \in \mathbb{R}^r$ be r real numbers. We let

$$t = \sum_{i=1}^{r} \nu_i. \tag{7.102}$$

Let

$$S(y) = \prod_{i=1}^{r} (y - A_i). \tag{7.103}$$

7.5.1. *Newton's polygon.* Let

$$\mathcal{P}(x,y) = \left(y - V'(x) + \sum_{i=1}^{r} \frac{\nu_i}{y - A_i} \right) S(y). \tag{7.104}$$

It has $r+1$ punctures, that we denote α_i, $i = 0, \dots, r$:

- at α_0, we have $X(\alpha_0) = \infty$, $Y(\alpha_0) = \infty$, $a_0 = 1$, $b_0 = \deg V' = d-1$, $r_0 = \deg V = d$.

$$x = \zeta_0^{-1}, \qquad y \sim \eta_+ \zeta_0^{1-d} \tag{7.105}$$

  The times at α_0 are

$$t_{\alpha_0,k} = \operatorname*{Res}_{\alpha_0} \zeta_0^{-k} Y dX = -t_k \qquad , k = 1, \dots, d \tag{7.106}$$

  and

$$t_{\alpha_0,0} = t = \sum_{i=1}^{r} \nu_i. \tag{7.107}$$

- at α_i for $i = 1, \dots, r$, we have $Y(\alpha_i) = A_i$ and $X(\alpha_i) = \infty$, and $a_i = a_{\alpha_i} = 1$, $b_i = b_{\alpha_i} = -1$, $r_i = r_{\alpha_i} = 0$. We have

$$t_{\alpha_i,k} = -\nu_i \delta_{k,0}. \tag{7.108}$$

  The moduli space is:

$$\mathcal{M} = \mathcal{P} + \left\{ S(y) \sum_{k=0}^{d-3} \sum_{i=1}^{r} c_{k,i} x^k / (y - A_i) \right\}, \quad \dim \mathcal{M} = (d-2)r. \tag{7.109}$$

7.5.2. *Boutroux Curve.* Consider now the Boutroux curve

$$P(x,y) = \mathcal{P}(x,y) + Q(x,y) \qquad = \text{Boutroux}. \tag{7.110}$$

The function $\phi(p) = \operatorname{Re} \int_o^p YdX$ is harmonic on $\Sigma \setminus \{\alpha_0, \dots, \alpha_r\}$. We use it to define the index $i(p) \in [0, \dots, r]$, and the spectral network graphs Γ of Section 6.

7.5.3. *Interpretation as matrix model.* This is related to a random matrix model as follows: let $N \in \mathbb{Z}$ an integer, let $n_i = N\nu_i$. Let

$$A = \operatorname{diag}(\overbrace{A_1 \dots, A_1}^{n_1}, \overbrace{A_2 \dots, A_2}^{n_2}, \dots, \overbrace{A_r \dots, A_r}^{n_r}) \tag{7.111}$$

a diagonal matrix with r degenerate eigenvalues, of total size $N = \sum_{i=1}^{r} n_i = Nt$. This can be interpreted as a diagonal matrix only if all n_i are positive integers. In fact for negative integers it could be interpreted as a fermionic diagonal matrix. However, the formalism presented here works for ν_i real.

The Boutroux curve below will be associated to the following matrix measure:

$$\frac{1}{Z_N(A)} e^{-\frac{N}{t} \operatorname{tr}(V(M)-MA)} dM, \tag{7.112}$$

where dM is the canonical measure on $H_N = H_N(\mathbb{R})$ (and generalizable to $H_N(\gamma)$ the set of normal matrices with eigenvalues on γ), with partition function

$$Z_N(A) = \int_{H_N(\gamma)} e^{-\frac{N}{t} \operatorname{tr}(V(M)-MA)} dM. \tag{7.113}$$

In the large N limit, the empirical density of eigenvalue of M is conjectured (only proved in very few cases with small r and small $d = \deg V$) to tend to a limit, called the "equilibrium density" $d\mu(x)$. Equivalently in the large N limit, the Stieltjes transform of the empirical density of eigenvalue of M will tend to a limit $W(x)$.

The conjecture is that

$$tW(x) = V'(x) - Y(x), \tag{7.114}$$

$$d\mu(x) = \frac{1}{2\pi i t} \left(Y(x_{\mathrm{left}}) - Y(x_{\mathrm{right}}) \right), \qquad \operatorname{supp}(d\mu) = \hat{\Gamma}, \tag{7.115}$$

where $P(x, y)$ is a Boutroux curve, $\hat{\Gamma}$ is the cellular graph of a maximal domain containing all tiles adjacent to the puncture α_0, and $Y(x)$ is the solution of $P(x, y) = 0$ in $\mathbb{C} \setminus \hat{\Gamma}$, and $d\mu$ is the measure supported on $\hat{\Gamma}$ given by the discontinuity of Y.

We believe that this should be provable by Deift-Zhou's steepest descent method [DZ92; Dei+99], using the g-function of the Boutroux curve as the g-function of [DZ92; Dei+99].

8. Conclusion

We have proved the existence of Boutroux curves in the moduli space of algebraic plane curves with prescribed punctured asymptotic behaviors.

This has many practical applications, like finding foliations of surfaces (Strebel's theorem), or finding spectral networks, and finding the g-functions useful for the Riemann-Hilbert method in asymptotic theory in random matrices, or in potential theory.

Possible generalizations

- All the method presented here can probably be extended to algebraic curves over a base curve $x \in \Sigma_0$ not necessarily $\mathbb{C}$ or $\mathbb{C}P^1$, but any compact Riemann surface Σ_0.
- Also instead of $Y \in \mathbb{C}$, the 1-form YdX could take its values in the cotangent space $T^*\Sigma_0$, and more generally in the adjoint bundle of a Higgs bundle for a Lie group G. In other words, this formalism should be extended to Hitchin spectral curves.

- Another generalization could replace $\mathbb{C}\times\mathbb{C}$ by $\mathbb{C}^*\times\mathbb{C}^*$. It would means replace $X\to\ln X$ and $Y\to\ln Y$, i.e. a polynomial equation $P(e^x,e^y)=0$ rather than $P(x,y)=0$. Many of the tools used here would be adaptable. Instead of subtracting poles at punctures, we would subtract logarithmic singularities, but almost all the rest would be similar.

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APPENDIX A. PUNCTURES AND NEWTON'S POLYGON

A.1. **Punctures and slopes.** It is well known that there is a 1-1 correspondence between punctures and minimal integer segments of the convex envelope of $\mathcal{N}$. I.e. a segment $\alpha=[(i_1,j_1),(i_2,j_2)]$ of the envelope, such that $(i_1,j_1)\in\mathcal{N}$, $(i_2,j_2)\in\mathcal{N}$ and the half plane strictly to the left of the segment α doesn't contain any point of $\mathcal{N}$, and "minimal" means that the segment α contains no other integer point in its interior.

- If the segment is horizontal ($j_2=j_1$), α corresponds to a pole of Y.

- If the segment is vertical ($i_2=i_1$), α corresponds to a pole of X.

- Otherwise, α is a pole of both X and Y such that in a local variable $\zeta\to 0$

$$X(z)\sim\zeta^{j_2-j_1},\quad Y(z)\sim\eta\zeta^{i_1-i_2}. \tag{A.1}$$

If α is not contained in a larger segment of the envelope, in a neighborhood of the puncture we have asymptotically

$$P_{i_1,j_1}x^{i_1}y^{j_1}+P_{i_2,j_2}x^{i_2}y^{j_2}=o(x^{i_1}y^{j_1}), \tag{A.2}$$

i.e.

$$y\sim x^{\frac{i_1-i_2}{j_2-j_1}}\left(-\frac{P_{i_1,j_1}}{P_{i_2,j_2}}\right)^{1/(j_2-j_1)}. \tag{A.3}$$

More generally, if $\alpha\subset\alpha'$ a maximal segment of the envelope, we have

$$\sum_{(i,j)\in\alpha'}P_{i,j}x^iy^j=o(x^{i_1}y^{j_1}) \tag{A.4}$$

i.e.

$$y\sim x^{\frac{i_1-i_2}{j_2-j_1}}\,C, \tag{A.5}$$

where C is some solution of

$$\sum_{(i,j)\in\alpha'}P_{i,j}C^{j-j_1}=0. \tag{A.6}$$

If the segment is horizontal, the values of X at the punctures are the zeros of

$$\sum_{i|\ (i,j)\in\alpha'}P_{i,j}x^i=0. \tag{A.7}$$

The canonical local coordinate in the neighborhood of a puncture α is:

- If α is a pole of X of some degree $-d_\alpha=j_1-j_2>0$:

$$\zeta_\alpha=X^{1/d_\alpha}. \tag{A.8}$$

- If α is not a pole of X, and is such that :

$$\zeta_\alpha = (X - X(\alpha))^{1/d_\alpha} \qquad d_\alpha = \mathrm{order}_\alpha\, X - X(\alpha). \tag{A.9}$$

It is well known that

Proposition A.1. *There is a 1-1 correspondence between the set of coefficients $P_{i,j}$ such that $(i,j) \in \partial\mathcal{N}$, and the independent times $t_{\alpha,k}$:*

$$\mathbb{C}[\partial\mathcal{N}] \sim \mathbb{C}^{-1\sum_\alpha (m_\alpha+1)}. \tag{A.10}$$

In other words, fixing the non-interior coefficients $P_{i,j}$, is equivalent to having fixed the times.

A.2. **Times and exterior coefficients.**

Proposition A.2 (Recovering the polynomial P from the times)**.** *First, for each puncture α, define a polynomial $P_\alpha \in \mathbb{C}[x,y]$ as follows:*

- *If $a_\alpha > 0$, let $j \in [1, a_\alpha]$ and let*

$$P_\infty(x,y) = \prod_{\alpha \in X^{-1}(\infty)} P_\alpha(x,y), \tag{A.11}$$

where

$$P_\alpha(x,y) := \prod_{j=1}^{a_\alpha} P_{\alpha,j}(x,y), \quad P_{\alpha,j}(x,y) := \left(y + \sum_{k=0}^{r_\alpha} \frac{t_{\alpha,k}}{a_\alpha} x^{\frac{k}{a_\alpha}-1}\, e^{2\pi i \frac{jk}{a_\alpha}}\right). \tag{A.12}$$

- *If $a_\alpha < 0$, let $j \in [1, |a_\alpha|]$ and let*

$$P_{\alpha,j}(x,y) := \left(y + \sum_{k=0}^{r_\alpha} \frac{t_{\alpha,k}}{a_\alpha} (x - X(\alpha))^{\frac{-k}{a_\alpha}-1}\, e^{2\pi i \frac{jk}{a_\alpha}}\right), \quad P_\alpha(x,y) := \prod_{j=1}^{|a_\alpha|} P_{\alpha,j}(x,y) \tag{A.13}$$

Then, let

$$P_{X_\alpha}(x) = -y^d + \prod_{\alpha \in X^{-1}(X_\alpha)} P_\alpha(x,y). \tag{A.14}$$

We have

$$P(x,y) = D(x)\Big(P_\infty(x,y) + \sum_{\alpha,\ a_\alpha<0} P_\alpha(x,y)\Big) + \mathbb{C}[\mathring{\mathcal{N}}]. \tag{A.15}$$

where

$$D(x) = \prod_{\alpha,\ X_\alpha \neq \infty} (x - X_\alpha)^{b_\alpha}. \tag{A.16}$$

This implies

Corollary A.1. *The non-interior coefficients $P_{i,j}$ of P, are polynomials of the times.*

Vice-versa, the times are algebraic functions of the non-interior coefficients of P.

Proof. Let

$$\tilde{P}(x,y) = \prod_{k=1}^{d} (y - Y_k(x)) = y^d + \sum_{l=1}^{d} (-1)^l y^{d-l} e_l(Y_1(x), \dots, Y_d(x)), \tag{A.17}$$

where $Y_k(x)$ are the zeros of $P(x,y)=0$, and e_l are the elementary symmetric polynomials:

$$\tilde{P}_l(x) = e_l(Y_1(x), \dots, Y_d(x)) = \sum_{1 \leqslant i_1 < \dots < i_l \leqslant d} Y_{i_1}(x) \dots Y_{i_l}(x). \tag{A.18}$$

$\tilde{P}_l(x) \in \mathbb{C}(x)$ is a rational function of x, and it can have poles only where at least one of the $Y_{i_j}(x)$ has a pole, i.e. only if $x = X(\alpha)$ for some puncture α.

We can decompose $\tilde{P}_l(x)$ into simple elements:

$$\tilde{P}_l(x) = \tilde{P}_{\infty,l}(x) + \sum_{q \in X(\text{finite punctures})} \tilde{P}_{q,l}(x) \tag{A.19}$$

where $\tilde{P}_{\infty,l}(x) \in \mathbb{C}[x]$ is a polynomial, and if $q \neq \infty$ $\tilde{P}_{q,l}(x) \in \mathbb{C}[1/(x-q)]$ is a rational function with poles only at $x = q$ or equivalently a polynomial of $1/(x-q)$ with no constant term.

• Consider $q = \infty$, and let $\alpha \in X^{-1}(\infty)$. For each such α we have

$$Y \sim \left(-\sum_{k=0}^{r_\alpha} \frac{t_{\alpha,k}}{a_\alpha} x^{\frac{k}{a_\alpha}-1} + O(x^{-1-1/a_\alpha}) \right) \tag{A.20}$$

This implies

$$\begin{aligned} \tilde{P}_{\infty,l}(x) &= e_l \left(\left(-\sum_{k=0}^{r_\alpha} \frac{t_{\alpha,k}}{a_\alpha} x^{\frac{k}{a_\alpha}-1} e^{2\pi i \frac{jk}{a_\alpha}} + O(x^{-2}) \right)_{\alpha \in X^{-1}(\infty),\ j=1,\dots,a_\alpha} \right) \\ &= e_l \left(\left(-\sum_{k=0}^{r_\alpha} \frac{t_{\alpha,k}}{a_\alpha} x^{\frac{k}{a_\alpha}-1} e^{2\pi i \frac{jk}{a_\alpha}} \right)_{\alpha \in X^{-1}(\infty),\ j=1,\dots,a_\alpha} \right) (1 + O(x^{-2})) \end{aligned} \tag{A.21}$$

The $O(x^{-2})$ are necessarily powers of x that are at least 2 less than the highest power, i.e. they correspond to point in the Newton's polygon that are strictly to the left of the convex envelope, they are interior points. This means that, up to interior points we can replace $\tilde{P}_\infty$ by P_∞.

• We redo the same for finite poles.

• eventually we multiply by the common denominator $D(x)$ so that $\tilde{P}(x)D(x)$ is a polynomial of x. This gives

$$D(x)\tilde{P}(x) = D(x) \left(P_\infty(x,y) + \sum_{\alpha, a_\alpha < 0} P_\alpha(x,y) \right) \quad \bmod \mathbb{C}[\mathring{\mathcal{N}}]. \tag{A.22}$$

Appendix B. Integrals over small circles

Lemma B.1. *We have*

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX = -\sum_{k=1}^{r_\alpha} \frac{|t_{\alpha,k}|^2}{k} R_\alpha^{-2k} + \sum_{k=1}^{\infty} k|\tilde{t}_{\alpha,k}|^2 R_\alpha^{2k} + 2t_{\alpha,0}^2 \ln R_\alpha - t_{\alpha,0} g_\alpha(p_\alpha) + \pi i t_{\alpha,0}^2 \tag{B.1}$$

where $\mathcal{C}_\alpha$ is the circle $\zeta_\alpha = R_\alpha e^{i\theta}$ with $\theta \in]-\pi,\pi]$, and p_α is the point of coordinate $\zeta_\alpha = R_\alpha e^{+i\pi}$.

Proof. We use $\zeta_\alpha = R_\alpha e^{i\theta}$ with $\theta \in]-\pi,\pi]$, and

$$YdX = \sum_{k=0}^{r_\alpha} t_{\alpha,k} \zeta_\alpha^{-k-1} d\zeta_\alpha + \sum_{k=1}^{\infty} k\tilde{t}_{\alpha,k} \zeta_\alpha^{k-1} d\zeta_\alpha \tag{B.2}$$

and therefore

$$g_\alpha = -\sum_{k=1}^{r_\alpha} \frac{t_{\alpha,k}}{k} \zeta_\alpha^{-k} + t_{\alpha,0} \ln \zeta_\alpha + \sum_{k=1}^{\infty} \tilde{t}_{\alpha,k} \zeta_\alpha^{k}. \tag{B.3}$$

This gives

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\alpha} \overline{g_\alpha}\, YdX$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(-\sum_{k=1}^{r_\alpha}\frac{\overline{t_{\alpha,k}}}{k}R_\alpha^{-k}e^{ik\theta}+t_{\alpha,0}(\ln R_\alpha - i\theta)+\sum_{k=1}^{\infty}\overline{\tilde t_{\alpha,k}}R_\alpha^{k}e^{-ik\theta}\Big)\\
&\quad\Big(\sum_{k=1}^{r_\alpha}t_{\alpha,k}R_\alpha^{-k}e^{-ik\theta}+t_{\alpha,0}+\sum_{k=1}^{\infty}k\tilde t_{\alpha,k}R_\alpha^{k}e^{ik\theta}\Big)d\theta\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+t_{\alpha,0}^2\ln R_\alpha\\
&\quad -\frac{it_{\alpha,0}}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=1}^{r_\alpha}t_{\alpha,k}R_\alpha^{-k}e^{-ik\theta}+t_{\alpha,0}+\sum_{k=1}^{\infty}k\tilde t_{\alpha,k}R_\alpha^{k}e^{ik\theta}\Big)\theta d\theta\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+t_{\alpha,0}^2\ln R_\alpha\\
&\quad -\frac{t_{\alpha,0}}{2\pi}\int_{-\pi}^{\pi}\theta\, d\Big(\sum_{k=1}^{r_\alpha}\frac{-1}{k}t_{\alpha,k}R_\alpha^{-k}e^{-ik\theta}+t_{\alpha,0}(\ln R_\alpha+i\theta)+\sum_{k=1}^{\infty}\tilde t_{\alpha,k}R_\alpha^{k}e^{ik\theta}\Big)\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+t_{\alpha,0}^2\ln R_\alpha\\
&\quad -\frac{t_{\alpha,0}}{2\pi}\int_{-\pi}^{\pi}\theta\, dg_\alpha(R_\alpha e^{i\theta})\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+t_{\alpha,0}^2\ln R_\alpha\\
&\quad -\frac{t_{\alpha,0}}{2\pi}(\pi g_\alpha(p_\alpha)+\pi(g_\alpha(p_\alpha)-2\pi i t_{\alpha,0}))+\frac{t_{\alpha,0}}{2\pi}\int_{-\pi}^{\pi}g_\alpha(R_\alpha e^{i\theta})d\theta\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+t_{\alpha,0}^2\ln R_\alpha\\
&\quad -t_{\alpha,0}g_\alpha(p_\alpha)+\pi i t_{\alpha,0}^2+\frac{t_{\alpha,0}^2}{2\pi}\int_{-\pi}^{\pi}(\ln R_\alpha+i\theta)d\theta\\
&= -\sum_{k=1}^{r_\alpha}\frac{|t_{\alpha,k}|^2}{k}R_\alpha^{-2k}+\sum_{k=1}^{\infty}k|\tilde t_{\alpha,k}|^2R_\alpha^{2k}+2t_{\alpha,0}^2\ln R_\alpha-t_{\alpha,0}g_\alpha(p_\alpha)+\pi i t_{\alpha,0}^2
\end{aligned}
\tag{B.4}
$$

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