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"PROBLEMES INFRAROUGES EN THEORIE
PERTURBATIVE DES CHAMPS"

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TABLE DES MATIÈRES

INTRODUCTION GENERALE

La dernière décennie a été incontestablement marquée par le rôle très important qu'ont joué les idées et les techniques de la théorie quantique des champs dans de nombreux domaines de la physique. Deux grandes percées sont à l'origine d'un tel essort : D'une part les théories de Jauge non abéliennes [1] se sont avérées être des théories quantiques sans doute cohérentes et elles ont simultanément permis de proposer des modèles séduisants et raisonnables des interactions fondamentales entre les particules élémentaires (que sont pour le moment les leptons et les quarks) [2]. Dans le même temps les idées "d'invariance d'échelle" et de "groupe de Renormalisation" [3] ont fourni une compréhension qualitative et quantitative des phénomènes critiques. De tels progrès ont été possibles en grande partie grâce à des outils développés initialement dans le cadre de la théorie perturbative des champs : Théorie de la Renormalisation, équations de Callan-Symanzik, développement de Wilson. Par ailleurs des efforts et des progrès considérables ont été faits pour développer d'autres méthodes d'étude en théorie quantique des champs et en mécanique statistique, en particulier à l'aide des méthodes semi-classiques et de l'étude des théories sur réseaux. Ces progrès ont porté à la fois sur l'obtention de résultats rigoureux et sur le développement de nouvelles méthodes d'approximations et d'études numériques. Ils ont également permis de préciser la nature des développements perturbatifs eux-mêmes. Les calculs perturbatifs sont néanmoins le seul moyen direct d'atteindre la "limite continue" encore à notre disposition et de nombreuses questions se posent encore concernant le statut de ces développements.

Les travaux réunis dans cette thèse traitent principalement de questions liées à la présence de divergences "infrarouges" dans les développements perturbatifs des fonctions de Green de certaines théories des champs de masse nulle. Il est bien connu que les développements perturbatifs se font en terme d'intégrales de Feynman associées aux graphes du même nom ; ces intégrales peuvent présenter des divergences lorsque certaines des impulsions internes sont nulles. On s'attend malgré tout à rencontrer beaucoup moins de problèmes infrarouges dans les calculs de fonctions de Green (c'est-à-dire d'intégrales de

Feynman "hors de la couche de masse") que dans les calculs d'amplitudes de diffusion (qui correspondent à des amplitudes "sur la couche de masse"). En particulier les théories renormalisables à quatre dimensions (ϕ^4, électrodynamique quantique, théories de jauge) ne présentent pas de telles divergences (pourvu que l'on ait renormalisé correctement les théories) [4]. Par contre l'étude des théories scalaires (comme ϕ^4) à 2 ou 3 dimensions, des modèles bidimensionnels ou des théories de Jauge à température finie conduit à des situations où de telles divergences empèchent de définir perturbativement les fonctions de Green. Dans certains cas on peut contourner le problème en introduisant un régulateur infrarouge et en extrapolant les propriétés de la théorie lorsque ce régulateur tend vers zéro, par exemple à l'aide des équations du Groupe de Renormalisation (c'est le cas pour la théorie $\phi^4_{4-\epsilon}$ et cette idée est à la base du célèbre développement en $4 - \epsilon$). Néanmoins, l'étude directe de la structure de ces divergences est importante pour plusieurs raisons qui nous l'espérons apparaîtront clairement dans ce travail.

Il est tout d'abord important de déterminer le statut mathématique des développements perturbatifs en théorie des champs, ne serait-ce que pour préciser leur domaine de validité.

Ensuite, de même que les problèmes infrarouges "sur la couche de masse" de l'Electrodynamique Quantique sont liés à la nature des états asymptotiques de cette théorie [5], les problèmes infrarouges "hors de la couche de masse" sont souvent liés aux propriétés dynamiques du "vide" d'une théorie.

Enfin, comme dans les problèmes de renormalisabilité, les symétries d'une théorie jouent souvent un rôle crucial dans ces problèmes infrarouges.

Après ces considérations générales, précisons donc le contenu de ce travail. L'unité de cette thèse est plutôt à chercher dans les techniques mathématiques utilisées (représentations intégrales des amplitudes de Feynman, méthodes de désingularisation, régularisation et renormalisation dimensionnelle) que dans les modèles et les problèmes considérés, qui correspondent à des motivations physiques variées. Nous reviendrons sur ces techniques à la fin de cette introduction.

La première partie est d'ailleurs consacrée à l'application de ces techniques de désingularisation à un problème de divergences "ultraviolettes" (à courte distance). Nous construisons explicitement un opérateur de soustraction des amplitudes de Feynman correspondant à la procédure de renormalisation dimensionnelle [6, 7].

La partie II est consacrée aux modèles Sigma non linéaires à deux dimensions d'espace-temps. Ces modèles, qui décrivent à basse énergie la dynamique des Bosons de Goldstone associés à une brisure de symétrie continue globale ont été beaucoup étudiés en raison de leurs analogies avec les théories de Jauge à quatre dimensions. Ils présentent à deux dimensions des divergences infrarouges dans leur développement perturbatif. Ces divergences sont reliées au théorème de Mermin-Wagner-Coleman [8, 9], qui interdit l'existence d'une telle brisure de symétrie à deux dimensions. Le principal résultat de cette section est la preuve d'une conjecture, due à S. Elitzur [10], affirmant que dans ces modèles les divergences infrarouges disparaissent totalement des développements perturbatifs des observables invariantes sous l'action du groupe de symétrie globale de ces modèles.

Dans la partie III nous étudions les modèles de surface aléatoire développés par D. Wallace et al [11, 12]. Nous construisons les observables invariantes de ces modèles et prouvons que près de leur dimension critique inférieure (qui est ici zéro), ces observables invariantes sont finies infrarouge. Nous montrons également que l'indice critique ν calculé par D. Wallace est en fait relié à la dimension de Hausdorff de la surface au point critique. Enfin, nous discutons les problèmes posés par le développement en $1/d$ de ces modèles (d étant la dimension de l'espace dans lequel est plongée la surface), ainsi que certaines relations de ce modèle avec les modèles de cordes duales.

La partie IV est consacrée à une étude générale des divergences infrarouges des théories de masse nulle super-renormalisables. L'un des buts de cette étude est de donner une formulation précise à une conjecture de G. Parisi [13] sur la structure de ces divergences. Nous précisons également le lien entre cette structure et l'apparition de termes non analytiques dans la constante de couplage (tels des logarithmes). De tels termes avaient été prédits à partir d'arguments non perturbatifs par K. Symanzik dans le cas de la théorie ϕ^4 à $d = 4 - \epsilon$ dimensions [14] et par différents auteurs dans les théories de Jauge à trois dimensions ou à température finie.

Enfin la partie V représente à notre avis un premier pas dans une étude rigoureuse des effets non perturbatifs présents dans les théories de masse nulle asymptotiquement libres. Les techniques développées dans la partie IV nous permettent d'étudier, à l'aide du développement en 1/N du modèle sigma non linéaire à deux dimensions, la structure de la transformée de Borel de son développement perturbatif dans la constante de couplage. Ces résultats indiquent qu'il existe une solution probable aux ambiguïtés (dans la sommation de Borel des séries perturbatives) dues à la présence de singularités infrarouges (Renormalons) sur l'axe réel positif de la transformée de Borel des modèles bidimensionnels et des théories de Jauge à quatre dimensions [15, 16]. Une telle solution permettrait également de donner un statut mathématique précis au formalisme baptisé "règles de somme de QCD" ou "Dualité de Shifman - Vainsthein - Zakharov" utilisé actuellement en physique des particules pour tenir compte des effets non perturbatifs des théories de Jauge [16].

Revenons maintenant sur les techniques de désingularisation utilisées au cours de ce travail ; en effet, les détails en sont dispersés dans les différents articles qui composent cette thèse et un lecteur non averti aura peut être du mal à en dégager le principe : ces techniques ont été développées par M. Bergère et Y. M. Lam [17] qui ramènent l'étude du comportement asymptotique d'une amplitude de Feynman I_G lorsque l'on dilate par un paramètre réel λ très grand certains de ses invariants à l'étude de la structure analytique de la transformée de Mellin :

$$M_G(x) = \int_0^\infty d\lambda \; \lambda^{-x-1} \, I_G(\lambda) \qquad (0.1)$$

L'intégrale (0.1) est en général convergente pour des valeurs de la variable de Mellin suffisamment grandes ($x > x_o$) et s'avère définir une fonction méromorphe dans le domaine $\{\mathrm{Re}\ x \leqslant x_o\}$. Chaque terme du développement de Laurent autour d'un pôle correspond à un terme (en λ^P Ln λ^Q) du développement asymptotique de $I_G(\lambda)$ quand $\lambda \to \infty$. On peut maintenant utiliser la représentation paramétrique de Schwinger - Symanzik pour écrire $M_G(x)$ sous la forme intégrale :

$$M_G(x) = \int_0^\infty d\alpha \; m_G(\alpha, x) \tag{0.2}$$

valable pour Re x > x_o et où l'intégration est effectuée sur ℓ paramètres positifs $\alpha = (\alpha_i \; ; \; i = 1, \ell)$ associés aux ℓ lignes du graphe G. Le caractère divergent de l'intégrale (0.2) pour $x \leqslant x_o$ provient uniquement du comportement de l'intégrand $m_G(\alpha, x)$ lorsque certains α tendent vers zéro.

Dans le cas qui nous intéresse ($\lambda \; \alpha$ $[\text{masse}]^{-2}$), ce comportement s'avère d'un type très particulier, baptisé FINE dans [18] (factorisé dans chaque secteur de Hepp) qui rend possible l'étude des singularités de M_G par des techniques similaires à celles employées dans la théorie de la renormalisation*. En particulier, un théorème dû à M. Bergère et à nous même [19] montre que, lorsque m_G est FINE, la représentation intégrale (0.2) s'étend pour toute valeur de x (pourvu que Re x soit différent d'un des pôles de M_G) sous la forme

$$M_G(x) = \int_0^\infty d\alpha \; R \, m_G(\alpha, x) \tag{0.3}$$

où R est un opérateur de soustraction (indépendant de x et de la topologie du graphe G), mais dont l'action sur m_G dépend bien sûr du comportement de m_G en 0, donc de x. Ce résultat fournit un moyen très général d'étude de la fonction M_G. En effet, la partie divergente de M_G en un pôle $x = x_i$ s'obtient, par exemple par des intégrales de Cauchy, en fonction de la discontinuité en Re(x) = x_i de l'intégrand R $m_G(\alpha, x)$. On se trouve donc ramené à l'étude des propriétés de l'intégrand, lorsque certains des α_i tendent vers zéro et en particulier à la détermina-

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* Une extension de ces techniques au cas non FINE, qui fournit une preuve générale de la méromorphie des transformées de Mellin, est donnée par C. de Calan et al. dans [18].

tion des sous-ensembles de α (c'est-à-dire des sous-graphes de G) divergents. Les problèmes posés par cette étude et par l'influence de la renormalisation sont traités en détail dans les différentes parties.

Finalement, un mot sur l'organisation de la thèse. Les travaux qui la constituent ont fait l'objet d'articles publiés ou en cours de publication qui sont suffisamment détaillés pour que nous les reproduisions tels quels en les accompagnant seulement d'une introduction et de commentaires. La partie III est plus détaillée, et présente des résultats nouveaux qui n'ont pas été encore publiés.

PREMIERE PARTIE

ÉTUDE DE LA RENORMALISATION DIMENSIONNELLE

I.1 - INTRODUCTION

On peut distinguer deux approches dans l'étude des problèmes de renormalisation en théorie quantique des champs. Ou bien on se place dans un cadre très général en s'affranchissant de tout schéma de soustraction particulier et en utilisant le principe de l'action quantique [20], ou bien on recherche une méthode de régularisation optimale pour le problème considéré. De ce deuxième point de vue, et parmi les nombreuses procédures actuellement disponibles, la régularisation (et la renormalisation) dimensionnelle [6, 7, 21-28] est une des plus populaires et a joué un rôle important dans l'étude des Théories de Jauge et celle des phénomènes critiques. Ses avantages sont son élégance formelle, les simplifications qu'elle introduit dans les calculs, en particulier pour les théories de masse nulle, et surtout elle respecte de manière "optimale" les symétries telles l'invariance de Lorentz ou les invariances internes globales ou locales (de jauge)(les anomalies liées à l'invariance conforme ou aux matrices γ^5 apparaissent aussi simplement dans ce cadre).

Nous résolvons ici un problème soulevé dans [29] : Existe t-il une procédure de soustraction explicite sur l'intégrand d'une amplitude de Feynman transformant l'intégrale divergente de départ en une intégrale convergente coïncidant avec l'amplitude renormalisée dimensionnellement (définie habituellement en calculant et soustrayant les pôles en $\frac{1}{d-4}$ de manière récursive en accord avec la procédure de B. P. H. [30]). Nous construisons une telle procédure dans la représentation paramétrique de Schwinger - Symanzik qui est particulièrement adaptée pour ce problème.

Ce résultat nous semble pouvoir être utile lors d'une étude systématique des propriétés des intégrales de Feynman renormalisées dimensionnellement, puisqu'il réalise pour les soustractions minimales ce que l'opération R de B.P.H.Z. réalise pour les soustractions à moments nuls [31, 32]. Un autre point est que la procédure de désingularisation construite ici illustre le principe exposé dans l'introduction. Ici, c'est une intégrale de Feynman interpolée en dimension $I_G(d)$ qui est étudiée et c'est la dimension d qui joue le rôle de la variable x.

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Integral Representation for the Dimensionally Renormalized Feynman Amplitude

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Abstract. A compact convergent integral representation for dimensionally renormalized Feynman amplitudes is explicitly constructed. The subtracted integrand is expressed as a distribution in the Schwinger α-parametric space, and is obtained by applying upon the bare integrand a new subtraction operator $\underset{\cdot}{R}'$ which respects Zimmermann's forest structure.

1. Introduction

Dimensional renormalization [1–9], first introduced by Speer and Westwater [1] and applied in the study of gauge theories by t'Hooft and Veltman [3], has proved to be an essential tool in quantum field theory. Indeed, it preserves gauge invariance, Lorentz invariance and avoids the infrared problem which appears when subtractions are performed at zeromomenta. Another advantage of this renormalization is that the Callan-Symanzik [10] function $\beta(g)$ is then independent of the dimension of space time (apart from a trivial $(D-4)\, g$ factor) and is also independent of the mass ratios which enter the theory.

In recent years, dimensional regularization and dimensional renormalization were established on firm ground as were other kinds of renormalization, and we refer to the literature [7, 9, 11]. According to Bogoliubov-Parasiuk-Hepp (BPH) recurrence, the usual method to calculate such an amplitude is first to renormalize the smaller divergent subgraphs by extracting their poles at $D=4$, then to introduce their finite parts into larger subgraphs and reproduce the same procedure in a recurrent way. This method becomes very difficult at high orders of perturbation, dealing with overlapping divergences, spinor, coupling derivatives and gluon propagators.

On the other hand, the existence of a compact expression which, for a given Feynman graph, gives directly the dimensionally renormalized integrand is still missing. Some authors in the study of the properties of dimensional renormalization come close to achieving this goal (for instance, the C_H operators of Breitenlohner and Maison [7] or the $\mathscr{P}\mathscr{P}_\lambda$ operators of Collins [6], organized in forests of divergent subgraphs). But the successive applications of these operators,

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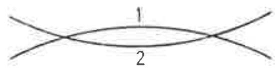

Fig. 1. The one loop graph attached to (1.1) and (1.2)

which include the implicit extraction of a pole singularity attached to a given subgraph, without giving an explicit algorithm to isolate this pole, remain to be made precise.

In this paper we solve this question and we construct explicitly a subtraction operator R', acting directly upon the bare integrand of any Feynman graph expressed in the α-Schwinger-Symanzik representation; this operator gives the dimensionally renormalized integral as a compact, convergent integral in the α-parametric space. The action of R' transforms the bare integrand into a distribution expressible in terms of θ, δ distributions and their derivatives. Such distributions are shown to exist and to be integrable over the α-parametric domain.

This paper is organized as follows: In the end of this introduction we describe, using the example of a one loop graph, the principle of the method which shall be used to obtain the dimensionally renormalized amplitude. Then, we recall the integral representation of a dimensionally regularized integrand [12]. In Sect. 2, we define the subtraction operator R'; we give two examples and we comment on our result. Section 3 is devoted to the proofs and is divided into three parts: first, we show that R' defines a renormalization which satisfies the recurrence of Bogoliubov-Parasiuk [13] and we describe the corresponding counterterms; then we show by recurrence that R'' corresponds, subgraph by subgraph, to the extraction of the pole singularities at $D=4$ and we prove the absolute convergence of the finite parts (via the introduction of a regulator in order to avoid distributions); finally, we remove the regulator and we prove the existence and the integrability of the distributions which describe the dimensionally renormalized integrand.

We now describe the principle of the method which shall be used to obtain the dimensionally renormalized amplitude by considering the simple example of the one-loop graph of Fig. 1 which diverges logarithmically at $D=4$. For $\operatorname{Re} D<4$, the amplitude of the graph is given by the integral representation

$$I_G(S,m,D)=\int_0^\infty d\alpha_1 d\alpha_2 \frac{\exp(-(\alpha_1+\alpha_2)m^2)\exp(-S(\alpha_1\alpha_2)/(\alpha_1+\alpha_2))}{(\alpha_1+\alpha_2)^{D/2}}. \tag{1.1}$$

For $4<\operatorname{Re} D<6$, it has been shown in [12] that

$$I_G(S,m,D)=\int_0^\infty d\alpha_1 d\alpha_2 \frac{\exp(-(\alpha_1+\alpha_2)m^2)\exp(-S(\alpha_1\alpha_2)/(\alpha_1+\alpha_2))-1}{(\alpha_1+\alpha_2)^{D/2}}. \tag{1.2}$$

The function $I_G(S,m,D)$ behaves like $\Gamma\left(2-\frac{D}{2}\right)$ ar large imaginary D and consequently a contour integral around the single pole $D=4$ can be seen as the integrals over two lines C_+ and C_- as shown in Fig. 2.

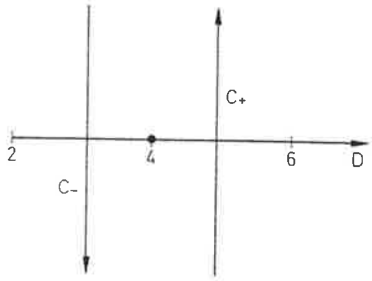

Fig. 2. Integration contour around the pole of the graph of Fig. 1

The residue of the pole is found to be

$$r = -\int_0^\infty d\alpha_1 d\alpha_2 \delta[-\tfrac{1}{2}\mathrm{Ln}(\alpha_1+\alpha_2)], \tag{1.3}$$

and defining the dimensionally renormalized amplitude of G as

$$I_G^{R'}(S, m, D) = I_G(S, m, D) - \frac{r}{D-4}, \tag{1.4a}$$

which is equivalent to

$$I_G^{R'}(S, m, D) = \frac{1}{2i\pi}\oint \frac{I_G(S, m, D')}{D'-D} dD', \tag{1.4b}$$

where the contour in the D' complex plane encircles the two points $D'=4$ and $D'=D$, we obtain

$$I_G^{R'}(S, m, D) = \int_0^\infty d\alpha_1 d\alpha_2 \frac{[e^{-(\alpha_1+\alpha_2)m^2} e^{-(S\alpha_1\alpha_2)/(\alpha_1+\alpha_2)} - \theta\{-\frac{1}{2}\mathrm{Ln}(\alpha_1+\alpha_2)\}]}{(\alpha_1+\alpha_2)^{D/2}}, \tag{1.5}$$

where $\theta(x)$ is $+1$ for $x>0$ and 0 for $x<0$.

The above integral, for $\mathrm{Re}\, D<6$, is absolutely convergent at $\alpha_1+\alpha_2 \sim 0$ because of the subtraction, and at α_1 or/and $\alpha_2 \sim \infty$ because of the absence of subtraction; moreover, as wanted, the variables m^2 and S are treated equally in the subtraction procedure. Another property of the above subtraction is that the zero mass limit of $I_G^{R'}(S, m, D)$ exists for $2<\mathrm{Re}\, D<6$.

The purpose of this paper is to generalize the above example to any graph G and thus to obtain in compact form a convergent integral representation for $I_G^{R'}$. Several difficulties are encountered:

The above procedure of subtracting away the negative powers of $(D-4)$ in the Laurent expansion destroys unitarity as soon as multiple poles occur because it cannot be implemented by a counterterm formalism. It is necessary to suppress these negative powers by using a forest (set of non-overlapping subgraphs) subtraction formula with the condition that for each subgraph we subtract the corresponding pole at $D=4$ and nothing else.

If we consider a graph with coupling derivatives or/and spinors, the dependence in D of the α-integrand contains a polynomial in D. Each power of D is obtained from a contraction $g_\mu{}^\mu$ generated by a pair of coupling derivatives or/and spinors. Now, when we calculate the residue at $D=4$ associated to a subgraph $\mathscr{S}$, one power of D should be included (excluded) in the calculation if the pair of coupling derivatives or/and spinors belongs (does not belong) to $\mathscr{S}$. In other words, when we proceed to the calculation of residues for a set of subgraphs $\mathscr{S}_1, \ldots, \mathscr{S}_n$, how are we going to decide how many powers of D are generated by each of the $\mathscr{S}_i$'s? This problem already exists for scalar amplitudes because the subtractions over divergent subgraphs generate coupling derivatives for the corresponding reduced subgraphs. A convenient solution to these difficulties has been proposed by Ashmore [8] who introduced a multidimensional formalism. This formalism attaches a separate dimension to each subgraph and is exposed in Appendix A.

The multidimensionally regularized Euclidian Feynman amplitude is given by the integral representation

$$I_G(p_i, m, D, \omega_{\mathscr{S}}) = \int_0^\infty \prod_{a=1}^{\ell} d\alpha_a Y_G(p_i, m, \alpha, D, \omega_{\mathscr{S}}), \tag{1.6}$$

where

$$Y_G(p_i, m, \alpha, D, \omega_{\mathscr{S}}) = \left\{ \exp\left(- \prod_{a=1}^{\ell} \alpha_a m_a^2 \right) S_G(p_i, \alpha, D, \omega_{\mathscr{S}}) \cdot \exp(-pd^{-1}(\alpha)p) P_G(\alpha)^{-D/2} \prod_{\mathscr{S} \subseteq G} P_{\mathscr{S}}(\alpha)^{-\omega_{\mathscr{S}}/2} \right\}. \tag{1.7}$$

The functions $P_G(\alpha)$, $P_{\mathscr{S}}(\alpha)$ and $pd^{-1}(\alpha)p$ are characteristic functions of the topology of the graph and of its subgraphs. The function $S_G(p_i, \alpha, D, \omega_{\mathscr{S}})$ describes the spin and coupling derivatives part of the amplitude. In (1.7) the dimension D is the dimension of space-time and the variables $\omega_{\mathscr{S}}$ are introduced according to Ashmore's formalism (Appendix A) to be the dimensions attached to every subgraph $\mathscr{S}$.

The integral (1.6) is absolutely convergent for $\{\mathrm{Re}\, D, \mathrm{Re}\, \omega_{\mathscr{S}}\}$ sufficiently small and defines by analytic continuation a meromorphic function of the variables D and $\omega_{\mathscr{S}}$. As a generalization of the result obtained in [12], the analytic continuation of $I_G(p_i, m, D, \omega_{\mathscr{S}})$ is given almost everywhere (that is away from those hyperplanes in D and $\omega_{\mathscr{S}}$ where $I_G(p_i, m, D, \omega_{\mathscr{S}})$ is singular) by the following absolutely convergent integral representation:

$$I_G(p_i, m, D, \omega_{\mathscr{S}}) = \int_0^\infty \prod_{a=1}^{\ell} d\alpha_a R Y_G(p_i, m, \alpha, D, \omega_{\mathscr{S}}). \tag{1.8}$$

The subtraction operator R is defined in [14] as

$$R = \prod_{\mathscr{S} \subseteq G} (1 - \tau_{\mathscr{S}}^{-2\ell(\mathscr{S})}) = \left[1 + \sum_{\mathscr{F}} \prod_{\mathscr{S} \in \mathscr{F}} (-\tau_{\mathscr{S}}^{-2\ell(\mathscr{S})}) \right], \tag{1.9}$$

where the generalized Taylor operators $\tau_{\mathscr{S}}^{-2\ell(\mathscr{S})}$ are defined in [14] and where the sum over $\mathscr{F}$ runs over all forests of "divergent" subgraphs.

2. The Subtraction Operator R'

We consider a graph G and its Feynman amplitude $I_G(p, m, D)$ as defined in (1.6–8) with all $\omega_{\mathscr{S}}$'s $=0$. Let us consider three consecutive poles of $I_G: D^- < D^* < D^+$ (if D^* is the smallest pole of I_G, $D^- = -\infty$). From now on, we denote by B^- and B^+ respectively the strips $D^- < \operatorname{Re} D < D^*$ and $D^* < \operatorname{Re} D < D^+$. We intend to define the dimensionally renormalized amplitude at D^*.

We note that the operator $\tau_{\mathscr{S}}^{-2\ell(\mathscr{S})}$ introduced in (1.9) subtracts differently whether we stand in the strips B^- or B^+. Let us call $\tau_{\mathscr{S}}^-$ (respectively $\tau_{\mathscr{S}}^+$) the generalized Taylor operator relative to $\mathscr{S}$ and subtracting minimally (of degree $-2\ell(\mathscr{S})$, where $\ell(\mathscr{S})$ is the number of internal lines in the subgraph $\mathscr{S}$) in the strip B^- (respectively B^+). If $\mathscr{S}$ develops no pole at D^*, $\tau_{\mathscr{S}}^- = \tau_{\mathscr{S}}^+$.

According to the requirements imposed by dimensional renormalization, namely – subtraction of the Feynman amplitude in agreement with a counterterm structure – extraction and subtraction of the singular part of the Laurent expansion around D^* for each divergent subgraph once its interior has been subtracted – we found in the strip $D^- < \operatorname{Re} D < D^+$ which contains D^* the following convergent integral representation for the renormalized amplitude:

$$I_G^{R'}(p_i, m, D) = \int_0^\infty \prod_{a=1}^{\ell} d\alpha_a R' Y_G(p_i, m, \alpha, D, \omega_{\mathscr{S}}). \tag{2.1}$$

The operator R' in (2.1) acts upon the function $Y_G(p_i, m, \alpha, D, \omega_{\mathscr{S}})$ in the following way:

First, it subtracts the amplitude according to a forest formula of divergent subgraphs

$$1 + \sum_{\mathscr{F}} \prod_{\mathscr{S} \in \mathscr{F}} (-\tau'_{\mathscr{S}}), \tag{2.2}$$

where $\tau'_{\mathscr{S}}$ are new subtraction operators defined as

$$\tau'_{\mathscr{S}} F(\alpha, \omega_{\mathscr{S}}) = \tau_{\mathscr{S}}^- F(\alpha, \omega_{\mathscr{S}}) + (\tau_{\mathscr{S}}^+ - \tau_{\mathscr{S}}^-) F(\alpha, \omega_{\mathscr{S}}) \theta(x_{\mathscr{S}}). \tag{2.3}$$

The function $\theta(x_{\mathscr{S}})$ is the Heaviside function and is introduced in order to perform the Cauchy integration around D^*. In this subtraction procedure the $\omega_{\mathscr{S}}$ are considered as small positive parameters which do not change the number of subtractions of τ^+ and τ^-.

Second, it replaces the variables $\omega_{\mathscr{S}}$ by the operators $\dfrac{\partial}{\partial x_{\mathscr{S}}}$ acting upon the θ's at $x_{\mathscr{S}} = 0$.

The latter operation performs for every $\omega_{\mathscr{S}}$ the complex integration

$$\frac{1}{2\pi} \int_{\substack{\sigma - i\infty \\ \sigma > \omega}}^{\sigma + i\infty} d(\operatorname{Im} z) \frac{A(z) e^{x(z-\omega)}}{z - \omega} = A\left(\omega + \frac{\partial}{\partial x}\right) \theta(x) \tag{2.4}$$

and generalizes to every subgraph the Cauchy extraction of the pole at D^* performed in the example of Sect. 1.

Once the operator (2.2) is applied on Y_G, the dependence of the integrand in any $\omega_{\mathscr{S}}$ appears in terms of the form:

$$N(\alpha, \omega_{\mathscr{S}}) Q(\alpha)^{-\omega_{\mathscr{S}}/2}, \tag{2.5}$$

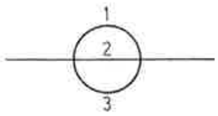

Fig. 3. The two loops quadratic diagram of Example 1

where N is a polynomial of $\omega_{\mathscr{S}}$ and Q some polynomial of the α's. So the second part above transforms (2.5) into either

$$N\left(\alpha,\frac{\partial}{\partial x_{\mathscr{S}}}\right)Q(\alpha)^{-1/2\frac{\partial}{\partial x_{\mathscr{S}}}}\theta(x_{\mathscr{S}})|_{x_{\mathscr{S}}=0}=N\left(\alpha,\frac{\partial}{\partial x_{\mathscr{S}}}\right)\theta[-\tfrac{1}{2}\operatorname{Log}Q(\alpha)+x_{\mathscr{S}}]_{x_{\mathscr{S}}=0}, \quad (2.6)$$

which generates for every subgraph $\theta,\delta,\delta',\ldots$ distributions in the α-space, or

$$N\left(\alpha,\frac{\partial}{\partial x_{\mathscr{S}}}\right)Q(\alpha)^{1/2\frac{\partial}{\partial x_{\mathscr{S}}}}\mathbb{1}=N(\alpha,0) \quad (2.7)$$

if there is no θ function.

To sum up the action of the operator R', we shall write

$$R'=\left[1+\sum_{\mathscr{F}}\prod_{\mathscr{S}\in\mathscr{F}}(-\tau'_{\mathscr{S}})\right]_{\Delta} \quad (2.8)$$

where Δ means the operation $\omega_{\mathscr{S}}\to\frac{\partial}{\partial x_{\mathscr{S}}}|_{x_{\mathscr{S}}=0}$ once all operators $\tau'_{\mathscr{S}}$ have been applied.

We now illustrate the rules given above by two examples:

Example 1. We consider the two loop quadratic diagram of Fig. 3 at $D^*=4$.

$$Y_G(\omega_{\mathscr{S}})=\frac{\exp\left(-(\alpha_1+\alpha_2+\alpha_3)m^2-p^2\cdot\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}\right)}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^{(D+\omega_G)/2}(\alpha_1+\alpha_2)^{\omega_{12}/2}(\alpha_2+\alpha_3)^{\omega_{23}/2}(\alpha_3+\alpha_1)^{\omega_{31}/2}}. \quad (2.9)$$

The divergent subgraphs at $\left\{\begin{matrix}D^*=4\\ \omega_{\mathscr{S}}=0\end{matrix}\right\}$ are: {123} quadratically divergent; {12}, {23}, {31} logarithmically divergent

$R'Y_G(\omega_{\mathscr{S}})$

$$=Y_G(0)-\frac{1-\left|(\alpha_1+\alpha_2+\alpha_3)m^2+p^2\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}\right|\theta\{-\frac{1}{2}\operatorname{Ln}(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)\}}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)^{D/2}}$$

$$-\frac{e^{-\alpha_1 m^2}\theta\{-\frac{1}{2}\operatorname{Ln}(\alpha_2+\alpha_3)\}}{\alpha_1^{D/2}(\alpha_2+\alpha_3)^{D/2}}+\text{circ. perm.}$$

$$+\frac{\theta\{-\frac{1}{2}\operatorname{Ln}(\alpha_2+\alpha_3)\}[1-\alpha_1 m^2\theta\{-\frac{1}{2}\operatorname{Ln}[\alpha_1(\alpha_2+\alpha_3)]\}]}{\alpha_1^{D/2}(\alpha_2+\alpha_3)^{D/2}}+\text{circ. perm.} \quad (2.10)$$

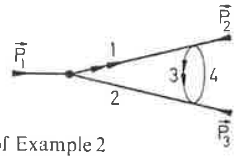

Fig. 4. The two loops diagram of Example 2

The above result differs from the usual R operation [14] (subtraction at zero external momentum) first by the presence of θ distribution, second by the fact that the mass m^2 terms are subtracted, the consequence of which is that the subgraphs $\{12\}$, $\{23\}$ and $\{31\}$ give non-zero subtraction terms although they are not generalized vertices.

Example 2 (Fig. 4). This example illustrates the difficulties incoming from coupling derivatives. We concentrate on the forest of the two logarithmically divergent subgraphs at $D^* = 2$, $G = \{1, 2, 3, 4\}$ and $\mathscr{S} = \{3, 4\}$.

$$Y_G(\omega_{\mathscr{S}}) = \frac{\partial}{\partial z_1^2} \frac{\partial}{\partial z_3^2} \Bigg[P(\alpha)^{-(D+\omega_G)/2} (\alpha_3 + \alpha_4)^{-\omega_{\mathscr{S}}/2}$$

$$\cdot \exp\left\{ -\sum_{a=1}^{4} \alpha_a \bar{m}_a^2 - \bar{p}_1^2 \frac{\alpha_1\alpha_2(\alpha_3+\alpha_4)}{P(\alpha)} - \bar{p}_2^2 \frac{\alpha_1\alpha_3\alpha_4}{P(\alpha)} - \bar{p}_3^2 \frac{\alpha_2\alpha_3\alpha_4}{P(\alpha)} \right\} \Bigg]_{z_1 = z_3 = 0}, \quad (2.11)$$

where

$$\bar{m}_a^2 = m_a^2 - \frac{z_a^2}{4\alpha_a^2} \qquad a = 1, 2, 3, 4 \quad (2.12a)$$

$$\bar{p}_1 = p_1 + \frac{z_1}{2\alpha_1} + \frac{z_2}{2\alpha_2} \quad (2.12b)$$

$$\bar{p}_2 = p_2 - \frac{z_1}{2\alpha_1} + \frac{z_3}{2\alpha_3} - \frac{z_4}{2\alpha_4} \quad (2.12c)$$

$$\bar{p}_3 = p_3 - \frac{z_2}{2\alpha_2} - \frac{z_3}{2\alpha_3} + \frac{z_4}{2\alpha_4} \quad (2.12d)$$

$$P(\alpha) = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_3\alpha_4 . \quad (2.12e)$$

In (2.12), the momentum $p_i \in \mathbb{R}^D$, the vectors z_1 and $z_2 \in \mathbb{R}^D \oplus \mathbb{R}^{\omega_G}$, and z_3 and $z_4 \in \mathbb{R}^D \oplus \mathbb{R}^{\omega_G} \oplus \mathbb{R}^{\omega_{\mathscr{S}}}$.

The derivatives $\frac{\partial}{\partial z_1^2}$ and $\frac{\partial}{\partial z_3^2}$ generate the following polynomial of ω:

$$\frac{D_1 D_3}{4} \frac{(\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_4)}{P^2(\alpha)} + \frac{D_3\alpha_4^2}{2P^2(\alpha)} + \frac{D_3 A^2(\alpha_1+\alpha_2+\alpha_4)}{2P(\alpha)}$$

$$+ \frac{D_1 B^2(\alpha_3+\alpha_4)}{2P(\alpha)} + \frac{2A.B\alpha_4}{P(\alpha)} + A^2B^2 , \quad (2.13)$$

with

$$A^\mu = \frac{-p_1^\mu \alpha_2(\alpha_3+\alpha_4)+p_2^\mu \alpha_3\alpha_4}{P(\alpha)}, \tag{2.14a}$$

$$B^\mu = \frac{-p_2^\mu \alpha_1\alpha_4+p_3^\mu \alpha_2\alpha_4}{P(\alpha)}, \tag{2.14b}$$

$$D_1 = D+\omega_G, \tag{2.14c}$$

$$D_3 = D+\omega_G+\omega_{\mathscr{S}}. \tag{2.14d}$$

Then,

$$\tau_G^+ \tau_{\mathscr{S}}^+ Y_G(\omega_{\mathscr{S}}) = \frac{D_1 D_3}{4}(\alpha_1+\alpha_2)^{-\left(\frac{D+\omega_G+2}{2}\right)}(\alpha_3+\alpha_4)^{-\left(\frac{D+\omega_G+\omega_{\mathscr{S}}+2}{2}\right)} \tag{2.15}$$

and finally

$$[\tau'_G \tau'_{\mathscr{S}}]_\Delta Y_G(\omega_{\mathscr{S}}) = [(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)]^{-\left(\frac{D+2}{2}\right)}$$
$$\cdot\left\{\frac{D^2}{4}\theta_{G,\mathscr{S}}\theta_{\mathscr{S}}+\frac{D}{2}\delta_{G,\mathscr{S}}\theta_{\mathscr{S}}+\frac{D}{4}\theta_{G,\mathscr{S}}\delta_{\mathscr{S}}+\tfrac{1}{4}\delta'_{G,\mathscr{S}}\theta_{\mathscr{S}}+\tfrac{1}{4}\delta_{G,\mathscr{S}}\delta_{\mathscr{S}}\right\}, \tag{2.16}$$

where the derivatives $\theta_{G,\mathscr{S}}^{(n)}$ means $\theta^{(n)}[-\frac{1}{2}\mathrm{Ln}\{(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\}]$ and $\theta_{\mathscr{S}}^{(n)}$ means $\theta^{(n)}[-\frac{1}{2}\mathrm{Ln}(\alpha_3+\alpha_4)]$.

To close this section, let us comment on our result. The dimensionally renormalized integrand is a distribution which is a sum of products of derivatives of the θ distribution. This fact raises a problem of existence of these products; it is shown in Sect. 3 and Appendix C that the manifolds in α which are the support of the $\delta^{(n)}$ distributions ($n=0, 1, 2, \ldots$) are neither tangent between themselves, nor tangent to the edges of the integration domain in α. To prove the integrability of these distributions, in Sect. 3 we introduce a regulator $a>0$ and we define the functions $\theta_a^{(n)}$ which tend toward $\theta^{(n)}$ when $a\to 0$. Then, we prove the absolute convergence of the integrals at $a>0$ and finally we introduce test functions to show that this limit is the result of integrating the integrand at $a=0$ in the sense of distribution.

In the strip $D^- < \mathrm{Re}\, D < D^*$, the integral representation (1.8) for the regularized function $I_G(p, m, D)$ develops singularities at $D=D^*$, because of divergences when some $\alpha'_s\to 0$. On the other hand, in the strip $D^* < \mathrm{Re}\, D < D^+$, the subtraction operator R in (1.8) in such that no divergences appear at D^* when some $\alpha'_s\to 0$, but the divergences appear when some $\alpha'_s\to\infty$ because the R operator also subtracts the mass term. The R' operator in (2.8) generates for each forest a product of θ distributions (amongst other distributions) which organize themselves in such a way that subtractions are present when $\alpha'_s\to 0$ and absent when $\alpha'_s\to\infty$, so that the amplitude remains finite at $D=D^*$.

When derivative couplings or spinors are present or when we have nested quadratic divergences, we generate in $S_G(p_i, \alpha, D, \omega)$ a polynomial in D of the type

$$\sum\left\{f(p_i,\alpha)\prod\left[D+\sum_{\mathscr{S}}\omega_{\mathscr{S}}\right]\right\};$$

this formulation tells what subgraphs are responsible for what power of D. The terms in D^q (without ω_s') generate only θ distributions. As mentioned above, these products of distributions are sufficient to make the α-integrals convergent at $D=D^*$ in a way which is implementable by counterterms in a Lagrangian; this finite part would violate field equations and Ward identities and another finite renormalization has to be performed to restore them. The corresponding finite counterterms are responsible for the $\delta^{(n)}$ distributions.

It is known that dimensional renormalization depends on a mass scale; this dependence is implicit in our renormalization operator R', since in the Schwinger representation, the α_s' have the dimension of the inverse of the square of a mass, and the subtractions are performed by functions $\theta^{(n)}[-\frac{1}{2}\operatorname{Ln}Q_{\mathscr{S}}(\alpha)]$.

In the massless case, the R' dimensionally renormalized amplitude exists at D^* provided that the dimensionally regularized amplitude exists in a neightborhood of D^*. This is known to be the case for strictly renormalizable field theories at D^*, at non-exceptional momentum and when all masses are nul [7, 15].

3. Construction of the Subtraction Operator R' and Convergence of the Renormalized Integral $I_G^{R'}(D)$

This section is devoted to the proofs of the assertions of Sect. 2. As explained in Sect. 2. to separate the problems of convergence and those of the distributions in the integrand, we regularize the $\theta^{(n)}$ distributions by introducing C^∞ functions $\theta_a^{(n)}$ given by (3.26). Then we define the regularized operator R_a' in a similar way to R' by a forest formula

$$R_a' = \left[1 + \sum_{\mathscr{F}} \prod_{\mathscr{S}\in\mathscr{F}} (-\tau_{a,\mathscr{S}}')\right]_\Delta , \tag{3.1}$$

where the generalized Taylor operator $\tau_{a,\mathscr{S}}'$ is defined from $\tau_{\mathscr{S}}'$ by regularizing the $\theta^{(n)}$ distribution. Obviously, when $a=0$, we recover the R' operator. The Taylor operator $\tau_{a,\mathscr{S}}'$ may be written

$$\tau_{a,\mathscr{S}}' = \tau_{\mathscr{S}}^- + U_{a,\mathscr{S}} = \tau_{\mathscr{S}}^+ + V_{a,\mathscr{S}} . \tag{3.2}$$

This section is then divided into three parts. In parts A and B we study the operator R_a' for $a \neq 0$. In part A we prove that the operator R_a', acting upon a regularized integrand, divides it into a sum of terms which, after integration, will determine the counterterms, according to BPH recurrence. In part B we prove the absolute convergence of the renormalized integrals

$$I_G^{R_a'}(p, m, D) = \int_0^\infty \prod d\alpha\, R_a' Y_G(p, m, \alpha, D, \omega) \tag{3.3}$$

in a neighbourhood of D^*, for any $a>0$. Simultaneously, we prove that the corresponding counterterms are given by the extraction of the poles of I_G at D^* via the modified Cauchy integral (3.18). Finally in part C we study the limit $a \to 0$. First we prove that the integral $I_G^{R_a}$ tends toward a limit, which is the dimensionally renormalized integral. Then, as explained in Sect. 2, the subtracted integrand

$R'Y_G = \lim_{a\to 0_+} R'_a Y_G$ appears as a sum of products of distributions in α space. We give a sense to this object as a distribution. Simultaneously we prove that the "integral of this distribution" $I_G^{R'}$ is perfectly meaningful and corresponds to the dimensionally renormalized integral.

A. The Counterterm Structure of the Operator R'_a. In this section we use the notations of Sect. 2. The counterterm structure will be proved if in each strip B^- and B^+, R'_a acts upon the regularized integrand $Y_G(p, m, \alpha, D, \omega)$ and gives the following characteristic decomposition

$$R'_a Y_G(p, m, \alpha, D, \omega) = \sum_{\{\mathscr{S},\chi\}} \left[\prod_{\mathscr{S}} C_{\mathscr{S}}^{\chi\pm}(\alpha, D)\right] R^{\pm} Y_{[G/U\mathscr{S}]_\chi}(p, m, \alpha, D), \tag{3.4}$$

where the sum runs over all (eventually empty) families $\{\mathscr{S},\chi\}$ of connected, one particle irreducible, disjoint subgraphs $\mathscr{S}$ which have a pole at D^* and over the families χ of $\omega_{D^*}(\mathscr{S})$ derivatives relative to momenta on external legs of $\mathscr{S}$ and to internal masses of $\mathscr{S}$ ($\omega_{D^*}(\mathscr{S})$ is the superficial degree of divergence of $\mathscr{S}$ at D^*). Here $R^{\pm} Y_{[G/U\mathscr{S}]_\chi}$ are the dimensionally regularized integrands of the reduced graph $[G/U\mathscr{S}]_\chi$, and are defined respectively in the strips B^+ and B^-. $C_{\mathscr{S}}^{\chi\pm}(\alpha, D)$ are functions of the α'_s relative to $\mathscr{S}$, and are defined respectively in the strip B^+ or B^-.

We prove this result in the strip B^-.

Theorem 1.

$$R'_a Y_G(p, m, \alpha, D, \omega) = \sum_{\{\mathscr{S},\chi\}} \prod_{\mathscr{S}} [\bar{R}'^{(-)}_a Y_{\mathscr{S}}^{\chi}(\alpha, D, \omega)] R^- Y_{[G/U\mathscr{S}]_\chi}(p, m, \alpha, D), \tag{3.5}$$

where the $\bar{R}'^{(-)}_a$ operator is given by a sum over all forests in $\mathscr{S}$ which do not contain the graph $\mathscr{S}$ itself,

$$\bar{R}'^{(-)}_a = -U_{a,\mathscr{S}}\left[1 + \sum_{\mathscr{F} \not\ni \mathscr{S}} \prod_{\mathscr{S}' \in \mathscr{F}} (-\tau'_{a,\mathscr{S}'})\right]_{\Delta}. \tag{3.6}$$

Proof. To prove this result, let us look at the difference between the two operators R'_a and R^-. In the proof, we shall forget the dependence on a, since we only look at algebraic rules. We have

$$R' - R^- = \sum_{\mathscr{F} \neq 0} \left[\prod_{\mathscr{S}\in\mathscr{F}} (-\tau'_{\mathscr{S}}) - \prod_{\mathscr{S}\in\mathscr{F}} (-\tau^-_{\mathscr{S}})\right]_{\Delta}. \tag{3.7}$$

For any given forest $\mathscr{F}$, we have the following identity

$$\prod_{\mathscr{S}\in\mathscr{F}} (-\tau'_{\mathscr{S}}) - \prod_{\mathscr{S}\in\mathscr{F}} (-\tau^-_{\mathscr{S}}) = \sum_{\{\mathscr{S}_i\}} \prod_{\mathscr{S} > \{\mathscr{S}_i\}} (-\tau^-_{\mathscr{S}}) \cdot \prod_i \left[(\tau^-_{\mathscr{S}_i} - \tau'_{\mathscr{S}_i}) \cdot \prod_{\mathscr{S} \subsetneq \mathscr{S}_i} (-\tau'_{\mathscr{S}})\right], \tag{3.8}$$

where the sum runs over all non-empty families $\{\mathscr{S}_i\}$ of disjoint elements of $\mathscr{F}$ (each $\mathscr{S}_i$ giving a pole at D^*). $\mathscr{S} > \{\mathscr{S}_i\}$ means that the graph $\mathscr{S}$ of $\mathscr{F}$ is either disjoint or contains some $\mathscr{S}_i$.

Let us now apply the operator (3.8) on the integrand $Y_G(D, \omega)$. (For simplicity of notation, we omit the dependence in p and m.) In (3.8) we may take the dimensions $\omega_{\mathscr{S}}$ equal to zero before applying the Taylor operators, if $\mathscr{S}$ does not belong to the involved forest $\mathscr{F}$. Since $(\tau^-_{\mathscr{S}_i} - \tau'_{\mathscr{S}_i})$ is equal to $(-U_{\mathscr{S}_i})$, we first apply the result of Appendix B, giving the action of the operator $(\tau^+_{\mathscr{S}_i} - \tau^-_{\mathscr{S}_i})$ on $Y_G(D, \omega)$. We have from (B.10)

$$(\tau^+_{\mathscr{S}_i} - \tau^-_{\mathscr{S}_i}) Y_G(D, \omega) = \sum_{\chi} Y_{[G\,\mathscr{S}_i]_\chi}(D, \omega) Y^\chi_{\mathscr{S}_i}(D + \sum_{\mathscr{S} \supsetneq \mathscr{S}_i} \omega_{\mathscr{S}}, \omega), \tag{3.9}$$

where the sum runs over all families of derivatives as in (3.4).

Noting by [ ] the operator (3.8), we obtain

$$[\,] Y_G(D, \omega) = \sum_{\{\mathscr{S}_i, \chi_i\} \mathscr{S} > \{\mathscr{S}_i\}} \prod (-\tau^-_{\mathscr{S}})$$

$$\prod_i \cdot \left[(-U_{\mathscr{S}_i}) \cdot \prod_{\mathscr{S} \subsetneq \mathscr{S}_i} (-\tau'_{\mathscr{S}})\right] \left\{\prod_i Y^{\chi_i}_{\mathscr{S}_i} \cdot Y_{[G\,\cup\mathscr{S}_i]_{\chi_i}}\right\}. \tag{3.10}$$

As in [12, 16], any $Y^{\chi_i}_{\mathscr{S}_i}$ is a homogeneous function of α of such degree that it may pass through the $\tau^-_{\mathscr{S}}$ operator by simply modifying its degree. We obtain

$$[\,] Y_G(D, \omega) = \sum_{\{\mathscr{S}_i, \chi_i\} \mathscr{S} > \{\mathscr{S}_i\}} \prod [(-\tau^-_{[\mathscr{S}\,\cup\mathscr{S}_i]}) Y_{[G\,\cup\mathscr{S}_i]_{\chi_i}}(D, \omega)]$$

$$\cdot \prod_{\mathscr{S}_i} \left\{(-U_{\mathscr{S}_i}) \prod_{\mathscr{S} \subsetneq \mathscr{S}_i} (-\tau'_{\mathscr{S}}) \left[Y^{\chi_i}_{\mathscr{S}_i}\left(D + \sum_{\mathscr{S} \supsetneq \mathscr{S}_i} \omega_{\mathscr{S}}, \omega\right)\right]\right\}. \tag{3.11}$$

Then, to obtain the action of $R' - R^-$ on Y_G, we have to sum (3.11) over all non-empty forests $\mathscr{F}$ and to perform the operation $\omega_{\mathscr{S}} \to \frac{\partial}{\partial x_{\mathscr{S}}}|_{x_{\mathscr{S}}=0}$. Reorganizing this sum as a sum over all non-empty families of disjoint divergent subgraphs and of corresponding derivatives $\{\mathscr{S}, \chi\}$, it is easy to obtain the identity (3.5). This ends the proof of Theorem 1.

We have a similar result in the strip B^+, whose proof can be performed exactly in the same way and where in (3.5–6), we change $\bar{R}'^{(-)}_a$, R^- and $U_{a_{\mathscr{S}}}$ respectively into $\bar{R}'^{(+)}_a$, R^+ and $V_{a_{\mathscr{S}}}$. The operators $\bar{R}'^{(+)}_a$ and $\bar{R}'^{(-)}_a$ differ from R'_a only by the last operator $U_{a_{\mathscr{S}}}$ and $V_{a_{\mathscr{S}}}$ relative to the entire graph; these operators do not subtract, but on the contrary retain the divergent part at D^* due to the graph $\mathscr{S}$.

We thus have proved the identity (3.4). The functions $C^{\chi\,\pm}_{\mathscr{S}}(\alpha, D)$, which are expected to give an integral representation of the counterterms in the strips B^+ and B^- respectively, are given by

$$C^{\chi\,\pm}_{\mathscr{S}}(\alpha, D) = \bar{R}'^{(\pm)}_a Y^\chi_{\mathscr{S}}(\alpha, D, \omega) \tag{3.12}$$

Before going to part (*B*), we prove the following result, which will be useful in part (*B*).

Theorem 2. *Let G be a divergent graph at D*. If we consider the right hand side of* (3.4), *where the sum is restricted to the families* $\{\mathscr{S}, \chi\}$ *(eventually empty) such that the subgraphs* $\mathscr{S}$ *are strictly contained in G, we have the following identity in both strips* B^+ *and* B^- *:*

$$\left[(1 - \tau^\pm_G)\left(1 + \sum_{\mathscr{F} \not\ni G} \prod_{\mathscr{S} \in \mathscr{F}} (-\tau'_{a_{\mathscr{S}}})\right)\right]_J Y_G(D, \omega)$$

$$= \sum_{\substack{\{\mathscr{S}, \chi\} \\ \mathscr{S} \neq G}} \left[\prod_{\mathscr{S}} \bar{R}'^{(\pm)}_a Y^\chi_{\mathscr{S}}(D, \omega)\right] R^\mp Y_{[G\,\cup\mathscr{S}]_\chi}(D) \tag{3.13}$$

In the left hand side we sum over all non-empty forests $\mathcal{F}$ which do not contain the graph G.

The proof is similar to the proof of Theorem 1. We apply the technique of Eq. (3.7) to $\left[1+\sum_{\mathcal{F}/G}\prod_{\mathcal{S}\in\mathcal{F}}(-\tau'_{a_{\mathcal{S}}})\right]$ so that the subgraphs $\mathcal{S}_i$ in (3.8) are all different from G. Multiplying by $(1-\tau_G^{\pm})$ we obtain (3.13).

B. Extraction of Poles and Convergence of Subtracted Integrals. We now consider the problems of convergence. We want to prove that the subtracted integrand $R'_a Y_G(\alpha, D, \omega)$ is absolutely integrable for $D^- < \operatorname{Re} D < D^+$ and corresponds to the extraction of the poles at D^* via BPH recurrence. Let us recall that, if the corresponding counterterms $C^{\chi}_{a_{\mathcal{S}}}(D)$ are known for any divergent subgraph $\mathcal{S}$ in G, the counterterm of the graph G itself is given by extracting the singular part at D^* of the function A_{a_G}, which is defined by

$$A_{a_G}(p,m,D)=I_G(p,m,D)+\sum_{\substack{\{\mathcal{S},\chi\}\\ \mathcal{S}\neq G}}\prod_{\mathcal{S}} C^{\chi}_{a_{\mathcal{S}}}(D)\cdot I_{(G/\cup\mathcal{S})_\chi}(p,m,D), \tag{3.14}$$

where the sum runs over all non-empty families of divergent subgraphs $\mathcal{S}$ of G different from G, as in Theorem 2.

We shall prove that the counterterms $C^{\chi}_{a_{\mathcal{S}}}(D)$ are meromorphic functions of D, whith a pole at D^*, and are given in the strips B^+ and B^- respectively by the following convergent integrals:

$$C^{\chi}_{a_{\mathcal{S}}}(D)=\int_0^\infty \prod d\alpha_a\{\bar{R}'^{(-)}_a Y^{\chi}_{\mathcal{S}}(\alpha,D)\} \quad \text{if} \quad D\in B^- \tag{3.15a}$$

$$=\int_0^\infty \prod d\alpha_a\{\bar{R}'^{(+)}_a Y^{\chi}_{\mathcal{S}}(\alpha,D)\} \quad \text{if} \quad D\in B^+ . \tag{3.15b}$$

This result will allow us to integrate over the α's the identity (3.4) of Theorem 1 in the strips B^- and B^+. We shall then obtain the counterterm expression of the subtracted integral

$$I^{R_a}_G(p,m,D)=I_G(p,m,D)+\sum_{(\mathcal{S},\chi)}\prod_{\mathcal{S}} C^{\chi}_{a_{\mathcal{S}}}(D) I_{[G/\cup\mathcal{S}]_\chi}(p,m,D), \tag{3.16.a}$$

where the sum $(\mathcal{S},\chi)$ contains $\mathcal{S}=G$, so that from (3.14)

$$I^{R_a}_G(p,m,D)=A_{a_G}(p,m,D)+\sum_{\chi} C^{\chi}_{a_G}(D) I_{[G/G]_\chi}(p,m). \tag{3.16b}$$

In (3.16b), we see explicitly how the pole at D^*, corresponding to the entire graph G, cancels.

Let us now set the following theorem:

Theorem 3. *For any $a>0$, the integral*

$$I^{R_a}_G(p,m,D)=\int_0^\infty \prod d\alpha R'_a\cdot Y_G(p,m,\alpha,D,\omega) \tag{3.17}$$

is absolutely convergent for any D such that $D^- < \operatorname{Re} D < D^+$.

The corresponding counterterms have poles at D^* and are given in the strips B^- and B^+ by the convergent integral representations (3.15a and b). Moreover, this subtraction operator corresponds to the extraction of the singular part of A_{aG} at D^* (defined by 3.16) via the Cauchy integral

$$I_G^{R'_a}(p,m,D)=\oint_c \frac{dz}{2i\pi}\,\frac{A_{aG}(p,m,z)e^{a(z-D)^2}}{z-D}, \tag{3.18}$$

where c is a complex contour containing the poles of $A_{aG}/z-D$ at $z=D$ and at $z=D^*$ (see Fig. 5).

Proof. As explained before, the function $e^{a(z-D)^2}$ in (3.18) is introduced to control the convergence of the integral when $|\mathrm{Im}\,z|\to+\infty$. To treat in a correct way the question of absolute convergence, we have to use the L_1 norm on α integrals. Let us note

$$\|I_G(D)\|_1=\int\prod d\alpha|R^-Y_G(D)| \quad \text{if} \quad D\in B^-, \tag{3.19a}$$

$$=\int\prod d\alpha|R^+Y_G(D)| \quad \text{if} \quad D\in B^+. \tag{3.19b}$$

Since in B^+ and B^- the number of subtractions is different, $\|I_G\|_1$ is analytic in B^+ and B^-, but not defined on the line $\mathrm{Re}\,D=D^*$.

Similarly, we define $\|C^{\chi}_{a_\mathscr{S}}(D)\|_1$ and $\|I_G^{R'_a}(p,m,D)\|_1$ in B^+ and B^- from the integral representations (3.15) and (3.17) (up to now, they are not proved to be finite). We now perform the BPH recursion on the number of loops $L(G)$ of a graph G to prove the theorem. The recursion hypothesis will be the following:

a) for any graph $\mathscr{S}$ such as $L(\mathscr{S})<L(G)$, Theorem 3 is satisfied.

b) Moreover, for any $\mathscr{S}$ divergent at D^* such as $L(\mathscr{S})<L(G)$, the function $\|C^{\chi}_{a_\mathscr{S}}(D)\|_1$ (which is finite in B^+ and B^- from hypothesis a) is polynomially bounded in B^+ and B^- as $|\mathrm{Im}\,D|\to+\infty$ for $\mathrm{Re}\,D$ fixed (of course for any $a>0$).

If $L(G)=0$, the hypothesis is trivially satisfied, since $R'_a\equiv 1$. Let us prove the recursion hypothesis in the next order. According to a), the function $A_{aG}(p,m,D)$ defined by (3.14) is given by the convergent integral representation in B^+ and B^- respectively

$$A_{aG}(D)=\int_0^\infty\prod_a d\alpha_a\sum_{\substack{\{\mathscr{S},\chi\}\\ \mathscr{S}\neq G}}\Big[\prod_{\mathscr{S}}\bar R_a'^{(\pm)}Y^\chi_\mathscr{S}(D,\omega)\Big]R^\pm Y_{[G\cup\mathscr{S}]_\chi}(D). \tag{3.20}$$

Using Theorem 2, this integral representation may be written:

$$A_{aG}(D)=\int_0^\infty\prod_a d\alpha_a[1-\tau_G^\pm]\Big[1+\sum_{\mathscr{F}\neq G}\prod_{\mathscr{S}\in G}(-\tau'_{a_\mathscr{S}})\Big]_\Delta Y_G(D,\omega). \tag{3.21}$$

We now perform the Cauchy integral (3.18) in order to remove from A_{aG} its singular part at D^*. We have to take for C a contour around the two poles at D and D^*; this is always possible since $D^-<\mathrm{Re}\,D<D^+$.

We know, from Appendix D, that any $\|I_{[G\,\mathscr{S}]_\chi}(D)\|_1$ is polynomially bounded as $|\mathrm{Im}\,D|\to+\infty$. This result and part b) of the recurrence hypothesis show that $\|A_{aG}(p,m,D)\|_1$ is polynomially bounded in B^+ and B^- as $|\mathrm{Im}\,D|\to+\infty$ ($\mathrm{Re}\,D$ being fixed). So, $\|A_{aG}(p,m,z)\|_1|e^{a(z-D)^2}|$ is exponentially decreasing as $|\mathrm{Im}\,z|\to+\infty$.

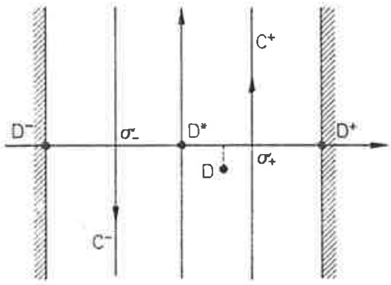

Fig. 5. Integration contour C_+ and C_-, defined in (3.22–23), for the integral (3.18) in the complex z plane

We may take for C (see Fig. 5) the two lines C_+ and C_- defined respectively by

$$C_+ = \{z = \sigma_+ + iy\} \quad \text{such that} \quad \mathrm{Sup}(D^*, \mathrm{Re}(D)) < \sigma_+ < D^+ , \tag{3.22}$$

$$C_- = \{z = \sigma_- - iy\} \quad \text{such that} \quad D^- < \sigma_- < \mathrm{Inf}(D^*, \mathrm{Re}\, D) . \tag{3.23}$$

The contour does not cross the line $\mathrm{Re}\, z = D^*$, so we may apply Fubini's theorem and invert the integrations in z and in α. Since the α-integrand of (3.21) on C^+ differs from the integrand on C^- by the subtraction operator $\tau_G^+ - \tau_G^-$, we obtain:

$$I_G^{R'_a}(p, m, D) = \int_0^\infty \prod_a d\alpha_a \left\{ \left[(1 - \tau_G^-)|_{z=D} - \frac{1}{2i\pi} \int_{C_+} \frac{dz}{z-D} e^{a(z-D)^2} (\tau_G^+ - \tau_G^-) \right] \right.$$
$$\left. \cdot \left[1 + \sum_{\mathscr{F} \not\ni G} \prod_{\mathscr{S} \in \mathscr{F}} (-\tau'_{\mathscr{S}}) \right]_\Delta Y_G(p, m, \alpha, z, \omega) \right\} . \tag{3.24}$$

We note that the term $(\tau_G^+ - \tau_G^-) \left[1 + \sum_{\mathscr{F} \not\ni G} \prod_{\mathscr{S} \in \mathscr{F}} (-\tau'_{\mathscr{S}}) \right]_\Delta Y_G(p, m, \alpha, z, \omega)$ is a function of z which is a sum, relative to the forests $\mathscr{F}$, of terms of the form: $N(z) Q^{-z/2}$, where $N(z)$ is a polynomial of z. The Q's are products of $P(\alpha)$ polynomials relative to reduced graphs $[\mathscr{S}]_{\mathscr{F}}$ of the forest. Using the relation

$$\frac{1}{2i\pi} \int_{C_+} \frac{dz}{z-D} \frac{e^{a(z-D)^2} N(z)}{Q^{z/2}} = \frac{N\left(D + \frac{\partial}{\partial x}\right)}{Q^{D/2}} \theta_a(-\tfrac{1}{2} \mathrm{Ln}\, Q + x)|_{x=0} , \tag{3.25}$$

where θ_a is the convolution product of the Heaviside function with a gaussian

$$\theta_a(x) = \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{4a}} * \theta(x) = \sqrt{\frac{\pi}{a}} \int_{-\infty}^{+\infty} dy\, e^{-\frac{y^2}{4a}} \cdot \theta(x-y) , \tag{3.26}$$

we deduce that the integrand in (3.24) is $R'_a Y_G(p, m, \alpha, D, \omega)$.

From Fubini's Theorem and (3.18), we have proved that the integral (3.17) is absolutely convergent. Then, subtracting $A_{a_G}(D)$ from $I_G^{R'_a}(D)$, we obtain for the counterterms relative to G the convergent integral representations (3.15a) and (3.15b) in the strips B^+ and B^-. Theorem 3 is proved for the graph G, that is part a) of the recursion hypothesis is achieved.

To verify part b) of the recursion, let us come back to the Cauchy integral (3.18). We denote by $B^{\pm}_{aG}(p,m,\alpha,D)$ the integrand of the convergent integral representation (3.21) of A_{aG}. The L_1 norm of $I^{R'_a}_G$ is given, inverting z and α integrations in (3.18), by

$$\|I^{R'_a}_G\|_1 = \frac{1}{2\pi}\int_0^\infty d\alpha \left|\int_C \frac{dz}{z-D}\, e^{a(z-D)^2} B^{\pm}_{aG}(p,m,\alpha,z)\right|, \tag{3.27}$$

where C is the union of the two lines C_+ and C_-. Denoting by ds the curvilinear absciss on C (that is to say $ds=|dz|$), we have the inequality

$$\|I^{R'_a}_G\|_1 \leqq \frac{cte}{2\pi}\int_0^\infty \prod d\alpha \int_C \frac{ds}{|z-D|}\, e^{-a|\mathrm{Im}(z-D)|^2} |B^{\pm}_{aG}(p,m,\alpha,z)|. \tag{3.28}$$

And according to Fubini's theorem

$$\|I^{R'_a}_G\|_1 \leqq \frac{cte}{2\pi}\int_C \frac{ds}{|z-D|}\, e^{-a|\mathrm{Im}(z-D)|^2} \|A_{aG}(p,m,z)\|_1. \tag{3.29}$$

$\|A_{aG}\|_1$ being by hypothesis polynomially bounded as $|\mathrm{Im}\,z|\to\pm\infty$, the convergence of (3.29) is ensured by the $\exp[-a|\mathrm{Im}(z-D)|^2]$, and $\|I^{R'_a}_G\|_1$ is obviously polynomially bounded as $|\mathrm{Im}\,D|\to\pm\infty$, as well as the counterterms relative to G, by (3.16b).

So we have proved the second part of the recursion hypothesis. This ends the proof of Theorem 3.

C. The Limit $a\to 0_+$. We now look at the limit $a\to 0$ in order to recover the counterterms corresponding to the operation R'. Two problems occur since in this limit the integrand appears as a sum of products of functions and distributions in α space:

First, the definition of such products: in Appendix C, we study the supports of the distributions $\theta^{(n)}[-\frac{1}{2}\mathrm{Ln}\,Q(\alpha)]\,(n\geqq 1)$ which appear in the renormalized integrand via Eq. (3.25). Each distribution is defined only if its support is a smooth algebraic manifold. Moreover, the supports may intersect each other and/or the limits of the integration domain. Products of the corresponding distributions are defined only if these manifolds are not "tangent." These two parts are made explicit and proved in Appendix C.

Second, the integration of $R'Y_G$ over the α-space: this means that the distribution $R'Y_G$ is applied over the test function 1 in the α space $\mathbb{R}^{\ell(G)}_+$. So it is not sufficient to define $R'Y_G$ as a distribution over the usual spaces $\mathscr{D}(\mathbb{R}^\ell)$ or $\mathscr{S}(\mathbb{R}^\ell)$ (see [17]) to give a sense to $\int_0^\infty d\alpha\, R'Y_G$.

Let us compactify $\mathbb{R}^\ell$ by imbedding it in the ℓ-dimensional sphere S_ℓ via the stereographic projection and let us take for the space of test functions the space $E=C^\infty(S_\ell)$. The following theorem proves that $R'Y_G$ is a distribution on the space E. This is sufficient for the integral $\int_0^\infty d\alpha\, R'Y_G$ to be meaningful, since E contains the function $\mathbb{1}$, and contains the usual space $\mathscr{S}(\mathbb{R}^\ell)$.

Theorem 4. *Given a graph G, for any function φ belonging to E, the limit as $a\to 0_+$ of the integral*

$$\int_0^{\prime} \prod_G d\alpha R'_a Y_G(p, m, \alpha, D, \omega)\cdot\varphi(\alpha) \tag{3.30}$$

exists for any D belonging to the strip $D^- < \mathrm{Re}\, D < D^+$; and defines a distribution on E. In particular, the renormalized amplitude $I_G^{R'}(p, m, D)$ defined by applying this distribution onto the function $\mathbb{1}$, corresponds to the usual dimensionally renormalized amplitude.

To prove the existence of the limit of (3.30), we construct recursively the integral (3.30) using a Cauchy integral generalizing (3.18), in order to control the limit $a\to 0_+$. Such a construction is the generalization of the construction performed in the two last sections but is rather lengthy and will not be given here; we refer to [18] for a complete proof. The existence of the limit (3.30) is sufficient to define a distribution in E; indeed, the space E (with the usual topology induced by the sup norm on its elements and all their partial derivatives) is a countably normed space; it follows that its dual E' is complete (see [17]), and consequently the distributions $R'_a Y_G$ converge towards a distribution in E', defined as $R' Y_G$. This ends the principle of the proof.

Appendix A. Multidimensional Regularization

In this appendix we remind and adapt to our notations the construction of the dimenionally regularized Feynman integrands of [8]. Let us first recall how to obtain the Schwinger parametric form at integer dimension N: Given a Feynman graph G with d_a derivative couplings $\{k^{\mu_1}, \ldots, k^{\mu_{d_a}}\}$ on each line a (μ are the Lorentz indices), each propagator is written as

$$\left(\prod_{i=a}^{d_a} k_a^{\mu_i}\right)\Big/(k_a^2+m_a^2)=\prod_i\left(-\frac{\partial}{\partial z_a^{\mu_i}}\right)$$
$$\cdot\left\{\int_0^\infty d\alpha_a \exp[-\alpha_a(k_a^2+m_a^2)-k_a z_a]\right\}_{z_a=0}, \tag{A.1}$$

where k_a and z_a are N dimensional vectors.

Integrating over internal momentum k_a and taking into account momentum conservation at each vertex, we obtain the usual Schwinger representation for the Feynman integral at dimension N

$$I_G(p_i, m_a)=\int_0^{\prime}\prod_{a=1}^{\ell(G)} d\alpha_a\cdot Y_G(p_i, m_a, \alpha_a), \tag{A.2}$$

where the Feynman integrand Y_G is given by

$$Y_G(p_i, m_a, \alpha_a)=\left\{\prod_{a=1}^{\ell(G)}\prod_{i=1}^{d_a}\left(-\frac{\partial}{\partial z_a^{\mu_i}}\right)Z_G(\overline{p_i}, \overline{m_a}, \alpha_a)\right\}_{z=0}, \tag{A.3a}$$

and where Z_G is given by:

$$Z_G(\overline{p_i}, \overline{m_a}, \alpha_a)=P_G(\alpha)^{-N/2}\exp\left[-\sum_a \alpha_a\overline{m_a^2}-\sum_{i,j}\overline{p_i}d_G^{-1}(\alpha)_{ij}\overline{p_j}\right]. \tag{A.3b}$$

In (A.3). $d_G(\alpha)_{ij}$ and $P_G(\alpha)$ are the Symanzik functions, characteristic of the topology of the graph: the dependence in the z_a's is reported in

$$\overline{m_a^2} = m_a^2 - z_a^2/4\alpha_a^2 ,$$
$$\overline{p_i} = p_i + \sum_a \varepsilon_{ia} z_a/2\alpha_a , \qquad \text{(A.4)}$$

where the p_i's are the external momenta incoming at a vertex i and where $[\varepsilon_{ia}]$ is the incidence matrix of the graph G and is defined as

$$\varepsilon_{ia} = \begin{cases} +1 \text{ if the oriented line } a \text{ points away from the vertex } i \\ -1 \text{ if the oriented line } a \text{ points toward the vertex } i \\ 0 \text{ if the line } a \text{ does not contain the vertex } i. \end{cases} \qquad \text{(A.5)}$$

In (A.3), the dimension N may become a complex dimension D to define Z_G as an analytic function of D (see [12] for the definition of scalar products in dimension D).

We now define a multidimensional amplitude [8] in the following way: To each subgraph $\mathcal{S}$ we associate an additional positive dimension $\omega_{\mathcal{S}}$ so that the internal momentum k_a belongs to the space $\mathbb{R}_D \bigoplus_{\mathcal{S}\ni a} \mathbb{R}_{\omega_{\mathcal{S}}}$, while the external momentum p_i belongs to the space $\mathbb{R}_D \bigoplus_{\mathcal{S}\ni i} \mathbb{R}_{\omega_{\mathcal{S}}}$. At each vertex we impose energy-momentum conservation in all space $\mathbb{R}_\omega = \mathbb{R}_D \bigoplus_{\mathcal{S}} \mathbb{R}_{\omega_{\mathcal{S}}}$

$$p_i = \sum_{a=1}^{\ell} \varepsilon_{ia} k_a . \qquad \text{(A.6)}$$

(For practical use (as in Sect. 2), the external momenta p_i may be chosen in the subspace $\mathbb{R}_D$.)

It is then possible to extend Schwinger representation (A.2–3) by introducing a vector z_a in $\mathbb{R}_D \bigoplus_{\mathcal{S}\ni a} \mathbb{R}_{\omega_{\mathcal{S}}}$ and by computing (for $\omega_{\mathcal{S}}$ integer) the integrals

$$\int \prod_{a=1}^{\ell} d^{D+\sum_{\mathcal{S}\ni a}\omega_{\mathcal{S}}} k_a \cdot \exp\left(-\sum_{a=1}^{\ell} \alpha_a\left[k_a^2 + m_a^2 + \frac{k_a \cdot z_a}{\alpha_a}\right]\right) \prod_{i=1}^{n} \delta^{D+\sum_{\mathcal{S}\ni i}\omega_{\mathcal{S}}}\left[p_i - \sum_{a=1}^{\ell} \varepsilon_{ia} k_a\right], \qquad \text{(A.7)}$$

where the scalar products in [ ] are taken in $\mathbb{R}_D \bigoplus_{\mathcal{S}\ni a} \mathbb{R}_{\omega_{\mathcal{S}}}$.

The above integrals may be factorized into contributions corresponding to each subspace $\mathbb{R}_D \oplus \mathbb{R}_{\omega_G}$ and $\mathbb{R}_{\omega_{\mathcal{S}}}, \mathcal{S} \neq G$, and (A.3) may be calculated in each of these subspaces. In each of these subspaces $\mathbb{R}_{\omega_{\mathcal{S}}}$, we integrate over the internal momenta of $\mathcal{S}$ (all internal momenta outside $\mathcal{S}$ have zero components in $\mathbb{R}_{\omega_{\mathcal{S}}}$). We obtain

$$I_G(p_i, m_a, \Omega_{\mathcal{S}}) = \int_0^\infty \prod_{a=1}^{\ell} d\alpha_a Y_G(p_i, m_a, \alpha_a, \Omega_{\mathcal{S}}), \qquad \text{(A.8a)}$$

with

$$Y_G(p_i, m_a, \alpha_a, \Omega_{\mathscr{S}}) = \prod_{a=1}^{l} \prod_{i=1}^{d_a} \left(\frac{-\partial}{\partial z_a^{\mu_i}} \right) \bar{Y}_G(\bar{p}_i, \bar{m}_a, \alpha_a, \Omega_{\mathscr{S}})|_{z_a=0}. \tag{A.8b}$$

The function $\bar{Y}_G$ is here

$$\bar{Y}_G(\bar{p}_i, \bar{m}_a, \alpha_a, \Omega_{\mathscr{S}}) = \prod_{T \subseteq G} Z_T(\bar{p}_i, \bar{m}_a, \alpha_a)|_{N=\Omega_T}, \tag{A.9}$$

where Z_T is defined in (A.3b): the quantities $\Omega_{\mathscr{S}}$ are $\omega_{\mathscr{S}}$ for $\mathscr{S} \neq G$ and Ω_G is $(D+\omega_G)\cdot \ln Z_T$, the scalar products are taken in $\mathbb{R}_{\Omega_T}$ so that

$$\begin{aligned} [\bar{m}_a^2]_G &= m_a^2 - [z_a^2]_{\Omega_G}/4\alpha_a \\ [\bar{m}_a^2]_T &= -[z_a^2]_{\Omega_T}/4\alpha_a \quad \text{for} \quad T \neq G \\ [\bar{p}_i]_T &= [p_i]_T + \sum_a \varepsilon_{ia}[z_a]_T/2\alpha_a. \end{aligned} \tag{A.10}$$

If the variables $\omega_{\mathscr{S}}$ become complex variables, the integrand in (A.8) becomes an analytic function of these variables and defines the multidimensional regularized integrand.

Let us recall that a Feynman amplitude is obtained by associating γ Dirac and internal group matrices to (A.1) and by contracting some Lorentz and internal indices, so that the dependence of (A.8) in the $\omega_{\mathscr{S}}$'s appears in two ways: – first, by the quantity $P_{\mathscr{S}}(\alpha)^{-\omega_{\mathscr{S}}/2}$, – second, by various contractions between Lorentz indices which give some $g^{\mu\nu}$, μ and ν being relative to some $\mathbb{R}_{\omega_{\mathscr{S}}}$ and $\mathbb{R}_{\omega_{\mathscr{S}'}}$. The final contraction is then,

$$\sum_\mu [g^\mu{}_\mu]_{\mathbb{R}^{\omega_{\mathscr{S}}}} = \mathrm{Tr}_{\mathbb{R}_{\omega_{\mathscr{S}}}} g = \omega_{\mathscr{S}}. \tag{A.11}$$

(We do not emphasize the problems of the γ matrices and of the γ^5 anomaly which have been extensively treated in the literature, especially in [7].)

With these algebraic rules, one defines the regularized Feynman integrand which appears to be of the form:

$$\begin{aligned} Y_G(p_i, m_a, \alpha_a, \Omega_{\mathscr{S}}) = \prod_{T \subseteq G} & P_T(\alpha)^{-\Omega_T/2} \cdot S_G(p_i, \alpha_a, \Omega_{\mathscr{S}}) \\ & \cdot \exp\left[-\sum_a \alpha_a m_a^2 - p_i d_G^{-1}(\alpha)_{ij} p_j \right], \end{aligned} \tag{A.12}$$

where $S_G(p_i, \alpha_a, \Omega_{\mathscr{S}})$ is a rational function of the variables α_a which depends only polynomially on the p_i's and on the $\Omega_{\mathscr{S}}$.

Let us consider the convergence of the integral (A.8a). The following theorem is proved in [8].

Theorem. *The integral* (A.8a) *is absolutely convergent for $Re(\Omega_{\mathscr{S}})$ sufficiently small, and defines by analytic continuation a meromorphic function of the variables $\Omega_{\mathscr{S}}$.*

We now apply to this object the results of [12], which allow us to construct an explicit convergent integral representation of the analytic continuation of (A.8a) in all variables $\Omega_{\mathscr{S}}$.

It is easy to prove that, following the definitions of [14], the integrand $Y_G(p_i, m_a, \alpha_a, \Omega_{\mathscr{S}})$ admits a Taylor expansion in every Hepp's sector for every $\{\Omega_{\mathscr{S}}\}$. So we can extend without difficulty the methods of [12] to obtain the following theorem:

Theorem. *The integral*

$$I_G(p_i, m_a, \Omega_{\mathscr{S}}) = \int_0^\infty \prod_a d\alpha_a \{R\, Y_G(p_i, m_a, \alpha_a, \Omega_{\mathscr{S}})\} \tag{A.13}$$

is absolutely convergent for $\{\mathrm{Re}(\Omega_{\mathscr{S}})\}$ *not belonging to one of the hyperplanes in* $\mathbb{R}^k$ *corresponding to a singularity of* $I_G(p_i, m_a, \Omega_{\mathscr{S}})$. *This integral defines the analytic continuation of* (A.8a) *almost everywhere.*

R is the subtraction operator defined in [14] and in Sect. 1.

Appendix B. Action of Taylor Operators upon a Multidimensional Regularized Integrand

In this appendix, we intend to extend the results of [16] and to give the expression for

$$\tau^n_{\mathscr{S}} Y_G(p_i, m_a, \alpha_a, \Omega_T), \tag{B.1}$$

where $\tau^n_{\mathscr{S}}$ is a generalized Taylor operator defined in [14] and where Y_G is defined in (A.8.9).

To obtain Y_G in (A.8), we performed the Gaussian integrations of the various components of the internal momenta k_a in each subspace $\mathbb{R}_{\Omega_T}$. We make here this integration in two steps as in [16]: first, integrate the internal momenta of the subgraph $\mathscr{S}$ (those momenta which have non-zero components in $\mathbb{R}_{\Omega_T}$, that is those corresponding to the lines of $\mathscr{S} \cap T$) and obtain a factor

$$Z_{T\cap\mathscr{S}}\left(\overline{p_i} - \sum_{a\in\frac{T}{T\cap\mathscr{S}}} \varepsilon_{ia}k_a, \overline{m_a}, \alpha_a, \Omega_T\right) \delta^{\Omega_T}\left[\sum_{i\in T\cap\mathscr{S}}\left(p_i - \sum_{a\in\frac{T}{T\cap\mathscr{S}}} \varepsilon_{ia}k_a\right)\right], \tag{B.2}$$

where scalar products are taken in $\mathbb{R}_{\Omega_T}$ and where, in fact, we should write a product of δ distributions over each connected component of $T\cap\mathscr{S}$. In (B.2), we denote by $\overline{p_i}$ the components in $\mathbb{R}_{\Omega_T}$ of $\left[p_i + \sum_{a\in S\cap T} \varepsilon_{ia}z_a/2\alpha_a\right]$. Then, we replace k_a for $a\in\frac{T}{T\cap S}$ by $v\left(-\frac{\partial}{\partial z_a}\right)$ and we integrate over the remaining momenta of T. The result of this procedure is that, for any subgraph $\mathscr{S}$, we have

$$\begin{aligned}\bar{Y}_G(\overline{p_i}, \overline{m_a}, \alpha_a, \Omega_T) &= \sum_{T\subseteq G} Z_{T\cap\mathscr{S}}\left[\overline{p_i} + \sum_{a\in\frac{T}{T\cap\mathscr{S}}} \varepsilon_{ia}\frac{\partial}{\partial z_a}, \overline{m_a}, \alpha_a, \Omega_T\right] Z_{\frac{T}{T\cap\mathscr{S}}}(\overline{p_i}, \overline{m_a}, \alpha_a, \Omega_T).\end{aligned} \tag{B.3}$$

We may now group all functions $Z_{T\cap\mathscr{S}}$ with the same subgraph $(T\cap\mathscr{S})$ on one side, and all functions $Z_{T\,T\cap\mathscr{S}}$ with the same subgraph $\left(\frac{T}{T\cap\mathscr{S}}\right)$ on the other side. Now if we take the product over all possible subgraphs T, we obtain

$$\bar{Y}_G(\overline{p_i}, \overline{m_a}, \alpha_a, \Omega_T) = \bar{Y}_{\mathscr{S}}\left(\overline{p_i} + \sum_{a\in\frac{G}{\mathscr{S}}} \varepsilon_{ia}\frac{\partial}{\partial z_a}, \overline{m_a}, \alpha_a, \Omega'_T\right)\bar{Y}_{G\,\mathscr{S}}(\overline{p_i}, \overline{m_a}, \alpha_a, \Omega''_T), \quad \text{(B.4)}$$

where

$$\Omega'_T = \sum_{\substack{V \text{ such as}\\ V\cap\mathscr{S}=T}} \Omega_V, \quad \text{(B.5a)}$$

$$\Omega''_T = \sum_{\substack{V \text{ such as}\\ V\,\mathscr{S}=T}} \Omega_V. \quad \text{(B.5b)}$$

Now, $\overline{Y_G}$ is factorized into one function of α_a for $a\in\mathscr{S}$ and one function of α_a for $a\in\frac{G}{\mathscr{S}}$. It is then possible to apply the operator $\tau^n_{\mathscr{S}}$; in Sects. 2 and 3, all dimensions ω_V are small and positive in such a way that the number of subtractions generated by $\tau^n_{\mathscr{S}}$ depends only on n and on the dimension D.

First, let us take care of the coupling derivatives on the graph G by applying the derivatives $\left(-\frac{\partial}{\partial z}\right)$ in each subspace $\mathbb{R}_{\Omega_T}$. We use for any subgraph Σ, the property

$$\prod_{a\in\Sigma}\left(-\frac{\partial}{\partial z_a}\right) Z_\Sigma[\overline{p_i}, \overline{m_a}, \alpha_a, \Omega]|_{z=0} = S_\Sigma(p_i, \alpha_a, \Omega) Z_\Sigma[p_i, m_a, \alpha_a, \Omega], \quad \text{(B.6)}$$

where S_Σ is a polynomial in p_i and Ω; each monomial of S_Σ satisfies the homogeneity relation

$$h(p) - 2h(\alpha) = h, \quad \text{(B.7)}$$

where $h(p)$ and $h(\alpha)$ are respectively the degree of homogeneity in the external momentum p_i and in all the variables α, and h is the number of coupling derivatives.

To apply the operator τ^n_Σ over $S_\Sigma Z_\Sigma$, we dilate by ϱ^2 all α'_S corresponding to the lines of Σ, and we apply τ^n_ϱ; since S_Σ and P_Σ in Z_Σ are homogeneous in ϱ^2, we have to apply $T^{n-2h(\alpha)+LD^*}_\varrho$ over

$$\exp\left\{-\left[\sum_a \alpha_a m_a^2 - \sum p_i d_{ij}^{-1} p_j\right]\varrho^2\right\},$$

where m_a and p_i are the internal masses and the external momenta of the graph Σ. A Taylor expansion in ϱ^2 of this exponential is also a Taylor expansion in m_a and p_i. Taking into account the polynomials of p_i in S_Σ we may write

$$\tau^n_\Sigma\{S_\Sigma Z_\Sigma\} = \sum_{k=0}^{n+2\ell(\Sigma)+\omega(\Sigma)} \frac{1}{k!}\chi_k\cdot\left[\frac{\partial}{\partial\chi_k}\{S_\Sigma Z_\Sigma\}\right]_{p_i=m_a=0}, \quad \text{(B.8)}$$

where $\ell(\Sigma)$ is the number of lines of Σ, $\omega(\Sigma)$ is the superficial degree of divergence of Σ at $D=D^*$, χ_k is a subset of k internal masses and external momenta of Σ and $\frac{\partial}{\partial \chi_k}$ is the k^{th} derivative in regards to the variables in χ_k; in (B.8) summation over all χ'_s is understood (as well as summation over Lorentz indices).

It is clear that the above homogeneity properties also hold over all functions Z_T for $T \subset \Sigma$ and with Ω'_T small enough. We denote by

$$Y_{\mathscr{S}}^{\chi_k}(\alpha, \Omega'_T) = \frac{1}{k!} \frac{\partial}{\partial \chi_k} Y_{\mathscr{S}}(p_i, m_a, \alpha, \Omega'_T)|_{p_i = m_a = 0} \tag{B.9}$$

and by $[G/\mathscr{S}]_{\chi_k}$ the reduced graph obtained by shrinking into a point the subgraph $\mathscr{S}$ and by attaching to the reduced vertex the masses of χ_k and the momenta of χ_k which are external momenta p_i of $\mathscr{S}$ and internal coupling derivatives of $[G/\mathscr{S}]$ generated by $\varepsilon_{ia} \frac{\partial}{\partial z_a}$ in (B.4).

The final result used in (3.9) can be read

$$\tau''_{\mathscr{S}} Y_G(p_i, m_a, \alpha_a, \Omega_T) = \sum_{k=0}^{n+2\ell(\mathscr{S})+\omega(\mathscr{S})} Y_{\mathscr{S}}^{\chi_k}(\alpha, \Omega'_T) Y_{[G/\mathscr{S}]_{\chi_k}}(p_i, m_a, \alpha, \Omega''_T). \tag{B.10}$$

Appendix C

We study here the properties of the distribution $\theta^{(n)}$ which appears in the dimensionally renormalized integrands. For simplicity we shall use a vectorial notation in α space: an element $\alpha = \{\alpha_a, a=1,\ell\}$ in $\mathbb{R}_+^\ell$ shall be noted $\boldsymbol{\alpha}$. If $f(\alpha)$ is a differentiable function of the α's, we shall denote the vector

$$\left\{\frac{\partial f}{\partial \alpha_a}, a=1,\ell\right\} \quad \text{by} \quad \frac{\partial f}{\partial \boldsymbol{\alpha}}.$$

We have seen in Sect. 2 that we have to define products of distributions of the form

$$\theta^{(n)}[-\tfrac{1}{2}\ln[Q_{\mathscr{S},\mathscr{F}}(\boldsymbol{\alpha})]] \quad n \geqq 0 \tag{C.1}$$

where $\mathscr{F}$ is a forest of subgraphs of G (excluding tree graphs which are never divergent), $\mathscr{S}$ a graph of $\mathscr{F}$ and where $Q_{\mathscr{S},\mathscr{F}}$ is a polynomial in α, defined as

$$Q_{\mathscr{S},\mathscr{F}} = \prod_{\substack{\mathscr{S}' \in \mathscr{F} \\ \mathscr{S}' \subset \mathscr{S}}} P_{[\mathscr{S}']_{\mathscr{F}}}. \tag{C.2}$$

In (C.2), $P_{[\mathscr{S}']_{\mathscr{F}}}$ is the Symanzik polynomial of the graph $[\mathscr{S}']_{\mathscr{F}}$ obtained by reducing to points in $\mathscr{S}'$ every graph $\mathscr{S}''$ strictly contained in $\mathscr{S}'$.

We first look at the existence of the distribution (C.1). This distribution is singular on the algebraic manifold $V_{\mathscr{S},\mathscr{F}}$ defined by

$$V_{\mathscr{S},\mathscr{F}} = \{\boldsymbol{\alpha} : Q_{\mathscr{S},\mathscr{F}}(\alpha) = 1\}. \tag{C.3}$$

According to [17], this distribution is defined if, on every point of $V_{\mathscr{S},\mathscr{F}}$ the gradient

$$\boldsymbol{\Delta}_{\mathscr{S},\mathscr{F}}=\frac{\partial}{\partial\boldsymbol{\alpha}}\mathrm{Ln}[Q_{\mathscr{S},\mathscr{F}}(\alpha)] \tag{C.4}$$

is not zero (this means that $V_{\mathscr{S},\mathscr{F}}$ has no singular points). This is implied by the following theorem:

Theorem C. 1. *On every point of $V_{\mathscr{S},\mathscr{F}}$, the vector $\boldsymbol{\Delta}_{\mathscr{S},\mathscr{F}}$ is different from zero.*

Proof. From (C.2), $Q_{\mathscr{S},\mathscr{F}}$ is an homogeneous polynomial of degree $L(\mathscr{S})$, where $L(\mathscr{S})$ is the number of loops of $\mathscr{S}$ (which is non-zero since $\mathscr{S}$ is not a tree graph). Applying the Euler relation to Q we obtain, if $\alpha\in V_{\mathscr{S},\mathscr{F}}$

$$L(\mathscr{S})=\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\boldsymbol{\alpha}}Q_{\mathscr{S},\mathscr{F}}=\boldsymbol{\alpha}\cdot\boldsymbol{\Delta}_{\mathscr{S},\mathscr{F}}\,. \tag{C.5}$$

This ensures that $\boldsymbol{\Delta}_{\mathscr{S},\mathscr{F}}$ is never null.

We now look at the product of such distributions. From Sect. 2, the integrand in $\mathbb{R}^{\ell}$ appears to be a sum of terms of the type

$$\prod_{\mathscr{S}\in\mathscr{F}}\theta^{(n_{\mathscr{S}})}[-\tfrac{1}{2}\mathrm{Ln}\,Q_{\mathscr{S},\mathscr{F}}(\boldsymbol{\alpha})]\cdot\prod_{a=1}^{\ell}\theta(\alpha_a)\,, \tag{C.6}$$

where $\mathscr{F}$ is a forest of subgraphs $\mathscr{S}(L(\mathscr{S})\neq 0)$ and the distributions $\theta(\alpha_a)$ are introduced to take into account the integration over the positive α_a's only. Such a product is defined if the various manifolds $V_{\mathscr{S},\mathscr{F}}$ and the hyperplanes P_a defined by

$$P_a=\{\boldsymbol{\alpha}:\alpha_a=0\} \tag{C.7}$$

are never "tangent" at their intersections. This is ensured by the following theorem.

Theorem C. 2. *Given any subforest $\mathscr{F}_I=\{\mathscr{S}_i, i\in I\}$ of the forest $\mathscr{F}$ and any subgraph $\mathscr{S}_0$ of G (eventually empty), if the intersection*

$$V=\Big(\bigcap_{i\in I}V_{\mathscr{S}_i,\mathscr{F}}\Big)\cap\Big(\bigcap_{a\in\mathscr{S}_0}P_a\Big) \tag{C.8}$$

is not empty, at every point of V the vectors $\boldsymbol{\Delta}_{\mathscr{S}_i,\mathscr{F}}(i\in I)$ and the vectors $\mathbf{n}_a$ defined as

$$\mathbf{n}_a=\frac{\partial}{\partial\boldsymbol{\alpha}}\alpha_a(a\in\mathscr{S}_0) \tag{C.9}$$

are linearly independent.

Proof. We first prove the theorem when the graph $\mathscr{S}_0$ is empty. Then, we first note that according to (C.2), every polynomial $Q_{\mathscr{S}_i,\mathscr{F}}$ may be written as a product of polynomials R_i relative to the graphs of the subforest $\mathscr{F}_I$ contained in $\mathscr{S}_i$:

$$Q_{\mathscr{S}_i,\mathscr{F}}=\prod_{j\in I:\mathscr{S}_j\subset\mathscr{S}_i}R_j\,, \tag{C.10}$$

where R_j is defined as the product of the polynomials $P_{[\mathscr{S}]_{\mathscr{F}}}$ relative to the subgraphs $\mathscr{S}$ of $\mathscr{F}$ which are contained in $\mathscr{S}_j$ but not contained in any subgraph $\mathscr{S}_k$ of $\mathscr{F}_I$ strictly contained in $\mathscr{S}_j$.

From (C.8) and (C.10), the point α belongs to V if and only if, for any $i\in I$, $R_i(\alpha)=1$. Then, if α belongs to V, we have

$$\Delta_{\mathscr{S}_i,\mathscr{F}}(\alpha)=\sum_{\substack{j\in I\\ \mathscr{S}_j\subset\mathscr{S}_i}}\frac{\partial}{\partial\alpha}R_j(\alpha). \tag{C.11}$$

By the same homogeneity argument used in Theorem C.1, the vectors $\frac{\partial}{\partial\alpha}R_i(\alpha)$ are non-zero; since they are orthogonal $\left[\text{that is } \frac{\partial}{\partial\alpha}R_i(\alpha)\cdot\frac{\partial}{\partial\alpha}R_j(\alpha)=0, \text{ if } i\neq j\right]$, they are linearly independent. This result, with (C.11), ensures that at every point of $\bigcap_{i\in I} V_{\mathscr{S}_i,\mathscr{F}}$, the vectors $\Delta_{\mathscr{S}_i,\mathscr{F}}(\alpha)$ are linearly independent.

We now consider the case where $\mathscr{S}_0$ is not empty. Since the vectors $\mathbf{n}_a$ are orthogonal, we may restrict ourselves to the subspace

$$E=\bigcap_{a\in\mathscr{S}_0} P_a=\{\alpha:\alpha_a=0:a\in\mathscr{S}_0\}. \tag{C.12}$$

To any subgraph $\mathscr{S}$ of $\mathscr{F}$ we associate the subgraph

$$\bar{\mathscr{S}}=[\mathscr{S}/\mathscr{S}\cap\mathscr{S}_0] \tag{C.13}$$

of $[G/\mathscr{S}_0]$ and we consider the forest

$$\bar{\mathscr{F}}=\{\bar{\mathscr{S}}:\mathscr{S}\in\mathscr{F}\}. \tag{C.14}$$

The restriction to the subspace E of the polynomials $P_{[\mathscr{S}]_{\mathscr{F}}}$ may be proved to be (see [16]):

$$\begin{aligned} &P_{[\mathscr{S}]_{\mathscr{F}}}(\alpha)=P_{[\bar{\mathscr{S}}]_{\bar{\mathscr{F}}}} \quad \text{if} \quad L([\bar{\mathscr{S}}]_{\bar{\mathscr{F}}})=L([\mathscr{S}]_{\mathscr{F}}),\\ &P_{[\mathscr{S}]_{\mathscr{F}}}(\alpha)=0 \quad \text{if} \quad L([\bar{\mathscr{S}}]_{\bar{\mathscr{F}}})<L([\mathscr{S}]_{\mathscr{F}}). \end{aligned} \tag{C.15}$$

The manifold V defined by (C.8) is not empty only if for any $\mathscr{S}$ of $\mathscr{F}$ contained in any $\mathscr{S}_{i\in I}$ we have $L(\mathscr{S}\cap\mathscr{S}_0)=0$.

In that case, we are reduced to the problem of the independence of the vectors $\Delta_{\bar{\mathscr{S}}_i,\bar{\mathscr{F}}}$ relative to the new forest $\bar{\mathscr{F}}$ and the new subforest $\bar{\mathscr{F}}_I=[\bar{\mathscr{S}}_i\}$ in the subspace E. The proof follows similar to the case $\mathscr{S}_0$ empty. This ends the proof of Theorem C.2.

Appendix D

This appendix is devoted to the proof of various properties of the dimensionally regularized integrals as $|\mathrm{Im}\,D|\to+\infty$, which are used in Sect. 3.B.

Let G be a Feynman graph (with internal non-zero masses). According to [12], its regularized integral has the following integral representation provided that $\mathrm{Re}\,D$ is different from any pole characteristic of the graph G.

$$I_G(p,m,D)=\int_0^\infty\prod_{a=1}^{l} d\alpha_a R\{Y_G(p,m,D,\alpha)\}. \tag{D.1}$$

According to the appendix C of [12], Eq. (D.1) may be decomposed in absolutely convergent integrals associated to different Hepp's sectors and different equivalent classes Γ of nests. A Hepp sector is given by an ordering of the lines

$$s=\{\alpha;\alpha_{a_\ell}\geqq\alpha_{a_{\ell-1}}\ldots\geqq\alpha_{a_1}\}. \tag{D.2}$$

We perform the change of variables into Hepp variables, that is to say

$$\alpha_{a_i}=\prod_{j=1}^{\ell}\beta_j. \tag{D.3}$$

Then, each contribution is of the form

$$\int_0^{+\infty} d\beta_\ell\beta_\ell^{a-\frac{DL}{2}-1}\int_0^1\prod_{i=1}^{\ell-1} d\beta_i\beta_i^{a_i-\frac{DL_i}{2}-1}\int_0^1\prod_{j\in J_i} d\chi_j^i(1-\chi_j^i)^{b_j^i}$$
$$\cdot\exp\left(-\prod_{i=K}^{\ell}\beta_i\chi_{r_i-1}^i[m_{i_K}^2+E(m,p,\beta,x)]\right)F(p,m,D,\beta_i\prod_{j\in J_i}\chi_j^i). \tag{D.4}$$

The variables χ_j^i, $j\in J_i$ (J_i is some subset of $\{1,\ldots,r_i-1\}$), are introduced to take into account the subtractions due to divergent subgraphs associated to the class Γ. K is some integer $\leqq\ell$.

The variables $\chi_{r_i-1}^i$ in $\exp(-\prod_{i=K}^{\ell}\beta_i\chi_{r_i-1}^i)$ are present if and only if the graph G is subtracted: if not, they are set equal to 1 in (D.4).

The exponents $a_j (j=1,\ell)$ and b_j^i are such that:

$$0<\mathrm{Re}\left(a_\ell-\frac{DL}{2}\right)<1, \quad \text{if} \quad G \text{ is subtracted}, \tag{D.5a}$$

$$\mathrm{Re}\left(a_\ell-\frac{DL}{2}\right)>0, \quad \text{if} \quad G \text{ is not subtracted}, \tag{D.5b}$$

$$a_i-a_\ell+\frac{D}{2}(L-L_i)>0, \quad K\leqq i<\ell, \tag{D.6a}$$

$$a_i-\frac{DL_i}{2}>0, \quad i<K, \tag{D.6b}$$

$$b_j^i\geqq 0. \tag{D.7}$$

The function $E(p,m,\beta_i)$ is a continuous non-negative function of the β_i's $(i=1,\ell-1)$.

The function $F(p,m,D,\beta_i)$ is a continuous function of the β_i's $(i=1,\ell=1)$ and is polynomially bounded as $|\mathrm{Im}\,D|\to+\infty$.

The conditions (D.5, D.6 and D.7) were sufficient to ensure the absolute convergence of the integral (D.4). We now study the limit $|\mathrm{Im}\,D|\to+\infty$.

Lemma 1. *For* Re D *fixed, different from any poles of* G, $I_G(p,m,D)$ *is exponentially bounded as* $|\mathrm{Im}\,D|\to\infty$ *: more precisely*

$$\forall\alpha>\frac{\pi\cdot L(G)}{4}, \quad \exists M>0:|I_G(p,m,D)|\leqq Me^{-\alpha|\mathrm{Im}\,D|}. \tag{D.8}$$

Proof. Let us integrate (D.4) over β_ℓ. We obtain

$$\Gamma\left(a_\ell - \frac{DL}{2}\right) \int_0^1 \prod_{i<K} d\beta_i \beta_i^{a_i - \frac{DL_i}{2} - 1} \int_0^1 \prod_{i=K}^{\ell-1} d\beta_i \beta_i^{a_i - a_\ell - D\frac{L-L_i}{2}} \int_0^1 \prod_{j \in J_i} d\chi_j^i$$

$$(1-\chi_j^i)^{b_j^i} \prod_{i=K}^{\ell} (\chi_{r_i-1}^i)^{\frac{DL}{2} - a_\ell} \cdot F(p, m, D, \beta\chi)(m_{i_K}^2 + E)^{\left(\frac{DL}{2} - a_\ell\right)} . \qquad \text{(D.9)}$$

At $\mathrm{Re}\,D$ fixed, away from a pole, the integrals over the β_i's and the χ_j^i's are convergent (from the conditions (D.5, D.6 and D.7). The modulus is majored by the integral of the modulus, which is polynomially bounded as $|\mathrm{Im}\,D| \to +\infty$. So it is easy to see that the integral is polynomially bounded.

From the asymptotic behaviour of the Γ function:

$$\Gamma(x+iy) \sim \sqrt{2\pi}\,|y|^{x-1/2} e^{-\frac{\pi}{2}|y|}, \quad \text{as} \quad |y| \to +\infty, \qquad \text{(D.10)}$$

we deduce the result of Lemma D.1.

Lemma 2. *For* $\mathrm{Re}\,D$ *fixed, different from any poles of* G, *the* L_1 *norm of the regularized integral* $\|I_G(p, m, D)\|_1$ *as defined in* (3.19) *is polynomially bounded as* $|\mathrm{Im}\,D| \to +\infty$.

Proof. Since $I_G(p, m, D)$ is a sum of integrals of the form (D.4), its L_1 norm is majored by the sum of the L_1 norms of the integrals (D.4). Those L_1 norms are integrals of the form

$$\int_0^{+\infty} d\beta_\ell \beta_\ell^{a_\ell - \frac{\mathrm{Re}\,D.L}{2} - 1} \int_0^1 \prod_{i=1}^{\ell-1} d\beta_i \beta_i^{a_i - \frac{\mathrm{Re}\,D.L_i}{2} - 1} \int_0^1 \prod_{j \in J_i} d\chi_j^i (1-\chi_j^i)^{b_j^i}$$

$$\cdot \exp\left(-\prod_{i=K}^{\ell} \beta_i \chi_{r_i-1}^i (m_{i_K}^2 + E)\right) \quad |F(p, m, D, \beta\chi)|^{\frac{\mathrm{Re}\,D.L}{2} - a_\ell} . \qquad \text{(D.11)}$$

$|F|$ is a continuous function of $\beta\chi$, polynomially bounded as $|\mathrm{Im}\,D| \to +\infty$. Since $|F|$ contains the only dependence in $|\mathrm{Im}\,D|$ of (D.11), and since the integral is absolutely convergent, the integral (D.11) is polynomially bounded as $|\mathrm{Im}\,D| \to +\infty$. The result of Lemma D.2 is then proved.

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DEUXIÈME PARTIE

DIVERGENCES INFRAROUGES DANS LES MODÈLES SIGMA NON LINÉAIRES A DEUX DIMENSIONS

II.1 - INTRODUCTION

Les modèles bidimensionnels appartenant à la classe des "modèles Sigma non linéaires" ont été beaucoup étudiés ces dernières années pour plusieurs raisons. Ces modèles ont été initialement introduits pour décrire la dynamique effective à basse énergie des Bosons de Goldstone d'une théorie présentant une brisure de symétrie continue. Considérons une telle théorie des champs (massive), c'est-à-dire une théorie telle que son Lagrangien soit invariant sous l'action d'un groupe de symétrie (de Lie compact connexe) G mais qui possède plusieurs états fondamentaux (vides) invariants seulement sous l'action d'un sous groupe (fermé) H de G. L'ensemble des vides $|\phi\rangle$ peut être alors paramétrisé par l'espace homogène quotient M = G/H et la théorie possède des modes de Goldstone de masse nulle [33]. Par raison de symétrie, la dynamique à basse énergie de ces modes devra être décrite par un Lagrangien phénoménologique non linéaire de la forme [34]

$$\mathcal{L} = \int d^d x \; \partial_\mu \phi^a g_{ab}(\phi) \, \partial^\mu \phi^b \qquad (2.1)$$

où le champ scalaire classique $\phi = (\phi^a) \in M$ décrit les modes de Goldstone dans une paramétrisation particulière de M et où $g_{ab}(\phi)$ est une métrique de l'espace M invariante sous l'action de G [35]. Une telle métrique dépend d'un nombre fini de paramètres, les "constantes de couplage" de la théorie. Les modèles les plus connus correspondent à des espaces symétriques où il n'y a qu'une constante de couplage. Citons :

- les modèles O(N), correspondant à $M = S^{N-1}$, la sphère de dimension (N-1), dont la version euclidienne est la limite continue des modèles de spins de Heisenberg à N composantes ;
- les modèles CP^{N-1} [36-38] et plus généralement les modèles Grassmanniens décrits dans [39] qui correspondent à G = U(N) et H = U(M) x U(N-M) ;

- les modèles "Chiraux" $G \otimes G$, où M est un groupe de Lie (semi simple) G [40].

De tels modèles sont définis par un Lagrangien effectif à basse énergie (grande distance) et ne sont pas renormalisables à quatre dimensions, comme le montre l'analyse dimensionnelle :

$$\text{dimension de } \mathcal{L} = [\text{masse}]^{2-d} \tag{2.2}$$

On voit par contre qu'à deux dimensions, le Lagrangien (2.1) est sans dimension et invariant conforme. Tout le problème est de savoir si on peut construire une théorie quantique renormalisée en préservant les symétries non linéaires (liées aux groupes G et H) qui définissent le modèle. La renormalisabilité du modèle O(N), étudiée à l'ordre d'une boucle par A. Polyakov [41], a été prouvée par E. Brezin, J. Zinn-Justin et J. C. Leguillou [42]. La renormalisation de ces modèles a été étudiée d'un point de vue très général par D. Friedan [43] (qui considère des modèles définis par l'action (2.1) sur un espace Riemannien quelconque). La renormalisation de ces modèles se traduit par une reparamétrisation de l'espace M sans signification physique, par une renormalisation de "fonction d'onde", c'est-à-dire une redéfinition de la mesure du champs ϕ sur M, et par une renormalisation de la "constante de couplage", c'est-à-dire de la métrique g sur M, préservant l'invariance sous l'action de G. Pour tous les modèles définis sur un espace symétrique, la fonction β est à l'ordre d'une boucle :

$$\beta(t) = -\frac{R}{2\pi n} t^2 + o(t^3) \tag{2.3}$$

R est la courbure scalaire et n la dimension de M. Si M est compact R est positif et ces modèles sont asymptotiquement libres, comme les théories de Jauge non abéliennes à quatre dimensions. On s'attend donc à ce qu'ils correspondent à des théories quantiques non triviales dont le comportement à grande distance (basse énergie) ait un caractère non perturbatif. De ce point de vue ces modèles apparaissent comme des laboratoires où l'on essaye de mettre au point des méthodes d'étude applicables aux théories de Jauge. Comme elles, ils possèdent également des solutions classiques non triviales d'action finie [37] (Instantons) ainsi que des solutions singulières, ici des vortex [40]. Mentionnons également tous les travaux liés à l'existence de lois de conservation non triviales dans ces modèles au niveau classique [44, 45] et quantique [46, 47] qui ont permis de calculer la matrice S du modèle O(N) [48].

Le caractère global de la symétrie de ces modèles rend plus facile la compréhension des phénomènes non perturbatifs qui s'y produisent. En effet, nous avons vu qu'ils décrivent classiquement la dynamique des bosons de Goldstone associés à une brisure de symétrie. A deux dimensions d'espace-temps, le théorème de Mermin-Wagner-Coleman [8, 9] interdit l'existence d'un tel phénomène en imposant aux fonctions de corrélation de décroître à l'infini. Considérons par exemple le modèle O(N), dont l'action s'écrit habituellement :

$$A = \frac{1}{2t} \int d^2x \; \partial_\mu \vec{S} \cdot \partial_\mu \vec{S} \qquad (2.4)$$

$\vec{S}(x)$ étant un champ scalaire à N composantes réelles contraint à appartenir à la sphère unitée S^{N-1} :

$$S^2(x) = 1 \qquad (2.5)$$

Le "vide" classique décrit une situation totalement ordonnée

$$\vec{S}(x) = \vec{S}_o \qquad (2.6)$$

mais pour toute valeur de la constante de couplage t, la symétrie est dynamiquement restaurée, l'état fondamental est invariant O(N) et les états excités de la théorie deviennent massifs*. Pour employer un lan-

* Ce dernier point, rendu plus que plausible par les arguments du groupe de renormalisation [41, 42], son développement en 1/N ainsi que par la matrice S [48], n'est en fait pas prouvé rigoureusement.

gage de mécanique statistique : "Les ondes de spin désorganisent le système à grande distance pour toute température".

A ce stade on rencontre une difficulté qui n'est pas présente dans les théories de Jauge à quatre dimensions. Les calculs perturbatifs de ces modèles sont divergents infrarouge ! Ces divergences sont en fait à la base de l'argument de Coleman [9] montrant qu'une symétrie continue ne peut pas être brisée : le propagateur d'un boson de masse nulle à deux dimensions

$$G(x-y) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik(x-y)}}{k^2+i\varepsilon} \tag{2.7}$$

diverge logarithmiquement en $k = 0$.

Les développements perturbatifs des modèles Sigma font intervenir ce propagateur divergent. De plus, les intégrations sur les impulsions internes des graphes de Feynman font apparaître des divergences supplémentaires. Pour cette raison, les premiers calculs perturbatifs ont été effectués en introduisant explicitement dans l'action un terme brisant la symétrie globale et rendant le propagateur (2.7) massif [39].

Néanmoins, A. Jevicki remarqua que le potentiel effectif du modèle Sigma linéaire était fini à l'ordre de deux boucles [49]. Ceci conduisit S. Elitzur à conjecturer que le développement perturbatif de toute observable <u>invariante</u> sous l'action du groupe O(N) du modèle Sigma non linéaire O(N) était fini infrarouge [10]. Cette conjecture, qu'il vérifia aux premiers ordres, fut étendue par A. Mc Kane et M. Stone au cas des modèles Chiraux $G \otimes G$ [50], et par D. Amit et G. Kotliar pour les modèles CP^{N-1} [51].

Un tel phénomène apparaît déjà dans l'exemple très simple suivant. Considérons simplement le champ libre scalaire de masse nulle. L'action

$$A = \int d^2x \, \frac{1}{2} \, \partial_\mu \phi \, \partial^\mu \phi \tag{2.8}$$

est invariante sous le groupe abélien des transformations

$$\phi(x) \longrightarrow \phi(x) + a \qquad (2.9)$$

et le Boson de Goldstone associé n'est autre que le champ ϕ lui même. Le propagateur de ce champ ϕ est divergent à deux dimensions. Par contre, l'observable suivante, invariante sous les transformations (2.9),

$$\langle 0 | [\phi(x) - \phi(y)]^2 | 0 \rangle = 2 \int \frac{d^2k}{(2\pi)^2} \, \frac{1 - e^{ik(x-y)}}{k^2 + i\varepsilon} \qquad (2.10)$$

est finie infrarouge (La divergence ultraviolette à grand k s'élimine par un produit normal : ϕ^2 : standard).

Les trois articles qui constituent les sections suivantes sont consacrés à l'étude et à la démonstration de la conjecture de S. Elitzur. Dans les deux premiers nous introduisons un terme de brisure explicite de la symétrie, H, et étudions le comportement des séries perturbatives lorsque H tend vers zéro. Le premier article (II.2) est consacré à une étude détaillée du problème dans le cas du modèle Sigma O (N). Dans un premier temps nous extrayons tous les termes divergents du développement d'une observable quelconque, à l'aide des méthodes dont nous avons parlé dans l'introduction. Le résultat de cette analyse est contenu dans le Lemme 2.1. Sa forme est assez similaire au développement en produit d'opérateur [52], les divergences infrarouges sont contenues dans des termes proportionnels à des valeurs moyennes dans le vide d'opérateurs locaux. Dans un deuxième temps nous montrons que les coefficients de ces termes divergents sont identiquement nuls lorsque l'on s'intéresse à des observables invariantes.

Dans le second article (II.3), nous étendons cette analyse aux modèles généraux définis plus haut à partir d'un espace homogène quelconque. Dans ces deux sections la procédure de régularisation ultraviolette utilisée est la régularisation dimensionnelle (principalement pour des raisons techniques). Cependant les résultats sont valides dans d'autres types de régularisation, pourvu qu'elles respectent l'invariance globale du modèle, en particulier pour les modèles sur réseaux.

Le troisième article (II.4) est consacré à une approche un peu différente de ces problèmes. Au lieu d'introduire un terme explicite de brisure de symétrie, nous quantifions ces modèles dans un volume fini (le tore T^2 pour simplifier) et étudions la nature des divergences infrarouges lorsque le volume tend vers l'infini. Ceci ne constitue pas une simple variante de la méthode précédente. En effet, pour quantifier de manière cohérente ces modèles dans un volume fini, il faut introduire une contrainte pour éliminer les modes zéro liés à la symétrie globale* (exactement comme pour les théories de Jauge) ; dans certaines jauges les divergences sont de nature très différente.

Nous proposons une jauge, très simple mais brisant l'invariance par translation, qui conduit seulement à un changement du propagateur libre et le rend fini infrarouge. Dans cette nouvelle jauge toutes les observables sont maintenant finies. Les divergences infrarouges des amplitudes de Feynman de ce nouveau développement perturbatif sont contenues dans des quantités très "globales" (des dérivées de la fonction de partition par rapport à la contrainte), et non plus locales comme précédemment. Cette nouvelle approche permet également de considérer les divergences infrarouges des modèles "non-standards" introduits par D. Friedan [43].

Finalement, quelles sont les conséquences de la finitude infrarouge des développements perturbatifs de ces modèles ? Un premier point, technique, est le suivant : les calculs perturbatifs, en particulier ceux des fonctions du groupe de renormalisation, deviennent plus simples lorsque l'on manipule des observables invariantes [50, 51].

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* En particulier, introduire un terme de brisure de symétrie explicite et faire tendre ce terme vers zéro ne conduit pas à des résultats cohérents à volume fini [53]. Un phénomène analogue a lieu pour les théories de Jauge [54].

En second lieu le fait que seules les observables invariantes soient définies est un signal clair du fait que la symétrie globale du modèle doive être restaurée. Cependant, ces développements perturbatifs ne sont finis que tant que l'on ne s'intéresse pas à des comportements à basse énergie ou à grande distance. Si certaines des impulsions externes tendent vers zéro, c'est-à-dire si l'on se place sur la "couche de masse perturbative", des divergences infrarouges apparaissent exactement comme dans les processus faisant intervenir des "gluons mous" dans les théories de Jauge [55]. Les mêmes questions se posent alors : Peut-on resommer ces divergences et obtenir des effets non perturbatifs comme la décroissance exponentielle des fonctions de corrélation ou d'autres effets non perturbatifs entrent t'ils en ligne de compte ? Quelle est la nature du développement perturbatif de ces modèles ? Nous revenons en partie sur ces importantes questions dans la partie V.

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Cancellations of Infrared Divergences in the Two-Dimensional Non-Linear σ Models

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Abstract. In the two-dimensional $O(N)$ nonlinear σ models, the expectation value of any $O(N)$ invariant observable is shown to have an infrared finite weak coupling perturbative expansion, although it is computed in the "wrong" spontaneously broken symmetry phase. This result is proved by extracting all infrared divergences of any bare Feynman amplitude at $D = 2 - \varepsilon$ dimension. The divergences cancel at any order only for invariant observables. The renormalization at $D = 2$ preserves the infrared finiteness of the theory.

1. Introduction

Two-dimensional σ-models have raised an increasing interest during the last years, owing to their similarity with four-dimensional gauge theories, their simpler structure and the development of powerful nonperturbative methods. In this paper we deal with the weak coupling perturbative approach. This approach suffers from the fact that the naive vacuum state is not the true one, as presumably is the case for four-dimensional gauge theories. Indeed, the perturbative expansion has to be made around a peculiar classical solution, i.e. in the spontaneously broken symmetry phase, although such a phase cannot exist in two-dimensional space [1, 2]. So the symmetry has to be dynamically restored for any positive coupling constant [3, 4, 5]. A drastic consequence of the fact that the perturbative expansion is made in the wrong phase is that this expansion has very important infrared divergences, since even the free propagator of a massless Goldstone boson does not exist at two dimensions. For this reason the first perturbative calculations have been performed by introducing a symmetry breaking term which makes the theory infrared (I.R.) finite (and then by setting this term to zero) [4, 5].

However, S. Elitzur, following a remark of A. Jevicki about the effective potential of the $O(N)\,\sigma$ model [6], conjectured that any $O(N)$ invariant observable has an infrared finite expectation value to any order in perturbation expansion [7], and checked the fact up to third order of the two-point function. Various computations have been made by some authors [8, 9] for the $O(N)$ and $G \otimes G$ chiral models which have verified the conjecture in many cases and used it to study these models. Moreover, this result is very similar to what is expected for four-dimensional gauge theories, a namely that some gauge invariant quantities should be

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I.R. finite, although the nature of the divergences, as of the invariant states, is much more complicated for gauge theories. Nevertheless, Elitzur's argument is very insufficient and may not be considered as a complete proof.

In this paper we present a general proof of Elitzur's conjecture for the $O(N)$ nonlinear σ model. The infrared problem is shown to be disconnected with the ultraviolet (U.V.) problem, as assumed in [7, 8]. We use dimensional regularization (which has many advantages for this purpose), and prove that the regularized vacuum expectation value of any $O(N)$ invariant observable is I.R. finite at any order of perturbation at dimension $D = 2 - \varepsilon$. The U.V. renormalization at $D = 2$ may then be performed.

The key to the proof is the explicit extraction of the I.R. divergent part of any integral present in the perturbative expansion of any observable of the fields. For this purpose we use and adapt the general method developed by M. C. Bergère and Y. M. P. Lam in [10, 11] for studying the asymptotic expansion of Feynman amplitudes, and we also use results of [12] about dimensional regularization. This analysis is very technical and uses general technics of Mellin transform and subtraction operators. Only its final result is needed to show the mechanism of cancellations of I.R. divergences, so for clarity we shall first present this last point.

This paper is organized as follows:

In Sect. 2 we briefly recall the perturbative expansion of the $O(N)$ σ model. We then present in Lemma 2.1 the result of Sect. 3, namely the I.R. behaviour of any Feynman integral at $2 - \varepsilon$ dimensions. Then this result is used to exhibit the mechanism of cancellation of I.R. divergences for invariant observables. We finally deal with the problem of renormalization.

Section 3 is devoted to the analysis of I.R. divergences of the $O(N)$ σ model. We first present (in part A) the method of [10, 11] of analysis of asymptotic behavior, and introduce the main tools which shall be used. This method allows the extraction of I.R. divergences of any graph at generic (nonexceptional) momenta. This is performed for the graphs of the σ model in part B.

Divergences remain at exceptional momenta (this is related to the distribution-like character of the amplitudes in two-dimensional momentum space). This problem is discussed in part C. One has to look at the limit of nonexceptional momenta tending toward exceptional ones. This may not be studied by the former methods. We do not give a complete solution of this problem but present arguments for the general decomposition. This decomposition is proved in particular for the 2-point function, and a complete proof shall be given elsewhere.

2. I. R. Divergences of the $O(N)$ σ Model and Their Cancellations

We consider the Euclidean $O(N)$ nonlinear σ model. The $O(N)$ invariant action is

$$A_0 = \frac{1}{2g} \int d^D x (\partial_\mu \vec{S})(\partial^\mu \vec{S}) \tag{2.1}$$

where $\vec{S}(x)$ is a N-component real field with the usual constraint

$$\| \vec{S} \| = \sum_i (S^i(x))^2 = 1. \tag{2.2}$$

To obtain in I.R. finite perturbative expansion around $g = 0$, let us note

$$S^i(x) = \sqrt{g}\pi^i(x) \qquad i = 1, N-1$$
$$S^N(x) = \sigma(x) = \sqrt{1 - g\vec{\pi}^2(x)} \tag{2.3}$$

and let us introduce a magnetic field H in the N direction, so that the action becomes:

$$A_H = \frac{1}{g}\int d^Dx\left[\frac{\partial_\mu \vec{S}\cdot\partial^\mu\vec{S}}{2} - H(\sigma(x) - 1)\right] \tag{2.4}$$

$$= \frac{1}{2}\int d^Dx\left[(\partial_\mu\vec{\pi})^2 + H\vec{\pi}^2\right] + \frac{1}{g}\left[(\partial_\mu(\sigma-1))^2 + H(\sigma-1)^2\right]. \tag{2.5}$$

The expectation value of any observable $\mathcal{O}(\vec{\pi})$ is given by the functional integral

$$\langle \mathcal{O} \rangle_H = \frac{\int \mathcal{D}(\vec{\pi})\mathcal{O}(\vec{\pi})e^{-A_H(\pi)}}{\int \mathcal{D}(\vec{\pi})e^{-A_H(\pi)}} \tag{2.6}$$

where $\mathcal{D}(\vec{\pi})$ is the invariant measure

$$\mathcal{D}(\vec{\pi}) = \prod_x \frac{d\vec{\pi}(x)}{\sqrt{1 - g\vec{\pi}^2(x)}} \tag{2.7}$$

Since the interaction term may be written $\frac{1}{g}[\sigma - 1](-\Delta_x + H)(\sigma - 1)$, we associate to each vertex an "interaction line" which gives the factor $(-\Delta_x + H)$ of the vertex (that is a factor $p^2 + H$ in impulsion representation, where p is the total impulsion incoming to the $(\sigma - 1)$: (see Fig. 1)). The graphs of the model are considered as a set of usual propagator lines and of "interaction lines".

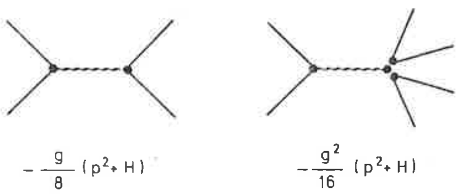

Fig. 1. Interaction vertices of first and second order. The wavy "interaction line" represents the term $p^2 + H$ where p is the impulsion carried through the line

We shall argue that the I.R. and the U.V. divergences of the model are completely disconnected. So, in order not to deal with U.V. divergences, we shall use dimensional regularization throughout this paper and study I.R. divergences at a dimension $D = 2 - \varepsilon$. This symmetry preserving regularization is much simpler than others (for instance lattice regularization used by Elitzur [7]) and was already used to study the I.R. problem of σ model in [8, 9]. In particular, it is well-known that with this procedure, the measure term in (2.7), which is proportional to

$\delta(0) = \int d^D k$, gives zero. Another advantage is that (as we shall see in the following), at $D = 2 - \varepsilon$ dimension, amplitudes diverge only as power of $H^{-\varepsilon}$ as $H \to 0$, and there is no more log H divergences (which make the analysis of I.R. divergences much simpler.

We now present the results of the analysis of the I.R. behaviour of any Feynman amplitude of the $O(N)$ σ model which is performed in Sect. 3. We shall show that those I.R. divergences may be cancelled by the introduction of I.R. counterterms (but at the price of strong modification of the functional integral). Then we shall exhibit identities relative to invariant operators that prove that these I.R. counterterms reduce to zero for these $O(N)$ invariant functions, so that the I.R. finiteness will be proved.

From Sect. 3, given a graph G, the subgraphs which give I.R. divergences are the "dominant subgraphs" E of G defined as:

Definition 2.1. A subgraph E of G is said "dominant" if:

(a) E contains all external vertices of G (that is vertices where external impulsions (or positions) are attached.

(b) E contains no disconnected part, that is has no connected part which does not contain any external vertex.

(c) There is no "interaction line" of $G-E$ attached to E.

A dominant subgraph may be disconnected (see Fig. 2).

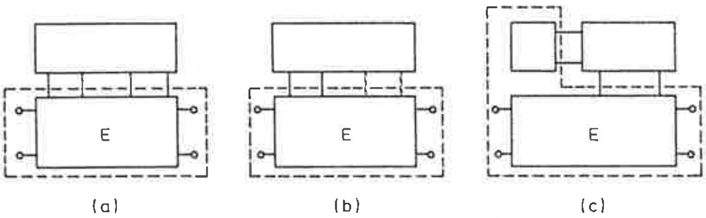

Fig. 2a a dominant subgraph E of a graph G. **b** an essential E of G which does not satisfy condition (**b**) of Definition 2.1. **c** an essential E of G which does not satisfy condition (**c**) of Definition 2.1

The main result of Sect. 3 is the following Lemma, which gives the I.R. behaviour of any amplitude.

Lemma 2.1. For any graph G, the I.R. behaviour of the amplitude $I_G(x, H)$ at $D = 2 - \varepsilon$ is given by a sum of contributions relative to dominant subgraphs of G:

$$I_G(x, H) = \sum_{\substack{E \subseteq G \\ \text{dominant}}} F_E(x) I_{[\widetilde{G/E}]}(H) + 0(H^{1-\varepsilon L(G)/2}). \tag{2.8}$$

The F_E's are finite functions of the external positions x (or well-defined distribution of the external momenta p).

$I_{[\widetilde{G/E}]}(H)$ is the regularized amplitude of the graph $[\widetilde{G/E}]$ obtained by shrinking into one vertex the whole dominant E. It diverges like a pure power of $H^{-\varepsilon/2}$, namely

$$I_{[\widetilde{G/E}]}(H) = \text{const } H^{-(\varepsilon/2)L([\widetilde{G/E}])} \tag{2.9}$$

where $L([\widetilde{G/E}])$ is the number of loops of the graph $[\widetilde{G/E}]$.

The asymptotic expansion (2.8) is valid for $|\mathrm{Re}\,\varepsilon| < \frac{2}{L(G)}$.

The finite term of this expansion is the term relative to the dominant G itself (which is the only dominant such that $L(E) = L(G)$), that is $F_G(x)$. It follows immediately from this lemma that for $D > 2(\varepsilon < 0)$, the amplitude is I.R. finite, that is

$$I_G(x, 0) = F_G(x) \qquad \text{if } \varepsilon < 0 \tag{2.10}$$

and that for $\varepsilon > 0$, $F_G(x)$ is the analytic continuation of $I_G(x, 0)$ in the half plane $\varepsilon > 0$. Of course, the F_E's and the $I_{[\widetilde{G/E}]}$'s always have ultraviolet poles at $\varepsilon = 0$, since we deal with regularized amplitudes.

The result of Lemma 2.1 is quite similar to the Wilson Operator Product Expansion [13], and indeed is obtained by the same methods, since in Sect. 3 we transform by homogeneity the problem of small masses ($H \to 0$) into a problem of large momenta (see Fig. 3).

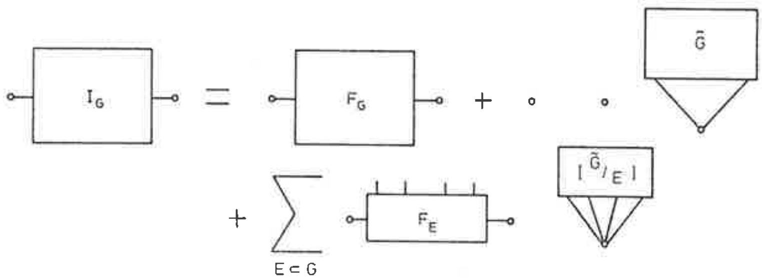

Fig. 3. A graphical interpretation of Lemma 2.1 (and of Eq. 3.43) giving the I.R. expansion of a graph of the 2-point function

We may invert (2.8) to express F_G as a function of the I_E's. A single recursive argument leads to:

Lemma 2.2.

$$F_G(x) = \sum_{\substack{E \subsetneq G \\ \text{dominant}}} I_E(x, H) A_{[\widetilde{G/E}]}(H) + 0(H^{1-(\varepsilon/2)L(G)}). \tag{2.11}$$

The coefficients $A_{[\widetilde{G/E}]}(H)$ diverge as $H^{-(\varepsilon/2)L(\widetilde{G/E})}$, like $I_{[\widetilde{G/E}]}(H)$, and are given by a sum over all nests of dominant subgraphs of $[\widetilde{G/E}]$, considered as a graph which appears in the perturbative expansion of $\langle(\vec{\pi}^2(x)^p\rangle$ (where $2p$ is the number of lines of $G-E$ attached to E).

The functions $A_{[\widetilde{G/E}]}(H)$ appears as I.R. counterterms to be added to the I.R. divergent amplitude according to (2.11) to obtain a finite amplitude F. The introduction of those counterterms corresponds to a strong (nonlocal) modification of the functional integral. We shall show that this modification reduces to zero for $O(N)$ invariant functions, so that the bare amplitude I_G may be replaced by the I.R. finite amplitude F_G in the perturbative expansion of these functions. So, having isolated I.R. singularities for any amplitude in Lemma 2.1. and 2.2., we now deal with the explicit mechanism of cancellation.

Let us consider a $O(N)$ invariant function $\mathcal{O}(\vec{\pi})$. In presence of a symmetry breaking term $-\frac{H}{g}(\sigma - 1)$ in the action, the vacuum expectation value of $\mathcal{O}$ is given by the functional integral (2.6).

Following an idea of Elitzur [7], let us perform an arbitrary rotation R of angle θ in a direction $\vec{u}$ of the tangent plane of the $\vec{\pi}$ ($|\vec{u}| = 1$), so that the fields are changed into

$$\vec{\pi} \longrightarrow {}^R\vec{\pi} = \vec{\pi} + \left[(\cos\theta - 1)(\vec{\pi}\cdot\vec{u}) + \frac{1}{\sqrt{g}} \sin\theta\sigma \right] \vec{u} \tag{2.12}$$

$$\sigma \longrightarrow {}^R\sigma = \cos\theta \cdot \sigma - \sqrt{g}\sin\theta(\vec{\pi}\cdot\vec{u}). \tag{2.13}$$

Since the measure $\mathcal{D}$ the function $\mathcal{O}$ and the invariant action A_0 are invariant under this rotation, the only term that is changed in the functional integral (2.6) is the symmetry breaking term which becomes

$$\frac{H}{g}(1 - {}^R\sigma) = \frac{H}{g}\left[1 - \cos\theta\cdot\sigma + \sqrt{g}\sin\theta(\vec{\pi}\cdot\vec{u})\right]. \tag{2.14}$$

Defining a parameter a by

$$a = \frac{1}{\sqrt{g}} \tan\theta \tag{2.15}$$

and rescaling H into

$$H \to H/\cos\theta = H\sqrt{1 + ga^2} \tag{2.16}$$

we obtain

$$\langle \mathcal{O}(\vec{\pi}) \rangle_{H\sqrt{1+ga^2}} = \frac{\int \mathcal{D}(\vec{\pi})\mathcal{O}(\vec{\pi}) \exp\left[-A_H(\vec{\pi}) - aH\int d^Dx(\vec{\pi}\cdot\vec{u})\right]}{\int \mathcal{D}(\vec{\pi}) \exp\left[-A_H(\vec{\pi}) - aH\int d^Dx(\vec{\pi}\cdot\vec{u})\right]}. \tag{2.17}$$

Eq. (2.17) corresponds to the following identity between vacuum expectation values for any $O(N)$ invariant observable $\mathcal{O}$ and any a.

$$\langle \mathcal{O}(\vec{\pi}) \rangle_{H\sqrt{1+ga^2}} = \frac{\langle \mathcal{O}(\vec{\pi}) \exp(-aH\int d^Dx(\vec{\pi}\cdot\vec{u}))\rangle_H}{\langle \exp(-aH\int d^Dx(\vec{\pi}\cdot\vec{u}))\rangle_H}. \tag{2.18}$$

To see the consequence of this identity at the perturbative level, let us develop in g and a both sides of (2.18). We shall note perturbative expansion of $\langle \mathcal{O}(\vec{\pi}) \rangle_H$

$$\langle \mathcal{O}(\vec{\pi}) \rangle_H = \sum_{N=0}^{\infty} g^N \mathcal{O}_N(H) \tag{2.19}$$

where $\mathcal{O}_N$ is the sum of the amplitudes of all graphs of order N which appear in the perturbative expansion of $\langle \mathcal{O} \rangle$. Developing $\sqrt{1 + ga^2}$ we get the following expansion of the l.h.s. of (2.18)

$$\langle \mathcal{O}(\vec{\pi}) \rangle_{H\sqrt{1+ga^2}} = \sum_{N=0}^{\infty} g^N \left[\mathcal{O}_N(H) + \sum_{P=1}^{N} a^{2P} \mathcal{P}_P\left(H\frac{\partial}{\partial H}\right) \mathcal{O}_{N-P}(H) \right] \tag{2.20}$$

where the $\mathscr{P}_P$ are polynomials of degree P and of valuation 1 (they have no term of order zero).

We now look at the r.h.s. of (2.18). $\mathcal{O}(\vec{\pi})$ being invariant is an even function of $\vec{\pi}$, so that, by parity, only even powers of a occur in the expansion. We obtain

$$\text{r.h.s. } (2.18) = \sum_{N=0}^{\infty} g^N \left[\mathcal{O}_N(H) + \sum_{P=1}^{N} a^{2P} \mathcal{O}_N^P(H) \right] \tag{2.21}$$

where the $\mathcal{O}_N^P(H)$ is sum of the graphs which occur in the perturbative expansion of $\langle \mathcal{O}(\vec{\pi})([H\int d^D x(\vec{\pi}\cdot\vec{u})]^{2P}/2P!\rangle$, that is graphs of $\mathcal{O}(\vec{\pi})$ with $2P$ insertions of $H(\vec{\pi}\cdot\vec{u})$ at zero momenta. Moreover, the presence of these insertions at the denominator of (2.18) ensures that not only vacuum diagrams disappear, but also diagrams with disconnected part where there are only insertions of $H(\vec{\pi}\cdot\vec{u})$ (see Fig. 4). (The factor $1/(2P)!$ is cancelled by a factor $(2P)!$ which comes from the contractions in Wick's theorem, since these $2P$ insertions are undiscernable). The insertions being made at zero momenta, the contribution of each line joining an insertion to the graph is $\frac{H}{H} = 1$, so that the graphs may be seen as graphs of $\mathcal{O}$ with "truncated insertions" of $(\vec{\pi}\cdot\vec{u})$ at zero momenta (see Fig. 4).

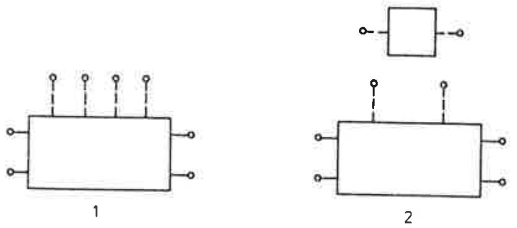

Fig. 4. Example of graphs with four insertions of $H(\vec{\pi}.\vec{u})$. Graph 1 contributes in (2.22), and graph 2 does not

Let us note by $\mathscr{G}_N^P(\mathcal{O})$ the set of all graphs that contribute to $\mathcal{O}_N^P$, and $C(E)$ the counting factor of a graph E of $\mathscr{G}_N^P(\mathcal{O})$ in $\mathcal{O}_N^P$. We have

$$\mathcal{O}_N^P(x, H) = \sum_{E \in \mathscr{G}_N^P(\mathcal{O})} C(E) I_G(x, H) \tag{2.22}$$

and we identify trivially $\mathcal{O}_N^0$ with $\mathcal{O}_N$.

Identifying term by term (2.20) and (2.21), we get the following identity, which is the perturbative version of identity (2.18):

$$\mathcal{O}_N^P(H) = \mathscr{P}_P\left(H\frac{\partial}{\partial H}\right)\mathcal{O}_{N-P}(H). \tag{2.23}$$

The crucial point of the mechanism of cancellation is that the graphs of the $\mathscr{G}_N^P(\mathcal{O})$ are exactly the "dominant subgraphs", as defined previously, that appear Lemma 2.1 and 2.2 in the I.R. expansion of the graphs of the perturbative expansion of $\langle \mathcal{O} \rangle$.

So let us come back to the result of Lemma 2.2.

Given a graph G of $\mathscr{G}_N^0(\mathcal{O})$, a dominant subgraph E of G appears to be a graph of some $\mathscr{G}_M^P(\mathcal{O})(M \leqq N, P \leqq M)$ where M is the order of the graph E and $2P$ is the number of lines of $G-E$ attached to E. Similarly, the reduced graph $[\widetilde{G/E}]$ appears to be a graph of order $N - M$ of the perturbative expansion of $(\vec{\pi}^2(0))^P$, that is a graph of $\mathscr{G}^0_{N-M}(\vec{\pi}^2(0)^P)$. It is shown in Appendix A that the counting factors are such that the sum over the graphs G of the decomposition (2.11) may be factorized into a sum over the dominant graphs E times a sum over the reduced graphs, so we have the following decomposition:

$$\sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) = \mathscr{O}_N(x, H) + \sum_{\substack{0\leq M\leq N\\ 1\leq P\leq M}} \mathscr{O}^P_M(x, H)\cdot \mathscr{A}_{P_{N-M}}(H) + O(H^{1-\varepsilon N/2}) \quad (2.24)$$

where the $\mathscr{O}^P_M(x,H)$ are given by (2.22) and where the $\mathscr{A}_{P_{N-M}}(H)$ are given by

$$\mathscr{A}_{P_{N-M}}(H) = \sum_{S\in\mathscr{G}^0_{N-M}(\pi^2(0)^P)} C(S).A_S(H). \quad (2.25)$$

The $\mathscr{A}_{P_Q}(H)$'s diverge as a power of $H^{-(\varepsilon/2)(P+Q)}$, except for the term $P = 0, Q = 0$, which corresponds to the graph reduced to a point and which is set equal to 1.

We now have all the elements needed to prove the I.R. cancellation. We have the theorem. :

Theorem 2.1. *The bare vacuum expectation value of any $\mathscr{O}(N)$ invariant function $\mathscr{O}$ is infrared finite at any order of perturbative expansion at dimension $D = 2 - \varepsilon$. The term of order N of the development of $\langle\mathscr{O}\rangle$, $\mathscr{O}_N$, is given by the sum of the finite part F_G of the amplitudes of the graphs G which appear in $\mathscr{O}_N$, namely*

$$\mathscr{O}_N(x, H) = \sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) + O(H^{1-\varepsilon N/2}). \quad (2.26)$$

Proof. Let us assume that the theorem is true at any order $M < N$. (This recursive hypothesis is trivially satisfied at order $N = 0$). Then, any derivative versus $\ln H$ of $\mathscr{O}_M(M < N)$ has a zero limit, namely

$$\left(H\frac{\partial}{\partial H}\right)^q \mathscr{O}_M = O(H^{1-\varepsilon M/2}) \quad \forall q \geqq 1, M < N. \quad (2.27)$$

From (2.23), we deduce immediately that the $\mathscr{O}^P_N$ have a zero limit.

$$\mathscr{O}^P_N = O(H^{1-\varepsilon(N-P)/2}) \quad \forall\, p \geqq 1. \quad (2.28)$$

This is also true for the $\mathscr{O}^P_M(M < N)$ by the recursive hypothesis. So, for ε small enough, we may add to $\mathscr{O}_N$ any linear combination of the $\mathscr{O}_{M,P}$, provided that their coefficients diverge as power of $H^{-\varepsilon/2}$. In particular, we may take the I.R. counter-terms $\mathscr{A}_{P_{N-M}}$. So (2.28) and (2.24) lead immediately to:

$$\sum_{G\in\mathscr{G}^0_N(\mathscr{O})} C(G)F_G(x) = \mathscr{O}_N(x, H) + O(H^{1-\varepsilon N/2}). \quad (2.29)$$

The l.h.s. of (2.29) being I.R. finite, the theorem is proved at order N.

We finally deal with the problem of renormalization. As claimed in the introduction, we have seen that regularized vacuum expectation values of any invariant function are I.R. finite, so that the I.R. problem is disconnected from the U.V. one. The renormalization of the σ-model in dimension two is performed in references [4, 5]. It is proved that, for soft invariant operators (that is for local functions of the fields without derivatives of the fields $\vec{\pi}$), only two counterterms are needed, corresponding to the renormalization of the fields and of the coupling constant [5]. The bare field $\vec{S}_0$ and the bare coupling constant g_0 are related to the renormalized ones $\vec{S}_R$ and g_R by

$$\vec{S}_0 = \sqrt{Z}\,\vec{S}_R \tag{2.30}$$

$$g_0 = Z_1 g_R. \tag{2.31}$$

The symmetry breaking term $\frac{1}{g}H\sigma$ has the dimension of the field $\vec{S}$ and so needs no additional counterterm. The "bare magnetic field" H_0 is related to the renormalized one by

$$H_0 = \frac{Z_1}{\sqrt{Z}} H_R. \tag{2.32}$$

The counterterms Z and Z_1 being independent of H, their perturbative expansions (in g_R or in g_0) are obviously I.R. finite (but of course U.V. divergent, and have poles at $\varepsilon = 0$). So we let the reader convince himself that the renormalization of any soft operator does not introduce any additional I.R. divergences. So the perturbative expansion of any renormalized $O(N)$ invariant soft operator is I.R. finite at any order of perturbation.

The renormalization of invariant operators of higher dimension (that is with derivatives of the fields) is more subtle. Because the $O(N)$ transformation laws of the fields are modified by renormalization, the invariant operators are mixed with what seems to be non-invariant ones [5]. The problem of the I.R. finiteness of those objects is discussed in [14] and it may be shown that they are also I.R. finite.

3. Extraction of I.R. Divergences of Regularized Amplitudes at $D = 2 - \varepsilon$

A. Introduction

We now present the analysis of the I.R. divergences of the regularized amplitudes at $D \leqq 2$. The general method of analysis of the asymptotic expansion of Bergère and Lam exposed in [10, 11] is adapted to study the I.R. limit of graphs at generic (nonexceptional) momenta. In our case, some simplifications occur, since we deal with regularized (instead of renormalized) amplitudes. However, we adapt their procedure of extraction of singularities of the Mellin transform to extract not only the dominant part of the asymptotic expansion, but subdominant ones. Then we give arguments, but not a complete proof, for the extraction of singularities at exceptional momenta. Let us first introduce the main tools which are used.

Schwinger representation of dimensionally regularized amplitude
The Feynman amplitude are written in the α-Schwinger representation.

Each propagator of a line a is written:

$$\frac{1}{p^2 + H} = \int_0^\infty d\alpha\, e^{-\alpha(p^2+H)} \tag{3.1}$$

and the contribution of the "interaction lines" introduced in Sect. 2 is written

$$(p^2 + H)\int_0^\infty d\alpha\left(\frac{\partial}{\partial\alpha}\right)^2 e^{-\alpha(p^2+H)}. \tag{3.2}$$

Performing the integration over internal momenta, we obtain the Schwinger representation for the amplitude of a graph G.

$$I_G(p, H) = \int_0^{+\infty} \prod_{a\in G} d\alpha_a \mathscr{D}_G\left[\exp\left(-\sum_{a\in G}\alpha_a H - p_i d_{G_{ij}}(\alpha)p_j\right)P_G(\alpha)^{-D/2}\right]. \tag{3.3}$$

$\mathscr{D}_G$ is the differential operator

$$\mathscr{D}_G = \prod_{a\in\mathscr{I}_G}\left(\frac{\partial}{\partial\alpha_\alpha}\right)^2 \tag{3.4}$$

where $\mathscr{I}_G$ is the set of "interaction lines" of the graph $G.P_G$ and $d_{G_{ij}}$ are the Symanzik functions, characteristic of the topology of the graph.

Dimensional regularization is performed by taking D complex in (3.3) [12]. This integral is U.V. convergent for Re D small enough, and has an analytic continuation in D meromorphic with poles at dimension D such that the superficial degree of divergence ω_S of a connected one-particle irreducible (1.P.I.) subgraph S is a positive or null integer. ω_S is defined in our case as:

$$\omega_S(D) = \frac{DL(S)}{2} - p(S) + i(S) \tag{3.5}$$

where $L(S)$ is the number of independent loops of S and $p(S)$ (respectively $i(S)$) is the number of propagator lines (respectively interaction lines) in S.

According to [12] when (2.10) is divergent, the regularized integral is given, for Re D away from the U.V. poles, by the convergent integral.:

$$I_G = \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\left[\mathscr{D}_G \exp\left(-\left[\sum_a \alpha_a H + pd_G p\right]\right)P_G^{-D/2}\right] \tag{3.6}$$

where $\mathscr{R}$ is the subtraction operator defined in [15] as a sum over nests of divergent subgraphs at the dimension D of products of Taylor operators:

$$\mathscr{R} = 1 + \sum_{\mathscr{N}}\prod_{S\in\mathscr{N}}(-\tau_S^{-\ell(S)}). \tag{3.7}$$

Each Taylor operator acts as follows:

The α'_a variables ($a\in S$) of the subgraph S are scaled by ρ. One writes the generalized Laurent expansion in ρ of a function f on which $\tau_S^{-\ell(S)}$ is applied as

$$f(\alpha, \rho) = \sum_{p=0}^{\infty} \rho^{p_0+p} f_p(\alpha). \tag{3.8}$$

(In our case p_0 is complex, and depends on D.) The Taylor operator τ only retains the terms with Re $(p_0 + p) \leqq -\ell(S)$, where

$$\ell(S) = p(S) + i(S) \tag{3.9}$$

is the number of lines of S, and then takes $\rho = 1$.

We recall that a nest $\mathcal{N}$ is a set of subgraphs S such that, given two subgraphs S_1 and S_2 of $\mathcal{N}$, either S_1 is included in S_2, or S_2 is included in S_1.

Applied on the integrand of (3.6), R may be rewritten as a sum over Zimmermann forests of connected 1.P.I. divergent subgraphs [15]. The action of $\mathcal{R}$ in (3.6) depends on the values of Re D and $\mathcal{R}$ performs subtractions at zero external momenta and internal mass H, so that it is not a renormalization. Indeed the singularities of I_G are always present, and appear where D tends towards a pole where the integral (3.6) is no more convergent.

Mellin transform and the I.R. *asymptotic expansion*

We now give the principle of the study of the I.R. behaviour, as exposed in [10, 11]. We want to study the limit as $H \to 0$ of the regularized amplitude $I_G(p, H)$ for a dimension D less than two (namely $D = 2 - \varepsilon$ with $0 < \varepsilon < \frac{2}{L(G)}$. Scaling α into α/H in (3.6) we get the homogeneity relation:

$$I_G(p, H) = H^{\omega_G} I_G\left(\frac{1}{\sqrt{H}} p, 1\right). \tag{3.10}$$

The Mellin transform of (3.10) is defined in [11] as

$$M_G(p, x) = \int_0^{+\infty} d\lambda (1 - \tau_\lambda^{-1}) \lambda^{-x-1} I_G\left(p, \frac{1}{\lambda}\right). \tag{3.11}$$

The integral (3.10) is convergent at infinity for x great enough (the τ_λ^{-1} ensures the convergence at zero). It is shown in [16] that using the integral representations (3.6) and (3.11), we may intervert integration in λ and α, to get the integral representation of M_G:

$$M_G(p, x) = \Gamma(-x - \omega_G) \int_0^{+\infty} \prod_G d\alpha\, \mathcal{R}\left\{ \mathcal{D}_G \exp\left(-\sum_G \alpha\right) P_G^{-D/2} (p d_G p)^{x + \omega_G} \right\}. \tag{3.12}$$

The integral in α is convergent for x great enough, $\mathcal{R}$ being given by (3.7). The singularities of the Γ function are related to the behaviour of I_G as $H \to +\infty$ and need not be considered. The integral in α which defines the function

$$F_G(x) = M_G(x)/\Gamma(-x - \omega_G) \tag{3.13}$$

may be analytically continued into a meromorphic function of x, with real poles in decreasing order $x_0 > x_1 > \ldots > x_i > \ldots$. If we know the Laurent expansion of $M_G(x)$ around these poles, namely

$$M_G(x) = \sum_{p=1}^{p_{\max}(N)} \frac{a_{N,p}}{(x - x_N)^p} + \text{regular part at } x = x_N, \tag{3.14}$$

we obtain by inverse Mellin transform the asymptotic expansion of $I_G(p, H)$, that is:

$$I_G(H) \simeq \sum_{N=0}^{\infty} H^{-x_N} \sum_{p=1}^{p_{\max}(N)} \frac{a_{N,p}}{\Gamma(p)} \ln^{p-1}(1/H). \tag{3.15}$$

So, we have to analyse the singularities of the function $F_G(x)$ which arise, as for

$U.V.$ singularities in (3.3), from the divergent behavior of the integrand of (3.12) when some subset of α tends towards zero. We now recall the notions of [11] which are used to analyse those divergences.

Definition 3.1. Given a graph G and a set of external momenta $\{p_i\}$, a subgraph S of G is *essential* if, setting all α's relative to S equal to zero, the function $pd_G p$ is set to zero. This is equivalent to the fact that all external vertices belong to S, and that the sum of the momenta attached to any connected part of S is zero.

The notion of essential subgraph depends on the external momenta $\{p_i\}$. At non-exceptional momenta, that is, in the euclidean case, if any partial sum of p_i's is different from zero, a subgraph is essential if and only if all external vertices of G belong to the same connected part of S.

Definition 3.2. A set of subgraphs of G is *misjoint* if they have no line in common, and if the number of loops of their union is equal to the number of loops of the individual subdiagrams. (The subdiagrams may have vertices in common).

Definition 3.3. A set ψ of subgraphs of G is called a *Q-extended forest* if it satisfies:

(a) Any subset of mutually noninclusive elements is misjoint.

(b) The union of nonessential elements of ψ is nonessential.

(c) Every essential element E of ψ has no disconnected part, that it has no subgraph E' such that E' and $E - E'$ are disconnected and such that $E - E'$ is still essential.

(In [11] Bergère and Lam do not consider the condition *c*) but always consider Q-extended forests with this condition.)

Definition 3.4. An essential E has an inactive part E' if E' and $E - E'$ are misjoint and if $E - E'$ is still essential.

We now come back to the function $F_G(x)$. It is shown in [16] that (D being fixed away from an U.V. pole) the poles of the function $F_G(x)$ are given by essential subgraphs of G, and are characterized by

$$x + \omega(G) - \omega(E) = -n \qquad n \text{ positive or null integer,} \tag{3.16}$$

E being an essential of G.

However, from [12] and [16], as for the dimensionally regularized integral (3.6), for Re x away from such poles, the integral (3.12) is absolutely convergent and defines the function $F_G(x)$, provided that the $\mathscr{R}$ operator is given by the nest formula (3.7), and consequently subtracts not only U.V. divergent subgraphs, but also divergent essential subgraphs.

We now extend the results of Bergère and Lam [10, 11]. It is proved that the nest formula (3.7) for $\mathscr{R}$ may be replaced, when acting upon a function like in (3.12), by a formula over Q-extended forests, namely:

$$\mathscr{R} = 1 + \sum_{\psi Q\text{ext.forests}} \prod_{S \in \psi} (-\tau_S^{-\ell_S}) \tag{3.17}$$

where the sum runs over all Q-extended forests of divergent nonessential connected 1.P.I. subgraphs and of essential subgraphs (with no disconnected part from the Definition 3.3) of G.

If we scale by ρ the α's of a subgraph S in the integrand of (3.12) the first term of the Laurent expansion in ρ is proved to be [11]

$$\rho^{-\omega_S-\ell_S}\{\mathcal{D}_S P_S^{-D/2}\times\mathcal{D}_{[G/S]}\exp\left(-\sum_{[G/S]}\alpha\right)P_{[G/S]}^{D/2}pd_{[G/S]}p^{x+\omega_G}\}\tag{3.18}$$

if S is nonessential, and

$$\rho^{x+\omega_G-\omega_S-\ell_S}\left\{\mathcal{D}_S P_S^{-D/2}pd_S p^{x+\omega_G}\times\mathcal{D}_{[G/S]}\exp\left(-\sum_{[G/S]}\alpha\right)P_{[G/S]}^{-D/2}\right\}\tag{3.19}$$

if S is essential.

Since $F_G(x)$ has singularities only at values of x such that the action of some Taylor τ_S has a discontinuity, we deduce immediately that:

1) $F_G(x)$ has singularities given by (3.16), only for the essential subgraphs with no disconnected part. (Other essentials give no singularities)

2) x being fixed away from these poles, the only forests which contribute in (3.17) are forests of nonessential connected 1.P.I. subgraphs S such that $\omega_S > 0$ (to deal with U.V. singularities) and of essential subgraphs with no disconnected part such that

$$\mathrm{Re}\, x < \omega_E - \omega_G.\tag{3.20}$$

So, (3.12) gives an integral representation of $M_G(x)$ around any poles x_i. To extract the Laurent expansion of M_G at x_i, we may perform explicitly a Cauchy integral around x_i to compute residues. The general method presented here is now applied to the Green's functions of the $O(N)$ σ model.

B. The Case of Nonexceptional Momenta

The I.R. behaviour of an amplitude is in general different at exceptional and at nonexceptional momenta (since the essential subgraphs are different). So we first deal with the I.R. divergences of the $O(N)$ σ model at nonexceptional momenta (that is when any partial sum of external momenta is nonzero).

Let us consider a connected graph G that appears in the perturbative expansion of a N point function at $D = 2 - \varepsilon$ (with $N \geqq 2$). At nonexceptional momenta, any essential subgraph E without disconnected part in connected and contains all external vertices. Then we have

$$\omega_E - \omega_G = \frac{\varepsilon}{2}[L(G) - L(E)] - n_i\tag{3.21}$$

where n_i is the number of "interaction lines" in $G - E$ attached to E. (See Fig. 2). From (3.22), only essentials with $n_i = 0$ give poles at $x \geqq 0$. Let us call such essentials "leading essentials." (The leading essentials are the connected dominant subgraphs of G).

The function $F_G(x)$ has singularities at

$$x_L = \frac{\varepsilon L}{2} \qquad L = 0, \ldots, L(G)\tag{3.22}$$

since the superficial degree of convergence of G is

$$\omega_G = (1 - N) - \frac{\varepsilon}{2} L(G). \tag{3.23}$$

The singularities of F_G do not interfere with those of the function $\Gamma(-x-\omega_G)$ in the definition of $M_G(x)$. Moreover, from the general analysis of (**A**), the following poles of F_G are smaller than $-1+\frac{\varepsilon}{2}L(G)$. (See Fig. 5.)

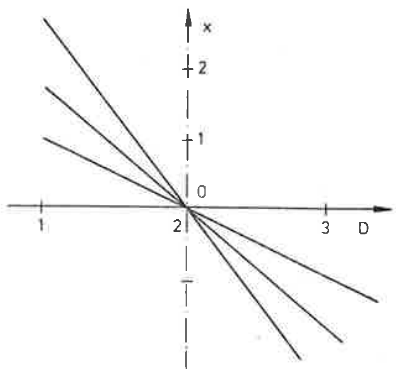

Fig. 5. Singularities in x of the Mellin transform $M_G(x)$ around $D = 2$

To extract the Laurent expansion around x_L

$$M_G(x) = \sum_{p=1}^{p_{\max}} a_p \frac{1}{\left[x - \frac{\varepsilon}{2}L\right]^p}, \tag{3.24}$$

we have only to write the Cauchy integral

$$a_p = \frac{1}{2\pi}\left[\int_{\substack{-\infty \\ \varepsilon L/2 < \sigma_+ < ((\varepsilon/2)(L+1))}}^{+\infty} dy M_G(\sigma_+ + iy)(\sigma_+ + iy - x_L)^{p-1} - \int_{\substack{-\infty \\ (\varepsilon/2)(L-1) < \sigma_- < \varepsilon L/2}}^{+\infty} dy M_G(\sigma_- + iy)(\sigma_- + iy - x_L)^{p-1}\right] \tag{3.25}$$

and to use the integral representation (3.12) of $M_G(x)$ in the bands

$$B_L = \left\{\frac{\varepsilon L}{2} < \operatorname{Re} x < \frac{\varepsilon}{2}(L+1)\right\} \tag{3.26a}$$

and

$$B_{L-1} = \left\{\frac{\varepsilon}{2}(L-1) < \operatorname{Re} x < \frac{\varepsilon}{2}L\right\}. \tag{3.26b}$$

From (3.17), in B_L, (respectively B_{L-1}) in the expression of $\mathscr{R}$, only Q-extended forests which contain leading essential subgraphs E such that $L(G) - L(E) > L$ (respectively $L(G) - L(E) \geqq L$) contribute. The Γ function in (3.12) ensures convergence at infinity of the integral (3.25). So, we invert integrations in y and α in (3.25), to obtain for a_p a convergent integral representation where only the Q-extended forests are present which contribute in B_{L-1} and do not in B_L, namely those which contain one leading essential such that $L(G) - L(E) = L$.

We get

$$a_p = -\int_0^{+\infty} \prod_G d\alpha \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy (\sigma_- + iy - x_L)^{p-1} \Gamma(-\sigma_- - iy - \omega_G)$$

$$\times \left(\sum_{\substack{\psi \text{ containing} \\ \text{a lead. ess. } E \\ \text{such that } L(G/E) = L}} \prod_{S \in \psi} (-\tau_S^{-\ell_S}) \left\{ \mathscr{D}_G \exp\left(-\sum_G \alpha \right) P_G^{-D/2} (pd_G p)^{\sigma_- + iy + \omega_G} \right\} \right). \tag{3.27}$$

Let us consider such a Q-extended forest ψ. It is obvious that it contains only one leading essential E such that $L(G/E) = L$, and that it is the greatest essential of ψ. The action of $\tau_E^{-\ell_E}$ onto the function $\{\ \}$ in (3.27) is given by (3.19); we get a factorization into a part relative to E and a part relative to $(G/E]$:

$$\tau_E^{-\ell_E}\{\ \} = [\mathscr{D}_E \cdot P_E^{-D/2} (pd_E p)^{\sigma_- + iy + \omega_G}] \times \left[\mathscr{D}_{[G/E]} \exp\left(-\sum_{[G/E]} \alpha \right) P_{[G/E]}^{D/2} \right] \tag{3.28}$$

where $[G/E]$ is the reduced graph where E is shrunk to a point in G. Reorganizing the sum order Q-extended forests as a sum over leading essentials E such that $L(G/E) = L$ and a sum over Q-extended forests containing E, and factorizing over forests of $[G/E]$ and of E we get finally for a_p, up to the problems of convergence of the integrals, which shall be discussed later:

$$a_p = \sum_{\substack{E \text{ leading ess.} \\ \text{such that} \\ L(G/E) = L}} \int_0^{+\infty} \prod_{[G/E]} d\alpha \left[1 + \sum_{\substack{Q \text{ ext} \\ \text{forests} \\ \text{in } [G/E]}} \prod_S (-\tau_S^{-\ell_S}) \right] \left[\mathscr{D}_{[G/E]} \exp\left(-\sum_{[G/E]} \alpha \right) P_{[G/E]}^{-D/2} \right]$$

$$\times \int_0^{+\infty} \prod_E d\alpha \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \, \Gamma(-\sigma_- - iy - \omega_G)(\sigma_- + iy - \omega(G/E))^{p-1}$$

$$\times \left\{ 1 + \sum_{\substack{Q \text{ ext. forests} \\ \text{in } E \text{ which do} \\ \text{not contain } E}} \prod_S (-\tau_S^{-\ell_S}) \right\} \left\{ \mathscr{D}_E P_E^{-D/2} (pd_E p)^{\sigma_- + iy + \omega_G} \right\}. \tag{3.29}$$

The term relative to $[G/E]$ is simply the regularized amplitude $I_{[G/E]}(H = 1)$. In the integral relative to E the sum runs over all Q-extended forests in E which do not contain E itself. Let us note this term R_E^p. To perform the integration over y, we have to make precise the dependence of the integrand in (3.29) on y. From the expansion properties (3.18) and (3.19), we have for a given Q-extended forest ψ:

$$\prod_{S \in \psi} (-\tau_S^{-\ell_S}) \mathscr{D}_E P_E^{-D/2} (pd_E p)^{\sigma_- + iy + \omega_G} = Z(\alpha, D) (pd_{[E_{\min}]_\psi^p})^{\sigma_- + iy + \omega_G} \tag{3.30}$$

where $E_{\min}$ is the smallest leading essential in ψ (if ψ does not contain a leading essential, we take $E_{\min} = E$), and where $[E_{\min}]_\psi$ is the reduced graph obtained by shrinking to a point every subgraph of ψ in $E_{\min}$.

We now perform the integration over y. Denoting $z = -\sigma_- - iy - \omega(G/E)$, we compute

$$\frac{1}{2i\pi}\int_{\mathrm{Im}\, z=-\infty}^{\mathrm{Im}\, z=+\infty} dz\,\Gamma(z-\omega_E)\left[pd_{[E_{\min}]^p_\psi}\right]^{-z+\omega_E} z^{p-1}$$
$$= (pd_{[E_{\min}]^p_\psi})\omega_E \quad G_p(pd_{[E_{\min}]^p_\psi}, \omega_E). \tag{3.31}$$

$G_p(x,\omega)$ is defined as

$$G_p(x,\omega) = \left(-x\frac{\partial}{\partial x}\right)^{p-1}(x^{-\omega}e^{-x}). \tag{3.32}$$

So finally for R^p_E we get the following integral representation

$$R^p_E = \int_0^\infty \prod_E d\alpha \sum_{\substack{\psi\ Q\text{-ext. forest}\\ \psi \not\ni E\\ (\psi\ \text{eventually empty})}} G_p(pd_{[E_{\min}]^p_\psi}, \omega_E)$$
$$\left[\prod_{S\in\psi}(-\tau_S^{-\ell_S})\right]\left[\mathscr{D}_E P_E^{-D/2}(pd_E\, p)^\omega_E\right]. \tag{3.33}$$

In (3.33) we sum over all Q-extended forests which do not contain E, including the empty forest. In fact, the R^p_E are null if $p > 1$. Indeed, let us scale α into $\lambda\alpha$ in the integral (3.33). We get the same integral, except that $G_p(pd_{[E_{\min}]^p_\psi}, \omega_E)$ is changed into $G_p(\lambda pd_{[E_{\min}]^p_\psi}, \omega_E)$. Differentiating with respect to λ and using (3.32) we get

$$\lambda\frac{\lambda}{\partial\lambda}\mathscr{R}^p_E = \mathscr{R}^{p+1}_E = 0, \tag{3.34}$$

so that only R^1_E is nonzero and shall be noted R_E in the following. The conclusion of this study is that $M_G(x)$ has a single pole at $x_L = \frac{\varepsilon}{2}L$. Its residue is given by

$$\mathrm{Res}_{\varepsilon L/2}\{M_G(x)\} = \sum_{\substack{\text{leading essential}\\ E\ \text{such that}\ L(G/E)=L}} I_{[G/E]}(H=1). \quad R_E(p). \tag{3.35}$$

This may be done at every pole $\frac{\varepsilon L}{2}$, so that we recover all leading essentials. By the inverse Mellin transform, we deduce immediately the exact asymptotic behaviour of $I_G(p,H)$ as $H \to 0$.

Theorem. 3.1. *Any regularized amplitude of the $O(N)$ σ model at nonexceptional momenta has the following I.R. asymptotic behaviour at $D = 2-\varepsilon$:*

$$I_G(p,H) = \sum_{n=L(G)}^{0} H^{-\varepsilon n/2} \sum_{\substack{\text{lead ess. } E\\ \text{such that } L(G/E)=n}} I_{[G/E]}(H=1)R_E(p) + 0(H^{1-\varepsilon L(G)/2}).$$

The divergent terms $H^{-\varepsilon L(G/E)/2} I_{(G/E)}$ (1) are relative only to reduced subgraphs $[G/E]$ and are, by homogeneity, equal to $I_{(G/E)}(H)$. The part $R_E(p)$ associated to a leading essential E is I.R. finite and is given by (3.33) with $p = 1$.

The finite part of the expansion (3.36) is the term at $n = 0$. The only leading essential such that $L(G/E) = 0$ is the graph G itself, so that the finite part of (3.36) is $R_G(p)$.

To complete the proof, we have yet to prove that the integral (3.33) which defines $R_E(p)$ is convergent. This is now obvious, because we know that

$$R_E(p) = \text{Residue at } x = 0 \quad \text{of } M_E(p, x).$$

$R_E(p)$ is directly given by the convergent integral representation (3.33) with $E = G, L = 0, p = 1$.

This suggests another representation of $R_G(p)$. We may extract the pole at $x = 0$ simply by scaling $\alpha \to \lambda\alpha$ in the integral representation (3.12) of $M_G(x)$. We thus get

$$\lambda^{-x} M_G(p, x) = \Gamma(-x - \omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\{\mathscr{D}_G e^{-\lambda\Sigma\alpha} p_G^{-D/2} (p d_G p)^{x+\omega_G}\}. \quad (3.37)$$

Differentiating with respect to λ and setting λ equal to one, we get

$$M_G(p, x) = \frac{1}{x}\Gamma(-x - \omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\left\{\mathscr{D}_G e^{-\Sigma_G \alpha} \left(\sum_G \alpha\right) p_G^{-D/2} (p d_G p)^{x+\omega_G}\right\}. \quad (3.38)$$

So we have

$$R_G(p) = \Gamma(-\omega_G) \int_0^{+\infty} \prod_a d\alpha_a \mathscr{R}\left\{\mathscr{D}_G e^{-\Sigma_G \alpha} \left(\sum_G \alpha\right) p_G^{-D/2} (p d_G p)^{\omega_G}\right\} \quad (3.39)$$

where $\mathscr{R}$ is given by (3.17)

The result of theorem 3.1 is clearly close to Lemma 2.1. But only connected dominants are present in the decomposition. We shall indicate in the next section how nonconnected dominants have to the introduced to take into account the problem at exceptional momenta.

We finally mention that in (3.36), some leading essentials give a zero contribution, that is to say $R_E = 0$. It is the case of the "essentials with inactive parts" (Definition 3.4) which may be proved not to contribute in the decomposition (3.36).

C. The Case of Exceptional momenta

However, we have not yet extracted all I.R. divergences. Indeed, the $R_G(p)$'s are well-defined functions of p at nonexceptional momenta, but they diverge at exceptional momenta, and so are not well-defined distributions of the p's. This difficulty occurs with the bare propagator:

The propagator $1/p^2$ is not a distribution at $D \leqq 2$, since the integral $\int d^D p \frac{1}{p^2}$ is not convergent at zero. As already mentioned in [8, 9], to obtain the well defined distribution $Fp\frac{1}{p^2}$ (Finite part of $\frac{1}{p^2}$, see [17]) we have to subtract a divergent term at $p^2 = 0$, since $Fp\left(\frac{1}{p^2}\right)$ is defined as

$$Fp\frac{1}{p^2} = \lim_{H \to 0} \frac{1}{p^2 + H} - \delta(p) \int d^D k \frac{1}{k^2 + H}. \tag{3.40}$$

This problem is not an academic one, since we have to define I.R. finite distributions in impulsion space if we want, for instance, to obtain a finite function in position space by the Fourier transform. So we have to analyze the behaviour of the functions $R_G(p)$ as the impulsions p tend toward some exceptional impulsions. This analysis needs different methods, since the methods used in 3. A only allow us to analyze the I.R. limit at fixed momenta. We shall not give a complete proof in the general case but shall discuss here two points:

—First the extraction of the singular part may be performed very easily for graphs of the 2-point function.

—Second we shall give arguments for the general form of decomposition which leads to Lemma 2.1.

Let us first look at the two-point function. By homogeneity, the function $R_E(p)$ relative to any leading essential E depends on the external impulsion p by a power

$$R_E(p) = \text{const}\,[p^2]^{-1-\varepsilon L(E)/2} \tag{3.41}$$

and is not a distribution. The finite part of $R_E(p)$ is defined as for the propagator by

$$F_E(p) = R_E(p) - \delta(p) \int d^D k R_E(k). \tag{3.42}$$

We incorporate this equation in the asymptotic expansion (3.36) of $I_G(p, H)$ (G being a graph of the two point function) and get

$$I_G(p, H) = \delta(p) \int d^D k I_G(k, H) + \sum_{\substack{\text{lead ess.} \\ E \subseteq G}} F_E(p) I_{[\widetilde{G/E}]}(H) + \ldots \tag{3.43}$$

The expansion is now made in terms of well defined distributions, the new term may be rewritten

$$\delta(p) \int d^D k I_G(k, H) = F_{S_0}(p) I_{[\widetilde{G/S_0}]}(H) \tag{3.44}$$

where $\delta(p) = F_{S_0}(p)$ is the contribution (I.R. finite) of the graph S_0 which contains only the two external vertices and where $[\widetilde{G/S_0}]$ is the graph obtained from G by reducing to a point S_0 (Fig. 3).

We now generalize this result. It follows from the general arguments of (A) that, at exceptional momenta, the essentials which give the leading I. R. singularities are not the "essential leadings" described in (B). More precisely, any set of exceptional momenta is characterized by a (unique) partition of the external vertices,

$$\{1 \ldots N\} = I_1 \cup \ldots \cup I_p \qquad (p \geqq 1) \tag{3.45}$$

such that the subfamily of external momenta attached to any element of the partition I_q has its sum equal to zero and any partial sum nonzero. Then the analysis of (B) may be performed.

The leading subgraphs are then subgraphs E of G with p disconnected parts $E_1 \ldots E_p$, each E_q being a "leading essential" for the set of momenta relative to I_q, that is to say (see Fig. 6):

—E_q is connected and contains the vertices of I_q.
—No interaction line of $[G/E]$ is attached to E.

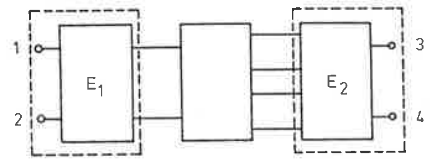

Fig. 6. Example of a leading essential $E = E_1 \cup E_2$ of a 4-point graph at exceptional moment $p_1 + p_2 = 0$

One may obtain an asymptotic expansion of the infrared divergences at exceptional momenta analog to (3.36), where the leading subgraphs previously defined play the role of the leading essential in (3.36). Those infrared divergences are of course more important than at nonexceptional momenta, since the amplitudes diverge at least as rapidly as a negative integer power of H. We expect that a quite similar expansion is obtained for $R_G(p)$ when the momenta p (nonexceptional) tend toward exceptional ones, and that, taking into account all cases of exceptional momenta, the I.R. distribution "Finite part" of $R_G(p)$ may be expressed in terms of all possible leading essentials of G that is precisely all "dominant subgraphs" of G (see Definition 2.1), and of I.R. divergent parts relative to the corresponding reduced graphs, as done for the two-point function. As explained before, a complete discussion of this problem should be much more cumbersome than the analysis of (B) and shall not be presented here. We present only the expected result, which appears to be the natural generalization of Theorem 3.1, where all possible leading essentials (in all cases of external momenta) are present.

The regularized amplitude of a graph G has the following I.R. asymptotic expansion in terms of finite distributions (in momentum space) relative to the dominant subgraphs of G:

$$I_G(p,H) = \sum_{\substack{E \subseteq G \\ \text{Dominant} \\ \text{Subgraphs}}} F_E(p) I_{[\widetilde{G/E}]}(H) + O(H^{1-\varepsilon L(G)/2}). \tag{3.46}$$

In (3.46) the sum runs over all dominant subgraphs E of G. $F_E(p)$ is a finite distribution of the external momenta, which is in fact the "finite part" of the function $R_E(p)$. $I_{[\widetilde{G/E}]}(H)$ is the amplitude of the graph obtained by shrinking the graph E to one point. As in Theorem 3.1, this term diverges as a pure power of $H^{-\varepsilon/2}$, namely $H^{-(\varepsilon/2)L(\widetilde{G/E})}$ (except for the graph G itself).

Equation (3.46) coincides with the Theorem 2.1 at nonexceptional momenta. Indeed, in that case, only connected leading essentials give a contribution and the distribution $F_E(p)$ coincides with the function $R_E(p)$.

This result is also in agreement with the decomposition of the two-point function given in (3.43). Indeed, at $p=0$, the only dominant subgraph which has to be taken into account is the graph S_0 (composed only of the two external vertices). The other dominant subgraphs necessarily have an inactive part (which is the subgraph itself) and so give a zero contribution, from the remark that ends part (B).

Of course, we have checked Eq. (3.46) on simple four point graphs. This equation is precisely the result of Lemma 2.1 (expressed in momentum space instead of position space).

4. Conclusion

In this paper we have presented a general proof of the perturbative I.R. finiteness of the vacuum expectation value of any $O(N)$ invariant function of the $O(N)$ nonlinear σ-model. The proof was performed by using general methods of extracting the I.R. divergences of *any* amplitude of the perturbative expansion. We recall that the extraction is made in Sect. 3 in the case of nonexceptional momenta and extended (but not completely proved) at exceptional momenta. The general I.R. behaviour allows us to exhibit in Sect. 2 the mechanism of cancellations.

For explicit perturbative computations, we have shown that we may replace the "bare" I.R. divergent amplitudes I_G by I.R. subtracted ones, the R_G given by Eq. (3.39), provided that we deal with invariant quantities. It may be shown that, for non-invariant quantities, this operation is equivalent to an average of the orientation of the symmetry breaking magnetic field H in (2.4). The amplitude of the magnetic field also has to be modified, and the average has to be performed with some weight over the sphere; this weight is related perturbatively to the I.R. counterterms of (2.11) in some complicated way, and so diverges as the symmetry breaking term tends towards zero.

The result of this paper is that, as expected, the perturbative expansion of any invariant function of the $O(N)$ two dimensional σ model is free of I.R. divergences although computed in the wrong phase where the symmetry is spontaneously broken. However, as argued in [8], the fact that the symmetry is dynamically restored so that there are no more long distance correlations, may not be seen at any order of the perturbative expansion (where there are always such correlations), but only by dealing with the full Green's functions.

Let us finally mention that the analysis of I.R. divergences at two dimensions presented here may be extended to other two-dimensional models, for instance the chiral models or the generalized σ-models [18], where such cancellations of I.R. divergences are also expected to occur.

Appendix A

Given a graph G belonging to $\mathscr{G}_N^0(\mathcal{O})$ and a dominant subgraph E in G, E belongs to some $\mathscr{G}_M^P(\mathcal{O})$ ($M \leqq N, P \leqq M$) where M is the order of E and $2P$ is the number of lines of $G-E$ attached to E. Similarly, let us consider the graph $G-E$ as a graph of order $(N-M)$ of the perturbative expansion of the operator:

$$\frac{1}{2P!}[\int d^D x \vec{\pi}.]^{2P}.$$

The counting factor $C(G)$ is the number of contractions which leads to G by applying Wick theorem in the expansion of $\langle \mathcal{O} \rangle$. Separating those contractions into contractions which lead to E plus contractions which lead to $G-E$ plus contractions between E and $G-E$ which lead to G we get the following relation between factors of G, E and $G-E$.

$$n_1 C(G) = n_2 n_3 C(E) C(G-E), \tag{A.1}$$

where n_1, n_2 and n_3 are defined as:

n_1 = number of ways to decompose G into E and $G-E$.

n_2 = number of ways to link the $2p$ lines of $G-E$ to E in order to reobtain G.

$n_3 = (N-1)^q$ where q is the number of lines carrying the $(N-1)$ internal indices of the field $\vec{\pi}$ closed by the former operation which get G from $G-E$ and E.

n_1 and n_2 are factors coming from Wick theorem, n_3 is a symmetry factor depending on the group $O(N)$.

Now, from (A.1) we may reorganize the sum over the G in (2.11) as a sum over the dominant subgraphs E times a sum over the graphs $G-E$, where we link all "free" lines of $G-E$ to a point in all the possible ways. We then obtain the reduced graphs $[\widetilde{G/E}]$ as graphs of the perturbative expansion of $(\vec{\pi}^2(0))^p$, with the corresponding counting factor. We thus obtain

$$\sum_{\mathcal{G}^0_N(\mathcal{O})} C(G) F_G = \sum_{\mathcal{G}^0_N(\mathcal{O})} C(G) I_G + \sum_{\substack{0 \leqq M \leqq N \\ 0 \leqq P \leqq M}} \Bigg(\sum_{\mathcal{G}^P_M(\mathcal{O})} C(E) I_E \Bigg) \Bigg(\sum_{\mathcal{G}^0_{M-N}(\vec{\pi}^{2P})} C(S) A_S \Bigg) \tag{A.2}$$

which gives (2.24).

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CANCELLATIONS OF IR DIVERGENCIES IN TWO-DIMENSIONAL CHIRAL MODELS

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For all two-dimensional chiral models which have a global symmetry, the invariant observables are proved to have an infra-red finite weak coupling perturbative expansion.

The two-dimensional σ-models (or chiral theories) have been extensively studied during the last years. Their geometrical structure leads, for a large class of models, to the existence of non-local [1] or local [2] classical conservation laws, which are proved to be preserved by quantization for the $O(N)\,\sigma$-model and lead to the factorization of the S matrix [3]. Another point of interest is the similarities of these models with four-dimensional gauge theories: asymptotic freedom, dynamical restoration of symmetry and the non-perturbative character of the particle spectrum.

In particular, the usual weak coupling expansion is performed in the spontaneously broken symmetry phase, and suffers from important IR divergencies. This is related to the fact that this phase cannot exist, from the Mermin–Wagner theorem [4]. It was conjectured by Elitzur [5], and proved by the present author [6], that, for the $O(N)$ σ-model, that those IR divergencies cancel for any $O(N)$ invariant observable (another analogy with what is expected in four-dimensional gauge theories).

In this paper we prove that this property is satisfied by all σ-models which possess a group of invariance, namely the models constructed on some homogeneous space. Such a space may be considered as the space G/H, where G is some Lie group and H some compact subgroup of G, or as a riemannian space E such that the metric is "the same" in the neighbourhood of every point of E (i.e., the group of transformations preserving the metric tensor sends any point of E in the whole space E). These two definitions are proven to be equivalent [7].

To deal with the most general models, we adopt the second, geometrical point of view. Given a riemannian space E, and considering some coordinate system $(\xi^i)_{i=1,N}$, a chiral field $\boldsymbol{\xi}$ with value into E may be constructed, whose euclidean action is

$$A_{HM} = \int \mathrm{d}^D x\, \tfrac{1}{2}\left[\partial_\mu \xi^i \partial_\mu \xi^j g_{ij}(\boldsymbol{\xi}) + HM(\boldsymbol{\xi})\right] , \tag{1}$$

g_{ij} is the metric tensor, $HM(\boldsymbol{\xi})$ is a "mass term" breaking geometrical invariance at the point $\boldsymbol{\xi} = 0$. We consider a general term of the form

$$M(\boldsymbol{\xi}) = M_{ij}\xi^i\xi_j + M_{ijk}\xi^i\xi^j\xi^k + \dots , \tag{2}$$

(M_{ij}) being symmetric positive definite. The vacuum expectation value of any observable $F(\boldsymbol{\xi})$ is given by the functional integral

$$\langle F\rangle_{HM} = \frac{1}{Z_0} \int \mathcal{D}[\boldsymbol{\xi}]\, F(\boldsymbol{\xi}) \exp(-t^{-2} A_{HM}) , \tag{3}$$

where $\mathcal{D}[\boldsymbol{\xi}] = \Pi_x\, \mathrm{d}\boldsymbol{\xi}(x)(|g(\boldsymbol{\xi})|)^{1/2}$ is the invariant measure and t the coupling constant.

The perturbative expansion is obtained by expanding the metric tensor around $\boldsymbol{\xi} = 0$

$$g_{ij}(\boldsymbol{\xi}) = g_{ij} + \xi^k g_{ij,k} + \xi^k\xi^l g_{ij,kl} + \dots , \tag{4}$$

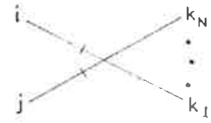
so that we get the free propagator $D^{ij}(x-y)$ as the inverse of the operator ‡1

$$D^{-1}_{ij} = -g_{ij}\Delta_x + HM_{ij} . \tag{5}$$

‡1 Diagonalizing M_{ij} in an orthonormal basis of g_{ij} we can express the propagator as a sum of usual scalar propagators: $D^{ij}(p) = \Sigma_{k=1}^{N} D^{ij}_{(k)}/(p^2 + Hm_k^2)$.

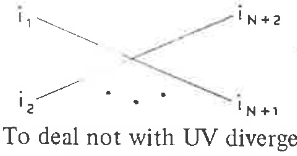
Volume 96B, numbers 3, 4 PHYSICS LETTERS 3 November 1980

We have two kinds of interaction vertices of order $t^N (N \geqslant 1)$ corresponding (a) to the expansion of g (g-vertices):



$$-\frac{t^N}{2} g_{ij,k_1 \ldots k_N}(p_i p_j) , \qquad (6a)$$

(b) to the expansion of M (m-vertices):



$$-\frac{t^N}{2} HM_{i_1 \ldots i_{N+2}} . \qquad (6b)$$

To deal not with UV divergencies we use dimensional regularization and calculate amplitudes at dimension $D = 2 - \epsilon$. Parametrization invariance is not broken and the measure terms are known to disappear.

We now analyse the IR divergencies of this general model. As the mass term is set to zero, the perturbative expansion diverges at dimension $D \leqslant 2$, since even the bare propagator becomes $g^{ij} p^{-2}$ and is not a well defined distribution [8]. In ref. [6], general methods of analysis of IR behaviour were developed and applied to the regularized O(N) model at dimension $D = 2 - \epsilon$. Those methods may be adapted to the general chiral case; indeed the IR structure of the graphs is the same, and only the algebra, related to the structure of the space E, is modified. We give only the final result of this study.

We consider a graph G which appears in the perturbative expansion of some function of the fields $O(\boldsymbol{\xi}(x))$. At dimension $D = 2 - \epsilon$, the amplitude $I_G(x, H)$ of the graph G has an IR asymptotic expansion in powers of $H^{-\epsilon/2}$ in terms of "dominant subgraphs" [6] E of G:

$$I_G(x, H) = \sum_{\substack{E \subseteq G \\ \text{dominant}}} F_E(x) \cdot I_{[\widetilde{G}/E]}(H) + O(H^{1-\epsilon L(G/2)}) , \qquad (7)$$

$F_E(x)$ is the finite part of the amplitude of the dominant E and so is IR finite. The $I_{[\widetilde{G}/E]}$'s are the IR divergent parts of the expansion (7) (They diverge like a power of $H^{-\epsilon/2}$) and are the amplitudes of the graphs $(\widetilde{G}/E)$ obtained by shrinking E into one vertex into G.

The dominant subgraphs E of G are the subgraphs of G which may be considered as graphs of the operator O with "connected truncated insertions" of the field ξ^i at zero momenta, that is graphs of the operator

$$O^{\mathrm{c}}_{i_1 \ldots i_P}(x) = \frac{1}{P!} O \left\{ \prod_{\alpha=1}^{P} HM_{i_\alpha j_\alpha} \int \mathrm{d}x \, \xi^{j_\alpha}(x) \right\}_{\mathrm{C}} , \qquad (8)$$

for some P. (The "C" means that graphs with disconnected part, where there are only insertions of ξ are not taken into account.) The reduced graphs $[\widetilde{G}/E]$ may be considered as a graph of the operator

$$D^{i_1 \ldots i_P} = \xi^{i_1}(x) \ldots \xi^{i_P}(x) , \qquad (9)$$

at some point x. From eq. (7) we construct the IR asymptotic expansion of the operator O at dimension $2 - \epsilon$.

$$\langle O(x) \rangle_{HM} = \sum_{P=0}^{\infty} \mathrm{f.p.} \langle O^{\mathrm{c}}_{i_1 \ldots i_P}(x) \rangle \times \langle D^{i_1 \ldots i_P} \rangle_{HM} . \qquad (10)$$

The "f.p." means the IR finite part of the operator $O^{\mathrm{c}}_{i_1 \ldots i_P}$ as $H \to 0$. The $\langle D^{i_1 \ldots i_P} \rangle_{HM}$ are the divergent parts of the expansion and diverge (perturbatively) as a power of $H^{-\epsilon/2}$. This expansion is valid at order N provided that $\epsilon < 2/N$, then only operators such that $P \leqslant N$ are present. (The summation over indices $i_1 \ldots i_P$ is understood.) Formula (10) is valid for any observable of any model. As pointed out in ref. [6], this result is quite similar to the Wilson operator product expansion [9].

An observable O will be IR finite only if the operators $O^{\mathrm{c}}_{i_1 \ldots i_P}(x)$ have a zero limit as $H \to 0$ for any $P > 0$, so that $\langle O \rangle = \mathrm{f.p.} \langle O \rangle$. We now prove that this is true only for invariant observables of models defined on homogeneous spaces. We consider some homogeneous riemannian space E and some coordinates ξ^i (with origin 0). To any point A of E we associate an isometric transformation τ_A on E which sends O into A and defines a new coordinate system $\bar{\xi}^i_A$ in a neighbourhood of A such that the metric tensor in the new system ($\bar{g}_A$) is the same as in the first one (g). We may then consider the coordinates ξ^i of a point of E as a function of its coordinates $\bar{\xi}^j_A$ in the new system and of the coordinates α^k of A in the first system.

$$\xi^i = \xi^i(\bar{\xi}^j_A, \alpha^k) . \qquad (11)$$

Moreover, the isometries τ_A are choosen such that, at the origin, this function is infinitely differentiable versus $\bar{\xi}^j_A$ and α^k. We now consider some invariant observable $O(g, \boldsymbol{\xi})$. (For instance the riemannian distance between the field at two points $\boldsymbol{\xi}(x)$ and $\boldsymbol{\xi}(y)$.)

The vacuum expectation value of O is expressed by the functional integral (3). If we made the change of variable $\xi \to \bar{\xi}_A$ into (3), the measure, the observable O and the free action are invariant, so that:

$$\langle O(\xi)\rangle_{HM} = \frac{1}{Z_0}\int \mathcal{D}[\bar{\xi}_A]\; O[\bar{\xi}_A]$$

$$\times \exp\left[-\frac{1}{t^2}\left(A_0[\bar{\xi}_A] + \int d^D x\; HM(\xi)\right)\right], \qquad (12)$$

the only change occurs in the symmetry breaking term:

$$M(\xi) = M(\bar{\xi}_A, \alpha)$$

$$= M(0,\alpha) + 2\bar{\xi}^i_A D_i(\alpha) + \bar{M}(\bar{\xi}_A, \alpha)\,. \qquad (13)$$

In eq. (13) we have separated the two first terms of the expansion of M around $\bar{\xi}_A = 0$, so that $\bar{M}$ has the expansion:

$$\bar{M}(\bar{\xi}_A, \alpha) = \bar{M}_{ij}(\alpha)\bar{\xi}^i_A\bar{\xi}^j_A + \bar{M}_{ijk}(\alpha)\bar{\xi}^i_A\bar{\xi}^j_A\bar{\xi}^k_A + \dots\,. \qquad (14)$$

Defining a new parameter $\bar{\alpha}$ as

$$\bar{\alpha}^i = D_k(\alpha)\bar{M}^{ki}(\alpha) = \alpha^i + O(\alpha^2)\,, \qquad (15)$$

we may invert the relation between the functions M and $\bar{M}$ and express the coefficients of the expansion of M, namely the $M_{i_1\dots i_k}$, as functions of $\bar{\alpha}$ and of the coefficients $\bar{M}_{i_1\dots i_n}$. We have in fact a linear relation between M and $\bar{M}$:

$$M_{i_1\dots i_n}(\bar{\alpha}) = \sum_{2\leqslant P\leqslant N} c^{j_1\dots j_P}_{i_1\dots i_N}(\bar{\alpha})\bar{M}_{j_1\dots j_P}\,, \qquad (16)$$

where the c's depend on $\bar{\alpha}$ (and are determined by the metric g and the isometry τ_A). We shall note the functional relation of M with $\bar{M}$ and $\bar{\alpha}$ by

$$M = M[\bar{M}, \bar{\alpha}]\,. \qquad (17)$$

Obviously, as $\bar{\alpha} = 0$, we have $M[\bar{M}, O] = \bar{M}$. With this notation, putting eq. (13) into eq. (12) and performing the same change of variables into Z_0, we get the fundamental identity valid for any invariant observable O:

$$\langle O(t\xi)\rangle_{H\bar{M}[M,t\alpha]}$$

$$= \frac{\langle O(t\xi)\exp[-H\alpha^i M_{ij}\int \xi^j\, dx]\rangle_{HM}}{\langle \exp[-H\alpha^i M_{ij}\int \xi^j\, dx]\rangle_{HM}}\,. \qquad (18)$$

In eq. (18) we have scaled $\xi \to t\xi$ and $\alpha \to t\alpha$ in order to obtain the usual weak coupling perturbative expansion, and inverted the notation M and $\bar{M}$. If we expand the r.h.s. of eq. (18) in powers of α, we note that the term of order P is precisely the vacuum expectation value of the operator $O^c_{i_1\dots i_p}$ defined in eq. (8).

$$\text{r.h.s.}\,(18) = \sum_{P=0}^{\infty} (-1)^P \alpha^{i_1}\dots\alpha^{i_P}\langle O^c_{i_1\dots i_P}\rangle_{HM}\,. \qquad (19)$$

Indeed, expanding the exponential at the numerator we get $1/p!$ times P insertions of $HM_{ij}\xi^j$ at zero momenta, and the disconnected insertions are eliminated by the denominator.

The l.h.s. of eq. (18) corresponds to a renormalization of the symmetry breaking term which depends on $t\alpha$. Then the term of order P in α ($P \geqslant 1$) and of order N in t is related to derivatives versus the symmetry breaking term (that is versus the $M_{ij\dots}$) of terms of order $N' < N$ of the perturbative expansion of $\langle O\rangle$.

So, if the terms of order $N' < N$ (in t) of $\langle O\rangle$ have been proved to be IR finite, and so independent of M as $H \to 0$, the terms of order $N' \leqslant N$ of $O^c_{i_1\dots i_p}$ are proved to have a zero IR limit as $H \to 0$ for $P \geqslant 1$. Their finite parts are then equal to zero and from eq. (10) (which gives the asymptotic IR behaviour of $\langle O\rangle$) we deduce that the term of order N of $\langle O\rangle$ is IR finite. So a recursive proof shows that any invariant observable of the regularized theory at ($D = 2 - \epsilon$) is IR finite at any order of perturbative expansion.

This result holds independently of the regularization (for instance for the lattice theory). We do not deal with the problems of renormalization at $D = 2$ (continuous limit) nor with the physics of such general models. The models defined on symmetric spaces are proved to be renormalizable and the counterterms are IR finite [10,11] so that renormalization preserves the IR finitness of invariant observables. These models include the CP^N models, the grassmannian models and the principal chiral fields. It seems very plausible that those IR cancellations are also present in the supersymmetric extensions of these σ-models [12].

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QUANTIZATION WITH A GLOBAL CONSTRAINT AND IR FINITENESS OF TWO-DIMENSIONAL GOLDSTONE SYSTEMS

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A method of quantizing Goldstone systems with a global continuous symmetry by introducing a global constraint is presented. This procedure is used to study the IR finiteness of the weak coupling expansion of such models at two dimensions. In this scheme, the propagator is IR finite at any dimension and all observables are proved to be IR finite at $d = 2$. The IR properties of non-standard models are elucidated.

1. Introduction

It was recently proved that the infrared divergences which occur in the weak coupling perturbative expansion of two-dimensional field theories and statistical systems with a global continuous symmetry disappear when looking at invariant quantities [1–3], although the expansion is performed in the spontaneously broken symmetry phase, which does not exist at two dimensions [4, 5]. The proof of this fact [2], and the previous calculations on such models [6–10], involve the introduction of an extra symmetry-breaking term which makes the perturbative expansion finite; this term being set to zero at the end of calculations in order to recover the IR limit.

In this paper we adopt a different approach to this problem. We show that it is possible to introduce a global constraint in the quantization of those models in such a way that the perturbative expansion of any quantity is IR finite at two dimensions. This new quantization procedure coincides with the usual one only for invariant observables, where we recover the usual IR-finite perturbative expansion.

If this approach seems simpler in principle than the direct proof of ref. [2], we shall use, in fact, the technical results of those references to prove the IR finiteness of the new perturbative expansion. However, this new expansion leads to more complicated calculations than the usual one, so we think that its interest lies in a new insight in the IR properties of two-dimensional Goldstone systems.

This paper is organized as follows: for simplicity, we treat the euclidean O(N) non-linear σ model in detail. In sect. 2 we present the quantization with a global constraint; for sake of rigor the study is performed on a lattice regularized theory. We prove that the introduction of our constraint leads only to a change in the propagator which becomes IR finite. The IR finiteness of the corresponding perturbative

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expansion is proved in sect. 3. Finally, we discuss the case of other Goldstone models in sect. 4. In particular, our approach allows us to study the IR structure of the non-standard models introduced by Friedan in refs. [11, 12] and to prove that it is always possible to define an IR-finite perturbative limit, but not in a standard way.

2. Quantization of the O(N) σ model with a global constraint

We consider the euclidean O(N) σ model on a two-dimensional square lattice of spacing a and size L with periodic boundary conditions. The action is

$$A = \frac{1}{2t} \sum_x a^2 [\nabla_\mu \boldsymbol{S}(x)]^2 , \tag{2.1}$$

where $\boldsymbol{S}(x)$ is an N-component real field restricted on the sphere S_{N-1} by the constraint

$$|\boldsymbol{S}(x)|^2 = 1 \tag{2.2}$$

and where ∇_μ runs for the finite difference between two nearest neighbours in direction μ ($\mu = 1, 2$).

The weak coupling perturbative expansion is obtained by setting

$$\begin{aligned} S^i(x) &= \sqrt{t}\pi^i(x) \qquad i = 1, N-1 \\ &= \sigma(x) = \sqrt{1 - t\boldsymbol{\pi}^2(x)} \qquad i = N , \end{aligned} \tag{2.3}$$

and by expanding in powers of t the integral generating the partition function, which reads, up to non-perturbative terms

$$Z(t) = \int \prod_x \mathrm{d}\pi^i(x) \exp\left[-\tfrac{1}{2} a^2 \sum_x (\nabla_\mu \pi^i)^2 - \frac{a^2}{2t} \sum_x (\nabla_\mu \sigma)^2 - \sum_x \ln \sigma \right] . \tag{2.4}$$

As long as the size of the system, L, is finite, we expect no IR divergences. However, we have always $N-1$ Goldstone modes which make the propagator undefined. Indeed, the quadratic part of the action is

$$A_0 = \frac{1}{2L^2} \sum_p \hat{\pi}^i(p) \left[\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} \right] \hat{\pi}^i(-p) , \tag{2.5}$$

where $\hat{\pi}(p)$ is the Fourier transform of $\pi(x)$:

$$\pi(x) = \frac{1}{L^2} \sum_p \mathrm{e}^{-ipx} \hat{\pi}(p) . \tag{2.6}$$

So A_0 has a zero eigenvalue corresponding to the $N-1$ modes $p = 0$ and is not invertible. As explained in sect. 1, this difficulty was usually removed by introducing a symmetry breaking term (this is equivalent to introduce a mass and makes the propagator well defined), then by taking the infinite volume limit and finally by using the fact that O(N) invariant quantities have an IR-finite limit when the symmetry-breaking term is set to zero [1, 2].

On the other hand, those Goldstone modes may be eliminated by introducing a global constraint which fixes the classical solution around which fluctuations are computed. The most simple way in our case is to fix the field $\boldsymbol{S}$ in a given direction at some arbitrary point x_0. So let us introduce the constraint

$$\boldsymbol{S}(x_0) = \boldsymbol{S}_0 \tag{2.7}$$

in the integral (2.4). We get

$$Z_{S_0}(t) = \int \mathscr{D}[\boldsymbol{S}]\delta_{S_0}[\boldsymbol{S}(x_0)] \exp -A[\boldsymbol{S}] . \tag{2.8}$$

$\delta_{S_0}(\boldsymbol{S})$ is the Dirac measure at the point $\boldsymbol{S}_0$ on the sphere S_{N-1}. $\mathscr{D}[\boldsymbol{S}]$ is the integration measure $\prod_x \mathrm{d}\pi^i(x)/\sigma(x)$. In the same way, we define the average value of any function of the field $F[\boldsymbol{S}]$ with constraint (2.7) as

$$\langle F \rangle_{S_0} = \frac{1}{Z_{S_0}} \int \mathscr{D}[\boldsymbol{S}]\delta_{S_0}[\boldsymbol{S}(x_0)]F[\boldsymbol{S}] \exp -A[\boldsymbol{S}] . \tag{2.9}$$

We now look at the corresponding perturbative expansion. Choosing $\boldsymbol{S}_0$ as the direction σ in (2.3), the constraint reads

$$\boldsymbol{\pi}(x_0) = 0 , \tag{2.10}$$

or equivalently in dual space

$$\sum_p \hat{\pi}^i(p)\, \mathrm{e}^{-ipx_0} = 0 . \tag{2.11}$$

If we eliminate the zero modes in (2.4) via (2.11), the quadratic part of the action is now given by (2.5), where the summation over p is restricted to $p \neq 0$ and is now invertible. Inverting A_0 and inserting constraint (2.11) in (2.6), we get for the propagator (in position space)

$$D(x, y) = \frac{1}{L^2} \sum_{p \neq 0} \frac{(\mathrm{e}^{-ipx} - \mathrm{e}^{-ipx_0})(\mathrm{e}^{ipy} - \mathrm{e}^{ipx_0})}{(4/a^2) \sum_\mu \sin^2(\frac{1}{2}ap_\mu)} . \tag{2.12}$$

The propagator is now well defined, but translation invariance has been broken by the constraint, so the propagator depends on x_0.

Moreover, it is easy to see that interaction terms in (2.4) are not modified by the constraint. So, in the perturbative expansion defined by constraint (2.7), the graphs are the same as in the usual perturbative expansion, only the propagator is modified and is given by (2.12).

The crucial point is that this propagator remains finite in the infinite volume limit. Indeed, as $L \to \infty$, the summation over p becomes an integration in the first Brillouin zone, but the integral is convergent at $p = 0$. So the new propagator $D(x, y)$ is really

IR finite. This procedure may, of course, be performed at any dimension $d \geq 1$ without changing this conclusion.

The previous study, performed on a lattice for sake of rigor, shows that the introduction of constraint (2.7) in the quantization leads only to a change in the propagator. This procedure may be performed whatever the ultraviolet regulator is. Indeed, from (2.12), the new propagator D is related to the usual IR-divergent one D_0 by

$$D(x, y) = D_0(x, y) - D_0(x, x_0) - D_0(x_0, y) + D_0(x_0, x_0) . \tag{2.13}$$

Eq. (2.13) may be generalized to any kind of regularized propagator D_0 and is sufficient to define an IR-finite propagator D (see fig. 1).

3. IR finiteness of the constrained perturbation at two dimensions

We now study the IR structure of this new perturbative expansion at two dimensions. As explained in sect. 1, for an arbitrary function $F[\boldsymbol{S}]$, its average value $\langle F\rangle_{S_0}$ appears to be IR finite, whether F is O(N) invariant or not.

This is not a surprising fact. Indeed, the constrained average value of F (F being a function of the field $\boldsymbol{S}$ at points $x_1 \dots x_p$) may be written as the unconstrained average value of some O(N) invariant function of $\boldsymbol{S}$ at the points $x_0, x_1 \dots x_p$ (x_0 being the point where the constraint is fixed). So, the IR finiteness of the constrained perturbative expansion is equivalent to the IR finiteness of invariant observables in the usual perturbative scheme.

However, it is interesting to look at the structure of the IR divergences and at their cancellations in this new perturbative scheme: IR divergences of any graph appear to be contained only in "disconnected parts" generated by the new propagator. Up to these "disconnected parts" which give divergences in power of the volume, the remaining "connected parts" of *any* graph are IR finite.

Moreover, a simple argument connects the disconnected parts to the partition function and shows that they combine into an IR-finite contribution depending only on geometrical properties of the model.

From (2.13), any graph computed with the new propagator D may be expressed in terms of graphs computed with the usual one D_0. More precisely, let us consider some function F and some graph G of the perturbative expansion of $\langle F\rangle$; we have

$$G[D] = \sum_{G \supseteq \bar{G}} (-1)^{n(\bar{G})} \bar{G}[D_0] , \tag{3.1}$$

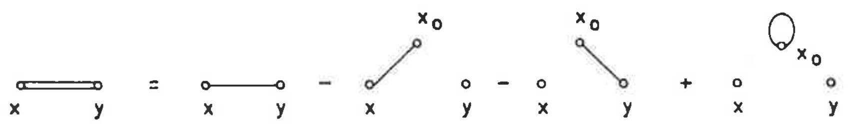

Fig. 1. The new propagator D expressed in terms of the usual one D_0.

where $G[D]$ ($G[D_0]$) is the integral of the graph G computed with the propagator D (D_0). The sum runs over all graphs $\bar{\mathrm{G}}$ obtained from G by disconnecting an arbitrary number $n(\bar{\mathrm{G}})$ of end points of lines of G and by attaching them at the point x_0; a factor (-1) is attached to each disconnected end point. An additional IR cut off is introduced in order to define D_0 (for instance we may take for D_0 its usual lattice form where we have eliminated the zero mode):

$$D_0(x-y)=\frac{1}{L^2}\sum_{p\neq 0}\frac{\mathrm{e}^{-ip(x-y)}}{(4/a^2)\sum_\mu \sin^2(\frac{1}{2}ap_\mu)}\,. \tag{3.2}$$

It is easy to see that this operation gives zero if a derivative coupling is attached to the end point of a propagator which is disconnected. So the former operation, which shall be denoted $\mathscr{D}$, has only to be performed on end points which are not attached to derivatives of the field.

As long as the graphs $\bar{\mathrm{G}}$ remain connected to the external vertices, the analysis of ref. [2] shows that $\bar{G}[D_0]$ would have logarithmic IR divergences*. A more serious problem comes from the graphs $\bar{\mathrm{G}}$ which have disconnected parts (not attached to external vertices), since those disconnected parts will give contributions in power of the volume. In order to treat those divergences, we first have to give a rigorous formulation to the above considerations.

Definition

Let $\bar{\mathrm{G}}$ be a graph obtained from G by the operation $\mathscr{D}$.

The disconnected part $\mathrm{V}(\mathrm{G},\bar{\mathrm{G}})$ associated to $\bar{\mathrm{G}}$ in G is defined as the greatest disconnected graph $\mathrm{V}\subset\mathrm{G}$ which may be obtained at some stage of the operation when disconnecting the lines of G to get $\bar{\mathrm{G}}$ in every possible order. (This defines $\mathrm{V}(\mathrm{G},\bar{\mathrm{G}})$ in an unique way).

Then, let us decompose the operation $\mathscr{D}(\mathrm{G}\to\bar{\mathrm{G}})$ into three steps (see fig. 2):

(a) First, we disconnect the $n(\mathrm{V})$ end points of lines of $(\mathrm{G}-\mathrm{V})$ attached to $\mathrm{V}(\mathrm{G},\bar{\mathrm{G}})$ so that we obtain the graph V and a graph C with $n(\mathrm{V})$ lines attached to x_0(C may be seen as the graph obtained by shrinking V into the vertex x_0 in G). If $\mathrm{V}=\emptyset$, $n(\mathrm{V})=0$ and $\mathrm{C}=\mathrm{G}$.

(b) Second, we disconnect the lines of V which are disconnected in $\bar{\mathrm{G}}$, so that we obtain a graph $\bar{\mathrm{V}}$.

(c) Third, we disconnect the remaining lines of C. In this operation we have disconnected and attached to x_0 $n(\bar{\mathrm{C}})$ end points of lines of C, which were not already

* When dealing with some UV regularization (in particular the lattice regularization), more serious IR divergences (in power of the IR cut off) are known to occur graph by graph and to cancel between different graphs (the measure terms being essential for such cancellations). It may be proved [13] that those divergences disappear in each graph when a first UV subtraction is performed (subtraction of quadratic divergences at zero momenta). The corresponding counterterms are in fact zero, so that the theory is not modified by those subtractions. In this section, such subtractions are assumed to be performed if necessary so that only logarithmic IR divergences are present and the analysis of ref. [2] may be performed without this additional difficulty.

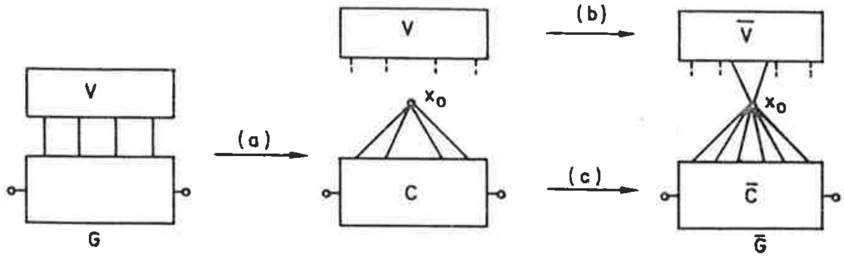

Fig. 2. The three steps of operation $\mathscr{D}$ which define the connected and disconnected parts of $\bar{\mathrm{G}}$.

attached to x_0 in C, and without generating a new disconnected part in order to obtain a graph $\bar{\mathrm{C}}$; let us call this operation $\mathscr{D}_{\mathrm{C}}$.

Definition

$\bar{\mathrm{V}}$ will be called the disconnected part of $\bar{\mathrm{G}}$, $\bar{\mathrm{C}}$ the connected part of $\bar{\mathrm{G}}$.

Each connected component of $\bar{\mathrm{C}}$ is attached to some external vertex; a connected component of $\bar{\mathrm{V}}$ may only be attached to x_0.

Now, we decompose the summation over all graphs $\bar{\mathrm{G}}$ in (3.1) as a summation over all possible $\mathrm{V} \subset \mathrm{G}$, times a summation over all $\bar{\mathrm{V}}$ obtained from V by operation $\mathscr{D}$, times a summation over all $\bar{\mathrm{C}}$ obtained from C by the operation $\mathscr{D}_{\mathrm{C}}$. The summation over all $\bar{\mathrm{V}}$ gives the integral associated to the disconnected part V, which is simply $V[D]$. Defining the integral associated to the connected part C, $C_{\mathrm{conn}}[D]$, as

$$C_{\mathrm{conn}}[D] = \sum_{\substack{\mathrm{C} \xrightarrow{} \bar{\mathrm{C}} \\ \mathscr{D}_{\mathrm{C}}}} (-1)^{n(\bar{\mathrm{C}})} \bar{C}[D_0] , \tag{3.3}$$

we finally get the decomposition into connected and disconnected parts:

$$G[D] = \sum_{(\mathrm{V},\mathrm{C})} (-1)^{n(\mathrm{V})} V[D] \cdot C_{\mathrm{conn}}[D] . \tag{3.4}$$

Now we can discuss the IR structure of $G[D]$. We first consider the connected parts. We have the following result.

Lemma 1

For any connected part C in some G, $C_{\mathrm{conn}}[D]$ is IR finite.

Proof

The graphs $\bar{\mathrm{C}}$ have the same IR structure as the usual graphs G of the model. So we can use the results of ref. [2], which give the IR structure of any graph computed with the propagator D_0. In our notation, lemma 2.1 of ref. [2] reads:

Given any graph $\bar{\mathrm{C}}$, the IR behaviour of $\bar{C}[D_0]$ is given by

$$\bar{C}[D_0] = \sum_{\substack{\mathrm{E} \subset \bar{\mathrm{C}} \\ \text{dominant}}} \mathrm{fp}\, E[D_0] \cdot [\widetilde{\bar{C}/E}][D_0] + \text{negl. terms} . \tag{3.5}$$

In (3.5), the sum runs over all "dominant subgraphs" E in $\bar{C}$ defined by the following conditions:

(D.1): E contains all external vertices of $\bar{C}$ (including x_0).

(D.2): E has no disconnected part (each connected component of E contains at least one external vertex).

(D.3): The end points of lines of $\bar{C}$–E attached to a vertex of E do not carry a derivative coupling.

fp $E[D_0]$ is the IR-finite part of the amplitude of the dominant E, and is a finite amplitude, independent of the IR cut off.

$[\widetilde{\bar{C}/E}][D_0]$ is the amplitude of the graph obtained by shrinking E into a vertex of $\bar{C}$, and diverges logarithmically as the IR cut off is set to zero (see fig. 3).

Now, for a given C, let us consider some $\bar{C}$ and some dominant E$\neq\bar{C}$ in $\bar{C}$. We consider the set $\mathscr{E}$ of the end points of lines of $\bar{C}$ which:

(a) do not belong to E,

(b) were attached in C to vertices different from x_0, and which belong to E in $\bar{C}$.

The set $\mathscr{E}$ is not empty, otherwise $\bar{C}$–E would be a disconnected part of $\bar{C}$, which contradicts the fact that $\bar{C}$ has no disconnected part. We then consider all graphs $\bar{C}'$ obtained from $\bar{C}$ by attaching the end points belonging to $\mathscr{E}$ either to x_0 or to their original vertex in E in all possible ways. The graphs $\bar{C}'$ are always obtained from C by the operation $\mathscr{D}_C$ and are present in (3.4). Moreover, E is a dominant subgraph of any $\bar{C}'$ and the graphs $[\widetilde{\bar{C}'/E}]$ coincide. So, from (3.5) the dominant E gives the same IR divergence in each $\bar{C}'$ (see fig. 3). When summing over every $\bar{C}'$ in (3.4), it is easy to see that the divergences associated to E cancel owing to the factor $(-1)^{n(\bar{C}')}$. This argument may be applied for any possible dominant in (3.4). This ensures the result of lemma 1.

We now consider the disconnected parts V. As mentioned above, for a given V, $V[D]$ is not *a priori* IR finite. We shall prove that, summing upon different V, we get an IR finite contribution.

Given a graph G of the perturbative expansion of some function $F[\pi]$, any possible disconnected part V in G is a graph of the observable

$$V_{\bar{n}} = \frac{1}{n!}\frac{\partial}{\partial a^{i_1}}\frac{\partial}{\partial a^{i_n}}\mathrm{e}^{V[\pi(x)-a]-V[\pi]}\Big|_{a=0}, \tag{3.6}$$

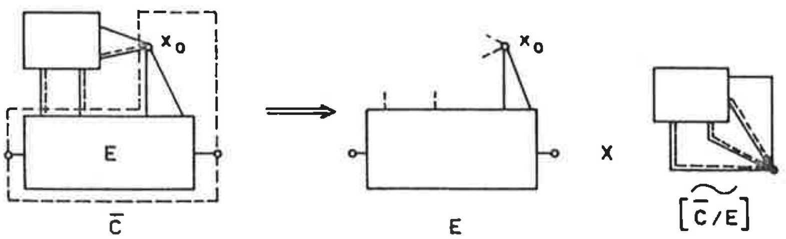

Fig. 3. The factorization of IR divergences of $\bar{C}$ in terms of essentials E as given by eq. (3.5). Lines whose end points belong to the set $\mathscr{E}$ are drawn as parallel full and dashed lines.

where $V[\pi]$ is the sum of all interaction terms in the functional integral (2.4)

$$V[\pi] = -\sum_x \left[\frac{a^2}{2t}(\nabla_\mu \sigma)^2 + \ln \sigma\right], \tag{3.7}$$

and $\bar{n}$ is the set of n indices, $\bar{n} = \{i_1 \dots i_n\}$. Similarly, the corresponding graph C is a graph of the observable

$$F\Delta^{\bar{n}}(x_0) = F[\pi]\pi^{i_1}(x_0) \dots \pi^{i_n}(x_0). \tag{3.8}$$

Counting factors and symmetry factors of the graphs manage in such a way that, when summing over all graphs G to obtain $\langle F \rangle_{S_0}$, eq. (3.4) can be generalized to

$$\langle F \rangle_{S_0} = \sum_{\bar{n}} \langle F\Delta^{\bar{n}}(x_0)\rangle_{S_0 \atop \text{conn}} \langle V_{\bar{n}} \rangle_{S_0}, \tag{3.9}$$

where $\langle F\Delta^{\bar{n}}(x_0)\rangle_{S_0 \atop \text{conn}}$ is the sum of the $C[D]_{\text{conn}}$ over all graphs C corresponding to the operator $F\Delta^{\bar{n}}(x_0)$, and where $\langle V_{\bar{n}}\rangle_{S_0}$ is the sum of the $V[D]$ over all graphs V corresponding to $V_{\bar{n}}$. From lemma 1, $\langle F\Delta^{\bar{n}}(x_0)\rangle_{S_0 \atop \text{conn}}$ is known to be IR finite graph by graph. We now prove that each $\langle V_{\bar{n}}\rangle_{S_0}$ is also finite.

Lemma 2

$\langle V_{\bar{n}}\rangle_{S_0}$ is simply given by

$$\langle V_{\bar{n}}\rangle_{S_0} = \frac{1}{n!} \frac{\partial^n}{\partial a^{i_1} \cdots \partial a^{i_n}} \frac{1}{\sqrt{1-ta^2}}\Bigg|_{a=0}. \tag{3.10}$$

Proof

Let us consider the generating function of the $\langle V_{\bar{n}}\rangle_{S_0}$,

$$V(\boldsymbol{a}) = \sum_{\bar{n}} a^{\bar{n}} \langle V_{\bar{n}}\rangle_{S_0}. \tag{3.11}$$

From (3.6), $V(\boldsymbol{a})$ is defined by

$$V(\boldsymbol{a}) = \frac{1}{Z_{S_0}} \int \prod_x \mathrm{d}\pi(x)\delta[\pi(x_0)]\, \mathrm{e}^{-A_0[\pi] + V[\pi - a]}. \tag{3.12}$$

Performing the change of coordinate $\boldsymbol{\pi} \to \boldsymbol{\pi} + \boldsymbol{a}$ on the sphere S_{N-1}, we get

$$V(\boldsymbol{a}) = \frac{1}{\sqrt{1-ta^2}} \frac{Z_{S(a)}}{Z_{S_0}}, \tag{3.13}$$

where $\boldsymbol{S}(a)$ is the point on the sphere defined as

$$\boldsymbol{S}(a) = \boldsymbol{S}_0(1-ta^2) + \sqrt{t}\boldsymbol{a}. \tag{3.14}$$

The factor $1/\sqrt{t - ta^2}$ comes from the constraint. Indeed, the Dirac measure on the sphere is, in our coordinate system,

$$\delta_{S(a)}(S)=\sqrt{1-ta^2}\delta^{N-1}(\pi-a)\,. \tag{3.15}$$

From O(N) invariance, $Z_{S(a)}=Z_{S_0}$. So $V(a)$ is obviously IR finite and (3.13) leads to lemma 2.

So, lemmas 1, 2 and (3.9) ensure the IR finiteness of any observable $F[S]$. An interesting point is that we have used the global symmetry of the model only in the proof of lemma 2, when identifying the partition functions computed with different constraints.

Finally let us discuss the relation between the results of our procedure and of the usual one. For O(N) invariant observables F, $\langle F\rangle_{S_0}$ is, in fact, independent of the constraint, and we recover the usual IR-finite result. For non-invariant observables, we have to average over all the constraints to get the physical average value of F. But it is easy to see that

$$\langle F\rangle=\int_{S_{N-1}}\mathrm{d}S_0\,\langle F\rangle_{S_0}=\langle\bar{F}\rangle_{S_0}\,, \tag{3.16}$$

where $\bar{F}$ is the projection of F upon the subspace of O(N) invariant functions defined as

$$\bar{F}[S]\int_{O(N)}\mathrm{d}R\,F[R^{-1}S]\,. \tag{3.17}$$

So, averaging a non-invariant observable over the constraint, we recover a finite invariant observable.

4. IR structure of general models

The arguments of sect. 3 may be applied without difficulties to the non-linear models which have a different global symmetry group. In this section we want to apply our approach to the IR structure of the general non-linear σ models discussed by Friedan in ref. [5]. Such models are constructed on a general (non-homogeneous) riemannian space M by the action

$$A[\phi]=\frac{1}{2t}\int\mathrm{d}^dx\;\partial_\mu\phi^i(x)g_{ij}(\phi)\,\partial_\mu\phi^j(x)\,, \tag{4.1}$$

where the field $\phi(x)$ is an element on M and where $g_{ij}(\phi)$ is the metric tensor on M at the point ϕ (in the coordinate system ϕ^i). In such general models there is no natural measure on the space of fields, so that we have to choose an *a priori* measure $d_M\phi$ on M:

$$\mathcal{D}[\phi]=\prod_x d_M\phi(x)\,. \tag{4.2}$$

So the parameters of the model are the metric g/t and the measure $d_M\phi$.

In refs. [12, 13], Friedan studied the renormalization properties of such models. He showed that those models are renormalizable at 2 dimensions and established the renormalization group equations (for the metric and the measure) at $d = 2 + \varepsilon$. Since there is no symmetry in these models, it is necessary to introduce a constraint to fix the point on M around which one computes fluctuations. The sorts of constraints we have presented in sect. 2 for the O(N) model (that is to fix the field at some point) have some advantages over the constraints used in ref. [12]: they are independent of the coordinate system chosen on M and it is not necessary to introduce ghost fields (the jacobian of the constraint is a constant).

So, let us consider the general model defined on M by (4.1) and (4.2). As in sect. 2, we introduce the constraint $\phi(x_0) = \phi_0$ (where ϕ_0 is some point on M), so that we define

$$Z_{\phi_0} = \int \mathscr{D}[\phi]\delta_{\phi_0}[\phi(x_0)]\, \mathrm{e}^{-A[\phi]}\,, \tag{4.3}$$

where δ_{ϕ_0} is the Dirac measure at ϕ_0, defined in a coordinate system ϕ^i as

$$\delta_{\phi_0}[\phi] = |g(\phi_0)|^{-1/2}\delta(\phi^i - \phi_0^i)\,. \tag{4.4}$$

Similarly, for any function of the fields $F[\phi]$, we define

$$\langle F\rangle_{\phi_0} = \frac{1}{Z_{\phi_0}}\int \mathscr{D}[\phi]\delta_{\phi_0}[\phi(x_0)]F[\phi]\, \mathrm{e}^{-A[\phi]}\,. \tag{4.5}$$

The conclusions of sect. 2 remain valid: the propagator is given by (2.13) and the interaction terms remain unchanged, the perturbative expansion is finite as long as the volume V is finite.

If we now take the infinite volume limit (the UV cut off being fixed), a difference with the case of models with a global symmetry appears. Indeed, the IR divergences in powers of the volume have no reason to be cancelled by the measure terms which are a free parameter of the model. (See remark in sect. 3). This is a consequence of the fact that, in general, fluctuations will (perturbatively) generate a mass of the order of the UV cut off while we expand around a massless theory. As explained in sect. 3, these volume terms are cancelled only if quadratic UV divergences are subtracted at zero momenta; power counting shows that the corresponding counterterms will appear as a renormalization of the measure $d_{\mathrm{M}}\phi$, which has to be performed in order to avoid perturbative generation of a mass. So the measure has to be adjusted order by order to keep a massless theory; the result at first order is

$$d_{\mathrm{M}}\phi = d\phi^i |g|^{1/2}\, \mathrm{e}^{tR/48 + O(t^2)}\,, \tag{4.6}$$

where R is the scalar curvature, which is in agreement with ref. [12].

If the measure is chosen in that way, volume terms disappear and we may apply the analysis of sect 3. We may define the operators $V_{\bar{n}}$ and $F\Delta^{\bar{n}}(x_0)$ by (3.6) and (3.8) (those operators depend on the coordinate system) and eq. (3.9) remains valid. Since lemma 1 involves only graphical arguments, the $\langle F\Delta^{\bar{n}}(x_0)\rangle_{\phi_0\,\text{conn}}$ are IR finite, but the $\langle V_{\bar{n}}\rangle_{\phi_0}$ are IR divergent. Indeed, the proof of lemma 2 runs in the general case up to eq. (3.13), which now reads

$$\langle V_{\bar{n}}\rangle_{\phi_0}=\frac{1}{n!}\frac{\partial^n}{\partial\phi^{i_1}\partial\phi^{i_n}}\left[\frac{|g(\phi)|^{1/2}}{|g(\phi_0)|^{1/2}}\frac{Z_\phi}{Z_{\phi_0}}\right]_{\phi^i=\phi_0^i}, \tag{4.7}$$

but in the non-standard case there is no symmetry principle which ensures that $Z_\phi=Z_{\phi_0}$. Since powers of the volume have been eliminated by the choice of the measure, it may be proved that Z_ϕ/Z_{ϕ_0} diverges as powers of ln V. However, from (3.9) and (3.12), we deduce that

$$\langle F\rangle_{\phi_0}=\frac{1}{|g(\phi_0)|^{1/2}Z_{\phi_0}}\sum_{\bar{n}}\langle F\Delta^{\bar{n}}x_0\rangle_{\text{conn}}\frac{1}{n!}\frac{\partial^{\bar{n}}}{\partial\phi^{\bar{n}}}(|g(\phi)|^{1/2}Z_\phi)_{\phi=\phi_0}. \tag{4.8}$$

So, the IR divergences of the model are entirely contained in the divergences of the partition functions Z_ϕ*.

In fact, we have yet some arbitrariness in the choice of the measure. Indeed we may add terms in $(1/V)\ln^p V$ in $d_M\phi$, so that we modify the logarithmic divergences Z_ϕ. In particular, it is possible to adjust the measure by *ad hoc* terms in $(1/V)\ln^p V$ to have

$$Z_\phi=Z_{\phi_0},\qquad\forall\phi\in\mathrm{M}. \tag{4.9}$$

From (4.8), it follows that any $\langle F\rangle_{\phi_0}$ is IR finite. The average value of F, defined as the sum over all possible constraints,

$$\langle F\rangle=\frac{\int_{\mathrm{M}}\mathrm{d}\phi_0|g(\phi_0)|^{1/2}Z_{\phi_0}\langle F\rangle_{\phi_0}}{\int_{\mathrm{M}}\mathrm{d}\phi_0|g(\phi_0)|^{1/2}Z_{\phi_0}}, \tag{4.10}$$

is also IR finite. The *ad hoc* measure to obtain (4.11) is for the square lattice model

$$d_{\mathrm{M}}(\phi)=\mathrm{d}\phi^i|g(\phi)|^{1/2}\exp\left\{tR(\phi)\left[\frac{1}{48}+\frac{1}{V}(\tfrac{1}{2}D_0(0)-\tfrac{1}{48})\right]+\mathrm{O}(t^2)\right\} \tag{4.11}$$

where $D_0(0)$ is given by (3.2) and diverges as ln V. But it is sufficient to adjust the measure in order to have Z_ϕ/Z_{ϕ_0} = an arbitrary function of ϕ and t (which is always possible) to get an IR-finite perturbative expansion for any observable. So, it is possible to adjust the measure in order to define an IR-finite limit, but this limit is not "natural", in that sense that it is not unique.

For the standard models defined on a homogeneous space M with the canonical invariant measure, eq. (4.9) is automatically satisfied, this ensures the IR finiteness, as for the O(N) model.

* Eq. (4.8) is very likely true even if the measure is not adjusted to (4.6), the $F\Delta^{\bar{n}}$ remaining finite and the divergences in power of the volume being contained in Z_ϕ.

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TROISIÈME PARTIE

DIVERGENCES INFRAROUGES
ET PROPRIÉTÉS CRITIQUES DES MODÈLES
DE SURFACES

III.1 - INTRODUCTION

Le modèle auquel nous nous intéressons maintenant a été introduit par D. Wallace et R. Zia[11] en vue d'étudier les propriétés de la surface de séparation entre les deux phases pures d'un système thermodynamique dans sa phase de basse température (par exemple : les parois de Bloch dans un système d'Ising, la surface de séparation liquide-vapeur, un fluide binaire). Leur idée est de supposer que les propriétés de la surface peuvent être décrites simplement par la compétition entre l'agitation thermique (qui tend à mélanger les deux phases en déformant la surface) et la tension superficielle (qui la stabilise). Une telle image conduit à introduire le modèle suivant :

La surface est décrite par les coordonnées de ses points $(x_1, \ldots, x_d)$ et elle est supposée être assez "proche" de sa position d'équilibre, qui correspond à une surface plane. Choisissant le système de coordonnées de manière à ce que ce plan corresponde à $x_d = 0$, les fluctuations transverses de la surface sont décrites par le champ ϕ

$$x_d = \phi(x_1, \cdots, x_{d-1}) \tag{III.1}$$

et l'énergie de la surface est proportionnelle à son aire

$$H = \frac{1}{t} \int d^{d-1}x \sqrt{1 + (\vec{\nabla}\phi)^2} \tag{III.2}$$

où $t = K_B T/\Sigma$ (Σ est la tension superficielle et K_B la constante de Boltzmann).

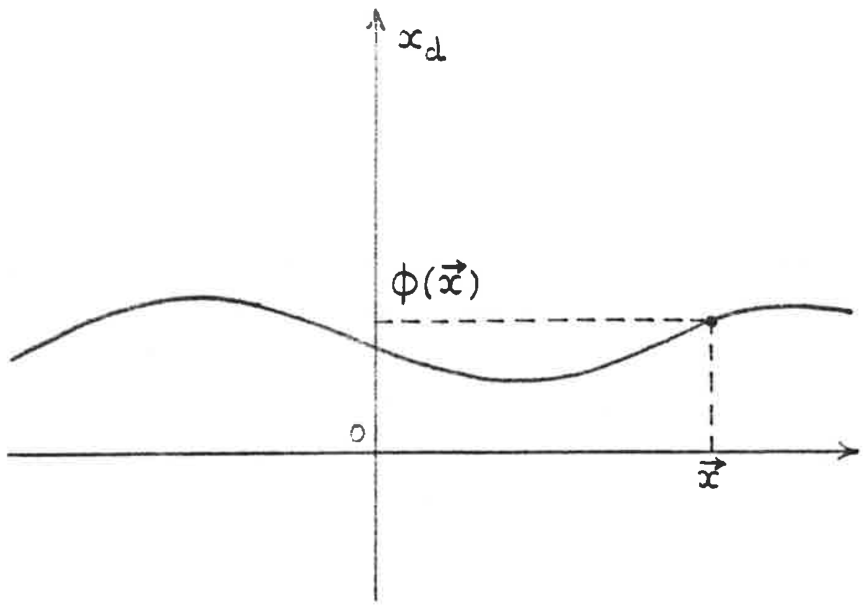

FIGURE 1

Sans entrer dans les détails de la dérivation de ce modèle, ni dans ceux de ses applications à la physique des interfaces [11, 56], nous voudrions faire plusieurs remarques :

i) Le Hamiltonien (III.2) correspond à une théorie euclidienne des champs à (d-1) dimensions. C'est une théorie effective en ce sens qu'elle ne décrit que les propriétés à grande distance de la surface. Si l'on veut tenir compte d'effets à courte distance, par exemple de l'épaisseur et de la rigidité de la surface de séparation entre deux phases, il convient d'ajouter des termes proportionnels à la courbure et à d'autres invariants de dimension plus élevée dans l'action (III.2), c'est-à-dire des termes d'interaction "non pertinents" au sens du Groupe de Renormalisation.

ii) En supposant que les déformations de la surface peuvent être décrites par (III.1) on fait deux approximations :

- on néglige le fait que la surface puisse engendrer des "surplombs", autrement dit déferle (cette approximation n'est pas trop grave, la paramétrisation (III.1), constitue simplement une "carte locale" dans l'espace des configurations au voisinage de la position d'équilibre) ;

- on néglige surtout les interactions de la surface avec elle-même et le fait qu'elle peut changer de topologie (engendrer des poignées, émettre des bulles, ... (Fig. 2)). Cette approximation est beaucoup plus grave et constitue une limitation essentielle du modèle. Notons que le modèle d'Ising à deux (respectivement trois) dimensions est équivalent à un modèle de chemin (respectivement de surface) aléatoire mais auquel est associé des degrés de liberté "fermioniques" qui tiennent compte de ces effets [57, 58].

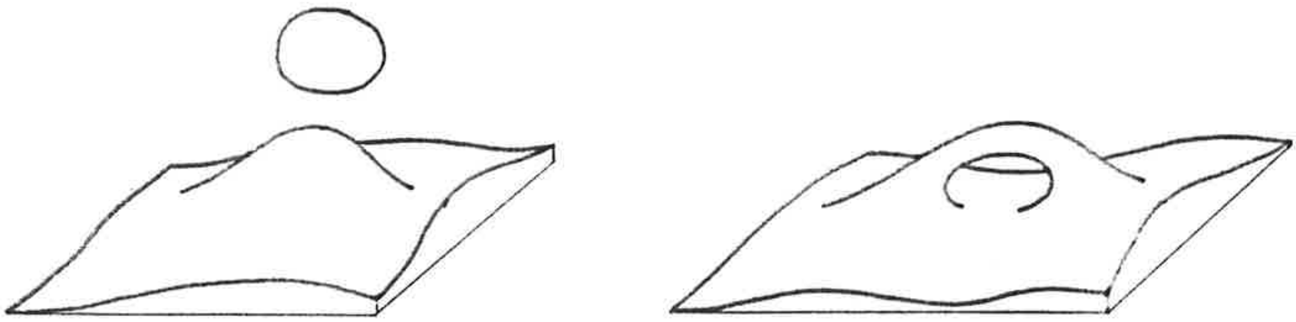

FIGURE 2

iii) L'action (III.1) est invariante sous l'action du groupe euclidien E(d) qui agit de manière non linéaire en mélangeant les coordonnées $\vec{x} = (x_1, \ldots, x_{d-1})$ et le champ $\phi = x_d$. Par contre le "vide classique" (c'est-à-dire une surface plane) décrit une situation où la symétrie E(d) est brisée et où ne subsiste qu'une symétrie E(d-1). Qui dit brisure de symétrie dit modes de Goldstone : ce sont ici les ondes de capillarité sur la surface de séparation, décrites par le champ ϕ, dont la dynamique effective est donnée par l'action (III.2). En effet développons H dans la constante de couplage t en redéfinissant $\phi = \sqrt{t}\varphi$; on obtient :

$$H = \int d^{d-1}x \ \frac{1}{t} + \frac{1}{2}(\vec{\nabla}\varphi)^2 + O(t) \qquad \text{(III.3)}$$

décrit bien des excitations de masse nulle en interaction.

Le modèle de surface décrit ci-dessus est donc assez similaire aux modèles σ non linéaires. Mais le groupe de symétrie E(d) n'est plus un groupe de symétrie interne agissant seulement sur les champs mais est un groupe de reparamétrisation globale agissant sur le champ ϕ et les coordonnées $\vec{x}$.

Comme pour les modèles σ, on peut se demander s'il existe une dimension critique inférieure où la symétrie E(d) sera restaurée dynamiquement et où l'action (III.1) pourra correspondre à une théorie des champs continue. L'analyse dimensionnelle montre immédiatement que la constante de couplage t est sans dimension à $d = 1$, ce qui correspond naturellement à la dimension critique inférieure où une symétrie discrète ne peut pas être brisée [59]. Mais alors le modèle est trivial puisque la surface de séparation se réduit à un point !

Néanmoins, Wallace et Zia montrèrent que, si le modèle était trivial à $d = 1$, il pouvait être défini perturbativement à $d = 1 + \epsilon$ dimensions et que sa renormalisation à $\epsilon = 0$ était non triviale. Une renormalisation de la constante de couplage t doit être effectuée (le champ ϕ, relié à $\vec{x}$ par l'invariance E(d), n'a pas à être renormalisé) et la fonction β est

$$\beta(t) = \epsilon t - 4t^2 + O(t^3) \qquad \text{(III.4)}$$

Le modèle est asymptotiquement libre à $\epsilon = 0$, et possède un point fixe ultraviolet non trivial pour $\epsilon > 0$

$$t_c = \frac{\epsilon}{4} + O(\epsilon^2) \qquad \text{(III.5)}$$

ainsi qu'un point fixe infrarouge trivial à $t = 0$. Les arguments standards du groupe de renormalisation indiquent que pour $0 < t < t_c$, on se trouve dans la phase ordonnée (existence des deux phases séparées par une interface bien définie) et que pour $t > t_c$ on se trouve dans la phase désordonnée (les deux phases se mélangent et la notion d'interface n'a plus de sens). L'indice correspondant

$$\nu = \left[\frac{d\beta}{dt}(t_c)\right]^{-1} = \frac{1}{\epsilon} + \cdots \qquad \text{(III.6)}$$

est identifié par Wallace et Zia à l'indice ν du modèle d'Ising (associé au comportement de la longueur de corrélation à la température critique).

Enfin, Lowe et Wallace [12] ont montré que le modèle décrivant maintenant une surface de dimension $\epsilon = d - n$ dans l'espace de dimension d (n quelconque) possédait exactement les mêmes propriétés.

Malgré ses aspects formels (il n'est définit que perturbativement pour $\epsilon \simeq 0$), ce modèle possède donc des propriétés intéressantes. Notons d'ailleurs qu'un modèle assez similaire est fourni par la théorie d'Einstein de la gravitation à $2 + \epsilon$ dimensions ! L'action classique

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{|g|}\, R \qquad \text{(III.7)}$$

est sans dimension à d = 2, mais triviale*, puisque proportionnelle à un invariant topologique. S. Weinberg a proposé d'étudier (toujours perturbativement à l'aide de l'interpolation dimensionnelle) la théorie à $d = 2 + \epsilon$. Il apparaît une renormalisation de la constante de Newton G non triviale et la théorie est également asymptotiquement libre à $\epsilon = 0$ [60]. Nous ne résistons pas à l'envie de classer les théories asymptotiquement libres (sans fermions) actuellement connues dans le tableau suivant :

Symétrie	Globale	Locale
Interne	modèles σ non linéaires d = 2	Théories de Jauge d = 4
Reparamétrisation	modèles de surface d = 0	Relativité générale d = 2

---

* du moins tant que l'on fixe la topologie de l'espace considéré.

Nous nous intéressons ici au problème suivant : ce modèle de surface jouit-il de propriétés de finitude infrarouge similaires à celles qui ont été mises en évidence dans les modèles Sigma non linéaires dans la partie II ? En effet, les développements perturbatifs de ce modèle possèdent des divergences infrarouges dès que la dimension de la surface $\epsilon = d - n$ est inférieure à deux. On peut montrer facilement que pour $0 < \epsilon \leqslant 2$ les seules divergences proviennent des propagateurs "externes" et disparaissent pour les observables invariantes par translation dans l'espace R^d (et apparaissent ainsi liées à la transition dite de "rugosité" des surfaces [61]). Par contre pour $\epsilon \leqslant 0$ les divergences proviennent aussi des intégrations sur les boucles internes des graphes et sont beaucoup plus importantes.

La question de savoir si ces divergences disparaissent pour les observables invariantes sous le groupe E(d) à ϵ "voisin" de 0 peut paraître formelle puisqu'à $\epsilon = 0$ le modèle est trivial mais est reliée au problème suivant que nous résolvons également : comment construire des observables invariantes pour ce modèle ?

La section (III.2) est constituée d'une lettre où nous répondons à ces questions : nous y caractérisons les observables invariantes, et montrons que leur développement perturbatif est bien fini infrarouge à $\epsilon \simeq 0$. Nous étudions ensuite les propriétés de renormalisation de ces observables. Ceci nous permet d'obtenir une définition intrinsèque (c'est-à-dire indépendante de l'éventuel lien de ce modèle avec des systèmes de type Ising) de l'indice ν donné par (III.6) à $d = n + \epsilon$ dimensions. A la température critique t_c (III.5) la surface devient un objet critique et ν est simplement l'inverse de la dimension de Hausdorff de cet objet.

Nous reviendrons sur cette interprétation dans la section (III.3) où nous discutons un certain nombre de résultats et de problèmes que nous avons rencontrés dans la limite $n = \infty$ de ce modèle et qui n'ont pas été publiés.

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INFRARED DIVERGENCES OF MEMBRANE MODELS

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The perturbative expansion of membrane models describing a $d - n$ dimensional membrane in a d-dimensional space with surface tension is proved to be free of infrared divergences for $d \approx n$ when looking at euclidean invariant quantities.

Membrane models have been introduced by Wallace and Zia [1] in order to study the critical behaviour of the interface between two pure phases of a thermodynamic system. Such models have been generalized by Lowe and Wallace [2] to the case of a $d - n$ dimensional membrane fluctuating in a d-dimensional space. The field $\boldsymbol{\phi}(x)$ is given by the n last coordinates of the membrane expressed (locally) as a function of the $d - n$ first coordinates $\boldsymbol{x}$. Keeping only relevant terms for the long-distance behaviour and from euclidean invariance the effective action has to be proportional to the hypervolume of the membrane

$$A = \frac{1}{T} \int \mathrm{d}^{d-n}x\, (\det g)^{1/2} + \tfrac{1}{2} m^2 \boldsymbol{\phi}^2 , \qquad (1)$$

The mass term is introduced in order to stabilize the membrane and acts as an infrared cutoff. $g(x)$ is the $n \times n$ matrix given by

$$g_{ab}(x) = \delta_{ab} + \partial_\mu \phi_a(x) \partial_\mu \phi_b(x) . \qquad (2)$$

The renormalization properties of this model have been studied in $d = n + \epsilon$ dimensions [1–3]. It has an UV fixed point $t_c = O(\epsilon)$, an IR fixed point at $t = 0$, and is asymptotically free at $\epsilon = 0$, as for non-linear σ models in $2 + \epsilon$ dimensions [4], or gauge theories in $4 + \epsilon$ dimensions. As for these models, there is a non-abelian symmetry group, here the euclidean group of displacements in the d-dimensional space, which acts in a non-linear way in the space of field configurations.

In this letter we establish another similarity with non-linear σ models [5–7]: near the critical dimension (here $d = n$), the weak coupling expansion of invariant quantities is infrared finite. In fact, since the field $\boldsymbol{\phi}$ is massless, the propagator itself is divergent for $d - n \leqslant 2$. The relationship of this divergence with the roughening transition has been discussed by Lüscher [4]. Much more important divergences arise from interaction terms of the action for $d \leqslant n$. Indeed, integration over internal loops gives integrals as $\int \mathrm{d}^\epsilon k \sim \int_0^\infty \mathrm{d}k^{\epsilon - 1}$ (where $\epsilon = d - n$) which diverge logarithmically at zero when $\epsilon = 0$. As for non-linear σ models, such divergences are related to the disappearance of the spontaneously broken phase (that is of the existence of a well-defined, although delocalized membrane) at $d = n$.

Let us first check the infrared finiteness of the model in a simple example of invariant observable. Since the euclidean group acts in a non-local way, mixing field and coordinates variables, an invariant observable has in general to be non-local. A two-points observable is for instance

$$\bar{O}[F] = \int \mathrm{d}^{d-n}y\, [\det g(y)]^{1/2} \times F\{(y - x)^2 + [\boldsymbol{\phi}(y) - \boldsymbol{\phi}(x)]^2\} , \qquad (3)$$

where $F(r^2)$ is some function with sufficient decrease at infinity. Taking for F the function $F(r^2) = \theta(r_0^2 - r^2)$, we get for $\bar{O}$ the volume of the membrane contained in the sphere of radius r_0 around the point $(\boldsymbol{x}, \boldsymbol{\phi}(x))$ of the membrane. Various n-points invariant

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quantities may be constructed in the same way, or by incorporating curvature or invariants of higher dimensionality. Now let us compute first orders of $\bar{O}[F]$ at a dimension d just below n ($d = n + \epsilon$ with ϵ negative). As previously explained, the propagator $G(x)$ is divergent as $m \to 0$ as

$$G(x) = (4\pi)^{-\epsilon/2}\Gamma(1-\epsilon/2)[m^{\epsilon-2} + (x^2/2\epsilon)m^{\epsilon}] + D(x) + O(m)\,, \quad (4)$$

$D(x)$ is the finite part of the massless propagator

$$D(x) = \tfrac{1}{4}\pi^{-\epsilon/2}\Gamma(\epsilon/2-1)|x|^{2-\epsilon}\,. \quad (5)$$

Computing $\bar{O}[F]$ at first order, divergences as $m^{\epsilon-2}$ cancel immediately between graphs and divergences as m^{ϵ} are proportional to the integral

$$\int d^{\epsilon}x\,[F(x^2) + (2x^2/\epsilon)F'(x^2)]\;, \quad (6)$$

which vanishes after integration by parts.

The same kind of cancellations occurs at second order, so that we get the infrared finite result

$$\langle \bar{O}(F)\rangle = \int d^{\epsilon}x\, F(x^2) - TF'(x^2)\,2nD(x) + T^2F''(x^2)\,2(n^2+\epsilon n)D^2(x) + O(T^3) \quad (7)$$

[the ultraviolet poles at $\epsilon = 0$ are contained in $D(x)$, see eq. (5)].

To prove the infrared finiteness at any order at dimension $d = n + \epsilon$ (with ϵ negative sufficiently close to zero), we have used technics developed in ref. [7] for studying two-dimensional non-linear σ models. We shall simply point out the main steps of the proof for the "interface model" ($n = 1$) which has some peculiar simplifications.

First, let us notice that in any invariant operator, the field ϕ appears only as a difference between two points, or as spacial derivatives.

Writing such a difference as

$$\phi(x) - \phi(y) = \int_y^x dx^{\mu}\,\partial_{\mu}\phi(x)\,, \quad (8)$$

any invariant operator may be decomposed into integrals (in position space) of products of local operators involving only derivatives of the field ϕ (that is of positive dimension). Short-distance divergences are eliminated by dimensional regularization.

The infrared behaviour of such a product of local operators $A(x_1 \ldots x_p)$ may be extracted as in ref. [7]. We get

$$A(x_1 \ldots x_p) = \sum_{K=0}^{\infty} F_{(\mu_i)}(x_1 \ldots x_p)\left(\prod_{i=1}^{K}\partial_{\mu_i}\phi(0)\right) + O(m)\,. \quad (9)$$

The operators $\Pi_{i=1}^{K}\,\partial_{\mu_i}\,\phi(0)$ are the divergent parts of A. The operator $F_{(\mu_i)}(x_1 \ldots x_p)$ is infrared finite and is defined by inserting the K operators $\int d^{\epsilon}x\; m^2 x_{\mu_i} \times \phi(x) (i\,1 \ldots K)$ in A (disconnected graphs where there are only such insertions being forbidden) and by retaining the infrared finite part of it, namely

$$F_{(\mu_i)}(x_1 \ldots x_p) = \text{finite part}\left(\frac{1}{K!}\prod_{i=1}^{K} m^2 \int d^{\epsilon}x\, x_{\mu_i}\,\phi(x)\right)_{\text{"conn"}} \times A(x_1 \ldots x_p). \quad (10)$$

From eq. (9), the finiteness of an invariant observable $\bar{O}$ will be proved if any such "connected insertion" into $\bar{O}$ gives zero.

Let us perform a finite rotation θ in the plane ($\boldsymbol{u}$, ϕ) ($\boldsymbol{u}$ is some unitary vector in $\boldsymbol{x}$). Coordinates and field are transformed, respectively, as

$$\boldsymbol{x} \to \boldsymbol{x} + (\cos\theta - 1)(\boldsymbol{x}\cdot\boldsymbol{u})\boldsymbol{u} + \sin\theta\,\phi\boldsymbol{u}\,,$$

$$\phi \to \cos\theta\,\phi + \sin\theta\,(\boldsymbol{x}\cdot\boldsymbol{u})\,. \quad (11)$$

After some partial integrations and elimination of terms independent of ϕ the quadratic term $\int d^{\epsilon}x\; \phi(x)^2$ is transformed into

$$\int d^{\epsilon}x\,\phi^2(x) \to \int d^{\epsilon}x\,[\cos\theta\,\phi^2(x) + 2\sin\theta\,\phi(\boldsymbol{x}\cdot\boldsymbol{u})]\;, \quad (12)$$

so that the mass term mixes with $m^2 \int d^{\epsilon}x\; x\phi(x)$. Performing such a transformation in the functional integral of an invariant observable $\bar{O}$ we get

$$\langle \bar{O}\rangle = \frac{\langle \bar{O}\exp[m^2\sin\theta\,\boldsymbol{u}\int d^{\epsilon}x\,\boldsymbol{x}\phi(x)]\rangle}{\langle\exp[m^2\sin\theta\,\boldsymbol{u}\int d^{\epsilon}x\,\boldsymbol{x}\phi(x)]\rangle}\,. \quad (13)$$

Expanding the rhs of eq. (13) in powers of sin θ it is easy to see that we generate the "connected insertions" of eq. (10). Eq. (13) ensures that such insertions into $\bar{O}$ give zero, and so that $\bar{O}$ is infrared finite.

In the general case ($n \neq 1$) the infrared divergences have the same structure but under a general euclidean rotation, the mass term ϕ^2 mixes not only with $x\phi$ but with other possible symmetry breaking terms such as $\phi^2\ \partial\phi$... . One has to consider the mixing between all possible such terms and there is no simple transformation law such as eq. (12).

Finally we discuss some consequences of this IR finiteness. Of course such a result has no direct implications at the critical dimension $\epsilon = 0$, since the model is then trivial. However, as for the non-linear σ model at $2 + \epsilon$ dimensions [8], it allows some simplifications in calculations of critical exponents at $d = n + \epsilon$. Let us consider the observable $\bar{O}$ defined by eq. (3). As shown by Wallace and Zia, there is a coupling constant but no wavefunction renormalization [11]. Using dimensional renormalization, the renormalized quantity $\bar{O}_R$ (expressed in terms of the renormalized coupling constant t) obeys to the renormalization group equation

$$[\mu\partial/\partial\mu + \beta(t)\partial/\partial t + \gamma(t)]\ \bar{O}_R = 0\ , \tag{14}$$

where $\gamma(t)$ is related to the β function via

$$\gamma(t) = \epsilon + \mathrm{d}\beta/\mathrm{d}t - 2\beta/t\ , \tag{15}$$

since there is only one renormalisation in t.

From the one-loop calculation of eq. (7), we get β and γ at order t^2, but eq. (15) gives us the third order of β,

$$\beta(t) = \epsilon t - 4nt^2 - 8n^2t^3 + \ldots\ , \tag{16}$$

so that we get the index ν at second order in ϵ

$$\nu = 1/\beta'(t_c) = 1/\epsilon - \tfrac{1}{2} + O(\epsilon)\ , \tag{17}$$

in agreement with known results [2,9]. Let us recall that ν is defined in ref. [1] from the "bulk correlation length" ξ, which diverges at t_c as

$$\xi \sim (t - t_c)^{-\nu}\ , \quad t \to t_{c_-}\ . \tag{18}$$

It is interesting to look at the long-distance behaviour of $\bar{O}$ (which is related to the volume of the membrane enclosed in a sphere of radius R). From eq. (14), since $t = 0$ is an IR stable fixed point, for $t < t_c$, $\bar{O} \sim R^{\epsilon}$ as $R \to \infty$ (that is, at large scale, the membrane is a well-defined object). But at the critical temperature t_c (that is at short distance) $\bar{O} \sim R^{1/\nu}$, so the membrane becomes a critical object with dimension

$$d_c = 1/\nu = \epsilon + \tfrac{1}{2}\epsilon^2 + \ldots\ . \tag{19}$$

Such a behaviour is very similar to the classical problem of a polymer with a chemical potential associated to monomers [10]. This similarity is enforced by results of the limit $n \to \infty$ [2], which shows that for a one-dimensional surface ($\epsilon = 1$), $\nu = \frac{1}{2} + O(1/n)$, which is the result for the polymers in a space of large dimension ($\geqslant 4$). However, in this model of a fluctuating membrane, effects of self interaction of the membrane or non-planar configurations (bubbles, ...) are not taken into account.

I am very indebted to D. Amit to have focused my attention on such models. I thank E. Brezin, B. Derrida, J.M. Luck and C. Itzykson for useful discussions and E. Brezin for a critical reading of this manuscript.

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III.3 - DISCUSSION ET PROBLEMES OUVERTS

a) La limite $d = \infty$ du modèle de surface

L'interprétation $1/\nu$ = dimension de Hausdorff de la surface est très séduisante et s'avère une généralisation naturelle de la dimension de Hausdorff d'un chemin aléatoire (mouvement brownien) qui est, comme chacun sait, $d_H = 2$. On voit que la formule (17) donne, pour ϵ petit

$$d_H = \epsilon + \frac{\epsilon^2}{2} + O(\epsilon^3) \qquad \text{(III.8)}$$

d_H est donc plus grand que la dimension "naturelle" ϵ de la surface. Il est tentant d'essayer d'aller plus loin et de voir si ce modèle permet d'étudier le cas $\epsilon = 2$ qui correspond à une véritable surface aléatoire et qui suscite beaucoup d'intérêt actuellement. En particulier si $\epsilon = 2$ le lecteur familier avec les théories des cordes duales pourra sans peine reécrire l'action (1), (2) comme l'action de Nambu pour une corde euclidienne*

$$S = \int_\Gamma d\xi_1 d\xi_2 \det\left(\frac{\partial \vec{X}}{\partial \xi_i} \cdot \frac{\partial \vec{X}}{\partial \xi_j}\right)^{\frac{1}{2}} \qquad \text{(III.9)}$$

dans la jauge particulière**

$$\frac{\partial X_a}{\partial \xi_i} = \delta_{a,i} \qquad a = 1, 2 \qquad \text{(III.10)}$$

---

* Sans préciser les conditions aux limites puisque l'on s'intéresse aux propriétés à courte distance.

** Cette jauge "axiale" ne nécessite pas l'introduction d'un déterminant de Faddeev Popov.

Dans cette manière de voir, la théorie de cordes donnée par l'action de Nambu apparaît comme une théorie effective à grande distance, non renormalisable mais "assurée asymptotiquement" (au sens de [60]) par l'existence d'un point fixe ultraviolet non trivial et émergeant d'une dynamique à courte distance inconnue (par exemple SU(∞) ?). S'il est bien sûr irréaliste d'extrapoler à $\epsilon = 2$ les développements en ϵ, Lowe et Wallace ont proposé d'étudier ce modèle dans une autre limite : faire tendre la dimension de l'espace d vers l'infini, la dimension de la surface ϵ étant fixée. A $d = \infty$ on peut calculer exactement l'indice ν et leur résultat est [12]

$$\nu = \frac{1}{\epsilon} - \frac{1}{2} \tag{III.11}$$

résultat que l'on peut facilement retrouver en considérant l'observable invariante $O[F]$ donnée par (3). En effet, dans la limite $d \to \infty$, $\epsilon = d - n$ fixé, et en changeant simultanément l'échelle de la constante de couplage nue en

$$T_o = T.N \quad \text{fixé} \tag{III.12}$$

les seuls graphes qui subsistent dans le développement de $O[F]$ sont ceux donnés par la figure 3, qui correspondent à

$$\begin{aligned} \langle O[F] \rangle &= \int d^\epsilon x \, \frac{1}{K!} F^{(K)}(x^2) \left[-2T_o D(x)\right]^K \\ &= \int d^\epsilon x \; F(x^2 - 2\, T_o D(x)) \end{aligned} \tag{III.13}$$

où $D(x)$ est donné par (5) et est négatif pour $0 < \epsilon < 2$.

$$D(x) = \frac{1}{4} \pi^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2} - 1\right) |x|^{2-\epsilon} \tag{III.14}$$

Prenant pour F la fonction :

$$F(x^2) = \Theta(R^2 - x^2) \tag{III.15}$$

(ce qui correspond à calculer le volume de surface dans la sphère de rayon R) on trouve bien que pour $0 < \epsilon < 2$:

$$\langle O[F] \rangle \sim R^{\epsilon} \qquad R \to \infty \tag{III.16}$$

et

$$\langle O[F] \rangle \sim R^{\frac{2\epsilon}{2-\epsilon}} \qquad R \to 0 \tag{III.17}$$

A grande distance la surface est bien définie mais à courte distance devient un objet de dimension $d_H = \frac{1}{\nu}$ correspondant à (III.11).

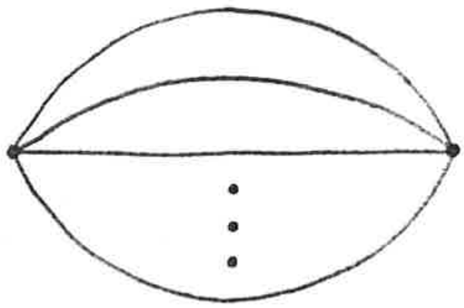

FIGURE 3

b) Surfaces aléatoires et surfaces browniennes

Pour $\epsilon = 1$ l'équation (III.11) donne bien $\nu = \frac{1}{2}$, c'est-à-dire le résultat d'un chemin brownien. J. des Cloizeaux a généralisé cette notion de chemin brownien à celle de surface brownienne de la manière suivante [62] :

On se donne une variété M de dimension ϵ compacte munie d'une métrique donnée g et on la plonge dans l'espace physique R^d avec le poids statistique :

$$\exp\left\{-\int_M d^{\epsilon}\xi \sqrt{g}\; g_{ab}(\xi)\left[\frac{\partial \vec{X}}{\partial \xi_a}\cdot\frac{\partial \vec{X}}{\partial \xi_b}\right]_{\Delta}\right\} \tag{III.18}$$

On introduit un régulateur à courte distance Δ sur la variété pour éviter les singularités à courte distance (par exemple en discrétisant le plongement par un maillage de la variété d'espacement moyen Δ).

Par analogie avec le cas des chemins ($\epsilon = 1$), on obtient ν en faisant tendre la "taille" de la variété M, S vers l'infini

$$S = \int_M d^\epsilon \xi \sqrt{g} \tag{III.19}$$

(Δ étant fixé) et en regardant comment varie le rayon de gyration, c'est-à-dire la distance moyenne entre deux points arbitraires sur la surface, R. ν est défini par :

$$R \sim S^\nu \qquad S \rightarrow \infty \tag{III.20}$$

Le poids statistique (III.18) étant gaussien, il est facile d'estimer ν et le résultat est :

$$\begin{aligned} \nu &= \frac{1}{\epsilon} - \frac{1}{2} && \text{si } 0 < \epsilon \leqslant 2 \\ \nu &= 0 && \text{si } \epsilon \geqslant 2 \end{aligned} \tag{III.21}$$

L'indice ν donné par le modèle de surface dans la limite $d = \infty$ coïncide donc avec celui donné par le modèle gaussien très simple de des Cloizeaux pour toute valeur de $0 < \epsilon \leqslant 2$.

c) Surfaces aléatoires et cordes duales

Si l'on s'intéresse maintenant au cas $\epsilon = 2$ on trouve $\nu = 0$, autrement dit, la dimension de Hausdorff d'une surface bidimensionnelle fluctuante devient infinie dans ce modèle ! Ce résultat peut paraître surprenant mais correspond en fait à des problèmes déjà rencontrés dans les théories des cordes duales.

Il est possible de construire une version discrétisée de l'"ancien" modèle des cordes duales qui correspond à un polymère de M + 1 particules massives interagissant par une force harmonique entre plus proches voisins, décrite par le Hamiltonien [63]

$$\mathcal{H} = M\alpha' \left[-\sum_{n=0}^{M} \frac{\partial^2}{\partial X_n^{i\,2}} + \frac{1}{2\pi\alpha'} \sum_{n=1}^{M} \left(X_n^i - X_{n-1}^i \right)^2 \right] \qquad \text{(III.22)}$$

Dans la limite continue ($M \to \infty$), la distance moyenne entre deux points voisins $\langle (X^i_{n+1} - X^i_n)^2 \rangle$ reste finie et non nulle, quelque soit la dimension de l'espace $\mathcal{D}$. Ceci correspond bien au fait que $d_{\text{Hausdorff}} = \infty$.

Alvarez [64] a étudié récemment la forme du potentiel statique donné par différents modèles de corde * à extrémités fixées : V(R). Il s'avère que ce problème peut être résolu dans la limite $\mathcal{D} \to \infty$, $\mathcal{D}/M^2$ fixé (M^2 est la tension de corde), ce qui est très similaire à la limite considérée par Lowe et Wallace. Le résultat est :

$$V(R) = M^2 R \left(1 - R_c^2/R^2 \right)^{1/2} \qquad \text{(III.23)}$$

Le potentiel est bien linéaire à grande distance mais devient singulier à une certaine distance critique R_c (dépendant de $\lambda = \mathcal{D}/M^2$). Une telle singularité à courte distance correspond à une "singularité de Landau ultraviolette" et peut s'expliquer dans le cadre de l'analyse précédente.

En effet, dans la limite $d = \infty$ du modèle de surface la fonction β obtenue par Lowe et Wallace correspond simplement à la relation suivante entre constante de couplage nue T_o et renormalisée t :

$$T_o = \mu^{-\epsilon}\, t \left(1 - t/\epsilon \right)^{\epsilon/2 - 1} \qquad \text{(III.24)}$$

(μ est l'échelle de renormalisation). Le point critique correspondant à $T_o = \infty$ est $t^* = \epsilon$. Par contre à $\epsilon = 2$ la relation (III.24) devient très singulière (voir figure 4)

---

* comprenant le modèle de Nambu.

$$t(T_o, \mu) = \begin{cases} \mu^2 T_o & \text{si} \quad o \leq T_o \leq t^* \mu^{-2} \\ t^* & \text{si} \quad T_o > t^* \mu^{-2} \end{cases} \tag{III.25}$$

Et si on définit une constante de couplage effective t(R) par la relation

$$t_{eff}(R) = t(t, R\mu) \tag{III.26}$$

on voit que $t_{eff}(R)$ se comporte en $1/R^2$ comme l'analyse dimensionnelle naïve l'indique jusqu'à un R_c critique :

$$R_c^2 = t/\mu^2 \tag{III.27}$$

où on tombe brutalement dans le domaine critique et où la corde cesse d'être définie. La brutalité de la transition qui l'apparente à une transition du 1^er^ ordre, est bien sûr liée au fait que $\nu = 0$, c'est-à-dire $d_H = \infty$.

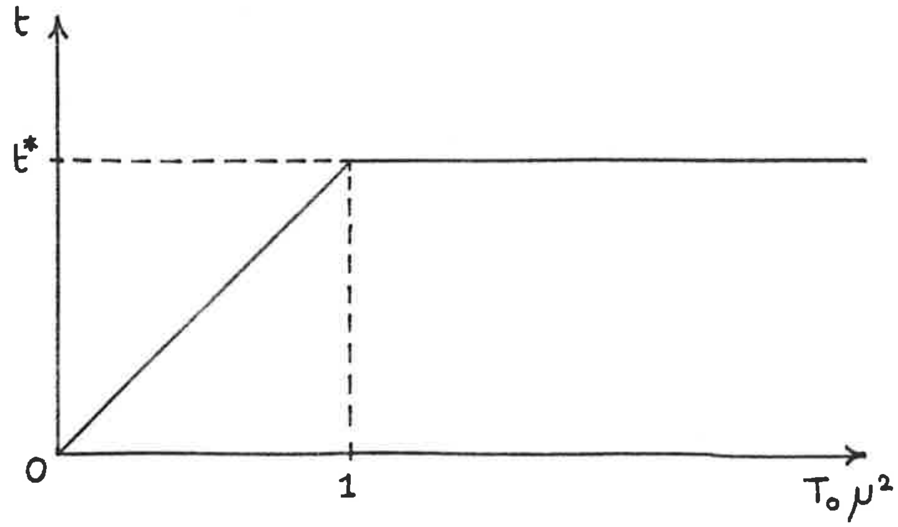

FIGURE 4

d) Corrections en 1/d

La limite d → ∞ considérée jusqu'à présent apparaît comme le premier terme d'un développement en 1/d analogue au développement en 1/N du modèle O(N) à 2 + ϵ dimensions [65]. Nous avons essayé de calculer la première correction en 1/d à l'exposant ν mais avons rencontré un problème inattendu.

On peut a priori calculer tous les termes du développement en 1/n (n = d − ϵ) du modèle de surface à l'aide d'une formulation fonctionnelle analogue à celle suggérée par Alvarez [64]. La fonction de partition initiale du modèle (donnée par (1) et (2) de (III.2)), peut s'écrire :

$$Z = \int \mathcal{D}[\vec{\phi}] \exp -\frac{n}{t} \int d^\epsilon x \det\left[\delta_{\mu\nu} + \partial_\mu \vec{\phi}\, \partial_\nu \vec{\phi}\right]^{1/2} \quad \text{(III.28)}$$

où le champ $\vec{\phi}$ décrit les n = d − ϵ coordonnées transverses de la surface. L'action dépend en fait uniquement de la matrice

$$\sigma_{\mu\nu}(x) = \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi} \quad \text{(III.29)}$$

et écrivant l'identité précédente à l'aide d'un multiplicateur de Lagrange $\alpha^{\mu\nu}(x)$, on reécrit Z comme

$$Z = \int \mathcal{D}[\vec{\phi}]\, \mathcal{D}[\sigma]\, \mathcal{D}[\alpha] \exp\Big\{ -\frac{n}{t} \int d^\epsilon x \det(1+\sigma)^{1/2} + \frac{in}{2t} \int d^\epsilon x\, \mathrm{Tr}\left[\alpha(\sigma - \partial\vec{\phi}\cdot\partial\vec{\phi})\right] \Big\} \quad \text{(III.30)}$$

L'intégration sur $\vec{\phi}$ est gaussienne et s'effectue facilement. On est ramené à une action effective dépendant seulement de α et σ

$$S_{eff}(\sigma,\alpha) = n \Big\{ \frac{1}{t} \int d^\epsilon x \left[\det(1+\sigma)^{1/2} - i\,\mathrm{Tr}\left(\frac{\alpha\cdot\sigma}{2}\right)\right] - \frac{1}{2} \mathrm{Tr}\, \mathrm{Ln}\, \Delta \Big\} \quad \text{(III.31)}$$

où l'opérateur Δ est

$$\Delta = -\frac{i}{t}\, \partial_\mu \left(\alpha^{\mu\nu}(x)\, \partial_\nu \right) \qquad \text{(III.32)}$$

La limite $n \to \infty$ du problème correspond à la recherche du minimum de S_{eff}, qui correspond (à condition de faire usage de la règle de l'interpolation dimensionnelle $\int d^{\epsilon}k.k^{2n} = 0$) à

$$\sigma_c(x) = 0 \qquad \alpha_c(x) = -i\,\mathbb{1} \qquad \text{(III.33)}$$

et les corrections en $1/n$ sont données par les contributions des fluctuations autour de cet extremum. La première correction en $1/n$ ne fait intervenir que le propagateur du champ α, $\mathbb{G}(p)$ qui, comme dans le cas du modèle O(N), correspond à sommer une série géométrique de "chaînes de bulles" du développement perturbatif en t (figure 5). Puisque α est un tenseur symétrique de rang deux, $\mathbb{G}$ sera un tenseur de rang quatre.

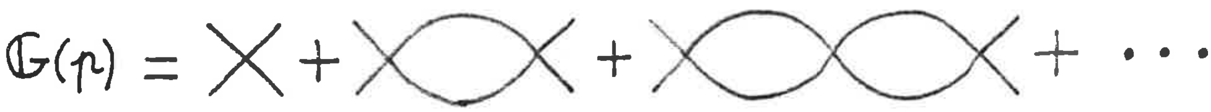

FIGURE 5

La somme précédente correspond à l'expression :

$$\mathbb{G}(p) = -\frac{t}{2}\left[\left(\mathbb{1} - \frac{\mathbb{P}}{2} \right)^{-1} + \frac{t}{4}\, \mathbb{B}(p) \right]^{-1} \qquad \text{(III.34)}$$

où $\mathbb{1} - \mathbb{P}/2$ correspond au vertex à quatre points (figure 6)

$$\left(\mathbb{1} - \mathbb{P}/2 \right)_{\mu\nu\rho\sigma} = \delta_{\mu\rho}\, \delta_{\nu\sigma} - \frac{1}{2}\, \delta_{\mu\nu}\, \delta_{\rho\sigma} \qquad \text{(III.35)}$$

$$\left[\mathbb{1}-\frac{\mathbb{P}}{2}\right]_{\mu\nu\rho\sigma} =$$

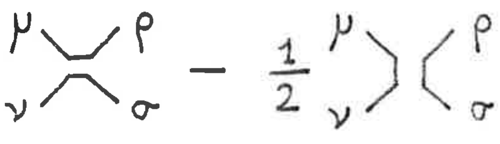

FIGURE 6

et où $\mathbb{B}(p)$ correspond à la "bulle", c'est-à-dire au graphe à une boucle.

$$\mathbb{B}(p)_{\mu\nu\rho\sigma} = \int \frac{d^\epsilon k}{(2\pi)^\epsilon} \frac{k_\mu (p+k)_\nu k_\rho (p+k)_\sigma}{k^2 (p+k)^2} \qquad \text{(III.36)}$$

$$\mathbb{B}_{\mu\nu\rho\sigma} =$$

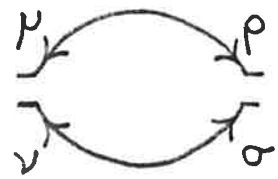

FIGURE 7

Le comptage de puissance naïf indique que $\mathbb{B}(p) \sim p^\epsilon$, si bien que pour $\epsilon > 0$, $\mathbb{G}(p)$ décroît comme $p^{-\epsilon}$ à grande impulsion. De ce fait, on s'attend à ce que les graphes du développement en $1/n$ ne divergent que logarithmiquement pour tout $0 < \epsilon < 2$, comme pour le modèle O(N) à $d = 2 + \epsilon$, c'est-à-dire que le développement en $1/n$ soit strictement renormalisable et que le calcul des contretermes permette de calculer l'indice ν ordre par ordre en $1/n$.

Malheureusement, une difficulté empêche de réaliser un tel programme ; en effet, le signe inverse $(^{-1})$ dans (III.34) signifie inverse dans l'algèbre des tenseurs de rang 4 (symétriques par les permutations, $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$ et $(\mu, \nu) \leftrightarrow (\rho, \sigma)$) muni de la multiplication

$$(A \cdot B)_{\mu\nu\lambda\tau} = A_{\mu\nu\rho\sigma}\, B^{\rho\sigma}{}_{\lambda\tau} \qquad \text{(III.37)}$$

malheureusement, si $\mathbb{G}(p)$ est bien inversible, le tenseur $\mathbb{B}(p)$ s'avère être un diviseur de zéro ! En effet un calcul long mais sans difficulté montre que :

$$\mathbb{B}(p) = a(\epsilon) p^{\epsilon} \left[2\mathbb{1} + \mathbb{P} + 2\epsilon\mathbb{A} - 4\mathbb{B} + \epsilon(\epsilon-2)\mathbb{C} \right] \qquad \text{(III.38)}$$

où les tenseurs $\mathbb{1}$, $\mathbb{P}$, $\mathbb{A}$, $\mathbb{B}$ et $\mathbb{C}$ forment la base des tenseurs homogènes en p de degré 0.

$$\mathbb{1} = \frac{1}{2}\left(g^{\mu\nu}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

$$\mathbb{P} = g^{\mu\nu}g^{\rho\sigma}$$

$$\mathbb{A} = \frac{1}{2}\left(\frac{p^\mu p^\nu}{p^2}g^{\rho\sigma} + g^{\mu\nu}\frac{p^\rho p^\sigma}{p^2}\right)$$

$$\mathbb{B} = \frac{1}{4}\left(\frac{p^\mu p^\rho}{p^2}g^{\nu\sigma} + \frac{p^\mu p^\sigma}{p^2}g^{\nu\rho} + \frac{p^\nu p^\rho}{p^2}g^{\mu\sigma} + \frac{p^\nu p^\sigma}{p^2}g^{\mu\rho}\right)$$

$$\mathbb{C} = \frac{1}{2}\,\frac{p^\mu p^\nu p^\rho p^\sigma}{p^4} \qquad \text{(III.39)}$$

et on peut vérifier que dans l'algèbre de dimension 5 engendrée par ces éléments $\mathbb{B}(p)$ n'est pas inversible. Il s'ensuit que le raisonnement précédent n'est pas applicable et que $\mathbb{G}(p)$ se comporte à grande impulsion comme p^0, c'est-à-dire que le développement en 1/n est non renormalisable ! Nous n'avons malheureusement pas trouvé de raison simple à l'existence d'un tel comportement et nous ne pouvons que faire des hypothèses sur sa signification, par exemple :

i) Il n'existe pas de correction en 1/d. Cette hypothèse est cohérente avec le fait que pour $\epsilon = 1$ (chemin aléatoire) $\nu = 1/2$ pourvu que $d \geqslant 4$ mais est contredite par le développement en ϵ de ν obtenu par Lowe et Wallace dans [12]

$$\nu = \frac{1}{\epsilon} - \frac{1}{2} + \frac{\epsilon}{2n} + O(\epsilon^2) \qquad \text{(III.40)}$$

qui montre qu'il existe une correction en $n = d - \epsilon$ pour ϵ petit.

ii) Le bon paramètre de développement n'est pas d^{-1} mais une puissance différente de d (par exemple $d^{-1/4}$ comme pour les théories de Jauge [66]).

iii) La limite $d = \infty$ n'est pas cohérente !

Comme on le voit, la limite $d = \infty$ soulève plus de problèmes qu'elle n'en résoud et nous ne savons toujours pas si l'on peut donner un sens au modèle de surface de Wallace pour un ϵ non infinitésimal. En particulier, on se retrouve confronté à des questions actuellement mal comprises : Peut-on construire un modèle continu de surfaces (à deux dimensions) aléatoires ? Existe-t-il une ou plusieurs classes de tels modèles ? Quelle est leur dimension de Hausdorff exacte ? (nous avons vu que le modèle de Wallace suggère $d_H = \infty$, mais un argument totalement différent de Parisi suggère $d_H = 4$ [67]). On peut espérer que les progrès récents apportés par le "nouveau" modèle de cordes duales de Polyakov [68, 69] et par des simulations numériques en préparation [70] apporterons une lumière nouvelle sur ces problèmes.

QUATRIÈME PARTIE

DIVERGENCES INFRAROUGES

DES THÉORIES SUPERRENORMALISABLES

IV.1 - INTRODUCTION

Revenons sur la forme des divergences infrarouges des modèles Sigma non linéaires telles que nous les avons obtenues dans la section II.2 (Lemme 2.1) et dans la section II.3 (Equations (8), (9) et (10)). Les divergences infrarouges d'une fonction de Green non invariante O(N) apparaissent contenues entièrement dans des valeurs moyennes dans le vide d'opérateurs locaux de dimension nulle de la théorie (les opérateurs D donnés par (9)) et proportionnelles à une quantité finie infrarouge qui s'interprète comme la "partie finie infrarouge" de la fonction de Green dans laquelle on a inséré un opérateur multi-local (donné par (8)) "dual" de l'opérateur local D. La structure finale de ce développement infrarouge apparaît très similaire au fameux développement à courte distance en produits d'opérateurs [52]. A la différence de ce dernier les termes singuliers de ce développement sont ici contenus dans les opérateurs locaux D et les termes réguliers dans les termes multi-locaux donnés par (8).

Ce résultat nous est apparu correspondre très précisément à une conjecture faite par G. Parisi [13] et concernant la structure des divergences infrarouges des théories superrenormalisables. Considérons une telle théorie, telle ϕ^4 à 2 ou 3 dimensions. Il est bien connu que puisque la théorie est superrenormalisable, la constante de couplage a la dimension d'une masse à une puissance positive, et qu'il s'ensuit, par simple analyse dimensionnelle, que les termes successifs du développement perturbatif dans la constante de couplage sont de plus en plus divergents lorsque la masse de la théorie est prise égale à zéro.

Un régulateur infrarouge permettant de "pénétrer" dans le domaine superrenormalisable $(d < 4)$ et souvent utilisé est fourni par l'interpolation (ou régularisation) dimensionnelle. Les amplitudes de Feynman de la théorie peuvent être définies pour $d > 4$ comme la limite de masse nulle (qui alors existe) des amplitudes massives régularisées, et sont des fonctions de d méromorphes avec des "pôles ultraviolets" (correspondant aux divergences ultraviolettes) dans le demi plan $\{\mathrm{Re}\, d > 4\}$. Ces fonctions peuvent être prolongées analytiquement dans le domaine Re $d \leqslant 4$, et apparaissent être méromorphes avec des pôles d'origine infrarouge à certaines dimensions rationnelles, reflettant l'existence des divergences infrarouges des intégrales "nues" de départ.

Dans [13], G. Parisi, en analysant la structure de ces pôles infrarouges aux premiers ordres de la théorie des perturbations (pour une théorie scalaire du type ϕ^4), fit donc la conjecture que ces singularités étaient telles qu'elles pouvaient être réabsorbées (c'est-à-dire soustraites) par l'insertion d'opérateurs multi-locaux, ces opérateurs étant en fait exactement les opérateurs "duaux" des opérateurs locaux du développement en produit d'opérateurs.

Cette conjecture suggérait donc que ce que nous avions obtenu dans le cas des modèles Sigma non linéaires n'était que le terme dominant d'un développement infrarouge systématique. Dans le travail qui suit nous généralisons les méthodes de la partie II pour obtenir ce développement à tous les ordres (toujours dans le cas d'une théorie scalaire). Nous avons vu que l'analyse du comportement d'une amplitude de Feynman I_G lorsque les masses tendent vers zéro se ramène à l'étude de sa transformée de Mellin. Si l'on considère cette amplitude comme dépendant de la dimension d'espace temps d (qui est un paramètre complexe), la transformée de Mellin est maintenant une fonction méromorphe des deux variables complexes z et d, $M_G(z, d)$, et l'on peut montrer [29] que cette fonction est singulière sur des hyper-plans (variétés complexes de dimension 1) dans $\mathbb{C} \times \mathbb{C}$ (voir la figure 2 de la section suivante). La détermination des résidus de M_G sur ces hyper-plans nous fournira à la fois le développement infrarouge des amplitudes et la structure des pôles infrarouges en d. En effet, le premier problème, comme on l'a vu, consiste à étudier les singularités de $M_G(z, d)$ à d fixé et le second consiste simplement à étudier ces singularités à z fixé (égal à zéro). La détermination de toutes les singularités infrarouges, obtenue dans la section suivante, est donc une réalisation complète du programme de M. Bergère et Y. M. Lam [17] (qui n'obtenaient que les termes dominants du développement) et nous permet de donner une formulation précise à la conjecture de G. Parisi et de la démontrer.

L'étude de la structure des divergences infrarouges des théories superrenormalisables pourrait paraître un problème quelque peu formel si les physiciens théoriciens ne s'étaient intéressés récemment à la thermodynamique des théories de Jauge, c'est-à-dire à l'étude des propriétés à l'équilibre de ces systèmes à température finie (essentiellement pour étudier les implications de ces théories dans les modèles cosmologiques du "Big-Bang" et dans les collisions à très hautes énergies entre ions lourds).

Il se trouve que l'étude à température finie d'une théorie des champs se ramène à l'étude de cette théorie dans une version euclidienne où le "temps euclidien" (it) est périodique de période $\beta = K_B T$. On voit qu'à très hautes températures (β petit) on est ramené à l'étude d'une théorie des champs euclidienne à trois dimensions et les calculs perturbatifs à température finie souffrent de problèmes infrarouges similaires à ceux des théories superrenormalisables. Récemment R. Jackiw et S. Templeton [71], ainsi que T. Applequist et R. Pisarski [72], montrèrent à l'aide d'arguments semi-perturbatifs basés sur une utilisation des équations de Dyson-Schwinger que dans les théories de Jauge à trois dimensions les divergences infrarouges se réabsorbaient dans des termes non perturbatifs proportionnels à des valeurs moyennes d'opérateurs locaux dans le vide, et conduisaient à l'apparition de termes non analytiques, comme des logarithmes de la constante de couplage, dans les premiers ordres des développements perturbatifs. En fait, l'apparition de termes non analytiques en g avait déjà été signalée par plusieurs auteurs, en particulier par K. Symanzik [14] précisément dans le cas de la théorie ϕ^4 de masse nulle à $4 - \epsilon$ dimensions. Dans ce cas K. Symanzik avait montré que la "masse nue" de la théorie était de la forme $g^{2/\epsilon}$ x(Fonction de ϵ) et qu'une telle dépendance suffisait déjà pour introduire des termes non analytiques dans les développements perturbatifs.

Notre approche nous permet de préciser les liens entre ces différents résultats. Nous montrons qu'un "ansatz" non perturbatif consistant à resommer les divergences infrarouges dans les valeurs moyennes d'opérateurs locaux dans le vide permet d'engendrer les "contres termes infrarouges" proposés par G. Parisi pour soustraire les pôles

I. R. en dimension et fournit un algorithme permettant de réobtenir et de généraliser à tous les ordres les résultats de [14, 71, 72]. Cette approche permet de préciser la structure de ces nouveaux développements perturbatifs (en particulier celle des termes en logarithmes de la constante de couplage) et de bien séparer ce qui est calculable en théorie des perturbations (qui correspond à des amplitudes de Feynman renormalisées et "soustraites infrarouges") de ce qui ne l'est pas (et qui correspond aux opérateurs locaux).

Précisons finalement que les outils que nous avons développés dans cette partie et qui permettent d'obtenir la structure de la transformée de Mellin d'une amplitude de Feynman jouent un rôle direct dans la partie V de cette thèse.

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Nonanalyticity of the Perturbative Expansion for Super-renormalizable Massless Field Theories

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The complete infrared expansion of Feynman amplitudes is established at any dimension d. The so called infrared finite parts develop poles at rational d. We prove a conjecture by Parisi by constructing an infrared subtraction procedure which defines finite amplitudes in such dimensions. The corresponding counterterms are associated to nonlocal operators and are generated in a nonperturbative way for super-renormalizable theories. We determine at all orders the perturbative expansion which contains powers and logarithms of the coupling constant.

I. INTRODUCTION

It is well known that the perturbative expansion of massless quantum theories with super-renormalizable interactions has very important infrared (I.R.) divergences (as indicated by naïve power counting arguments) which forbid the definition of a perturbative theory in the usual sense. However, in many cases, a massless quantum theory is expected to exist in its own. Let us mention the massless ϕ^4 theory which is proved to exist at two [1] and three [2] space-time dimensions (via constructive field theory) and which may be studied without I.R. problems in the $1/N$ expansion [3], or in the renormalization group approach and the ε expansion [4].

This situation was explained by Symanzik [5] who showed that in the ϕ^4 theory at $d = 4 - \varepsilon$ dimensions, the bare mass was nonzero and nonanalytic in the coupling constant as the physical mass was set equal to zero. As a consequence the weak coupling expansion of the massless ϕ^4 theory contains nonanalytic terms in the coupling constant. These terms cannot be obtained from perturbative calculations, and this explains the failure of the usual analytic perturbation theory.

The appearance of nonanalytic terms in the coupling constant was also discovered by Jackiw and Templeton [6] in a quite different approach (the use of Dyson–Schwinger integral equations) in the study of three dimensional gauge theories. They get at first orders both perturbative and nonperturbative terms, the last being associated to some matrix elements of composite operators, the leading logarithms being entirely perturbatively computable [7]. However, the arguments of Symanzik are irrelevant to this case since there is no mass renormalization in such theories.

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Another way to deal with those I.R. divergences was suggested by Parisi [8]: It is possible to define massless finite amplitudes via dimensional interpolation, by taking the massless limit in a large enough space-time dimension and analytically continuing the result into the super-renormalizable domain. In this domain the massless amplitudes have I.R. poles at some rational dimensions. Parisi conjectured that at these dimensions it was possible to construct an I.R. subtraction operator (corresponding to the insertion of counterterms associated to nonlocal operators) subtracting those I.R. singularities in order to obtain a well defined massless theory. However, he gave no arguments to justify the existence of such I.R. counterterms.

This paper is devoted to the clarification of these points. We prove Parisi's conjecture and identify the nonperturbative mechanism which generates such I.R. counterterms. We show that it leads to an I.R. finite nonanalytic perturbative expansion which we construct at all orders, and which generalizes the results of Jackiw and Templeton [6]. This paper deals explicitly with scalar theories but our results are very general and may be applied to others theories as in [6, 7].

In Section II we study the I.R. behaviour of Feynman amplitudes at any space-time dimension d. We adapt a general method developed by one of us and Lam [9, 10] to get the *complete* asymptotic expansion of any massive Feynman amplitude as the mass goes to zero at general d.

The massless regularized amplitudes defined as the zero mass limit of the amplitudes in a large enough space-time dimension are the finite parts of the asymptotic expansion and have I.R. poles at smaller rational dimensions. In Section III we construct explicitly an I.R. subtraction operator which extracts those I.R. poles and prove Parisi's conjecture [8]. We discuss some properties of the obtained I.R. subtracted amplitudes. This allows us to generalize the results of Section II to rational or entire dimensions d.

In Section IV we exhibit the mechanism which generates such I.R. counterterms: The I.R. divergent parts of the Green's functions obtained from Section II and III correspond to composite operators which may be assumed to have a nonperturbative finite massless limit. This mechanism generates in a self consistent way a nonanalytic perturbative expansion which we describe at all orders. In this section we consider the case where there is no mass renormalization.

Finally in Section V we discuss the case of the three dimensional ϕ^4 theory, where there is a mass renormalization. We show that the mechanism proposed by Symanzik [5] (the nonanalyticity of the bare mass) mixes with the previous one to generate additional logarithms of the coupling constant and additional nonperturbative terms.

II. I.R. Singularities of a Feynman Amplitude at Generic d

A. *Introduction*

In this section we describe a general method which determines at all orders the zero mass asymptotic expansion of a dimensionally regularized scalar Feynman amplitude. This method is based on the study of the analytic structure of the Mellin transform in regard to the mass and has been first introduced in Refs. [9, 10]. We shall generalize here to all nonleading singularities the technique of extraction of the leading singularities used in Ref. [11]. For justification and technical details we refer the reader to these references.

In what follows we have to handle Feynman amplitudes where arbitrary derivatives with respect to some of their external momenta have been taken. In order to simplify the notations we introduce the concept of *g-graph* to describe such objects in a general way:

Let G be a connected scalar Feynamn graph, its external momenta being splitted into two subsets $\{p_i\}$ and $\{k_i\}$. $\{p_i\}$ describes some observable we are interested in, while $\{k_i\}$ is a set of L variables upon which we apply some differential operator at all $k_i = 0$.

DEFINITION. Given some family $\chi = \{\lambda_i ; i = 1, L\}$ of L families λ_i of $|\lambda_i|$ Lorentz indices $\lambda_i = \{\mu_{i,1} \cdots \mu_{i,|\lambda_i|}\}$ we define a g-graph $\bar{G}$ as the pair $\{G, \chi\}$.

The amplitude of $\bar{G}$ is defined as

$$I_{\bar{G}}(p, m, d) = \prod_{i=1}^{L} \frac{1}{|\lambda_i|!} \prod_{j=1}^{|\lambda_i|} \left(\frac{\partial}{\partial k_i}\right)_{\mu_{i,j}} I_G(p, k, m, d)\,|_{k_i=0}, \tag{II.1}$$

where I_G is the Feynamn amplitude of the scalar graph G defined by dimensional regularization in an arbitrary dimension d (m is the mass of the propagators).

More precise definitions and properties of such g-graphs are reported in Appendix A. The expression of $I_{\bar{G}}(p, m, d)$ is in the Schwinger–Symanzik parametric representation

$$I_{\bar{G}}(p, m, d) = \int_0^{\infty} \pi\, d\alpha\, R\left[e^{-\Sigma \alpha m^2} Y_{\bar{G}}(p, m, d, \alpha)\right] \tag{II.2}$$

with

$$Y_{\bar{G}}(p, m, d, \alpha) = \left[\sum_{q=0}^{n(\chi)} S_q(p, \alpha)\right] e^{-p d_G p} P_G(\alpha)^{-d/2}, \tag{II.3}$$

where $d_G(\alpha)$ and $P_G(\alpha)$ are characteristic functions of the topology of the graph G and where the S_q take into account the derivatives with respect to k. More precisely, each $S_q(p, \alpha)$ is homogeneous in the p_i's of degree q and homogeneous in the α's of degree $(q + n(\chi))/2$, where $n(\chi)$ is the total number of derivatives in χ (Appendix B). The subtraction operator R in (II.2) ensures the absolute convergence of the integral for Re d away from the ultraviolet poles [12] and is defined in terms of generalized Taylor operators $\tau_s^{-2l(s)}$ as a sum over all nests $\mathcal{N}$ of subgraphs

$$R = \sum_{\mathcal{N}} \prod_{S \in \mathcal{N}} (-\tau_s^{-2l(s)}). \tag{II.4}$$

When applied upon the integrand of (II.2), R may be written as a sum over Zimmermann's forests of connected one particle irreducible divergent subgraphs.

Finally, the superficial degree of divergence of $\bar{G}$ is defined as

$$\omega(\bar{G}) = \frac{dL(G)}{2} - l(G) - \frac{n(\chi)}{2}, \tag{II.5}$$

where L and l are, respectively, the number of independent loops and lines of G.

B. *The Mellin Transform*

The zero mass expansion of $I_{\bar{G}}(p, m, d)$ may be obtained from the meromorphy of the Mellin transform of $I_{\bar{G}}$ with respect to $1/m^2$, which is

$$M_{\bar{G}}(p, z, d) = \int_0^{\infty} d\left(\frac{1}{m^2}\right)(1 - \tau^{-1}_{1/m^2})\left[\left(\frac{1}{m^2}\right)^{-z-\omega(\bar{G})-1} I_{\bar{G}}(p/m, 1, d)\right]. \tag{II.6}$$

The operator τ^{-1}_{1/m^2} is such that (II.6) exists away from $\mathrm{Re}\, z = -\omega(\bar{G}) + k$, $k \in \mathbb{N}$. We recall that each pole of $M_{\bar{G}}$ of the type $(1/(z - z_0)^p)$ corresponds to a definite power of m and of logarithm of m in the asymptotic expansion of $I_{\bar{G}}$, namely, $(1/\Gamma(p))\, m^{-2z_0} \mathrm{Log}(1/m^2)^{p-1}$. Using (II.2), we get for $M_{\bar{G}}$ the parametric representation

$$M_{\bar{G}}(p, z, d) = \int_0^{\infty} \pi d\alpha\, R[e^{-\Sigma\alpha} B_{\bar{G}}(p, z, d, \alpha)] \tag{II.7}$$

with

$$B_{\bar{G}}(p, z, d, \alpha) = \sum_{q=0}^{n(\chi)} \Gamma\left(-z - \omega(\bar{G}) + \frac{q}{2}\right) S_q(p, \alpha)(p\, d_G\, p)^{z+\omega(\bar{G})-q/2} P_G(\alpha)^{-d/2}. \tag{II.8}$$

By generalization of the results of Ref. [13], the function $M_{\bar{G}}(p, z, d)$ is found to be meromorphic in both variables z and d. The poles of the Γ functions in (II.8) describe the behaviour of $I_{\bar{G}}(p, m, d)$ as $m \to \infty$ and do not have to be considered. The remaining manifolds of singularities (coming from the behaviour of (II.8) as the α's go to zero) describes the ultraviolet and the infrared divergences and are found to be

$$\omega(s) = k, \quad k \in \mathbb{N} \quad \forall S \text{ nonessential,}^1 \tag{II.9}$$

$$z + \omega(G) - \omega(S) = -k, \quad k \in \mathbb{N} \quad \forall S \text{ essential.} \tag{II.10}$$

Integral (II.7), with the R operation defined by (II.4), is absolutely convergent for $(\mathrm{Re}\, x, \mathrm{Re}\, d)$ away from the manifolds (II.9–10). When applied upon the integrand of

[1] An essential subgraph S of G is a subgraph such that the reduced graph $|G/S|$ has zero external momenta at all its vertices. In momentum space, this notion depends upon the external momenta $\{p_i\}$.

(II.7), R may be written as a sum over Q-extended forests [10] of connected one. P.I. nonessential subgraphs and of essential subgraphs of G without inactive parts[2]

$$R = \sum_{\substack{Q \text{ ext forest} \\ \omega}} \prod_{S \in \omega} (-\tau_s^{-2l(s)}). \tag{II.11}$$

The remarks we want to make about the singularities of $M_{\bar{G}}$ are the following:

— Singularities (II.9) are the U.V. poles of the dimensional regularization; singularities (II.10) due to the essentials are of I.R. nature and depend on z.

— Each essential subgraph S gives a dominant pole at $z = \omega(S) - \omega(G)$, and an infinity of "daughter" poles at $z = \omega(S) - \omega(G) - k$, $k \in \mathbb{N}$. The dominant behaviour of $I_{\bar{G}}\ (p, m, d)$ is obtained from the pole

$$z_0 = \text{Sup}\{-\omega(G) + \omega(S); S \text{ essential}\}.$$

— The intersections of the I.R. singularities (II.10) are necessarily at a rational dimension $d = 2r/s$, $0 \leqslant s \leqslant L(G)$. It is only at those rational ones that U.V. poles (II.9) may occur. Away from those rationals, the singularities of $M_{\bar{G}}$ are single poles because in a Q-extended forest, only one essential subgraph may contribute to this singularity.

— Consequently, away from those rational dimensions, the expansion of $I_{\bar{G}}(p, m, d)$ as $m \to 0$ is an infinite sum of increasing powers of m starting from m^{-2z_0}.

— For any scalar graph $\bar{G}$, there is some dimension $d_{\text{IR}} > 2$ such that, for $d > d_{\text{IR}}$, $z_0 = 0$ (which is obtained for the essential $S = G$ itself). Consequently the massless limit $I_{\bar{G}}(p, 0, d)$ exists only for $d > d_{\text{IR}}$. For $d < d^{\text{IR}}$ there exists at least one essential $S \neq G$ such that $\omega(S) > \omega(G)$.

Finally, to illustrate the first part of this section we give on Figs. 1 and 2 an explicit example of the graph and the analytic structure of its Mellin transform.

C. *Extraction of the Single Poles of the Mellin Transform*

We now show how to compute explicitly the residues of the single poles of the Mellin transform $M_{\bar{G}}(p, z, d)$ at generic dimension $d \neq 2r/s$. Let us consider such a poles z_0. The residue of $M_{\bar{G}}$ at z_0 is given by the Cauchy integral

$$\text{Res}[M_{\bar{G}}; z = z_0] = \frac{1}{2i\pi} \oint_C dz \int_0^{\infty} \pi\, d\alpha\, R[e^{-\Sigma\alpha} B_{\bar{G}}(p, z, d, \alpha)], \tag{II.12}$$

where the contour C encircles the pole z_0. The integrand $R[\cdot]$ in (II.12) is nonanalytic in z along the line $\text{Re}\, z = z_0$ although the integral is analytic away from

[2] An essential subgraph S of G is said to have an inactive part S' if S' is attached to $S - S'$ by one vertex and if $S - S'$ is still an essential subgraph of G.

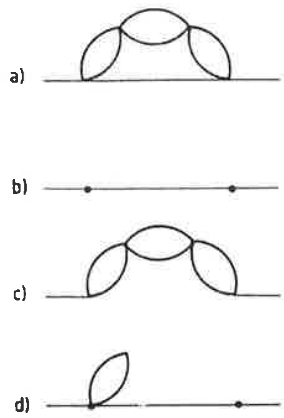

FIG. 1. (a) a 4 loop graph of scalar ϕ^4 theory. (b) The essential subgraph giving the leading singularity of $M(z, d)$ at $z + 2d - 6 = 0$. (c) An essential subgraph giving a subleading singularity at $z + d/2 - 1 = 0$. (d) An essential subgraph with inactive part which does not contribute to the I.R. singularities.

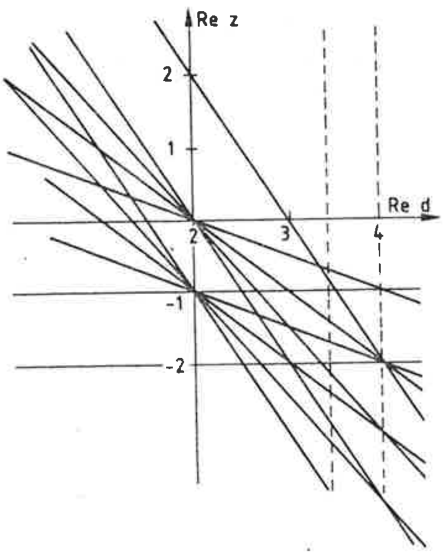

FIG. 2. The singularities of the Mellin transform of the graph of Fig. 1. Straight lines are the I.R. singularities, dotted lines are the first U.V. singularities at $d = 7/2$ and $d = 4$.

the pole z_0. However, the presence of Γ functions in $B_{\bar{G}}$ (II.8) permits us to choose for C the lines C_+ and C_- defined as

$$C_{\pm} = \{z = z_0 \mp (\varepsilon + iy), y \in \mathbb{R}\} \tag{II.13}$$

with $\varepsilon > 0$ small enough (Fig. 3). On each of the lines C_+ and C_-, the integrand is analytic in x and we may interchange α- and z-integrations to get

$$\operatorname{Res}[M_{\bar{G}}; z_0] = \int_0^{\infty} \pi\, d\alpha \frac{1}{2i\pi} \oint_{C^+ \cup C^-} dz\, R[e^{-\Sigma \alpha} B_{\bar{G}}(p, z, d, \alpha)]. \tag{II.14}$$

Now the operator R subtracts differently on C_+ and C_- (say, R^+ and R^-); if we move the contour C_- into C_+ for the integral dz we may write for a given Q-extended forest ψ

$$\oint_{C^+ \cup C^-} dz \prod_{S \in \psi} (-\tau_S^{-2l(S)})[e^{-\Sigma \alpha} B_{\bar{G}}(p, z, d, \alpha)] = \int_{C^+} dz \left[\prod_{S \in \psi} (-\tau_S^+) - \prod_{S \in \psi} (-\tau_S^-)\right]$$
$$\times [e^{-\Sigma \alpha} B_{\bar{G}}(p, z, d, \alpha)], \tag{II.15}$$

where $\tau_S^+ = \tau^{-2l(S)}$, $\tau_S^- = \tau_S^{-2l(S)-\delta}$ with $\delta = 1$ if S contributes to the pole z_0, i.e., if S is an essential satisfying (II.10) at $z = z_0$, and $\delta = 0$ otherwise. If ψ does not contain any essential S such that $\delta = 1$, (II.15) is zero. If ψ does not contain any essential S such that $\delta = 1$, (II.15) is zero. If ψ contains such an essential S_0, then S_0 is unique (for d generic, i.e., $d \neq 2r/s$, $0 \leqslant s \leqslant L(G)$); we obtain for (II.15) by standard manipulations

$$\int_{C^+} dz \prod_{S \triangleleft S_0} (-\tau_S^+)(\tau_{S_0}^- - \tau_{S_0}^+) \prod_{S \subsetneq S_0} (-\tau_S^+)[e^{-\Sigma \alpha} B_{\bar{G}}(p, z, d, \alpha)]. \tag{II.16}$$

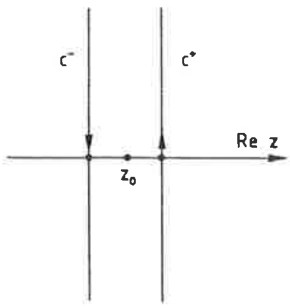

FIG. 3. The Cauchy contour for integral (II.12) defined by (II.13).

In Appendix C we calculate the action of $(\tau^{+}_{S_0} - \tau^{-}_{S_0})$ on the integrand. We get the result

$$(\tau^{+}_{S_0} - \tau^{-}_{S_0})\, e^{-\sum_G \alpha} B_{\bar{G}}(p, z, d, \alpha) = \sum_{(\chi, n)} \frac{1}{n!} \left(-\sum_{S_0} \alpha\right)^n B_{\bar{S}_0}(p, z - z_0 - n, d, \alpha) \times \exp\left[-\sum_{G/S_0} \alpha\right] Y_{[\bar{G}/\bar{S}_0]}(d, \alpha), \qquad \text{(II.17)}$$

where χ are families of derivatives with respects to external lines of S_0 (internal to G or already present in $\bar{G}$). $\bar{S}_0$ is the g-graph $\bar{S}_0 = \{S_0, \chi\}$. The sum runs over all χ and all postive integers n such that

$$n = -z_0 - \omega(\bar{G}) + \omega(\bar{S}) \in \mathbb{N} \qquad \text{(II.18)}$$

and such that $\bar{S}_0$ is a g-subgraph of $\bar{G}$ (see Appendix A). The g-graph $[\bar{G}/\bar{S}_0]$ is defined in Appendix A (Definition A.3) as the g-graph obtained by shrinking each connected component of S_0 into a vertex in G, and by adding at those vertices the derivative couplings corresponding to χ. $Y_{[\bar{G}/\bar{S}_0]}$ is the integrand of $[\bar{G}/\bar{S}_0]$ in the α-parametric representation. The function $(-\sum_{S_0} \alpha)^n B_{\bar{S}_0}(p, z - z_0 - n, d, \alpha)$ is homogeneous in all α's of S_0 of degree $[z - z_0 - l(S_0)]$, so that it may be taken through the Taylor operators $\tau_S^{-2l(S)}$ relative to the subgraphs $S \supset S_0$, which become $\tau_{S/S_0}^{-2l(S/S_0)}$. Equation (II.15) becomes

$$-\left[\sum_{S \triangleleft S_0} (-\tau_{S/S_0}^{-2l(S/S_0)})\, e^{-\sum_{G/S_0} \alpha} Y_{[\bar{G}/\bar{S}_0]}(d, \alpha)\right] \times \int_{C^+} dz \prod_{S \subsetneq S_0} (-\tau_S^{-2l(S)}) \times \left[\frac{1}{n!}\left(-\sum_{S_0} \alpha\right)^n B_{\bar{S}_0}(p, z - z_0 - n, d, \alpha)\right]. \qquad \text{(II.19)}$$

Summing over all Q-extended forests ψ we get a sum over g-subgraphs $\bar{S}$ of $\bar{G}$ satisfying (II.18),

$$-\sum_{\bar{S}} R[e^{-\sum_{G/S} \alpha} Y_{\bar{G}/\bar{S}}(d, \alpha)] \int_{C^+} \frac{dz}{2i\pi} \bar{R} \left[\frac{(-\sum \alpha)^n}{n!} B_{\bar{S}}(p, z - z_0 - n, d, \alpha)\right], \qquad \text{(II.20)}$$

where the subtraction operator $\bar{R}$ differs from R by the fact that in (II.11) the sum is taken only over all Q-extended forests in S which do not contain S itself. Performing the α-integrations we reconstruct on one side the regularized Feynman amplitude of the graph $(\bar{G}/\bar{S})$ at $m = 1$ and on the other side an amplitude relative to $\bar{S}$

$$R_{\bar{S}, n} = -\int_0^{\infty} \pi\, d\alpha \int_{C^+} \frac{dz}{2i\pi} \bar{R} \frac{1}{n!} \left(-\sum_S \alpha\right)^n B_{\bar{S}}(p, z - z_0 - n, d, \alpha). \qquad \text{(II.21)}$$

The final result for (II.14) is contained in

LEMMA 1.

$$\operatorname{Res}(M_{\bar{G}}(p, z, d); z_0) = \sum_{\bar{S}} R_{\bar{S},n}(p, d)\, I_{[\bar{G}/\bar{S}]}(m = 1, d), \tag{II.22}$$

where the sum runs over every essential g-subgraph $\bar{S}$ *of* $\bar{G}$ *such that* $\omega(\bar{S}) - \omega(\bar{G}) - z_0 = n \in \mathbb{N}$.

In (II.21) the integration over z may be performed in a way which respects the subtraction $\bar{R}$, so that we get for $R_{\bar{S},n}(p, d, \alpha)$ the following integral representation:

$$R_{\bar{S},n}(p, d, \alpha) = \int_0^\infty \pi\, d\alpha\, e^{-\Sigma_S \alpha} \left(\sum_S \alpha\right) \bar{R} \left[\frac{1}{n!}\left(-\sum_S \alpha\right)^n B_{\bar{S}}(p, -n, d, \alpha)\right]. \tag{II.23}$$

It is also possible to note that when applying Lemma 1 at $z_0 = -n$ ($n \in \mathbb{N}$), only the essential $\bar{S} = \bar{G}$ contributes to the residue, so that we have

$$R_{\bar{G},n}(p, d) = \operatorname{Res}[M_{\bar{G}}(p, z, d), z = -n]. \tag{II.24}$$

Extracting the pole of $M_{\bar{G}}$ at $z = -n$ by using homogeneity relations in its integral representation (II.7), we get for $R_{\bar{G},n}$ another representation

$$R_{\bar{G},n}(p, d) = \int_0^\infty \pi\, d\alpha\, R \left[e^{-\Sigma \alpha} \left(\sum \alpha\right)^{n+1} \frac{(-1)^n}{n!} B_{\bar{G}}(p, -n, d, \alpha)\right]. \tag{II.25}$$

The equality between (II.23) and (II.25) is due to the homogeneity properties of $B_{\bar{G}}$. Form (II.23) defines $R_{\bar{S},n}$ only around the dimension we have been considering, while (II.25) is the analytic continuation of (II.23) for all d.

D. *The Singularities of the Mellin Transform in Position Space*

The previous analysis was performed in momentum space at fixed external momenta $\{p_i\}$. In particular, the result of Lemma 1 is different if we are at nonexceptional momenta (i.e., when no partial sum of p_i vanishes) or not. At nonexceptional momenta, the essential subgraphs S of G are connected subgraphs containing all external vertices, while at exceptional momenta they may be disconnected (provided the sum of external momenta incoming to every connected part is zero). In particular, the residues $R_{\bar{G},n}(p, d)$ given by (II.23) are regular functions of the p_i's at nonexceptional momenta, but are singular when the p_i's tend towards exceptional ones, so that the $R_{\bar{G},n}$ are not integrable functions of the p_i's (in the sense of integration in a complex dimensional space of dimension d [12–14]) and in particular the $R_{\bar{G},n}$ have no Fourier transform in position space.

As a consequence Lemma 1 is not valid to extract the I.R. singularities of the Mellin transform $M_{\bar{G}}(x_i, z, d)$ in position space (for simplicity $M_{\bar{G}}$ will denote the Mellin transform (II.6) and its Fourier transform). The problem of generalizing Lemma 1 in position space is very technical and has been discussed more precisely in some very peculiar case (the leading I.R. singularities of the two points function) in Ref. [11]. This suggests a very simple generalization, but a complete proof for the nonleading singularities would be a very cumbersome task. So let us only express this generalization without complete justification.

LEMMA 2. *The Mellin transform* $M_{\bar{G}}(x_i, z, d)$ *(in position space) has I.R. poles* z_0 *given by* (II.10)

$$z_0 + \omega(\widetilde{\bar{G}/\bar{S}}) = -n, \qquad n \in \mathbb{N}, \tag{II.26}$$

for any g-subgraphs $\bar{S}$ *of* $\bar{G}$ *essential for some peculiar choice of external momenta (i.e., S may be connected or not, S may even be only the set of external vertices of G, which is essential if all* p_i*'s are zero). The reduced g-graph* $[\widetilde{\bar{G}/\bar{S}}]$ *is defined in Appendix A and is obtained by shrinking* $\bar{S}$ *into* one vertex *in* $\bar{G}$ *even if S is disconnected.*

The residue of $M_{\bar{G}}$ at z_0 is given by a sum over every such g-subgraph $\bar{S}$ satisfying (II.26)

$$\mathrm{Res}(M_{\bar{G}}(x_i, z, d); z_0) = \sum_{\bar{S}} F_{\bar{S},n}(x_i, d) \cdot I_{[\widetilde{\bar{G}/\bar{S}}]}(1, d), \tag{II.27}$$

where $F_{\bar{S},n}(x_i, d)$ is the Fourier transform of the Hadamard's finite part [15] of the function $R_{\bar{S},n}(p_i, d)$, which is a distribution obtained from $R_{\bar{S},n}$ by unambiguous subtraction prescriptions at exceptional momenta.

Since the finite parts $F_{\bar{S},n}$ are distributions in momentum space we have no explicit integral representation. However, we have the implicit representation (by applying (II.27) to $z_0 = -n$)

$$F_{\bar{G},n}(x_i, d) = \mathrm{Res}[M_{\bar{G}}(x_i, z, d); z = -n]. \tag{II.28}$$

E. *The Zero Mass Asymptotic Expansion at Generic d*

From Lemmas 1 and 2 we can get the *complete* asymptotic expansion of the dimensionally regularized amplitude of any g-graph $I_{\bar{G}}(x_i, m, d)$, as the mass m goes to zero, for d away from the rationals $2r/s$, $0 \leqslant s \leqslant L(G)$ (where the Mellin transform may have multiple I.R. poles and U.V. singularities).

First we define the I.R. *analytic part* of $\bar{G}$ as the series

$$F_{\bar{G}}(x_i, m, d) = \sum_{n \in \mathbb{N}} F_{\bar{G},n}(x_i, d)\, m^{2n}. \tag{II.29}$$

This has to be understood as a formal asymptotic series in m^2. However, it appears on simple examples that series (II.29) defines an *entire* function of m^2 in the complex plane; a general proof of this remarkable property has not been found.

With this notation the general result is

THEOREM 1. *The zero mass asymptotic expansion fo $\bar{G}$ is a sum over all g-essentials $\bar{E}$ in $\bar{G}$:*

$$I_{\bar{G}}(x_i, m, d) \cong \sum_{\bar{E} \subset \bar{G}} F_{\bar{E}}(x_i, m, d)\, I_{[\widetilde{\bar{G}/\bar{E}}]}(1, d)\, m^{2\omega(\widetilde{\bar{G}/\bar{E}})}. \tag{II.30}$$

The above expansion contains entire and nonrational powers of m^2 (for d nonrational) from some minimal value to $+\infty$. To a given power of m^2 corresponds in the expansion a finite *number of g-essentials $\bar{E}$ and of terms in expansion* (II.29) *of $F_{\bar{E}}$.*

For d large enough the graph $\bar{G}$ is I.R. convergent and all powers of m^2 are positive or null, and the massless limit is given by $F_{\bar{G}}(x_i, 0, d)$ which is the I.R. finite part of $I_{\bar{G}}$. We see that the analytic part $F_{\bar{G}}(x_i, m, d)$ is given by the sum of all entire powers of m^2 in expansion (II.30).

We see that expansion (II.30) splits into two kinds of contributions: I.R. regular ones associated to the essentials $\bar{E}$ (the $F_{\bar{E}}(x_i, m, d)$) and I.R. singular parts associated to the reduced graph $[\widetilde{\bar{G}/\bar{E}}]$. Homogeneity gives besides

$$I_{[\widetilde{\bar{G}/\bar{E}}]}(1, d)\, m^{2\omega(\widetilde{\bar{G}/\bar{E}})} = I_{[\widetilde{\bar{G}/\bar{E}}]}(m, d) \tag{II.31}$$

so that these parts are simply the dimensionally regularized amplitudes of $[\widetilde{\bar{G}/\bar{E}}]$.

This expansion may be seen as a special case of the short distance operator product expansion [16–18] given here explicitly at all orders at the level of Feynman amplitudes. Such an expansion may be summed over graphs and expressed in terms of operators. This will be more precisely discussed in Section IV.

Finally let us discuss what happens to expansion (II.30) when d tends toward a rational value where several infrared singularities coincide. Then we observe the appearance of poles in d in some coefficients of the expansion together with the fact that the corresponding powers of m coincide. The overall result (which is simply the consequence of the appearance of multiple poles in the Mellin transform) is the appearance of powers of logarithms of the mass (with finite coefficients) in the expansion. This case will be treated in the next section.

III. I.R. SINGULARITIES AT RATIONAL DIMENSIONS

A. *Subtraction of the I.R. Poles*

As we mentioned in Section I, the I.R. finite part for some graph G, $F_G(x_i, 0, d)$ given by (II.29) (which is the analytic continuation for small d of the massless amplitude of G defined only for d great enough) is considered by many authors as an "I.R. regularized massless amplitude" obtained by dimensional regularization [3–5, 8]. From Section II, this I.R. finite part is meromorphic in d with I.R. poles at some rational dimensions d^* (in addition to the usual U.V. poles), where the Mellin transform $M_G(z, d^*)$ has multiple poles in z.

It appears that the techniques of extraction of single poles of the Mellin transform used in Section II may not be easily applied to the extraction of multiple poles. Instead, we shall desingularize the I.R. finite part $M_G(x_i, m, d)$ at such an I.R. pole d^* by defining an I.R. subtracted massless amplitude S_G, finite at d^*. We shall show that this S_G is obtained from the I.R. finite part F_G by the insertion of nonlocal operators, according to the conjecture of Parisi [8].

This I.R. subtraction procedure depends on an arbitrary mass parameter μ which fixes the scale of the I.R. subtractions. In part B we get a differential equation with respect to μ for the subtracted amplitudes. The solutions for the subtracted amplitudes are polynomials in $\mathrm{Ln}(\mu)$.

As an application of those results we give in C the complete zero mass expansion of any Feynman amplitude $I_G(x_i, m, d^*)$ at the rational dimension d^*.

Not to deal with the U.V. problem we consider a $\underline{g}$-graph G (in the following we always deal with g-graphs and forget the subscript $\bar{G}$) which has I.R. poles but no U.V. ones at some fixed rational dimension d^*.

Let us consider $F_{G,n}(x_i, d)$ for some $n \in \mathbb{N}$. From (II.28), it has poles at d^* related to the singularities of $M_G(x_l, z, d)$ which meet at $(z = -n, d = d^*)$. These singularities are lines in the (z, d) plane of equations:

$$z_i(d) = -n - (d - d^*)\frac{i}{2}, \qquad i = 0, \dots, i_{\max}(G). \tag{III.1}$$

$z_i(d)$ is associated to the essential E in G such that

$$n - \omega^*(\widetilde{G/E}) \in \mathbb{N} \qquad \text{and} \qquad L(\widetilde{G/E}) = i, \tag{III.2}$$

where L is the number of loops and ω^* the superficial degree of divergence (given by (II.5)) at $d = d^*$. $F_{G,n}(x_l, d)$ was obtained by a Cauchy contour in the z plane around the pole $z_0(d)$; as d goes to d^*, this contour is pinched between z_0 and z_1 so that $F_{G,n}$ is singular.

A simple way to obtain a finite amplitude at d^* is to choose a Cauchy contour $\mathscr{C}$ which encircles all the z_i's given by (III.1) (and only those poles) as d is close enough to d^*.

DEFINITION. Let $\mathscr{C}$ be such a contour (see Fig. 4). We define a μ-I.R. *subtracted amplitude* at d^* by

$$T_{G,n}(x_i, d; \mu)_{d^*} = \frac{1}{2i\pi}\oint_{\mathscr{C}} dz\, M_G(x_i, z, d)\, \mu^{-2(z+n)}, \tag{III.3}$$

where μ is some positive parameter (which has engineering dimension of a mass).

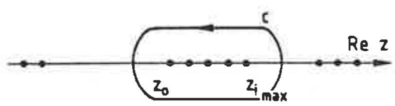

FIG. 4. The Cauchy contour used in (III.3) to define the I.R. subtracted amplitude T_G.

From the choice of $\mathscr{C}$, $T_{G,n}$ is obviously regular at d^*. Using Cauchy formula, the rhs of (III.3) is the sum of the residues of $M_G(x_i, z, d)\,\mu^{-2(z+n)}$ at the z_i's given by (III.1). Using Lemma 2 we get for $T_{G,n}$

$$T_{G,n}(x_i, d; \mu)_{d^*} = \sum_{n-\omega^*(\widetilde{G/E})=n_E} 4\, F_{E,n_E}(x_i, d)\, I_{[\widetilde{G/E}]}(1, d) \cdot \mu^{(d-d^*)L(\widetilde{G/E})}, \quad \text{(III.4)}$$

where the sum runs over g-essentials $E \subset G$ such that $n - \omega^*(\widetilde{G/E}) = n_E$ is a positive entire number.

To rewrite this result in a compact way for any n we define $T_G(x_i, m, d; \mu)_{d^*}$ as the (formal) series in m^2:

$$T_G(x_i, m, d; \mu)_{d^*} = \sum_{n\in\mathbb{N}} m^{2n} T_{G,n}(x_i, d; \mu)_{d^*}. \quad \text{(III.5)}$$

Then using (II.31), we get

LEMMA 3.

$$T_G(x_i, m, d; \mu)_{d^*} = \sum_{\substack{E\subseteq G\\ \omega^*(\widetilde{G/E})\in\mathbb{Z}}} F_G(x_i, m, d)\, I_{[\widetilde{G/E}]}(1, d)(\mu/m)^{(d-d^*)L(\widetilde{G/E})}, \quad \text{(III.6)}$$

where the sum runs over any g-esentials E in G such that $\omega^*(\widetilde{G/E})$ is a positive or negative entire number.

We now describe this result. The I.R. subtracted amplitude T_G can be considered as obtained from the "analytic part" F_G by I.R. subtractions relative to any g-essential E in G giving an I.R. pole at d^*. The corresponding counterterm is the bare massive Feynman emplitude $I_{[\widetilde{G/E}]}$ of the reduced graph $[\widetilde{G/E}]$, which may have U.V. poles at d^* ($[G/E]$ is superficially U.V. divergent when $\omega^*(\widetilde{G/E} \in \mathbb{N}$; it may in addition have divergent subgraphs). μ fixes the scale of those I.R. subtractions.

There is an unpleasant technical feature in the I.R. subtraction defined by (III.6): the presence of graphs $[\widetilde{G/E}]$ such that $\omega^*(\widetilde{G/E}) < 0$ means that the subtraction procedure defined above by the Cauchy contour (III.3) contains "I.R. oversubtractions." We now define "minimally subtracted amplitudes," where only essentials such that $\omega^*(\widetilde{G/E}) \geqslant 0$ are taken into account.

DEFINITION. The minimally I.R. subtracted amplitude $S_G(x_i, m, d; \mu)_{d^*}$ is defined as the formal power series in m^2:

$$S_G(x_i, m, d; \mu)_{d^*} = \sum_{n\in\mathbb{N}} m^{2n} S_{G,n}(x_i, d; \mu)_{d^*}. \quad \text{(III.7)}$$

It is defined from F_G by a sum over any g-essentials E in G such that $\omega^*(\widetilde{G/E})$ is a positive or null integer:

$$S_G(x_i, m, d; \mu)_{d^*} = \sum_{\substack{E\subseteq G\\ \omega^*(\widetilde{G/E})\in\mathbb{N}}} F_G(x_i, m, d)\, I^{(-)}_{[\widetilde{G/E}]}(m, d)(\mu/m)^{(d-d^*)L(\widetilde{G/E})}. \quad \text{(III.8)}$$

The counterterms $I^{(-)}_{[\widetilde{G/E}]}$ are defined in Appendix B (Definition B.2) as the Feynman amplitudes of the graphs $[\widetilde{G/E}]$ with U.V. subtractions only relative to subgraphs S in $[\widetilde{G/E}]$ which are more divergent than $[\widetilde{G/E}]$ itself ($\omega^*(S) > \omega^*(\widetilde{G/E})$). Since the graph $[\widetilde{G/E}]$ is itself divergent at d^* and is not subtracted, $I^{(-)}_{(\widetilde{G/E})}(1, d)$ is still singular at d^*. The fact that S_G is not singular is not obvious when looking at (III.8).

THEOREM 2. *For any $\mu > 0$, $S_G(x_i, m, d; \mu)_{d^*}$ is I.R. finite at d^*. The I.R. subtracted amplitude T_G is related to S_G by*

$$T_G(x_i, m, d; \mu)_{d^*} = \sum_{\substack{E \subset G \\ \omega^*(\widetilde{G/E}) \in Z}} S_E(x_i, m, d; \mu)_{d^*} \, I^{R,0}_{[\widetilde{G/E}]}(1, d)\, \mu^{(d-d^*)L(\widetilde{G/E})} m^{2\omega^*(\widetilde{G/E})}, \quad \text{(III.9)}$$

where $I^{R,0}_{[\widetilde{G/E}]}$ is an U.V. subtracted Feynman amplitude, finite at d^*, obtained by the minimal renormalization scheme $R, 0$ defined in Appendix C.

Since S_G and T_G differ from one another by finite counterterms at d^*, it is easy to conclude from the I.R. finiteness of T_G that S_G is also I.R. finite. Identity (III.9) is a consequence of the definition of $I^{R,0}_{[\widetilde{G/E}]}$ given by (C.1),

$$I^{R,0}_{[\widetilde{G/E}]}(1, d) = \sum_{\substack{S_0 \subset [\widetilde{G/E}] \\ \omega^*(S_0) \in N}} I_{[\widetilde{G/E}]/S_0}(1, d)\, K^0_{S_0}(d), \quad \text{(III.10)}$$

where $K^0_S(d)$ is the U.V. counterterm associated to the divergent subgraph S defined by (C.2), and of the following theorem, which expresses F_G in term of S_G.

THEOREM 3.

$$F_G(x_i, m, d) = \sum_{\substack{E \subset G \text{ essential} \\ \omega^*(\widetilde{G/E}) \in N}} S_E(x_i, m, d; \mu)_{d^*} \, K^0_{[\widetilde{G/E}]}[d]\, \mu^{(d-d^*)L(G/E)} m^{2\omega^*(\widetilde{G/E})}. \quad \text{(III.11)}$$

The proof of this theorem is obtained by inverting (III.7) and by expressing K^0 in terms of the $I^{(-)}$; it is reported in detail in Appendix D.

B. *A Differential Equation in μ for the I.R. Subtracted Amplitudes*

The subtracted amplitude $S_G(x_i, m, d^*; \mu)$ (from now on we omit the subscript d^*) defined above depends of a parameter μ which fixes the scale of the I.R. subtractions. This dependence can be explored systematically by constructing a differential equation similar to the well known U.V. renormalization group equation. This equation shows that S_G is a polynomial in $\operatorname{Ln} \mu$ at $d = d^*$; this result will be used in part C to obtain the zero mass asymptotic expansion of any bare Feynman amplitude at d^* (which generalizes Theorem 1 at nongeneric d), and will be useful in Section IV.

Let us report (III.11) into (III.8) for two different values μ and μ' of the scale parameter. We get

$$S_G(x_i, m, d^*; \mu) = \sum_{\substack{E \subset G \\ \omega^*(\widetilde{G/E}) \in \mathbb{N}}} S_E(x_i, m, d^*; \mu') \, J_{[\widetilde{G/E}]}(\mu/\mu', d^*) \, m^{2\omega^*(\widetilde{G/E})}, \quad \text{(III.12)}$$

where for any S, $J_S(\eta, d)$ is defined as a sum over U.V. divergent subgraphs $S' \subset S$ such that $\omega^*(S')$ is an integer $\leqslant \omega^*(S)$:

$$J_S(\eta, d) = \sum_{\substack{S' \subset S \\ \omega^*(S) \geqslant \omega^*(S') \in \mathbb{N}}} K^0_{S'}(d) \, I^{(-)}_{[S/S']}(1, d) \, \eta^{(d-d^*)L(S')}. \quad \text{(III.13)}$$

For any $\eta > 0$, $J_S(\eta, d)$ is finite at d^*. Differentiating (III.12) with respect to μ and setting $\mu = \mu'$ we get the linear differential equation

$$\mu \frac{\partial}{\partial \mu} S_G(x_i, m, d^*; \mu) = \sum_{\substack{E \neq G \\ \omega^*(\widetilde{G/E}) \in \mathbb{N}}} S_E(x_i, m, d^*; \mu) \, C_{[\widetilde{G/E}]}(d^*) \, m^{2\omega^*(\widetilde{G/E})}, \quad \text{(III.14)}$$

where the coefficients $C_{[\widetilde{G/E}]}$ do not depend on μ and are defined as

$$C_S(d^*) = \eta \frac{\partial}{\partial \eta} J_S(\eta, d^*) \bigg|_{\eta = 1}. \quad \text{(III.15)}$$

It is easy to integrate the differential equation (III.14) from $\mu = 1$.

THEOREM 4. *Each term $S_{G,n}$ of the formal series $S_G(x_i, m, d^*; \mu)$ in m is a polynomial in* Ln μ, *whose coefficients are given by integrating* (III.14) *from* 1 *to* μ. *We get a sum over all nests of essentials in G, $\mathbb{N} = \{E_0 \subset E_1 \cdots \subset E_N = G\}$, containing G as greatest element and such that $\{\omega^*(\widetilde{G/E_i}), i = 0, N\}$ is a decreasing sequence of positive integers.*

The final result reads

$$S_G(x_i, m, d^*; \mu) = \sum_{\mathbb{N}} S_{E_0}(x_i, m, d^*; 1) \frac{\prod_{i=1}^{N} C_{[\widetilde{E_i/E_{i-1}}]}(d^*)}{N!} m^{2\omega^*(\widetilde{G/E_0})} (\text{Ln}\, \mu)^N. \quad \text{(III.16)}$$

Let us note that an infinite number of nests have to be taken into account in (III.16), but only a finite number of nest are present at each order in m^2; so each $S_{G,n}$ is a well defined polynomial in Ln μ. It is easy to see that the order of this polynomial may not be greater than $L + N - \chi$, where L is the number of loops of G, N the number of external legs and χ the number of connected components of G.

C. *The Zero Mass Expansion of a Feynman Amplitude at Some I.R. Pole $d = d^*$*

As a straightforward application of those results, we can generalize the result of Theorem 1 at those rational dimensions d^*, where several powers of m coalesce while their coefficients are I.R. singular in (II.30). For this purpose, one has simply to use the result of Theorem 3 to express the finite parts F_G in (II.30) in terms of the subtracted amplitudes S_G. We get the asymptotic expansion

$$I_G(x_i, m, d) \underset{m \to \infty}{\cong} \sum_E S_E(x_i, m, d; \mu)_d \cdot I^{R,0}_{[\widetilde{G/E}]}\left(d, \frac{m}{\mu}\right) \mu^{2\omega(\widetilde{G/E})}, \qquad \text{(III.17)}$$

where $I^{R,0}_{[\widetilde{G/E}]}$ is the U.V. subtracted amplitude of $[\widetilde{G/E}]$ at d^* (see Appendix B). Now the lhs is finite at d^*. To get the complete asymptotic expansion at d^* we take $\mu = m$ and use Theorem 4. We get

THEOREM 5. *The zero mass asymptotic expansion I_G at d^* is a sum over all nests $\mathcal{N} = \{E_0 \subset \cdots \subset E_N\}$ of essentials in G such that $\{\omega^*(\widetilde{E_i/E_0}), i = 0, N\}$ is an increasing sequence of positive integers*

$$I_G(x_i, m, d^*) \underset{m \to 0}{\cong} \sum_{\mathcal{N}} S_{E_0}(x_i, m, d; 1) \prod_{i=1}^{N} C_{[\widetilde{E_i/E_{i-1}}]}(d^*)\, I^{R,0}_{[\widetilde{G/E_N}]}(1, d^*)\, m^{\omega^*(\widetilde{G/E_0})} \frac{(\text{Ln}\, m)^N}{N!}. \qquad \text{(III.18)}$$

Let us discuss this result. Equation (III.18) holds at any rational dimension d^*, provided that there are no U.V. poles in the dimensionally regularized amplitude $I_G(d)$. In the opposite case, the coefficients S_E, $C_{[\widetilde{E_i/E_{i-1}}]}$ and $I^{R,0}_{[\widetilde{G/E}]}$ still develop U.V. poles at d^* and some extra renormalization has to be performed.

By homogeneity this theorem provides also the complete asymptotic expansion of the amplitude I_G at large momenta ($p \to \lambda p$; $\lambda \to \infty$) (in (III.18) change m in $1/\lambda$ and multiply by a factor $\lambda^{\omega^*(G)}$). Equation (III.18) generalizes at all orders the relsult obtained in [9] for the leading power in λ of any convergent Feynman amplitude.

IV. Nonperturbative Summation of I.R. Divergences

We now look at the I.R. divergences of super-renormalizable massless field theories. We shall only consider for simplicity the case of the scalar ϕ^4_d theory ($d < 4$) as a test for our ideas, while other massless field theories will be discussed after.

The ϕ^4 theory in a d-dimension Euclidian space-time is defined by the bare Lagrangian density

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi \partial_\mu \Phi) + \frac{1}{2} m^2 \Phi^2 + \frac{g}{4!} \Phi^4. \qquad \text{(IV.1)}$$

m and g are the bare mass and the bare couping constant and have, respectively, for dimensions 2 and $\varepsilon = 4 - d$. Below 4 dimensions, g differs from the renormalized coupling constant g_R (usually defined at some normalization point) by some finite renormalization factor. On the contrary m differs from the physical renormalized mass m_R by an additive renormalization factor [5] (singular in general for $2 \leqslant d$) which may be written

$$m^2 = m_R^2 + \Delta^2. \tag{IV.2}$$

In dimensional regularization Δ has U.V. poles at $d = 4 - 2/n$ ($n = 1, 2, 3, \ldots$) but is finite away from those poles. In other words, if the bare mass m is set equal to zero, the model develops a mass by itself and we are not dealing anymore with a "true" massless theory.

However, as pointed out in [6], in theories where some symmetry principle forbids the perturbative generation of a mass, it is sufficient to set the bare mass equal to zero to get a massless theory. This is the case for

— pure gauge theories (gauge invariance makes the gauge field massless);

— gauge field + massless fermions (Chiral symmetry keeps the fermions massless);

— nonlinear σ models (the global $O(N)$ invariance), etc.;

— this is also the case for the ϕ^4 theory with $O(N)$ symmetry (ϕ is then an N-component real vector field) when N is (formally) set equal to -2 [19]. Indeed the irreducible 2-points function Γ_2 is $\Gamma_2 = p^2 + m^2$ at all orders. So there are neither mass nor field renormalizations and $m_R^2 = m^2$.

Although in the case of ϕ_d^4 there exists a mass renormalization, in this section we shall for simplicity forget this problem and study the ϕ^4 theory when the bare mass m is set equal to zero. Our result will be directly generalizable to those theories which do not develop a mass. It is a well defined problem to study the bare massless theory in a dimension d where there are no U.V. divergences before looking at the physical massless theory.

A. *Perturbative I.R. Expansion of Green's Functions*

We denote by

$$\mathscr{G}_M(x_i, m, g, d) = \langle 0|\, \phi(x_1) \cdots \phi(x_M)\, |0\rangle \tag{IV.3}$$

the value of the bare M-points Green's function at the points $\{x_i; i = 1 \cdots M\}$, which is defined perturbatively as the sum of the bare amplitudes $I_G(x_i, m, d)$ (given by (II.2)) with their corresponding powers of g and symmetry factors.

From Theorems 1 and 5, the zero mass asymptotic epansion of I_G is expressed in term of essential subgraphs $\bar{E}$ in G and of reduced subgraphs $[G/\bar{E}]$. As shown in Appendix A, any essential $\bar{E} = \{E, \chi\}$ of G corresponds to the insertion of the nonlocal operator $\mathscr{O}_\chi$,

$$\mathscr{O}_\chi = \prod_j \int d^d y_j \frac{(iy_j)^{\lambda_j}}{|\lambda_j|!} (-\Delta + m^2)\, \phi(y_j), \tag{IV.4}$$

in the Green's function $\mathscr{G}_M$. In the same way, the reduced graph $[\widetilde{G/E}]$ is a graph of the vacuum expectation value for the composite operator $\hat{\mathcal{O}}_\chi$ dual to $\mathcal{O}_\chi$:

$$\hat{\mathcal{O}}_\chi = \prod_j \left(i \frac{\partial}{\partial x_j}\right)^{\lambda_j} \phi(x_i) \bigg|_{x_j=0} . \tag{IV.5}$$

Summing the result of Theorem 1 over all graphs G in $\mathscr{G}_M$ we get a factorization in terms of a sum over operators $\mathcal{O}$.

1. *I.R. Asymptotic Expansion at Nonrational d*

At nonrational d, the Green's function $\mathscr{G}_M$ has the I.R. expansion

$$\mathscr{G}_M(x_i, m, g, d) \underset{m\to 0}{\cong} \sum \mathscr{F}_{M,\mathcal{O}}(x_i, m, g, d) \cdot \hat{\mathcal{O}}(m, g, d), \tag{IV.6}$$

where the sum runs over the infinite set of operators $\mathcal{O}$ (of form (A.5)). $\mathscr{F}_{M,\mathcal{O}}$ is the I.R. *analytic part* of the Green's function with insertion of the operator $\mathcal{O}$, and is obtained by replacing the bare amplitude $I_{\bar{G}}(x_i, m, d)$ by its analytic part $\mathscr{F}_{\bar{G}}(x_i, m, d)$ defined by (II.29),

$$\mathscr{F}_{M,\mathcal{O}}(x_i, m, g, d) = \sum_{\bar{G}} g^{n(\bar{G})} F_{\bar{G}}(x_i, m, d), \tag{IV.7}$$

and is a double series in g and m^2.

$\hat{\mathcal{O}}(m, g, d)$ is the bare vacuum expectation value of the dual operator $\hat{\mathcal{O}}$ (IV.5) and is defined perturbatively as a sum of massive bare amplitudes $I_S(m, d)$. The sum in (IV.6) is performed according to the increasing canonical dimension (IV.9) of the operator $\hat{\mathcal{O}}$.

Expansion (IV.6) splits $\mathscr{G}_M$ into the quantities $\mathscr{F}_{M,\mathcal{O}}$'s which are I.R. regular, and into the operators $\hat{\mathcal{O}}$ which are nonanalytic powers of m. Indeed, when looking at the result of Theorem 1 (or by naive power counting arguments), the term of order g^n ($n \geqslant 0$) of $\hat{\mathcal{O}}(m, g, d)$ is a pure power of the mass, namely,

$$\hat{\mathcal{O}}(m,g,d) = \sum_n g^n \mathcal{O}_n(d)\, m^{[\hat{\mathcal{O}}] - \varepsilon n}, \tag{IV.8}$$

where $\varepsilon = 4 - d$ and where $[\hat{\mathcal{O}}]$ is the canonical dimension of the operator $\hat{\mathcal{O}}$,

$$[\hat{\mathcal{O}}] = (1 - \varepsilon/2)\, L + n(\chi), \tag{IV.9}$$

where L is the number of external Φ fields and $n(\chi)$ the number of derivatives in (IV.5). Since we are in the super-renormalizable domain ($\varepsilon > 0$), $\hat{\mathcal{O}}$ becomes more and more I.R. divergent as the perturbation order increases.

2. *I.R. Asymptotic Expansion at Rational Dimension d**

From Section III, at any rational dimension ($d^* = p/q$), the terms of expansion (IV.6) develop I.R. poles. Using Eq. (III.17), where we have eliminated the I.R. poles at d^* from the massless expansion of a Feynman graph, we obtain by summation over all graphs

$$\mathcal{G}_M(x_i, m, g, d^*) \underset{m \equiv 0}{\equiv} \sum_{\mathcal{O}} S_{M,\mathcal{O}}(x_i, m, g, d^*; \mu)\, N_0'[\mathcal{O}] \left(\frac{m}{\mu}, g\mu^{-\varepsilon}, d^*\right) \mu^{[\mathcal{O}]}, \quad \text{(IV.10)}$$

where $S_{M,\mathcal{O}}(\ldots; \mu)$ is the μ-I.R. subtracted part of $\mathcal{G}_{M,\mathcal{O}}$, obtained now by using the I.R. subtracted amplitudes $S_{\bar{G}}(x_i, m, d; \mu)$ defined by (III.8)

$$S_{M,\mathcal{O}}(x_i, m, g, d^*; \mu) = \sum_{\bar{G}} g^{n(\bar{G})} S_{\bar{G}}(x_i, m, d; \mu)_{d^*}. \quad \text{(IV.11)}$$

$N_0'[\mathcal{O}]$ is the expectation value of the renormalized operator $\mathcal{O}$ following the normal product algorithm (NPA) at d^* defined in Appendix C, and is the sum of the renormalized amplitudes $I_S^{R,0}(m/\mu, d^*)$ at d^*. μ is an arbitrary mass parameter. The singular I.R. terms of the expansions are now powers and logarithms of m and are contained in the $N_0'[\mathcal{O}]$'s. (Taking $\mu = m$ in (IV.10), the Ln(m) are recast in $S_{M,\mathcal{O}}$ and are given by Theorem 4.)

As already discussed at the end of Section II, the I.R. expansions (IV.6) and (IV.10) are quite similar to the short distance operator product expansion [16–18], here obtained explictly at all subdominant orders. In our case the I.R. divergent parts of the expansion are associated to the composite operators $\mathcal{O}$ and do not depend on the considered observable.

B. *Nonperturbative Summation of the I.R. Divergences*

Expansions (IV.6–10) strongly suggest that if a massless theory exists, some nonperturbative summation of the I.R. divergences of the operators $\mathcal{O}$ should occur and lead to I.R. finite massless Green's functions. We may consequently assume the two following properties:

(i) Expansions (IV.6–10) are valid for the full (nonperturbative) Green's functions.

(ii) The composite operators $\mathcal{O}(g, m)$ (resp. $\mathcal{N}_0'(g, m)$) have a finite massless limit as $m \to 0$.

Of course we have no proof of the above assumptions. Assumption (i) remains completely formal as long as we stand at the perturbative level.[3] Assumption (ii) seems to be reasonable provided we expect that the massless theory exists.

1. *Nonrational Dimension d*

Let us first treat the case of nonrational dimensions. We assume that in (IV.6) we are allowed to sum up the I.R. singular parts $\hat{\mathcal{O}}(m, g, d)$ to get I.R. finite nonpertur-

[3] In the case of the ϕ^4 theory, the $1/N$ expansion provides a way to handle in a nonperturbative way the massless theory and to justify assumptions (i) and (ii).

bative contributions $\hat{\mathcal{O}}(0, g, d)$. The I.R. finite parts $\mathcal{F}_{M,\mathcal{O}}$ are regular at $m = 0$ and have a perturbative massless limit $\mathcal{F}_{M,\mathcal{O}}(x_i, 0, g, d)$. Since now we are in a bare massless theory, where the only mass scale is given by the coupling constant, the dependence in g of $\hat{\mathcal{O}}(0, g, d)$ is a simple power obtained from dimensional considerations

$$\hat{\mathcal{O}}(0, g, d) = \hat{\mathcal{O}}(0, 1, d) \cdot g^{[\hat{\mathcal{O}}]/\varepsilon}, \qquad \text{(IV.12)}$$

where $\hat{\mathcal{O}}(0, 1, d)$ is a pure number not computable by perturbative methods.

The massless Green's function reads

$$\mathcal{G}_M(x_i, 0, g, d) = \sum_{\mathcal{O}} \mathcal{F}_{M,\mathcal{O}}(x, 0, g, d)\, \hat{\mathcal{O}}(0, 1, d) \cdot g^{[\hat{\mathcal{O}}]/\varepsilon}. \qquad \text{(IV.13)}$$

Equation (IV.13) provides a new perturbative expansion in g for $\mathcal{G}_M$, containing analytic parts, the $\mathcal{F}_{M,\mathcal{O}}(x_i, 0, g, d)$ computable from perturbation theory, and nonanalytic powers of g, $g^{[\mathcal{O}]/\varepsilon}$. Expansion (IV.13) is organized following the increasing dimensions of $\hat{\mathcal{O}}$.

2. *Rational Dimension d^**

Applying the same procedure at $d = d^*$ rational, we sum up the renormalized operator $\mathcal{N}_0'[\hat{\mathcal{O}}](m/\mu, g\mu^{-\varepsilon}, d)$ in (IV.10) to get its I.R. nonperturbative massless limit $\mathcal{N}_0'[\hat{\mathcal{O}}](0, g\mu^{-\varepsilon}, d)^*$. We get at rational d^*

$$\mathcal{G}_M(x_i, 0, g, d^*) = \sum_{\mathcal{O}} S_{M,\mathcal{O}}(x_i, 0, g, d^*; \mu)\, N_0'[\hat{\mathcal{O}}](0, g\mu^{-\varepsilon^*}, d) \cdot \mu^{[\hat{\mathcal{O}}]} \qquad \text{(IV.14)}$$

(with $\varepsilon^* = 4 - d^*$). The dependence in g of $N_0'[\hat{\mathcal{O}}]$ may be eliminated by choosing $\mu = g^{1/\varepsilon^*}$in (IV.14) ($\mu$ is the scale of I.R. subtraction in $S_{M,\mathcal{O}}$ and is arbitrary). We get

$$\mathcal{G}_M(x_i, 0, g, d^*) = \sum_{\mathcal{O}} S_{M,\mathcal{O}}(x_i, 0, g, d^*; g^{1/\varepsilon^*})\, N_0'[\hat{\mathcal{O}}](0, 1, d)\, g^{[\hat{\mathcal{O}}]/\varepsilon^*}. \qquad \text{(IV.15)}$$

Equation (IV.15) defines a perturbative expansion in the coupling constant g which contains not only analytic and nonanalytic powers of g, but also logarithms of g. Indeed, each pertubative term of $S_{M,\mathcal{O}}(x_i, 0, g; \mu)$ is a polynomial in $\mathrm{Ln}\, \mu$, given by (III.16). In (IV.15), these $\ln \mu$ become $1/\varepsilon^*\, \mathrm{Ln}\, g$.

C. *Discussion*

Identities (IV.13) and (IV.15) show that, if we assume that the I.R. divergencies may be summed up in a nonperturbative way, the massless super-renormalizable theory has a weak coupling perturbative expansion containing not only analytic powers, but in general nonanalytic powers and logarithms of the coupling constant, which we characterize at all orders.

Equation (IV.13) provides a nice understanding of the conjecture by Parisi [8]. In the sum over the operators $\mathcal{O}$, the first term is associated to the operator $\mathcal{O} = \mathbb{1}$

(identity) and is nothing else than the "I.R. finite part" $\mathscr{F}_M(x_i, 0, g, d)$ of the Green's function $\mathscr{G}_M$. Equation (IV.13) means that the massless Green's function $\mathscr{G}_M$ differs from its finite part $\mathscr{F}_M$ by "I.R. counterterms" $\hat{\mathscr{O}}(0, 1, d)\, g^{[\hat{\mathscr{O}}]/\varepsilon}$ nonanalytic in g corresponding to the insertion of the nonlocal operator $\mathscr{O}$ in $\mathscr{F}_M$.[4] At rational dimensions d, $\mathscr{F}_M$ has I.R. poles which are cancelled by these "I.R. counterterms" to give a finite perturative expansion (IV.15) with additional logarithms of the coupling constant. However, these "I.R. counterterms" are generated by the theory itself; there is no need to introduce nonlocal counterterms in the Lagrangian to get an I.R. finite theory, as it was proposed by Parisi [8].

From this point of view the occurrence of logarithms of the coupling constant at $d = 2$ or $d = 3$ dimensions is rather clear. Let us look at the case $d = 3$ in more detail. In the case of the ϕ^4 theory, using Theorem 4, it may be shown that each term of order g^n for some $S_{M,\mathscr{O}}(x_i, m, g, 3; g)$ occurring in (IV.15) is a polynomial in $\operatorname{Ln} g$ of order $\leqslant n/2$. So we get for the massless Green's functions an expansion of the form

$$\mathscr{G}(g) = \sum_{n=0}^{\infty} \sum_{p=0}^{n/2} a_{n,p}\, g^n \operatorname{Ln} g^p \tag{IV.16}$$

whose coefficients a_{np} are linear combinations of perturbative terms (the $S_{M,\mathscr{O}}$) with nonperturbative coefficients (coming from the operators $\hat{\mathscr{O}}$ such that $[\hat{\mathscr{O}}] \leqslant n$).

Moreover the *leading logarithms* ($p = E(n/2)$, where E is the entire part) in (IV.16) come only from the I.R. subtracted part of $\mathscr{G}_M$, S_M; they are associated to $\mathscr{O} = \mathbb{1}$, and so are entirely computable by perturbation theory. Comparing (IV.15) and (IV.10), we note that the coefficients of the leading logarithms of the coupling constant are simply those of the leading logarithms of the mass in the I.R. asymptotic expansion (IV.10) of the massive Green's functions. This result was already noted in [6–7]. However, the subleading logarithms ($p < E(n/2)$) are nonperturbative, since they contain some $\hat{\mathscr{O}}$.

The arguments exposed above in the simple example of the scalar ϕ^4 theory may be extended to more complicated models, such as the three dimensional gauge theories [6, 7]. The technical point is that one has to look at gauge invariant observables and it is preferable to use a gauge invariant I.R. cutoff (such a cutoff is proposed in [20]). We expect an I.R. finite nonanalytic weak coupling expansion for the expectation values of various observables with nonperturbative contributions associated to all possible invariant composite operators which generalizes at all orders the results of [6, 7].

V. Mass Renormalization and Massless ϕ^4_d; $d < 4$

We now discuss the modifications brought in our scheme by the presence of a mass renormalization, as this is case for the massless ϕ^4 theory. Indeed, the bare

[4] The nonlocal operators conjectured by Parisi [8] are not of the form (IV.4). However, a straightforward use of the equations of motion relates our operators to Parisi's ones, up to contact terms which he had not considered.

mass m in the bare Lagrangian (IV.1) of the ϕ^4 theory differs from the physical mass, and a renormalization of the mass has to be performed in order to keep the physical mass to zero and to subtract the U.V. poles which appear at $2 \leqslant d < 4$. In order to perform this renormalization we define the renormalized Green's functions $\mathscr{G}^R_N(x_i, m_R, g, d)$ by the renormalization prescription that the 2-points irreducible function Γ_2 is equal to m_R^2 (where m_R is the renormalized mass) at zero momenta:

$$\Gamma_2^R(p = 0, m_R, g, d) = m_R^2. \tag{V.1}$$

This corresponds to the use of the bare mass m^2,

$$m^2 = m_R^2 + \Delta(g, m_R), \tag{V.2}$$

in the bare Green's function $\mathscr{G}_N$, where the counterterm Δ is of the form

$$\Delta(g, m_R) = \sum_{n=1}^{\infty} g^n m_R^{2-\varepsilon n} \Delta_n(\varepsilon). \tag{V.3}$$

Each $\Delta_n(\varepsilon)$ has a single pole at $\varepsilon = 2/n$ $(\varepsilon = 4 - d)$ and is perturbatively computable [13].

The renormalized Green's functions G^R_N are related to the bare ones $\mathscr{G}_N$ by

$$\mathscr{G}^R_N(x_i, m_R, g, d) = \mathscr{G}_N(x_i, m, g, d) \tag{V.4}$$

so that we may use the results of Sections II and IV to get the I.R. asymptotic behaviour of $\mathscr{G}^R_N$ as the renormalized mass m_R goes to zero as long as there are no U.V. poles $(d \neq 4 - 2/n,\ n \in \mathbb{N})$. Using (IV.6), this expansion reads at nonrational dimension d

$$\mathscr{G}^R_N(x_i, m_R, g, d) \cong \sum_{\mathscr{O}} \mathscr{F}_{N,\mathscr{O}}(x_i, m, g, d)\, \hat{\mathscr{O}}(m, g, d), \tag{V.5}$$

where the sum runs over all nonlocal operators $\mathscr{O}$ of the form (IV.4), as in Section IV. The analytic parts of $\mathscr{G}_N$, $\mathscr{F}_{N,\mathscr{O}}$, defined by (IV.7), are now taken at the bare mass $m(g, m_R)$ given by (V.2–3), and so contain singular powers of the mass m_R; $\hat{\mathscr{O}}(m, g, d)$ is the vacuum expectation value of the bare operator $\hat{\mathscr{O}}$ at the bare mass m, and so is equal to $\hat{\mathscr{O}}^R(m_R, g, d)$; from (IV.8) and (V.2, 3), its perturbative expansion is of the form

$$\hat{\mathscr{O}}^R(m_R, g, d) = \sum_n g^n \mathscr{O}^R_n(d)\, m_R^{[\mathscr{O}] - \varepsilon n}. \tag{V.6}$$

Following Symanzik [5], we assume that the bare mass m has a finite limit as the renormalized mass m_R goes to zero, of the form

$$\lim_{m_R \to 0} m = g^{1/\varepsilon} F(\varepsilon). \tag{V.7}$$

The function $F(\varepsilon)$ is nonperturbative (and is obtained in [5] by renormalization group arguments at $4 - \varepsilon$ dimensions), and it has U.V. poles at $\varepsilon = 2/n$ $(n \in \mathbb{N})$, whose residues are equal to those of Δ (V.3) at $m_R > 0$. The power $g^{2/\varepsilon}$ is given simply by dimensional considerations.

This assumption is not sufficient to get massless Green's functions. As in Section IV, we have also to make the nonperturbative hypothesis (i) and (ii), that is, to assume that the composite operators $\hat{\mathcal{O}}^R(m_R, g, d)$ have a nonperturbative massless limit in (V.5). This I.R. limit has to be of the form

$$\lim_{m_R \to 0} \hat{\mathcal{O}}^R(m_R, g, d) = \hat{\mathcal{O}}^R(0, g, d) = g^{[\hat{\mathcal{O}}]/\varepsilon} \hat{\mathcal{O}}^R(0, 1, d). \tag{V.8}$$

Using these hypotheses we get for the massless Green's functions at nonrational dimension d the weak coupling expansion

$$\mathcal{G}_N^R(x_i, 0, g, d) = \sum_{\mathcal{O}} \mathcal{F}_{N,\mathcal{O}}(x_i, g^{1/\varepsilon} F(\varepsilon), g, d)\, \hat{\mathcal{O}}^R(0, 1, d)\, g^{[\hat{\mathcal{O}}]/\varepsilon}. \tag{V.9}$$

We recall that from the definition of $\mathcal{F}_{N,\mathcal{O}}$, $\mathcal{F}_{N,\mathcal{O}}(x_i, g^{1/\varepsilon}F(\varepsilon), g, d)$ is in fact a double series in g and $g^{2/\varepsilon}F^2(\varepsilon)$.

This expansion differs from (IV.13), valid when there is no mass renormalization, by the fact that nonperturbative terms are associted not only to the composite operators $\hat{\mathcal{O}}$, but also to the bare mass. Equation (V.9) extends to all orders the results of Symanzik [5].

We now briefly discuss what happens at rational dimensions $d = 4 - 2/n$ ($n \geqslant 1$) in particular at $d = 2$ and 3. In (V.9), the $\mathcal{F}_{N,\mathcal{O}}$ have I.R. poles at rational dimensions which should be cancelled by the U.V. poles of the $\hat{\mathcal{O}}^R$ (the mass is renormalized but not the composite operator $\hat{\mathcal{O}}$ itself). In addition, the $\mathcal{F}_{N,\mathcal{O}}$ have U.V. poles at $\varepsilon = 2/n$ which should be cancelled by the poles of the mass counterterm $g^{1/\varepsilon}F(\varepsilon)$.

By the same mechanism as that in Section IV, each cancellation of poles gives a $\log(g)$. We get for the massless Green's functions a weak coupling expansion involving powers and logarithms of the coupling constant, with nonperturbative terms associated to the composite operators $\hat{\mathcal{O}}$ and to the renormalization of mass. Moreover, the $\log(g)$ are associated not only to the I.R. divergences (i.e., to the essential divergent subgraphs in the perturbative expansion) as in Section IV, but also to the U.V. divergences (i.e., to the U.V. divergent subgraphs) at d^*.

Appendix A

In this appendix we precise our notations and give some properties of the graphs with derivatives with respects to the momenta of some truncated legs.

Let us consider a scalar graph G (not necessary connected) with L truncated legs corresponding to some observable

$$\prod_{i=1}^{N} M_i(x_i) \prod_{j=1}^{L} \{(-\Delta_{y_j} + m^2)\,\phi(y_j)\}_{\text{conn}}, \tag{A.1}$$

where each M_i is a monomial in the field ϕ and its derivatives in x_i and where the $\{\ \}_{\text{conn}}$ means that each $\phi(y_j)$ is inserted in a connected way, that is: there is no connected part of G which contains only truncated legs $\{j\}$ and no $\{x_i\}$ vertex. The Feynman amplitude of G is $I_G(p_1,..., p_N; k_1,..., k_L)$, where the momenta p_i and k_j are, respectively, associated to x_i and y_j by Fourier transform.

DEFINITION A.1. Given some family $\chi = \{\lambda_1,..., \lambda_L\}$ of L families λ_i $(i = 1, L)$ of $|\lambda_i|$ Lorentz indices $\lambda_i = \{\mu_{i,1},..., \mu_{i,|\lambda_i|}\}$ we define the *g-graph* $\bar{G} = \{G, \chi\}$ as the graph corresponding to the Feynman amplitude

$$I_{\bar{G}}(p_1,..., p_N) = \left\{\prod_{j=1}^{L} \frac{1}{|\lambda_j|!} \left(\frac{\partial}{\partial k_j}\right)^{\lambda_j}\right\} I_G(p_1, p_N; k_1 \cdots k_L)\Bigg|_{k_j=0} \tag{A.2}$$

which corresponds to the observable

$$\prod_{i=1}^{N} M_i(x_i) \left\{\prod_{j=1}^{L} \frac{1}{|\lambda_j|!} \int d^d y_j (iy_j)^{\lambda_j} (-\Delta_{y_j} + m^2)\, \phi(y_j)\right\}_{\text{conn}}. \tag{A.3}$$

In (A.2) and (A.3), the notation $(iy_j)^{\lambda_j}$ and $(\partial/\partial k_j)^{\lambda_j}$ is a short writing for

$$\prod_{r=1}^{|\lambda_j|} (iy_j)_{\mu_{j,r}}; \quad \text{and} \quad \prod_{r=1}^{|\lambda_j|} (\partial/\partial k_j)_{\mu_{j,r}};$$

Eq. (A.2) is defined for positive integer dimensions d, but later on when all tensors will be contracted into scalars, the result will be analytically continued to all complex d via a well known procedure [12–14].

DEFINITION A.2. We now consider a g-graph $\bar{G} = \{G, \chi_G\}$, a subgraph $S \subseteq G$ and a family $\chi_S = \{\lambda'\}$ of derivatives in regards to the external legs of S (which may be external or internal to G). The g-graph $\bar{S} = \{S, \chi_S\}$ is called a *g-subgraph* of $\bar{G}$.

DEFINITION A.3. Given a g-graph $\bar{G}$ and a g-subgraph $\bar{S}$, the reduced g-graph $[\bar{G}/\bar{S}]$ is the g-graph

$$[\bar{G}/\bar{S}] = \{[G/S]_{\chi_S}, \chi_G\}, \tag{A.4}$$

where $[G/S]_{\chi_S}$ is defined as follows:

(i) Shrink any connected component S_i of S into one vertex v_i in G.

(ii) From (i), any external line (j) of S becomes a line attached to some v_i, carrying some incoming momenta q_j (this line may be internal or external to G/S). Then, insert on the line j the derivative couplings $(q_j)^{\lambda'_j}$.

In the definition of $\bar{G}/\bar{S}$, χ^G means that the amplitude $I_{[\bar{G}/\bar{S}]}$ is obtained from $I_{[G/S]\chi_S}$ by application of

$$\left\{\prod_{j=1}^{L}\left(\frac{\partial}{\partial k_j}\right)^{\lambda_j}\right\} \quad \text{at all } k_j = 0. \tag{A.5}$$

Let us consider any external leg of S which is also an external leg of G. If the corresponding λ' of χ_S and λ of χ_G are different, then $I_{[\bar{G}/\bar{S}]} = 0$ since $(1/\lambda!)(\partial/\partial q)^{\lambda}\, q^{\lambda'} = \delta_{\lambda\lambda'}$. If, for all external legs of S which are also external legs of G, $\lambda = \lambda'$, then χ_S is said to be compatible with χ_G and we say that $\bar{S} \subset \bar{G}$ (although χ_S is not a subfamily of χ_G).

Definition A.4. With the same notations as those in Definition A.3, we define the reduced g-graph $[\widetilde{\bar{G}/\bar{S}}]$ as the g-graph

$$[\widetilde{\bar{G}/\bar{S}}] = \{[\widetilde{G/S}]_{\chi_S}, \chi_G\}, \tag{A.6}$$

where the definition of $[\widetilde{G/S}]_{\chi_S}$ differs from that of $[G/S]_{\chi_S}$ simply by the fact that in (i), one shrinks the whole graph S into a single vertex in G, even if S is disconnected. The remainder of the definition remains the same.

We now extend to g-graphs the notion of essential [9].

Definition A5. Let $\bar{S} = \{S, \chi_S\}$ be a g-subgraph of $\bar{G} = \{G, \chi_G\}$. $\bar{S}$ will be called a g-essential of $\bar{G}$ if

— $\bar{S} \subset \bar{G}$,

— S is an essential subgraph of G; i.e., the reduced graph $[G/S]$ has zero external momenta at all its vertices.

We now precise the operators to which are related the g-graphs defined above in the case of a scalar theory.

Definition A.6. Let $\chi = \{\lambda_i; i = 1, L\}$ some family of derivatives. We denote by $\mathcal{O}_\chi$ the nonlocal operator associated to χ in (A.3):

$$\mathcal{O}_\chi = \prod_{j=1}^{L} \frac{1}{|\lambda_j|!} \int d^d y_j (iy_j)^{\lambda_j} (-\Delta_{y_j} + m^2)\, \phi(y_j). \tag{A.7}$$

From (A.3), $\bar{G} = \{G, \chi\}$ is a graph of the observable

$$\prod_{j=1}^{N} M_i(x_i) \cdot \{\mathcal{O}_\chi\}_{\text{conn}}. \tag{A.8}$$

DEFINITION A.7. We call "dual operator" of $\mathcal{O}_\chi$ and denote $\hat{\mathcal{O}}_\chi$ the composite operator

$$\hat{\mathcal{O}}_\chi(x_0) = \left\{ \prod_{j=1}^{L} \left(i \frac{\partial}{\partial x_j} \right)^{\lambda_j} \phi(x_j) \right\}_{x_j = x_0}. \tag{A.9}$$

Let $\bar{S} = \{S, \chi_S\}$ be a g-essential of $\bar{G} = \{G, \chi_G\}$. The reduced graph $[\bar{G}/\bar{S}]$ defined by (A.6) is a graph of the observable

$$\hat{\mathcal{O}}_{\chi_S}(x_0)\{\mathcal{O}_{\chi_G}\}_{\text{conn}}. \tag{A.10}$$

Finally, let us extend the notion of superficial degree of divergence to g-graphs:

DEFINITION A.8. The superficial degree of divergence of a g-graph $\bar{G} = \{G, \chi\}$ is equal to

$$\omega(\bar{G}) = \omega(G) - \tfrac{1}{2} \sum_{j=1}^{L} |\lambda_j|, \tag{A.11}$$

where the superficial degree of divergence $\omega(G)$ of G is given by

$$\omega(G) = \frac{d}{2} L(G) - l(G) + \frac{\delta(G)}{2}, \tag{A.12}$$

where $L(G)$, $l(G)$ and $\delta(G)$ are, respectively, the number of independent loops, of lines and of derivative couplings of G.

APPENDIX B

In this appendix we compute the quantity

$$[\tau_S^{-2l(S)} - \tau_S^{-2l(S)-1}]\{e^{-\Sigma_G a} B_{\bar{G}}(p, z, d, \alpha)\} \tag{B.1}$$

for S essential in the g-graph $\bar{G}$.

The basic property which has to be used is the following result proved in Ref. [9]: Given a scalar Feynman graph G, its integrand in the α-representation is

$$Y_G(p, d, \alpha) = e^{-p d_G(\alpha) p} P_G(\alpha)^{-d/2}. \tag{B.2}$$

If we dilate the α's relative to a subgraph S into $\rho^2\alpha$, we may write

$$Y_G(p, d, \alpha, \rho^2\alpha) = e^{-(p+k)\rho^2 d_S(\alpha)(p+k)} \cdot \rho^{-L_S d} \cdot P_S^{-d/2}(\alpha) \cdot Y_{[G/S]}(p, d, \alpha), \tag{B.3}$$

where d_S is the quadratic form in the external momenta of S, which may be external momenta of G and S (denoted p) or external momenta of S internal to the reduced graph $[G/S]$ (denoted k): in (B.3), those internal momenta k have to be understood as operators acting on $Y_{[G/S]}$ to add derivative couplings at the reduced vertex in $[G/S]$. The expansion in k of (B.3) gives

$$\sum_{n=0}^{\infty} \rho^{2n-L(S)d} Y_{[G/S]}(p,d,\alpha) \frac{(-k\,d_S k - 2k\,d_S\,p)^n}{n!} e^{-\rho^2 \cdot p d_S p} P_S(\alpha)^{-d/2}. \tag{B.4}$$

We may rewrite (B.4) as a sum over the family χ_S of derivatives with respect to external lines of S internal to G (as defined in Appendix A),

$$\sum_{\chi_S}^{\infty} \rho^{n(\chi_S)-L(S)d} Y_{[G/S]_{\chi_S}}(p,d,\alpha)\, Y_S^{\chi_S}(\rho p, d, \alpha), \tag{B.5}$$

where $Y_S^{\chi_S}$ is obtained by differentiating $Y_S(p,k)$ with respect to k's according to χ_S at $k=0$. That is to say

$$Y_S^{\chi_S}(p,d,\alpha) = D_{\chi_S}(p,k,d,\alpha)\,|_{k=0}, \tag{B.6}$$

where D_{χ_S} is the differential operator associated to χ_S present in (A.2) which we write in a straightforward way (without precising our notations)

$$D_{\chi_S} = \frac{1}{\chi_S!} \prod_{\chi_S} \left(\frac{\partial}{\partial k}\right). \tag{B.7}$$

According to the definitions of Appendix A, $Y_S^{\chi_S}(p,d,\alpha)$ is simply the integrand $Y_{\bar{S}}(p,d,\alpha)$ of the g-graph $\bar{S} = \{S, \chi_S\}$.

We now generalize (B.5) to the case of a g-graph $\bar{G} = \{G, \chi_G\}$. According to (A.2), its α-integrand $Y_{\bar{G}}$ is given by

$$Y_{\bar{G}}(p,d,\alpha) = D_{\chi_G}(p,k,d,\alpha)\,|_{k=0}. \tag{B.8}$$

When dilating the α's relative to S as above we get from (B.5)

$$Y_G(p,k,d,\alpha,\rho^2\alpha) = \sum_{\chi'_S} \rho^{n(\chi'_S)-L(S)d} Y_{G/S]_{\chi'_S}}(p,k,d,\alpha)\, Y_S^{\chi'_S}(\rho p, \rho k, d, \alpha). \tag{B.9}$$

Now we split the family χ_G into the two families $\chi_{G/S}$ and χ''_S relative to external momenta of G/S and S, respectively. Applying D_{χ_G} on $(B.9)$ we get

$$D_{\chi_G} Y_G(p,k,d,\alpha,\rho^2\alpha)\,|_{k=0} = \sum_{\chi'_S} \rho^{n(\chi'_S)+n(\chi''_S)-L(S)d} Y^{\chi_{G/S}}_{[G/S]_{\chi'_S}}(p,d,\alpha) \cdot Y_S^{\chi'_S \cup \chi''_S}(\rho p, d, \alpha). \tag{B.10}$$

The sum over χ'_S may be rewritten as a sum over families $\chi_S = \chi'_S \cup \chi''_S$.

Noting that only the χ_S which are compatible with χ_G may contribute in sum (B.10) (see Appendix A) and that from this notion of compatibility

$$Y^{\chi_{G/S}}_{[G/S]_{\chi'_S}} = Y^{\chi_G}_{[G/S]_{\chi_S}},$$

we get

$$Y_{\bar{G}}(p, d, \alpha, \rho^2\alpha) = \sum_{\chi_S} \rho^{n(\chi_S) - L(S)d} Y_{[\bar{G}/\bar{S}]}(p, d, \alpha) \cdot Y_{\bar{S}}(\rho p, d, \alpha) \tag{B.11}$$

which is a simple generalization of (B.5).

When applying the operator D_χ given by (B.7) onto integrand (B.2) it is easy to see that the integrand $Y_{\bar{G}}$ is of the form

$$Y_{\bar{G}}(p, d, \alpha) = \left[\sum_{\substack{q=0 \\ q+n(\chi)\text{even}}}^{n(\chi)} S_{\bar{G},q}(p, \alpha, d) \right] e^{-pd_G p} P_G(\alpha)^{-d/2}, \tag{B.12}$$

where each $S_{\bar{G},q}$ is a rational function of the α's, homogeneous in the α's of degree $(q + n(\chi))/2$, and is a polynomial in the p's, homogeneous of degree q.

We now go to the corresponding expansion for the Mellin integrand $B_{\bar{G}}(p, z, d, \alpha)$ when dilating the α's relative to an *essential subgraph* S in G, $(\alpha \to \rho^2\alpha)$. From (II.6), $B_{\bar{G}}$ is the Mellin transform of $Y_{\bar{G}}$,

$$B_{\bar{G}}(p, z, d, \alpha) = \int_0^\infty d\lambda (1 - \tau_\lambda^{-1})\, \lambda^{-z-\omega(\bar{G})-1} Y_{\bar{G}}(p/\sqrt{\lambda}, d, \alpha), \tag{B.13}$$

so that using (B.12) we get representation (II.8). Now we have simply to put expansion (B.11) into (B.13) and to notice that since S is essential, $Y_{G/S}(d, \alpha)$ does not depend on p, to get

$$B_{\bar{G}}(p, z, d, \alpha, \rho^2\alpha) = \sum_{\chi_S} \rho^{n(\chi_S) - L(S)d} Y_{[\bar{G}/\bar{S}]}(d, \alpha) \times \int_0^\infty d\lambda (1 - \tau_\lambda^{-1})\, \lambda^{-z-\omega(\bar{G})-1} Y_{\bar{G}}(\rho p/\sqrt{\lambda}, d, \alpha). \tag{B.14}$$

The last integral in λ is simply (using (B.13))

$$\rho^{2z + 2\omega(\bar{G})} B_{\bar{S}}(p, z - \omega(\bar{G}/\bar{S}), d, \alpha). \tag{B.15}$$

More generally, expanding now $e^{-\Sigma_G \alpha} B_{\bar{G}}$ we get

$$e^{-\Sigma_G \alpha} B_{\bar{G}} = \sum_{n=0}^{\infty} \sum_{\chi_S} \rho^{2z + 2\omega(\bar{G}) + n(\chi_S) - L(S)d + 2n} e^{-\Sigma_{G/S} \alpha} \times Y_{[\bar{G}/\bar{S}]}(d, \alpha) \frac{(-\sum \alpha)^n}{n!} B_{\bar{S}}(p, z - \omega(\bar{G}/\bar{S}), d, \alpha). \tag{B.16}$$

If we apply the Taylor operator $[\tau_S^{-2l(S)} - \tau_S^{-2l(S)-1}]$ onto $e^{-\Sigma_G \alpha} B_{\bar{G}}$, we simply pick up in the above expansion the power $\rho^{-2l(S)}$, i.e., the terms such that

$-2l(S) = 2z + 2\omega(\bar{G}) + n(\chi_S) - L(S)\,d + 2n$. From the definition of $\omega(\bar{S}) = dL(S)/2 - l(S) + n(\chi_S)/2$, we get the condition

$$n = -z - \omega(\bar{G}) + \omega(\bar{S}). \tag{B.17}$$

This proves Eq. (II.17) with condition (II.18).

Appendix C

In this appendix we consider the renormalization (at some fixed rational dimension d^*) of the composite operators $\hat{\mathcal{O}}$ defined by (A.9). Such operators are present in the reduced graphs $[G/E]$ which are shown in Appendix A to correspond to the observable $\hat{\mathcal{O}}_{\chi_E}\{\mathcal{O}_{\chi_G}\}_{\text{conn}}$. In this appendix we treat only the case of such a reduced graph S (in the framework of a salar field theory). The massive amplitude $I_{\bar{S}}[m, d]$ of $\bar{S}$ may be U.V. divergent at d^* if the composite operator $\hat{\mathcal{O}}$ has a positive or null canonical dimension at d^*, which is always the case in the super-renormalizable domain. We assume, as in Section III, that the only U.V. divergences come from this composite operator $\hat{\mathcal{O}}$ (i.e., there are no subgraphs S' in S which do not contain the vertex v corresponding to $\hat{\mathcal{O}}$ and such that $\omega^*(S') \in \mathbb{N}$).

Those U.V. divergences may be subtracted by the standard Normal Product Algorithm [17], but such a procedure is not well suited to our case; in particular we deal only with the poles at d^*, and this procedure does not emerge from the results of Section III. Here we define another subtraction prescription, which is given directly in term of Feynman amplitudes. Let us first note that since the only divergent subgraphs of S contain the vertex v, any forest of such graphs is necessarily a nest:

Definition C.1. We define the Rn renormalized amplitude $I_S^{Rn}[m, d]$ of S at d^* as

$$I_S^{Rn}[m, d] = \sum_{\substack{S_0 \ni v \\ n + \omega^*(S_0) = n_0 \in \mathbb{N}}} I_{[S/S_0]}[m, d] \cdot m^{2\omega^*(S_0)} K_{S_0}^n[d], \tag{C.1}$$

where the sum runs over every connected one particle irreducible g-subgraphs S_0 in S such that $\omega^*(S_0)$ is an integer $\geqslant -n$. The counterterm $K_{S_0}^n[d]$ is defined by a sum over every nest $\mathcal{N} = \{S_1 \subset \cdots \subset S_N = S_0\}$ of C.1 P.I. g-subgraphs S_i containing v and such that $n + \omega^*(S_i)$ is an integer $\geqslant 0$, and containing S_0 as greatest element, namely,

$$K_{S_0}^n[d] = \sum_{\substack{\mathcal{N} = \{S_1 \subset \cdots \subset S_N = S_0\} \\ n + \omega^*(S_i) \in \mathbb{N}}} \prod_{i=1}^{N} [-I_{[S_i/S_{i-1}]}(m = 1, d)]. \tag{C.2}$$

The integer n is the number of oversubtraction attached at the vertex v. In sum (C.1) the case $S_0 = \{v\}$ has to be included, and by definition $K_{\{v\}}^n = 1\ \forall n \in \mathbb{N}$.

To prove the finiteness of $I_S^{R,n}$, the simplest way is to show that it differs from another standard subtraction procedure by finite counterterms. The closest one is the "minimal subtraction at d^*" (dimensional renormalization) [21, 22] which subtracts only poles at d^* and so only C.1 P.I. S_0 such that $\omega^*(S_0) \in \mathbb{N}$. We give simply the final result.

LEMMA C1. *For any $n \geqslant 0$ (and any $m > 0$), $I_S^{R,n}[m, d]$ is finite at d^*.*

This renormalization prescription for $\hat{\mathcal{O}}(x_0)$ corresponds to some normal product algorithm differing from that of [17] by finite counterterms. We denote the corresponding renormalized composite operator

$$\mathcal{N}'_n[\hat{\mathcal{O}}(x_0)]. \tag{C.3}$$

Let us note that in the case $n=0$, if the bare amplitudes in (C.1) $(I_{[S/S_0]}(m, d)\, m^{2\omega^*(S_0)})$ have a massless limit as $d \simeq d^*$, since the counterterms K_S^n are independent of the mass, then the renormalized amplitude $I_S^{R,n}$ has also a massless limit at d^*. This is the case for the ϕ^4 theory at $d^* = 4$.

This renormalization prescription may be extended without difficulty to products of composite operators, but this is not needed here.

Let us finally mention that Definition C.1 runs also in the case $n < 0$. In this case of ultraviolet "undersubtraction" the amplitudes $I_S^{R,n}$ remain singular at d^*. In particular, we use in Section III the case $n = -\omega^*(S) - 1$ (the subgraphs more divergent that S itself and only these subgraphs are subtracted).

DEFINITION C2. Let S be a divergent subgraph at d^* (such that $\omega^*(S) \in \mathbb{N}$); we define the undersubtracted amplitude $I_S^{(-)}(m, d)$ by

$$I_S^{(-)}[m, d] = I_S^{R,n}[m, d]$$

with

$$n = -1 - \omega^*(S). \tag{C.4}$$

APPENDIX D

This appendix is devoted to the proof of Theorem 3. We want to prove that the rhs of (III.11) is equal to $F_G(x_i, m, d)$. Using (III.7), which defines S_E, we get for the rhs of (III.11)

$$\sum_{\substack{E_2 \subset G \\ \text{essential} \\ \omega^*(\widetilde{G/E_2}) \in \mathbb{N}}} \; \sum_{\substack{E_1 \subset E_2 \\ \text{essential} \\ \omega^*(\widetilde{E_2/E_1}) \in \mathbb{N}}} F_{E_1}(x_i, m, d)\, I^{(-)}_{[\widetilde{E_2/E_1}]}(1, d)\, K^0_{[\widetilde{G/E_2}]}(d) \times \mu^{(d-d^*)L(\widetilde{G/E_1})} m^{2\omega^*(\widetilde{G/E_1})}. \tag{D.1}$$

If for any essential $E_1 \subset G$ such that $E_1 \neq G$ and $\omega^*(\widetilde{G/E_1}) \in \mathbb{N}$, we can show that

$$\sum_{\substack{E_2 \text{ essential} \\ E_1 \subset E_2 \subset G \\ \omega^*(\widetilde{E_2/E_1}) \in \mathbb{N} \\ \omega^*(\widetilde{E_2/E_1}) \leqslant \omega^*(\widetilde{G/E_1})}} I^{(-)}_{[\widetilde{E_2/E_1}]}(1, d)\, K^0_{[\widetilde{G/E_1}]}(d) = 0 \tag{D.2}$$

then Theorem 3 will be proved, since the only term in (D.1) which is nonzero corresponds to $E_1 = E_2 = G$, and is $F_G(x_i, m, d)$. From the definitions of $I^{(-)}$ and K^0 given in Appendix C, it is not very difficult to show that (D.2) is nothing else that the renormalized amplitude of the graph $[\widetilde{G/E_1}]$, $I^{R,0}_{[\widetilde{G/E_1}]}(1, d)$, defined by (C.1). Since the subtraction prescription $R, 0$ corresponds to subtract at zero momenta, mass unity, every subgraphs S such that $\omega^*(S) \in \mathbb{N}$, and since the graph $[\widetilde{G/E_1}]$ is itself divergent at d^* $(\omega^*(\widetilde{G/E_1}) \in \mathbb{N})$ and is taken at $m = 1$, we have

$$I^{R,0}_{[\widetilde{G/E_1}]}(1, d) = 0. \tag{D.3}$$

This ends the proof.

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CINQUIÈME PARTIE

EFFETS NON PERTURBATIFS ET RENORMALONS INFRAROUGES

V.1 - INTRODUCTION

Il est maintenant clairement établi que les séries perturbatives définissant les fonctions de Green en théorie quantique des champs ont un rayon de convergence nul et ne peuvent être au mieux que des séries asymptotiques. Une telle idée date d'une remarque de E. Dyson [73] : en électrodynamique quantique (par exemple) le vide devient métastable si la constante de couplage g est négative. En conséquence, il apparaît une singularité essentielle en g = 0 et une coupure sur l'axe réel négatif.

La question se pose donc de savoir s'il existe néanmoins un moyen de reconstruire les fonctions de Green à partir de ces séries perturbatives divergentes. Mathématiquement, de tels procédés existent pourvu que les fonctions possèdent des propriétés d'analyticité et d'asymptoticité assez fortes au voisinage de g = 0. En particulier la sommation de Borel [74] joue un rôle important en physique mathématique et s'est avérée être extrêmement utile pour l'étude de ces problèmes en théorie quantique des champs.

Considérons une fonction G(g) dont le développement asymptotique en g = 0 est donné par une série entière formelle divergente :

$$G(g) = \sum_{n=0}^{\infty} a_n g^n \qquad (V.1)$$

Sa transformée de Borel est définie par la série

$$B(b) = \sum_{n=0}^{\infty} \frac{a_n}{n!} b^n \qquad (V.2)$$

Si les termes de la série (V.1) ne croîssent pas plus vite que n ! la fonction B(b) est analytique dans un voisinage de l'origine. Supposons que B(b) s'avère en fait analytique le long de l'axe réel positif et qu'elle ne croîsse pas plus vite qu'une loi de puissance à l'infini. Alors la transformée de Borel inverse

$$G'(g) = \frac{1}{g}\int_0^\infty db\, e^{-b/g} B(b) \qquad (V.3)$$

définit pour tout g positif assez petit une fonction dont le développement asymptotique en g = 0 coïncide avec celui de G.

Réciproquement, si la fonction G satisfait des critères d'asymptoticité à son développement en série en g = 0, qui sont donnés par le théorème de Nevanlinna - Sokal [75] (une version plus forte du célèbre théorème de Watson), alors la somme de Borel G'(g) est bien définie et coïncide avec la fonction G(g).

Dans un certain nombre de cas, qui correspondent en fait à des théories des champs à une dimension (mécanique quantique) [76] ou à 2 ou 3 dimensions (théories superrenormalisables) [77, 78] il s'avère que les fonctions de Green sont bien Borel-sommables. Dans ces cas la discontinuité sur l'axe réel négatif prédite par E. Dyson peut être estimée à partir de méthodes semi-classiques [79]. Les solutions classiques non triviales des équations du mouvement (Instantons) dominent l'intégrale fonctionnelle et permettent de déterminer le comportement aux grands ordres des séries perturbatives [80 - 84]. Ces résultats ont été utilisés avec beaucoup de succès pour calculer par exemple les indices critiques à trois dimensions à partir des calculs perturbatifs [85].

Malheureusement, la situation apparaît être beaucoup moins claire dans le cas des théories des champs à quatre dimensions. Bien qu'il n'existe pratiquement pas de résultats rigoureux dans ce cas [86], un certain nombre de problèmes ont été mis en évidence.

Les théories de Jauge à quatre dimensions (et les modèles Sigma non linéaires à deux dimensions) possèdent des solutions classiques des équations du mouvement non triviales d'action finie réelles (et complexes) pour toute valeur positive de la constante de couplage [37 - 87]. Ces Instantons sont des points cols de l'intégrale fonctionnelle et vont en principe donner des contributions non perturbatives d'ordre exp(- A/g) (où A est l'action de l'instanton) aux fonctions de Green. Simultanément il va leur correspondre des coupures sur l'axe réel positif de la transformée de Borel de la série perturbative "classique". Un effort considérable à été fait pour classifier ces solutions et estimer leurs contributions. Un certain nombre de problèmes qui leur sont liés apparaissent être du ressort d'une "méthode du col en dimension infinie" et sont déjà présents dans les problèmes de mécanique quantique [88].

De plus, on a affaire à des théories invariantes conformes et on rencontre dans les calculs d'instantons des divergences d'origine ultraviolette et infrarouge qui valident l'approximation dite du "gaz dilué d'instantons" et qui mettent en cause les approximations semi-classiques elles mêmes. Ces divergences sont en fait liées aux problèmes des "Renormalons" dont nous allons maintenant parler.

Un deuxième problème est le suivant : dans le cas des théories asymptotiquement libres à grande distance, telles l'électrodynamique quantique ou ϕ_4^4, il apparaît une "singularité de Landau" [89] dans les fonctions de Green à grandes impulsions, typiquement à

$$\mathrm{Ln}(p/\mu) = 1/g\beta_2 \qquad \text{(V.4)}$$

($\beta_2 > 0$ est le premier coefficient de la fonction β de Callan-Symanzik)

$$\beta(g) = \beta_2 g^2 + O(g^3) \qquad \text{(V.5)}$$

En repassant dans l'espace des positions cette singularité de Landau se transforme en une discontinuité dans la constante de couplage sur l'axe réel positif. Dès 1974 D. Gross et A. Neveu [90] notaient, qu'à cette discontinuité correspondait une singularité de la transformée de Borel à

$$b = {}^{2}/_{\beta_2} \qquad (V.6)$$

Ce type de singularité fut redécouvert par plusieurs auteurs [15, 91, 92] et baptisé "Renormalon ultraviolet". Bien qu'il existe une conjecture très précise sur la nature de ces singularités, due à G. Parisi [92], leur signification physique et en particulier leur lien avec la "trivialité" de ces théories n'est, à notre avis, pas encore éclairci.

Dans le cas des théories asymptotiquement libres à courte distance, telles les théories de Jauge non abéliennes, β_2 est négatif. Les renormalons UV sont situés à des valeurs de b négatives et ne nuisent pas à l'éventuelle sommabilité de Borel de la théorie (par contre ils contribuent à la discontinuité pour les g négatifs). Hélas, si la théorie est de masse nulle, il apparaît une singularité de Landau "infrarouge" et des "renormalons infrarouges" correspondant pour des $b > 0$ (voir figure 8).

En effet, considérons la constante de couplage "effective" $\bar{g}\,(p,g)$ telle qu'elle est définie à l'ordre d'une boucle en intégrant l'équation du groupe de renormalisation

$$\left[p\frac{\partial}{\partial p} + \beta(g)\frac{\partial}{\partial g}\right]\bar{g}(p,g) = 0 \qquad (V.7)$$

En ne gardant que le premier terme de la fonction β (V.5) on trouve immédiatement

$$\bar{g}(p,g) = \frac{g}{1 - g\beta_2 \operatorname{Ln}(p/\mu)} \qquad (V.8)$$

La constante de couplage effective est définie pour $p > p^* = \mu \exp(\frac{1}{g\beta_2})$ et diverge pour $p = p^*$.

La présence d'une telle "singularité de Landau infrarouge" (pour employer l'expression de D. Gross et A. Neveu [90]) ne correspond pas à une singularité "physique". Elle indique simplement qu'une description "perturbative", même améliorée à l'aide du groupe de renormalisation, n'est valide qu'à des distances plus petites que $L^* = \frac{2\pi}{p^*}$. Au delà on s'attend à ce que des effets "non perturbatifs" importants (comme le confinement, la brisure de la symétrie chirale) doivent être pris en compte. Par contre, il va apparaître une singularité infrarouge dans la transformée de Borel [15, 93]. Un argument simple a été donné par G. Parisi [13] pour prédire la position de ces singularités. Considérons la transformée de Borel (V.2) $\overline{B}$ (p,b) de la constante de couplage effective (V.8). Elle se comporte à petites impulsions comme

$$\overline{B}(p,b) \sim \left(\frac{p}{\mu}\right)^{b\beta_2} , \quad p \to 0 \qquad \text{(V.9)}$$

et donc diverge de plus en plus lorsque b augmente (β_2 est négatif). Cette quantité doit jouer un rôle important dans la transformée de Borel des équations de Dyson-Schwinger définissant la théorie ($\bar{g}$ est un vertex effectif resommé) où l'on trouvera des intégrales de la forme

$$\int d^4p \; \overline{B}(p,b) \cdot (\text{Fonction de } p) \qquad \text{(V.10)}$$

Ainsi, on s'attend à des divergences infrarouges (en p = 0) de la transformée de Borel pour

$$b = -\frac{4}{\beta_2} , -\frac{6}{\beta_2} , \ldots \qquad \text{(V.11)}$$

De plus, les caractéristiques de ces singularités (baptisées "Renormalons infrarouges" par analogie avec les "renormalons U. V.") ne semblent pas modifiées si on prend en compte les corrections dues aux termes suivants de la fonction $\beta(g)$.

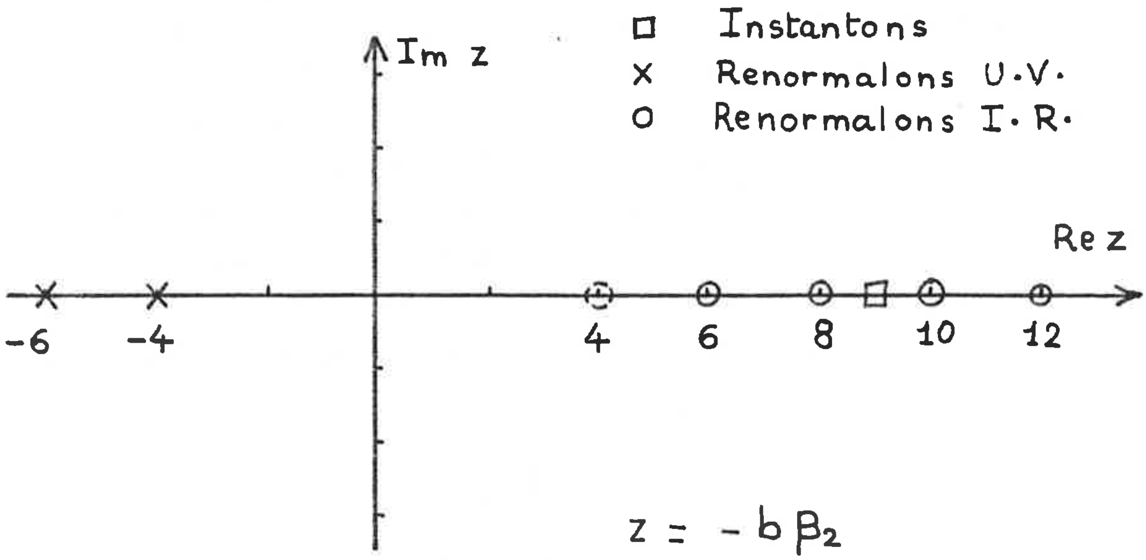

FIGURE 8

Singularités présumées de la transformée de Borel dans le cas d'une théorie de Jauge pure SU(3). Chaque singularité est en fait un point de branchement (d'après G.'t. Hooft).

Avant de revenir sur ces singularités I.R., mentionnons un dernier problème, soulevé par G. 't. Hooft [90] et N. Khuri [94]. Pour qu'une fonction soit sommable de Borel, il faut qu'elle soit analytique dans un disque tangent à l'origine, typiquement (figure 9)

$$\mathrm{Re}(1/g) > 1/R \tag{V.12}$$

Pour une théorie asymptotiquement libre, on s'attend à n'avoir analyticité que dans un "cornet"(figure 10)

$$|\mathrm{Im}(1/g)| < \pi |\beta_2| \tag{V.13}$$

La raison en est la présence de singularités "images" par le groupe de renormalisation des coupures de Landau dues à l'existence de résonances pour des $p^2 << 0$ minkowskiens, pour $\mathrm{Im}\ 1/g = \pi\ \beta_2(2n + 1)$. En conséquence, on s'attend à ce que la transformée de Borel B(b) (si elle existe) croisse comme $\Gamma(-b\beta_2)$ et que la somme de Borel (V.3) diverge à l'infini. G.'t. Hooft a proposé de resommer de Borel la transformée de Borel inverse elle-même [13]. A notre connaissance aucune suggestion ou étude plus précise n'a été faite sur ce problème.

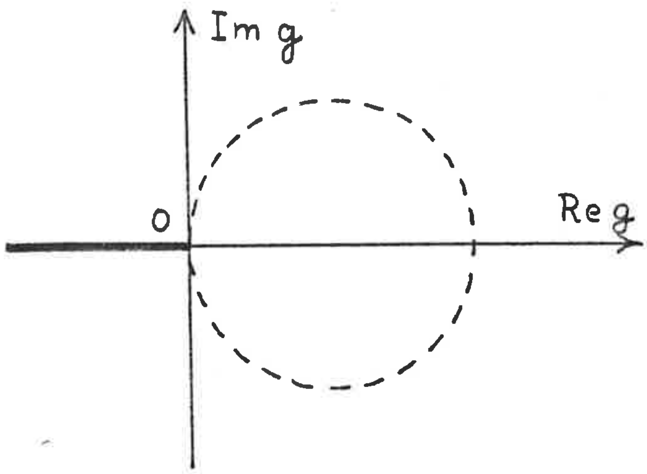

FIGURE 9

Domaine d'analyticité "minimal" d'une fonction G(g) sommable de Borel.

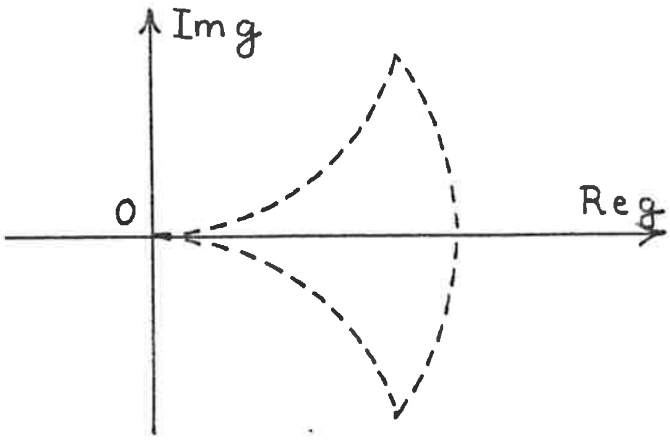

FIGURE 10

Domaine d'analyticité attendu pour une théorie asymptotiquement libre.

Dans cette partie nous nous intéressons donc au problème des "Renormalons Infrarouges", et en particulier à une conjecture de G. Parisi sur la forme de ces singularités [13] et leur relation avec ce que l'on appelle désormais "le formalisme des règles de somme de Q.C.D.". Cette conjecture était la suivante : G. Parisi remarquait que le comportement de $\bar{B}(p)$ (donné par V.9) était très analogue à celui de la "constante de couplage effective" d'une théorie de masse nulle super-renormalisable, qui, par simple analyse dimensionnelle, se comporte comme

$$\bar{g}(p) \sim \left(\frac{p}{\mu}\right)^{-\epsilon} , \quad p \to 0 \quad , \text{ avec } \epsilon = 4 - d > 0 \qquad \text{(V.14)}$$

C'est alors qu'il faisait la conjecture dont nous avons parlé (et que nous avons démontrée) dans la partie IV : les pôles infrarouges en ϵ de ces théories sont proportionnels à des opérateurs multi-locaux, "duaux" des opérateurs locaux du développement en produit d'opérateur de Wilson. Par analogie les singularités infrarouges (en b) de la transformée de Borel des théories de Jauge seront proportionnelles à ces mêmes opérateurs multi-locaux.

Un tel argument peut paraître extrêmement indirect et on trouve dans ce même article (très exactement dans la dernière phrase de l'appendice B !) un second argument plus direct.

M. Shifman, A. Vainshtein, M. Voloshin et V. Zakharof avaient suggéré peu de temps auparavant de tenir compte des effets non perturbatifs dans les calculs de chromodynamique quantique (Q.C.D.) à l'aide du développement en produit d'opérateurs [16, 95] de la manière suivante :

Le produit chronologique de deux courants de quarks lourds (par exemple) se développe à courte distance (x → 0) en

$$T J^A(x) J^B(0) = \sum_n C_n^{AB}(x) \cdot O_n \qquad \text{(V.15)}$$

La somme porte sur tous les opérateurs locaux O_n et les coefficients $C_n^{AB}(x)$, qui sont définis comme des séries perturbatives en g^2, sont de moins en moins singuliers quand x tend vers zéro lorsque la dimension des opérateurs O_n augmente. Ces auteurs proposaient d'admettre que la formule (V.15) restait valable lorsque l'on calculait la valeur moyenne de cette observable dans le vide

$$\langle 0 | T J^A(x) J^B(0) | 0 \rangle = \sum_n C_n^{AB}(x) \langle 0 | O_n | 0 \rangle \qquad \text{(V.16)}$$

les quantités $\langle 0|O_n|0\rangle$ étant supposées non nulles et paramétrisant les effets non perturbatifs de Q.C.D. (En théorie des perturbations, ces quantités sont nulles à tous les ordres, sauf bien sûr l'opérateur $O_o = \mathbb{1}$). Le terme $C_o^{AB}(x)$ (n = 0) est simplement prédit par la théorie perturbative habituelle. Le premier terme non perturbatif (n = 1) est proportionnel au "condensat de Gluons"

$$\langle 0 | Tr(G^{\mu\nu} G_{\mu\nu}) | 0 \rangle \neq 0 \qquad \text{(V.17)}$$

S.V.Z. et de nombreux auteurs ont utilisé cette théorie des perturbations modifiée en la comparant avec les résultats prédits à l'aide des relations de dispersion dans l'étude des spectres de résonance de Q.C. D. (Charmonium). En fait on ne retient que les premiers termes non perturbatifs (associés aux opérateurs de dimensions 4 ou 6) et les premiers termes (au plus une boucle) des séries perturbatives C_n^{AB}. On doit "sacrifier" une partie des données expérimentales pour ajuster la valeur des condensats $\langle 0|O_n|0\rangle$. De plus il apparaît que parmi les procédures possibles de sommation de la série (V.16), celle qui semble donner les meilleurs résultats est basée sur l'utilisation de la transformée de Borel.

Ce formalisme, dit des "règles de somme de Q.C.D.", est maintenant largement utilisé et semble donner de bons résultats. Malgré cela, il repose sur une utilisation non perturbative d'un résultat perturbatif (le développement en produit d'opérateurs) et n'est pas en fait mathématiquement cohérent. Ceci nous ramène aux Renormalons Infrarouges et à la remarque de G. Parisi.

En effet, s'il semble plausible que des opérateurs locaux O_n acquièrent une valeur moyenne dans le vide non nulle, les équations du groupe de renormalisation prédisent un comportement en

$$\langle 0|O_n|0\rangle = C_n\, g^{\delta_n} \exp\left(\frac{d_n}{\beta_2 g}\right)\left(1+o(g)\right) \qquad (V.18)$$

où d_n est la dimension canonique de O_n ($d_n = 4, 6, ..$) et δ_n une puissance fixée (dépendant de la dimension anormale de O_n et de second terme de $\beta(g)$. Seul C_n, constante d'intégration de ces équations, est a priori arbitraire. G. Parisi fit remarquer que l'apparition de ces termes non perturbatifs exponentiellement petits devait se traduire par des ambiguïtés de la série des perturbations habituelle à ces ordres $\exp\,(d_n/\beta_2 g)$; $d_n = 4, 6, \ldots$, et que les renormalons infrarouges étaient simplement les reflets de ces ambiguïtés.

Cette interprétation semble a priori séduisante, et permet de comprendre pourquoi les renormalons seraient proportionnels aux coefficients C_n de (V.16) (les C_n s'interprètent comme insertions des opérateurs multi-locaux duaux des O_n). Cependant un problème se pose alors : s'il existe un renormalon en $b = -4/\beta_2$, le terme perturbatif C_o dans (V.16) est ambigu (avec une partie imaginaire d'ordre $\exp\,(4/\beta_2 g)$) et il n'est pas cohérent d'y ajouter la première contribution non perturbative (qui elle n'est pas ambigüe). Autrement dit, le formalisme de S.V.Z. prédit l'existence des renormalons infrarouges mais ces renormalons semblent alors rendre ce formalisme incohérent !

Le travail qui suit a pour but de répondre à ces questions :

- Peut-on aller au delà des arguments qualitatifs de G. Parisi et étudier de manière précise la structure de ces renormalons infrarouges ?
- Le formalisme des règles de somme de QCD est-il plus qu'un ansatz phénomènologique permettant de tenir compte de manière approximative des "effets non perturbatifs" des théories de Jauge ?
- Quelle relation exacte existe-t-il entre ces deux problèmes ?

Pour cela, nous étudions le modèle Sigma non linéaire O(N) à deux dimensions. Nous avons vu dans II que ces types de modèles étaient à la fois plus simples et assez similaires aux théories de Jauge. En particulier, on s'attend à ce qu'ils possèdent également des renormalons infrarouges à

$$b = -\frac{2}{\beta_2} \,,\ -\frac{4}{\beta_2} \,,\ \dots \tag{V.19}$$

le premier opérateur local invariant étant de dimension 2.

Bien qu'apparemment ceci n'ait pas été envisagé dans la littérature, on doit s'attendre à l'existence de "condensats" pour ces modèles. Le modèle O(N) présente un avantage supplémentaire ; il possède un développement en 1/N qui permet de resommer partiellement les séries perturbatives et avec lequel les phénomènes non perturbatifs tels l'apparition d'une masse, peuvent être facilement étudiés.

De plus, dans la limite $N \to \infty$, gN fixé, les renormalons restent à une distance finie dans le plan de Borel tandis que les singularités éventuelles dues aux instantons "partent" à l'infini. Autrement dit, le développement en 1/N "découple" complètement les problèmes des instantons de ceux qui nous intéressent.

Le principe de l'étude est donc simple. Toute observable <u>invariante</u> G(x) se développe en

$$G(x) = \sum_{k=0}^{\infty} \frac{1}{N^k} \, G_{(k)}(x, g) \tag{V.20}$$

chaque terme $G_{(k)}$ apparaît comme une somme d'amplitudes renormalisées associées à de nouveaux graphes de Feynman caractéristiques du développement en 1/N et est une fonction non triviale de la constante de couplage g. Nous analysons la structure de la transformée de Borel $\hat{G}_{(k)}(x, b)$ de chaque $G_{(k)}$.

Il apparaît que cette transformée de Borel par rapport à la constante de couplage g n'est rien d'autre qu'une transformée de Mellin des amplitudes du développement en 1/N par rapport à la masse intervenant dans leurs propagateurs. Cette masse m est celle apparaissant par transmutation dimensionnelle à l'ordre $N = \infty$

$$m = \mu \exp\left(-\frac{2\pi}{g}\right) \tag{V.21}$$

Pour cette raison, on peut appliquer les techniques générales d'étude des transformées de Mellin développées dans la partie VI de cette thèse. En fait un certain nombre de modifications importantes doivent y être apportées et un autre outil, la représentation de "Mellin complet" des amplitudes de Feynman [96] joue également un grand rôle.

Pour terminer cette longue introduction, résumons les conclusions de cette étude, valables à tous les ordres du développement en 1/N de toute observable invariante du modèle O(N) :

(i) La transformée de Borel de la série perturbative possède bien des renormalons infrarouges qui sont bien proportionnels à certains coefficients du développement en produit d'opérateur.

(ii) Le premier renormalon I. R. n'est pas en $b = 4\pi$ (comme le prédit l'argument de G. Parisi) mais en $b = 8\pi$.

(iii) Il existe des termes non perturbatifs proportionnels à des valeurs moyennes dans le vide d'opérateurs invariants, qui s'organisent formellement dans le développement en produit d'opérateur de S.V. Z., chaque terme pris séparément possède également des renormalons I. R.

(iv) Il existe une procédure de sommation de Borel qui rend ce développement bien défini. Dans cette procédure, les renormalons I. R. disparaissent entre les différents termes du développement en produit d'opérateur.

Les conséquences de ce phénomène sont discutées à la fin de l'article qui suit. Nous conjecturons qu'un tel résultat est valable au-delà du développement en 1/N et fournit ainsi une solution générale au problème des renormalons infrarouges et à celui du status mathématique des "règles de somme de Q.C.D."

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NON PERTURBATIVE EFFECTS AND INFRARED RENORMALONS WITHIN THE 1/N EXPANSION OF THE O(N) NON LINEAR SIGMA MODEL

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ABSTRACT

We analyze the structure of the Borel Transform of the two dimensional O(N) non linear σ model within its 1/N expansion. We check the existence of I.R. singularities (I.R. renormalons) and the presence of non perturbative terms which organize themselves in an Operator Expansion à la Shifman-Vainshtein-Zakharov. We prove that renormalons cancel between the different terms of the Operator Expansion, so that there is a well defined resummation procedure of the perturbative series. We suggest that this mechanism provides a general solution of the I.R. renormalons problem for massless U.V. free field theories.

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1. Introduction

In a well known series of papers, Shifman, Vainshtein and Zakharov (S.V.Z.) [1] suggested that it was possible to take into account the large distance, non perturbative, effects of Q.C.D. with the help of the Operator Product Expansion. The basic idea was the following ; the product of two currents j^A, j^B (for instance) may be expanded into

$$T\,\{j^A(x),\, J^B(0)\} \underset{x \to 0}{=} \sum_n C_n^{A\,B}(x)\; \mathcal{O}_n \tag{1.1}$$

where the sum runs over local operators $\mathcal{O}_n$ (properly renormalized) and the $C_n^{A\,B}(x)$ depend on x [2] . In [1] , S.V.Z. assumed that the expansion (1.1) holds beyond perturbation theory, so that one may write

$$<0|T\{j^A(x), j^B(0)\}|0> \;=\; \sum_n C_n^{A\,B}(x)\;.\;<0|\mathcal{O}_n|0> \tag{1.2}$$

where the vacuum expectation values $<0|\mathcal{O}_n|0>$ are allowed to be non zero, and parametrize the non-perturbative effects of Q.C.D. (In perturbation theory, $<0|\mathcal{O}_n|0> = 0$ at all orders). The coefficients $C_n^{A\,B}(x)$ are given by perturbation theory.

Practically, S.V.Z. and coworkers used the operator expansion (O.E.) (1.2) by matching it with dispersion relations (and by retaining the first operators of dimension ≤ 6 and the first order (s) in α_s for the $C_n^{A\,B}$) to look for instance at the meson spectrum of Q.C.D. This "Q.C.D. sum rules formalism" has now a firm phenomenological status and the existence of "quark and gluon condensates" is recognized as an essential feature of Gauge Theories. Those condensates $<0|\mathcal{O}_n|0>$ have to be estimated from experimental data, Monte Carlo computations or semi-classical evaluations.

However, the validity of this expansion is not obvious, and was extensively discussed by S.V.Z. in [1] . Indeed, it relies finally on a (somewhat phenomenological) extension of perturbative arguments. A related and important problem concerns the mathematical consistence of the expansion (1.2). The non perturbative quantities $<0|\mathcal{O}_n|0>$ are exponentially small, typically

$$<0|\mathcal{O}_n|0> \;\sim\; \exp(d_n/\beta_2\alpha_s) \tag{1.3}$$

where β_2 is the first term of the β function and d_n the canonical dimension of O_n. It is not consistent to incorporate such terms as long as the perturbative series in α_s, $C_n^{AB}(x,\alpha_s)$,have not been properly summed. But those series are expected to be divergent so that even if there is a domain of physical parameters where the first perturbative corrections are smaller than the first non perturbative ones (as argued in [1]) this situation disappears at large orders in α_s. Thus a resummation procedure for the perturbative series has to be defined. However, according to our present knowledge, difficulties are expected in the usual program of Borel summation of the perturbative series. Indeed, even disregarding the problems of Instantons and of the behaviour of the Borel transform at infinity, one expects the presence of Infrared (I.R.) singularities on the positive real axis of the Borel Transform [3]. A simple argument to locate such singularities, usually called I.R. Renormalons, has been given by G. Parisi [4]. In a massless U.V. free theory such as QCD ($\beta_2 < 0$), the Borel transformed effective coupling constant $\tilde{g}(p,b)$ should behave at small momenta as

$$\tilde{g}(p,\alpha_s) \sim |p|^{b\beta_2} \tag{1.4}$$

(b is the Borel variable). Inserting (1.4) in the (Borel-transformed) Dyson Schwinger integral equations should give I.R. singularities at $b = -\frac{2n}{\beta_2}$, $n \in \mathbb{N}$.

Moreover, G. Parisi argued in [4] that (by analogy with I.R. divergences below 4 dimensions) these singularities could be classified in terms of the coefficients C_n of the O.E. (1.1), and so were related to the appearance of the non perturbative expectation values (1.3). This relationship between the existence of an "I.R. tachyonic Landau singularity" (which corresponds to (1.4)) and some "vacuum instability" was previously noted by D. Gross and A. Neveu [5] and by P. Olesen [6]. However, it seems to us that this relationship has not received a more quantitative formulation and that the problem of the summation of the pertubative series of Gauge Theories has still to be understood.

In this paper we shall look at these points within the 1/N expansion of the O(N) non linear Sigma Model at two dimensions. As Gauge Theories it is asymptotically free [9] and its perturbative expansion is made around a "wrong vacuum" since the classical theory describes N-1 interacting Goldstone Bosons but the Mermin-Wagner-Coleman theorem [7] ensures the dynamical restoration of the O(N) symmetry and the non perturbative generation of a mass gap for any positive coupling constant. In particular, the perturbative expansion has I.R. divergences which cancel only for the "physical" O(N) invariant observables [8]. The 1/N expansion takes into account these non perturbative effects and is a powerful tool to study the theory, since it allows partial infinite resummation of the usual perturbative series [9]. Moreover there are no instantons (N > 3), thus the structure of non perturbative effects is expected to be simpler.

Our purpose is to characterize the analytic structure of the Borel Tranform (with respect to the coupling constant) of any O(N) invariant observable at an arbitrary order of the 1/N expansion. The result is stated in Theorem A (sec. 3.C) and proves that, within the 1/N expansion :

- there are non-perturbative terms which organize themselves formally in an Operator Expansion but have I.R. renormalons ;
- nevertheless, there is a Borel summation prescription for those terms which makes the S.V.Z. Operator Expansion unambiguous and gives the right result ; with such a prescription, I.R. renormalons are cancelled between the different non perturbative terms.

Moreover, we shall argue that this mecanism goes beyond the 1/N expansion and may provide a general solution for the problem of I.R. renormalons.

This paper is organized as follows :

In section 2 we introduce the O(N) model and its 1/N expansion (§ A) and show that one recovers the Operator Expansion at leading order $N = \infty$ (§ B) and more generally at the order of tree diagrams (§ C), i.e. in cases where the perturbative series are convergent (no Borel transform is needed).

In section 3 we analyse the Borel Transform of any order of the 1/N expansion. We adapt desingularization technics of Bergère-Lam and the author [10-11]. Basic definitions are given in § A. For technical reasons one first have to look at the "Bare" theory below 2 dimensions (§ B) and then to take the limit $d \to 2$ (§ C), where the complete analytic structure of the Borel Tranform (in the first sheets) is obtained in theorem A. The result is discussed in § D.

In section 4 we discuss the possible validity of our result beyond the 1/N expansion and for other models. Various implications are examined.

Finally in Appendix A the technicalities of the Borel Transforms are recalled. In Appendix B we discuss the obtention of the coefficients of the Operator Expansion in perturbation theory for the O(N) model. Appendix C is devoted to another integral representation for the 1/N expansion needed in section 3 and used in Appendix D for explicit computations of I.R. renormalons at first 1/N order.

2. *The structure of the O(N) model for $N = \infty$.*

A. *The 1/N expansion of the O(N) sigma model* :

First we briefly recall how to obtain the 1/N expansion of the O(N) non linear σ model. The generating functional reads

$$Z[J] = \int \mathcal{D}[\vec{S}]\, \mathcal{D}[\alpha] \exp\left\{-\frac{N}{g_B}\int d^dx \left\{\frac{1}{2}(\partial_\mu\vec{S}.\partial_\mu\vec{S}) + \frac{1}{2}\alpha(x)[\vec{S}^2(x) - 1]\right\}\right.$$
$$\times \exp\left\{ d^dx\, \vec{J}(x)\, \vec{S}(x)\right\} \qquad (2.1)$$

where $\vec{S}(x)$ is a N-component real vector field defined in the d-dimensional Euclidian space ; the Lagrange multiplier $\alpha(x)$ fixes the constraint

$$\vec{S}^2(x) = 1 \qquad \forall\, x \qquad (2.2)$$

g_B is the bare coupling constant and $\vec{J}(x)$ the source term. Integrating over the $\vec{S}$ field we get :

$$Z[J] = \int \mathcal{D}[\alpha] \exp\left\{-\frac{N}{2} S_{eff}[\alpha]\right\}. \exp\left\{\frac{g_B}{2N}\int \langle x| \frac{1}{-\Delta+\alpha(x)}|y\rangle \vec{J}(x)\vec{J}(y)\,dxdy\right\}$$
$$(2.3)$$

with

$$S_{eff}[\alpha] = \mathrm{Tr\, Ln}\,[-\Delta + \alpha(x)] - \frac{1}{g_B}\int d^dx\, \alpha(x) . \qquad (2.4)$$

The limit $N = \infty$ is obtained by taking the constant saddle point of S_{eff}, $\alpha(x) = \alpha_c$, given by

$$\langle x| \frac{1}{-\Delta+\alpha_c}|x\rangle = \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2+\alpha_c} = \frac{1}{g_B} \qquad (2.5)$$

The integral (2.5) is U.V. divergent for $d \geq 2$. Using dimensional interpolation [12] and taking $d = 2-\varepsilon$ (Re $\varepsilon > 0$), (2.5) makes sense and gives

$$\alpha_c = \left[g_B \, \Gamma(\tfrac{\varepsilon}{2}) \, (4\pi)^{\frac{\varepsilon-2}{2}} \right]^{2/\varepsilon} . \tag{2.6}$$

At $N = \infty$ only the connected 2-points function survives and is

$$G_2(p) = \frac{g_B}{p^2 + \alpha_c(g_B)} \tag{2.7}$$

so that $\alpha_c(g_B)$ is the square of the physical mass at $N = \infty$.

At $d = 2$ a wave function and a coupling constant renormalization are needed. We define the renormalized coupling constant g by

$$\frac{1}{g_B} = \frac{1}{g} Z(g) \quad ; \quad Z(g) = 1 + g\mu^{-\varepsilon}\Gamma(\tfrac{\varepsilon}{2})(4\pi)^{\frac{\varepsilon}{2}-1} \tag{2.8}$$

(where μ is the subtraction mass scale), so that at $d = 2$:

$$\alpha_c = \mu^2 e^{-\frac{4\pi}{g}} \tag{2.8}$$

The renormalized 2-points function is

$$G_2^R(p) = Z.G_2(p) = \frac{g}{p^2 + \alpha_c} \tag{2.10}$$

Computing the fluctuations $\alpha = \alpha_c + \tilde{\alpha}$ around α_c in (2.3) we get the $\frac{1}{N}$ expansion. Its perturbative rules are illustrated in (fig.1). The two propagators are the $\vec{S}$ propagator (1.a) given by

$$D(p) = \frac{1}{p^2 + \alpha_c} \tag{2.11}$$

and the $\tilde{\alpha}$ propagator (1.b) which is $-\frac{1}{N} G(p)$ with

$$G(p) = \left[\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \alpha_c)((p+k)^2 + \alpha_c)} \right]^{-1} \tag{2.12}$$

The factors associated to $\vec{S}.\vec{S}.\tilde{\alpha}$ vertices and to internal $\vec{S}$ loops are illustrated in (1.c, 1.d). The internal $\vec{S}$ loops with only one or two $\tilde{\alpha}$ insertions are fordidden (1.e,1.f).

The renormalizability of the 1/N expansion was shown in [13] within the B.P.H.Z. scheme (see also [14]). This point will be discussed in sect 3.

B. *The analytic structure at $N = \infty$.*

As a first step let us discuss the analytic structure (in the coupling constant) of the leading order $N = \infty$. In a manner analoguous to the one used in sect. 3, we first look at the bare theory at $d = 2-\varepsilon$ (Re $\varepsilon > 0$). Using the technics of [11], the 2-points Green function given by (2.7) has the expansion

$$G_2(p) = \sum_{n=0}^{\infty} \left\{ (-1)^n g_B \text{ P.F.} \left[\frac{1}{(p^2)^{n+1}} \right] \cdot \alpha_c^n \right.$$

$$\left. + a_n(d)\, \Delta^n \delta(p) \cdot g_B \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^n}{(k^2 + \alpha_c)} \right\} \qquad (2.13)$$

"P.F." runs for the "Hadamard's finite part" [15] (P.F. $\frac{1}{(p^2)^{n+1}}$ is unambiguously defined for $-\frac{d}{2} + n+1 \notin \mathbb{N}$) ; $a_n(d)$ is a combinational factor ($a_n(d) = \Gamma(\frac{d}{2})/\Gamma(n+\frac{d}{2})\, 2^n (n!)^2$) ; Δ is the laplacian operator with respect to p. (2.13) is an expansion in powers of α_c since, using (2.5) and integration rules of dimensional interpolation [12,16] , we get

$$g_B \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^n}{(k^2 + \alpha_c)} = (-\alpha_c)^n \qquad (2.14)$$

The point is that this expansion may be easily interpreted as an Operator Expansion : indeed α_c^n corresponds to $\langle\alpha^n\rangle$ at $N = \infty$ and the integral (2.14) is simply equal to the (bare) vacuum expectation value of $[\vec{S}(x)\ (-\Delta)^n \vec{S}(x)]$ for $N = \infty$. So we may rewrite (2.13) as

$$G_2(p) = \sum_{n=0}^{\infty} F_{1,n}(p, g_B) . \langle\alpha^n\rangle + F_{2,n}(p)\ \langle\vec{S}(-\Delta)^n \vec{S}\rangle \qquad (2.15)$$

This is in fact an explicit realization of the Operator Expansion into non analytic terms for massless superrenormalizable theories (here $d = 2-\varepsilon$) [4,11,17,18]. The operator α is related to the operator $\partial_\mu \vec{S}.\partial_\mu \vec{S}$ via the equations of motion [9], namely

$$\alpha^n = (-\partial_\mu \vec{S}.\partial_\mu \vec{S})^n + \text{contact terms.} \qquad (2.16)$$

The limit d = 2 of (2.13) is performed by taking into account the renormalization with (2.8) and (2.10). The composite operators α^n and $\vec{S}(-\Delta)^n\vec{S}$ need no additional renormalization at leading order in 1/N. We get for the renormalized two points function

$$G_2^R(p,g) = \sum_n (-1)^n g \; S[(p^2)^{-1-n} ;\mu] \; . \; \langle\alpha^n\rangle$$

$$+ \sum_n a_m(d) \; \Delta^m\delta(p) . \; \langle\vec{S}(-\Delta)^m\vec{S}\rangle \qquad (2.17)$$

where the distribution $S[(p^2)^{-1-n}; \mu]$ is obtained by subtracting the pole of P.F. $(p^2)^{-1-n}$ at $\varepsilon = 0$;

$$S[(p^2)^{-1-n};\mu] = \lim_{\varepsilon \to 0} \left[\text{P.F.}(p^2)^{-1-n} + \mu^{-\varepsilon}\Gamma(\tfrac{\varepsilon}{2}) a_n(d)\Delta^n\delta(p) \right] \qquad (2.18)$$

and is a finite distribution at d = 2.

The expansion (2.17) is *exactly* the V.S.Z Operator Expansion (1.1). The ordinary perturbative part is given by the terms n = 0, m = 0 in (2.17), namely

$$G_2^R(p,g)_{pert} = \delta(p) + g \; S\,[\frac{1}{p^2} \;;\mu] \qquad (2.19)$$

but non perturbative terms proportional to the v.e.v.

$$\langle (-\partial_\mu\vec{S}\partial_\mu\vec{S})^n \rangle_{N=\infty} = \mu^{2n} \, e^{-\frac{4\pi n}{g}}$$

$$\langle(\vec{S} \; \Delta^m \; \vec{S})\rangle_{N=\infty} = \mu^{2m} \, e^{-\frac{4\pi m}{g}} \qquad (2.20)$$

have to be taken into account. One may check that the coefficients of the expansion coincide with those obtained from the perturbative O.E. (Appendix B).

In the following we shall rescale the renormalized coupling constant

$$g \to g' = \frac{g}{4\pi} \qquad (2.21)$$

in order not to deal with (4π) factors.

The (modified) Borel Transform (A.1) is a Mellin Transform with respects to the squared mass α_c/μ^2 (2.9). In this paper we often look first at the bare theory at d = 2-ε (Re ε > 0) not to deal with renormalization. Nevertheless we shall be interessed into the analytic structure of the Mellin Tranform with respects to α_c

$$\hat{f}(s) = \int_0^{cte} d\alpha_c \ \alpha_c^{s-1} \ f(\alpha_c) \tag{2.22}$$

which is no more the Borel Tranform with respects to g (or g_B) at d = 2-ε. For simplicity we shall use the term "Borel Transform" for the Mellin transform (2.22), keeping in mind that it coincides with the Borel Tranform (A.1) only for d = 2, but that the inverse representation (A.2) always holds

$$f(\alpha_c) = \int_c \frac{ds}{2i\pi} \ (\alpha_c)^s \ \hat{f}(s) \tag{2.23}$$

The Borel Transform of the propagator D (2.11) is then in momentum space for $d \leq 2$ (forgetting the distribution-like character of D and its singularities at p = 0, and integrating up to $\alpha_c = \infty$ in (2.22)):

$$\hat{D}(p,s) = (p^2)^{-s-1} \ \Gamma(s+1)\Gamma(s) = (p^2)^{-s-1} \ \hat{D}(s) \tag{2.24}$$

The poles at s = -1,-2,... are irrelevant and given by the behaviour at $\alpha_c = \infty$. The relevant poles of Γ(s) at s = 0,1,... give simply the Taylor expansion of D around $\alpha_c = 0$. (see Fig.2).

C. *The analytic structure of the G propagator*

Let us now look at the $\tilde{\alpha}$ propagator $\frac{-1}{N}$ G(p) (2.12). Since $G = (D*D)^{-1}$ (where * stands for the convolution product), using the expansions (2.13 - 2.17) of D we may expand G in terms of the operators α and $\vec{S}(-\Delta^m)\vec{S}$. The final result is

$$G(p) = G_o(p) \ [1 + \Sigma(p) \ G_o(p)]^{-1} \tag{2.25}$$

where $G_o(p)$ is the "perturbative" $\tilde{\alpha}$ propagator and is given at d = 2-ε by

$$G_o(p) = \left[\frac{1}{g_B} \ \frac{1}{p^2} + \frac{1}{2} \ F_o(p)\right]^{-1} \tag{2.26}$$

and at d = 2 by

$$G_o(p) = \left[\frac{1}{g} \frac{1}{p^2} + \frac{1}{2} S_o(p;\mu) \right]^{-1} \tag{2.27}$$

and where $\sum(p)$ contains the powers of α_c and is at $d = 2-\varepsilon$ of the form

$$\sum(p) = \frac{1}{g_B} \sum_{\substack{n,m \geq 0 \\ n+m \neq 0}} \langle\alpha\rangle^n_{N=\infty} \langle\vec{S}(-\Delta)^m \vec{S}\rangle_{N=\infty} D_{n,m}(p) + \frac{1}{2} \sum_{p>0} \langle\alpha^p\rangle_{N=\infty} F_p(p) \tag{2.28}$$

and at d = 2

$$\sum(p) = \frac{1}{g} \sum_{\substack{n,m \geq 0 \\ n+m \neq 0}} \langle\alpha\rangle^n_{N=\infty} \langle\vec{S}(-\Delta)^m \vec{S}\rangle_{N=\infty} D_{n,m}(p) + \frac{1}{2} \sum_{p>0} \langle\alpha^p\rangle_{N=\infty} S_p(p;\mu) \tag{2.29}$$

In (2.26-29),

$$D_{n,m}(p) = (-1)^n \, a_m(d) \, \Delta^m (p^2)^{-n-1} \tag{2.30}$$

and $F_p(p)$ (respect. $S_p(p;\mu)$) is the "I.R. finite part " (respect. the I.R. subtracted part") of the 1 loop graph with p mass insertions [11](see Fig.3).

So, G(p) may also be written as an Operator Expansion of the form

$$G(p) = \sum_{\mathcal{O}} G_{\mathcal{O}}(p,g) \quad \langle\mathcal{O}\rangle_{N=\infty} \tag{2.31}$$

where the composite operators $\mathcal{O}$ are now of the form

$$\mathcal{O} = \alpha^n \prod_{j=1}^{J} (\vec{S} \, \Delta^{m_j} \vec{S}) \quad ; \quad n \text{ and } J \geq 0, \quad m_j > 0 \tag{2.32}$$

and where $G_{\mathcal{O}}(p,g)$ is a series with a finite radius of convergence (depending on p) in g at d = 2 (resp. in g_B at $d = 2-\varepsilon$). For $\mathcal{O}$ given by (2.32) we have $\langle\mathcal{O}\rangle_{N=\infty} = \alpha_c^{d_{\mathcal{O}}/2}$ where $d_{\mathcal{O}} = 2\left(n + \sum_{j=1}^{J} m_j\right)$ is the dimension of $\mathcal{O}$.

From (2.31), we may write the Borel Tranform $\hat{G}$ of G as

$$\hat{G}(p,s) = (p^2)^{1+\frac{\varepsilon}{2}-s} \hat{G}(s) \tag{2.33}$$

where $\hat{G}(s)$ has the following analytic structure :

At $d = 2-\varepsilon$, $\hat{G}(s)$ has single poles at s of the form

$$s_{n,k} = n + (1+k) \frac{\varepsilon}{2} \quad , n,k \in \mathbb{N} \qquad (2.34)$$

Each series of poles at fixed n corresponds to the expansion in g_B of the $G_\mathcal{O}$'s such that $d_\mathcal{O} = n$ (see Fig. 4a)*.

At $d = 2$, $\hat{G}(s)$ has now branch points at each $s = n \in \mathbb{N}$*.

According to Appendix A, the discontinuity along the n^{th} cut is given by the "ordinary" Borel transform $\tilde{G}_\mathcal{O}(p,s)$ of the $G_\mathcal{O}(p,g)$ for $d_\mathcal{O} = n$ (see (A.4)). However, the fact that these series are convergent implies that each discontinuity is *analytic* in the whole complex s plane. An important consequence is that one meets such a cut at $s = n$ with *the same discontinuity* in the different Riemann sheets generated by the previous branch points at $p < n$. (see Fig. 4.b).

3. *The general structure of the 1/N expansion :*

The result of the section 2 is that the Operator Expansion holds at the tree order of the 1/N expansion. Now we intend to understand whether this remains true , and how, within the next terms of this expansion. For that purpose we shall investigate the analytic structure of the Borel Transform of an arbitrary amplitude of the 1/N expansion by using the desingularization technics of [10-11].

A. *An α parametric representation for the M.B. Transform :*

First we need a Schwinger-Symanzik representation for those amplitudes. We recall that the usual propagator D(p) (2.11) may be written :

$$D(p,\alpha_c) = \int_0^\infty d\alpha \; e^{-\alpha(p^2+\alpha_c)} \qquad (3.1)$$

Similarly, we write the propagator G(p) (2.12) as

$$G(p,\alpha_c) = \int_0^\infty d\alpha \; M(\alpha,\alpha_c) \; \left(\frac{\partial}{\partial\alpha}\right)^2 e^{-\alpha p^2} \qquad (3.2)$$

---

*In addition $\hat{G}(s)$ has irrelevant single U.V. poles at $s = (1-n+\frac{\varepsilon}{2})$ corresponding to the behaviour of G as $\alpha_c \to \infty$.

with

$$M(\alpha,\alpha_c) = \int_{-i\infty}^{+i\infty} \frac{ds}{2i\pi} \alpha_c^s \frac{\hat{G}(s)}{\Gamma(1+s-\frac{\varepsilon}{2})} \alpha^{s-\frac{\varepsilon}{2}} \tag{3.3}$$

where $\varepsilon = 2-d$, $\hat{G}(s)$ is the Borel Tranform of G (2.33) and where the derivatives with respects to α in (3.2) are introduced in order to make the integral convergent at $\alpha = 0$.

Let G be some graph of the 1/N expansion. We denote $\mathcal{D}(G)$ (respectively $\mathcal{G}(G)$) the set of D (respectively G) propagators of G. Using (3.1) and (3.3), the integrations over internal momenta may be performed in the standard way to get for the amplitude I_G of G the representation

$$I_G(p,\alpha_c) = \int_0^\infty \prod_a d\alpha_a \, M_a(\alpha_a,\alpha_c) \, \mathcal{D}_G \left[\exp\{-pd_G(\alpha)p\} \, P_G(\alpha)^{-d/2}\right] \tag{3.4}$$

where each α_a is associated to a line a of G, $M_a = \exp(-\alpha_a\alpha_c)$ if $a \in \mathcal{D}(G)$ and M_a is given by (3.3) if $a \in \mathcal{G}(G)$ and where

$$\mathcal{D}_G = \prod_{a \in \mathcal{G}(G)} \left(\frac{\partial}{\partial\alpha_a}\right)^2 \tag{3.5}$$

$pd_G(\alpha)p$ and $P_G(\alpha)$ are the usual Symanzik functions of G. (3.4) holds if the amplitude I_G is U.V. convergent. In order to make power counting rules simple we associate to each line a,

$$\text{an I.R. degree} \quad \underline{\delta}_a = -1 \quad \text{if } a \in \mathcal{D}(G)$$
$$= 1 + \frac{\varepsilon}{2} \quad \text{if } a \in \mathcal{G}(G) \tag{3.6}$$

$$\text{and U.V. degrees} \quad \bar{\delta}_a = -1 - n \; ; \; n \in \mathbb{N} \, , \quad \text{if } a \in \mathcal{D}(G)$$
$$= 1 - n - \frac{\varepsilon}{2} k \; ; \; n,k \in \mathbb{N} \, , \text{ if } a \in \mathcal{G}(G). \tag{3.7}$$

The I.R. superficial degree of G is

$$\underline{\omega}(G) = \frac{d}{2} L(G) + \sum_a \underline{\delta}_a + \frac{\Delta - N}{2} \tag{3.8}$$

and the U.V. degrees of G are defined as

$$\bar{\omega}(G) = \frac{d}{2} L(G) + \sum_a \bar{\delta}_a + \frac{\Delta - N}{2} \tag{3.9}$$

where L(G) is the number of internal loops of G ; Δ and N are respectively the number of derivative couplings in G and of derivatives with respects to some external momenta of G, if needed.

The main problem is that we cannot apply the standard technics of [10-11] to study the Borel transform of $I_G(p,\alpha_c)$ at two dimensions for two basic reasons :

(*a*) First those technics apply when the α integrand of (3.4) is FINE [19], that is has a "Generalized Taylor Expansion in every Hepp's sectors". This is not the case for the function $M_a(\alpha,\alpha_c)$ for the G-propagator, which contains infinite series of $\frac{1}{\mathrm{Ln}\alpha}$ as $\alpha \to 0$ coming from the cuts of $\hat{G}(s)$ in the representation (3.3).

(*b*) Second the amplitude has to be subtracted because of UV divergent (sub)graphs. The point is that we need a subtraction scheme in the 1/N expansion which corresponds to a definite subtraction scheme in the usual weak coupling (perturbative) expansion. This is not the case for the BPH subtractions at zero momenta which were used in [13-14] ; indeed such subtractions are known to give I.R. divergences in the perturbative expansion which describes a massless theory. The Modified Soft Mass Renormalization Scheme of [20] avoids that problem but introduces additional non-analyticity in the counterterms and is very difficult to handle explicitly*.

For those reasons we choose (as already done in section 2) to work in two steps :

- First we look at the (bare) dimensionally regularized theory at $d = 2-\varepsilon$ ($\mathrm{Re}\varepsilon > 0$), where the α-integrands are in fact FINE. This avoids point (a) and defines amplitudes meromorphic in the half plane $\mathrm{Re}d < 2$.

- Then we take into account renormalization and perform the limit $d \to 2$ by using dimensional renormalization [16] (the minimal subtraction scheme), which is known to respect the Ward Identities of the $\mathcal{O}(N)$ invariance of the model [9].

So we first study the bare amplitude I_G at $d = 2-\varepsilon$ ($\mathrm{Re}\,\varepsilon > 0$). Then the functions M_a are FINE and have the following expansion at $\alpha = 0$:

---

*Moreover, the Ward Identities of the model have to be restored by finite counterterms in the usual way.

if $a \in \mathcal{D}(G)$ we have obviously

$$M_a(\alpha,\alpha_c) = \sum_{n=0}^{\infty} \alpha^n \alpha_c^n d_n, \quad d_n = \frac{(-1)^n}{n!} \tag{3.10}$$

and if $a \in \mathcal{G}(G)$, from (3.3) and (2.34)

$$M_a(\alpha,\alpha_c) \simeq \sum_{n,k} \alpha^{n+k\frac{\varepsilon}{2}} \alpha_c^{n+(k+1)\frac{\varepsilon}{2}} g_{n,k} \tag{3.11}$$

where

$$g_{n,k} = \frac{-1}{\Gamma(1+n+k\frac{\varepsilon}{2})} \text{Res} \left\{\hat{G}(s) \; ; \; s_{n,k} = n+(k+1)\frac{\varepsilon}{2}\right\} \tag{3.12}$$

One can show that the amplitude I_G defined by (3.4) for d small enough is meromorphic in the half plane {Red < 2} with poles at any d such that there is some connected one particle irreducible (C.1.P.I.) subgraph S in G such that, for some set of $\{\bar{\delta}_a\}$ given by (3.7)

$$\bar{\omega}(s) = 0 \tag{3.13}$$

This leads to discrete series of poles at rational d with a point of accumulation at d = 2. So d=2 will be in general an essential singularity, and there is a cut and other singularities on the real axis d > 2 (Fig.5). For Red < 2 away from those discrete UV poles the convergent integral representation holds [21] :

$$I_G(p,\alpha_c) = \int_0^{\infty} d\alpha \; R \left\{\prod_a M_a(\alpha_a,\alpha_c) \; \mathcal{D}_G \; [e^{-pd_G p} \; P_G(\alpha)^{-d/2}]\right\} \tag{3.14}$$

where the subtraction operator R is a sum of products of Taylor operators over all nests N of divergent subgraphs S of G [22]

$$R = \sum_{N} \prod_{S \in N} (-\tau_s^{-\ell(s)}) \tag{3.15}$$

Using (2.22) and the homogeneity properties of the integrand of (2.14), the Borel Transform $\hat{I}_G(p,s)$ of I_G has the following integral representation

$$\hat{I}_G(p,s) = \Gamma(s-\underline{\omega}(G)) \int_0^{\infty} d\alpha \; R \left\{\prod_a M_a(\alpha_a,1) \; \mathcal{D}_G[(pd_G p)^{\underline{\omega}(G)-s} . P_G(\alpha)^{-d/2}]\right\} \tag{3.16}$$

B. *Structure of the bare amplitudes at $d = 2-\varepsilon$*

The fact that the integrand of (3.16) is now FINE implies that the Borel Tranform $\hat{I}_G(p,s)$ is meromorphic in s and that the representation (3.16) is convergent for Re s different from its poles. More precisely, one can extend the technics used in [11] to classify those poles in terms of essential g-subgraphs E of G^*. The result is the following :

Proposition 1.

Let G be some graph and $\varepsilon \neq$ the U.V. poles of G given by (3.13). The Borel Transform of G, $\hat{I}_G(p,s)$, is meromorphic in the positive half plane with single poles at values of s such that there is some g-essential $E \subset G$ and some choice of $\{\bar{\delta}_a, a \in E\}$ such that

$$s = \underline{\omega}(G) - \bar{\omega}(E) \tag{3.17}$$

This structure corresponds to the following expansion of I_G

$$I_G(p,\alpha_c) \simeq \sum_{E \subseteq G} F_E(p,\alpha_c)\, I_{(G/E)}(\alpha_c) \tag{3.18}$$

where the sum runs over the (infinite) set of g-essentials E in G.

$$I_{(G/E)}(\alpha_c) = (\alpha_c)^{\underline{\omega}(G/E)}\, I_{(G/E)}(1) \tag{3.19}$$

is the bare amplitude of the reduced graph $(G/E)^{**}$

$F_E(p,\alpha_c)$ is the "I.R. finite part" of $I_E(p,\alpha_c)$ and is a (formal) series in $\{\alpha_c^{\,n+\frac{\varepsilon}{2}k}\ ;\ n,k \in \mathbb{N}\}$ of the form

$$F_E(p,\alpha_c) \simeq \sum_{\{\bar{\delta}_a\}} (\alpha_c)^{\sum_a(\underline{\delta}_a - \bar{\delta}_a)}\, f_{E,\{\bar{\delta}_a\}}(p) \tag{3.20}$$

---

* According to [11], an essential g-subgraph of G is (at non exceptional momenta) a (connected) subgraph E of G containing all its external vertices plus a family of derivatives versus external momenta of E internal to G.

** (G/E) is obtained by reducing E to one vertex v in G and by putting on lines going to v the corresponding coupling derivatives. The amplitude of (G/E) does not depend on the p's, so that we get the homogeneity relation (3.19).

obtained from I_E by :

(i) Insert the expansions (3.10-11) of the functions M_a into the integral representation (3.14) of I_E.

(ii) Take the "Finite part" of each term of this expansion, which gives the $f_{E,\{\bar{\delta}\}}$'s.

Comment :

The proof of this proposition is a straightforward extension of technics of [11] and will not be given here. In our case the poles given by (3.17) have necessarily Re (s) > 0. So any g-essential E gives series of sequences of poles at

$$s = \underline{\omega}(G/E) + n + \frac{\varepsilon}{2} k \; ; \; n, k \in \mathbb{N} \quad . \tag{3.21}$$

We now have to check that this diagrammatic expansion corresponds to an Operator Expansion as for the N = ∞ order (2.15).

Proposition 2.

Let $G(p_i)$ be some bare Green Function of the model at d = 2-ε . Order by order within the 1/N expansion, the following expansion over all composite operators $\mathcal{O}_n$ (product of derivatives of α and $\vec{S}$ fields at the same point) holds

$$G(p_i, \alpha_c) \simeq \sum_n G_n(p_i, \alpha_c) \quad \langle \mathcal{O}_n \rangle \tag{3.22}$$

where each term of the A/N expansion of $\mathcal{O}_n$ is a series in $\varepsilon_c^{\alpha/2} \propto g_B$ and so corresponds to a perturbative series in g_B and where each term of the 1/N expansion of the vacuum expectation value of $\mathcal{O}_n$; $\langle \mathcal{O}_n \rangle$ is proportional to $\alpha_c^{d_n}$ (where d_n is the canonical dimension of $\mathcal{O}_n$).

This expansion reflects the meromorphic structure of the Borel transfrom of G , which has (order by order) series of single poles at

$$s = d_n + \frac{\varepsilon}{2} k \; ; \; k \in \mathbb{N} \tag{3.23}$$

Proof

Diagrammatically, the reduced graphs (G/E) of the expansion (3.18) are obviously related to various composite operators $\mathcal{O}_n$ at different 1/N orders.

Moreover, the sequences of poles (3.21) given by some E with $n \neq 0$ (which give α_c^n contribution in F_E) come from the terms with $n \neq 0$ of the expansion (3.10-11) of the functions M_a. Those terms come from the operator expansions (2.15) and (2.31) of the propagators D and G, which involve vacuum expectation values (at order $N = \infty$) of operators of the form (2.32). When summing upon all graphs present in the 1/N expansion of $\mathcal{G}$, it is possible to reorganize all those contributions into an expansion of the form (3.22) (this needs a careful but not difficult analysis which will not be given here).

An explicit example is given in Fig. 6. Let us note that many essentials contribute to the leading term ($n = 0$) since the reduced graph may correspond to the observable $(\vec{S})^2 = \mathbb{1}$

C. *Structure of the renormalized amplitudes at d = 2*

As already discussed in 3 (A), the subtraction schemes at zero momenta of [13,14,20] are not suited to our study of the renormalized theory at d = 2. For that reason we shall use dimensional renormalization (the Minimal Subtraction Scheme or M.S.) [16] which is known to lead to an I.R. finite perturbative expansion [8] and which respects the Ward Identities [9] (the O(N) invariance) and the "Quantum Chirality Identities"[13,20] ($S^2=1$) in the perturbative phase. Unfortunately, there is no corresponding explicit subtraction scheme in the 1/N expansion ; indeed, we have seen in (A) that d = 2 may be a branch point and/or an essential singularity, which cannot be subtracted as a pole (via some Cauchy integration for instance). For that reason we must define the M.S. scheme implicitly in the following way.

Lemma

a) Let G be some graph of the 1/N expansion. The renormalized amplitude $I_G^{M.S.}$ of G is defined at $d = 2-\varepsilon$ (Re $\varepsilon > 0$) by

$$I_G^{M.S.}(p,\alpha_c) = \sum_{\{S\}} I_{G/US}(p_i,\alpha_c) \prod_S (\alpha_c)^{\underline{\omega^*(S)}} K_S(g;\mu) \quad (3.24)$$

where :

- The sum is performed over all families (eventually empty) of disjoint C.1.P.I. divergent subgraphs S of G (at d = 2).

- Each counterterm $K_S(g;\mu)$ is defined as a series in the renormalized coupling constant g (given by (2.6) and (2.8) as a function of α_c) of the form

$$K_S(g;\mu) = \sum_n g^n \mu^{-\varepsilon n} k_{S,n}(\varepsilon) \qquad (3.25)$$

obtained by summing the counterterms (of the M.S. scheme) of the perturbative expansion corresponding to S (consequently each $k_{S,n}(\varepsilon)$ is a polynomial in $1/\varepsilon$). The series (3.25) has a finite radius of convergence as long as (Re $\varepsilon > 0$) and so defines an analytic function of g (or α_c).

- μ is the usual subtraction mass scale (the same than in 2.8)
- In (3.24) and in the following $\underline{\omega}^*(s)$ runs for $\underline{\omega}(s)$ at $\varepsilon = 0$. S is divergent at $d = 2$ iff $\omega^*(s) \in \mathbb{N}$.

b) The amplitude $I_G^{M.S}$ has a limit as $\varepsilon \to 0$ provided that

$$\text{Arg } \varepsilon \in]-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}[\qquad .$$

Comments

The fact that (3.24) defines a finite amplitude at d = 2 is not obvious: indeed the counterterms are defined perturbatively and the bare amplitudes contain also non perturbative terms (Propositions 1 and 2). One may introduce an I.R. cut off (finite volume or external symmetry breaking term) which eliminates the non perturbative terms in (3.22). The counterterms of the minimal subtraction scheme do not depend on the I.R. cut off so that one may sum up the perturbative expansion order by order to get an 1/N expansion which is now U.V. finite, then, set the I.R. cut off to zero and recover (3.24). A complete and rigorous proof is rather delicate and will not be given here.

With this subtraction scheme we can now go to $\varepsilon = 0$. The main result is :

Theorem A.

a) At each order of the 1/N expansion, the Borel Transform of any renormalized Green function $\hat{G}^{MS}(p_i,s)$ at two dimensions is analytic in s away from the positive real axis and has branch points at each entire point $s \in \mathbb{N}$.

b) Each discontinuity $\Delta_p \hat{G}^{MS}$ at $s = p$ *in the first sheet* (Fig.7.a) may be written as a sum over all operators $\mathcal{O}_n$ of dimension $d_n = p$

$$\Delta_p \hat{G}^{MS}(s) = \sum_{\substack{\mathcal{O}_n \\ d_n = p}} \tilde{G}_n^{MS}(s) * \tilde{\mathcal{O}}_n^{MS}(s) \tag{3.26}$$

where $\tilde{G}_n^{MS}(s)$ is the ordinary Borel Transform (see Appendix A) of the term dual to $\mathcal{O}_n$, G_n^{MS}, in the formal Operator Expansion obtained by the technics exposed in Appendix B. More precisely, each G_n^{MS} is at each order of the $1/N$ expansion a (divergent) series of the form

$$G_n^{MS}(g) = \sum_k g^k \, G_{n,k} \tag{3.27}$$

and $\tilde{G}_n^{MS}(s)$ is given by the convergent series*

$$\tilde{G}_n^{MS}(g) = \sum_k \frac{s^{k-1}}{\Gamma(k-1)} G_{n,k} \tag{3.28}$$

Similarly $\tilde{\mathcal{O}}_n^{MS}$ is the ordinary Borel Transform of $\langle \mathcal{O}_n^{MS} \rangle$, and is given by the discontinuity at $s = p$ of the Borel transform $\hat{\mathcal{O}}_n^{MS}(s)$**, $*$ is the Borel Convolution Product which reads

$$\tilde{g} * \hat{\mathcal{O}}(s) = \int_p^s du\, \tilde{\mathcal{O}}(u)\, \tilde{G}(s-u) \tag{3.29}$$

c) the discontinuities at $s = p$ ($p \geq 2$) in the different sheets corresponding to the branch points at $q < p$ are in general different from the first one given by (3.26).

<u>*Proof*</u> :

The principle of the proof is the following. We first look at the analytic structure of the Borel Tranform of G^{MS} at $d = 2-\varepsilon$. Using Prop. 2 and Lemma 1, one can show that, order by order in the $1/N$ expansion, $\hat{G}^{MS}(p_i, s)$, is meromorphic like $\hat{G}$ with infinite series of single poles at $s = n + \frac{\varepsilon}{2}k$ ($n,k \in \mathbb{N}$) corresponding to the new Operator Expansion

$$G^{MS}(p_i, \alpha_c) = \sum_n G_n^{MS}(p_i, \alpha_c) \langle \mathcal{O}_n^{MS} \rangle \tag{3.30}$$

----

* The term $k = 0$ has to be understood as $\delta(s)$

** $\hat{G}_n^{MS}(s)$ has in fact a single branch point at $s = p$ so that $\tilde{\mathcal{O}}_n^{MS}(s)$ is analytic on the real axis $s > d_n$.

where $\langle \mathcal{O}_n^{MS} \rangle$ is now the v.e.v. of the renormalized operator $\mathcal{O}_n$ and is (in the 1/N expansion) of the form

$$\langle \mathcal{O}_n^{MS} \rangle = \alpha_c^{d_n} \{\text{series in } g_B\}$$

and where each $G_n^{MS}(p_i, \alpha_c)$ is a perturbative series in g_B (and so in g) convergent for Re $\varepsilon > 0$; each term of the corresponding series in g, $G_{n,k}(\varepsilon)$ corresponds to I.R. and U.V. subtracted amplitudes of the perturbation theory and so has a limit as $\varepsilon \to 0$ which is the $G_{n,k}$ of (3.27).

We now have to take the limit $\varepsilon \to 0$ with Arg $\varepsilon = \theta$ fixed. The crucial point is that each series of poles at $s = n + \frac{\varepsilon}{2} k$, n fixed, coalesces to give a cut along $s = n + \lambda$, Arg $\lambda = \theta$, and that no other singularity appears as $\varepsilon \to 0$. This may be shown by the following argument, developed in appendix C.

The amplitudes subtracted at zero momenta according to the usual Zimmermann Scheme may be represented by a "Complete Mellin Representation" [23]. In this representation, (given in Appendix C) general arguments show that there are no other singularities that the above cuts at $\varepsilon = 0$. Then one can argue that the finite counterterms needed to recover the M.S. subtracted amplitude do not destroy the structure and simply modify the discontinuities. So we get part a).

Starting now from the fact that those cuts are the limit of the series of poles, one use (3.30) to get (3.26). Indeed, for $\varepsilon \neq 0$, Arg $\varepsilon = \theta$, the discontinuity along the line $\{s = p + e^{i\theta} x\}$ may be written in the form (3.26), but now $\tilde{G}_n^{MS}$ is defined on the line Arg $s = \theta$ as a sum of Dirac distributions at $s = \frac{\varepsilon}{2} k$, $k \in \mathbb{N}$ and $\tilde{\mathcal{O}}_n^{MS}$ on the line Arg $(s-p) = \theta$ as a sum of Dirac distribution at $s = p + \frac{\varepsilon}{2} k$, $k \in \mathbb{N}$.

Now the distribution $\tilde{G}_n^{MS}$ with discrete support at $\varepsilon > 0$ tends toward (3.28) as $\varepsilon > 0$; indeed, using (2.6) and (2 .8), to the term g^k in (3.27) corresponds the distribution

$$\Delta_\varepsilon^{(k)} = \sum_{m=0}^{\infty} \Gamma(\tfrac{\varepsilon}{2})^{-k} \frac{(k+m-1)!}{m!(k-1)!} \delta(s - (k+m)\tfrac{\varepsilon}{2}) \qquad (3.31)$$

which tends toward $s^{k-1}/\Gamma(k)$ as $\varepsilon \to 0$. $\tilde{\mathcal{O}}_n^{MS}$ being equal to the discontinuity of $\hat{\mathcal{O}}_n^{MS}$ for any $\varepsilon \geq 0$, we finally get part b) of the theorem.

The arguments developed here do not permit us to look at the singularities in other sheets than the first one. The C.M. Representation is a more adequate tool for that problem. In Appendix D we use it to look explicitly at point c) at order $(\frac{1}{N})^1$.

D. The status of the Operator Expansion and I.R. renormalons

Now we can see how the Operator Expansion makes sense. Using (3.26) and the inverse Borel transform (2.23) we get that the Operator Expansion

$$G^{MS}(p_i,g) = \sum_n G_n^{MS}(p_i,g) \langle O_n^{MS}(g)\rangle \qquad (3.32)$$

is *exact* at each order of the $1/N$ expansion with the following resummation prescriptions which make it unambiguous.

- All $G_n(p_i,g)$ are defined as the Borel sum (A.6) of (3.27) with the *same* prescription of integration on a complex line above (or under) the positive real axis.

- The non-perturbative quantities $\langle O_n^{MS}(g)\rangle$ are defined without ambiguities since each ordinary Borel Transform $\tilde{O}_n^{MS}$ is analytic for Re $s > d_n$.

The remarkable point is that we have obtained that result quite independently of the presence and of the nature of the I.R. renormalons on the positive real s axis of the ordinary Borel transform $\tilde{G}_n$. Those singularities come obviously from point (c) of Theorem A. However, since the discontinuities $\Delta_p\hat{G}$ given by (3.26) are real for $p < s < p+1$, they are equal in the first sheet above or under the positive real axis; one concludes that the first I.R. renormalon of the perturbative series $\tilde{G}_o^{MS}$ is at $s = 2$ (and not at $s = 1$) and corresponds to the operators of dimension 4 (see Fig.7).

Similarly, the first I.R. renormalon of $\tilde{G}_1^{MS}$ (corresponding to the operator $(\partial\vec{S})^2$) is at $s = 1$. From (3.32) when adding the two contributions associated to the operators $\mathbb{1}$ and $(\partial S)^2$, the corresponding singularities at $s = 2$ must cancel, leaving the next renormalon at $s = 3$. This mechanism holds at all orders ; namely, when adding the terms corresponding to the operators of dimensions $\leq P$ in the Operator Expansion, all renormalons at $s \leq P+1$ cancel .

Of course one would like to have more details on the nature of those singularities in the 1/N expansion. Unfortunately, the technics developed here cannot cope with that problem in general. We have checked in Appendix C the presence of such renormalons at the order 1/N. At that order those singularities of the Borel Transform $\tilde{G}_o(s)$ are single poles at s = 2,3,4,... and are checked to be proportional to the terms $\tilde{G}_n(s)$ dual to some of the operators of the corresponding dimension. The renormalon at s = 2 is for instance dual to the operator $\alpha.\alpha$, the renormalons at s = 3 dual to $\alpha.\alpha.\alpha$ and $\partial_\mu\alpha.\partial_\mu\alpha$, etc... one expects that such a feature remains at next orders.

4. General Discussion

1) Let us first outline the results of this paper : At any arbitrary order of the 1/N expansion, we have shown that in the $\mathcal{O}(N)$ σ model.:

- There are I.R. renormalons at $s = 4\pi n$, $n \geq 2$ on the positive real axis of the Borel Transform of the perturbative expansion.

- Non perturbative terms proportional to vacuum expectation values of all invariant operators are present.

- Those terms are organized in an Operator Expansion à la S.V.Z. at all orders.

- There is a well defined Borel summation prescription of the perturbative series which deals with I.R. renormalons and makes the Operator Expansion unambiguous. Basically, the Operator Expansion manages to cancel the I.R. renormalons of its different perturbative parts, as explained in (3.D).

2) The technics of this paper may be applied without difficulties to the two-dimensional U(N) Gross-Neveu Model [5] . One can show similarly that the vacuum expectation values of the Fermion Condensates associated to the spontaneous breakdown of the (discrete) $\mathbb{Z}_2$ Chiral symmetry organize in a S.V.Z. operator expansion which cancels the corresponding I.R. renormalons.

3) It seems reasonable to think that those results are not modified when summing the 1/N expansion to get the σ model for finite N (the 1/N expansion

is likely Borel summable [24-25]). One expects that the branch points of the Borel Transform are only shifted to $s = 2n/\beta_2$ with $\beta_2 = \frac{N-2}{2\pi N}$ but that their structure remains the same. The cancellation of I.R. renormalons which takes place is in fact the only possible way to make the S.V.Z. Operator Expansion consistent with the Borel resummation procedure. So we conjecture that the same phenomenon occurs in four dimensional Gauge Theories (with massive or massless fermions). It is interesting to note that S.V.Z. found heuristically that an optimal summation procedure for the Operator Expansion was their "Borel Improvement" [1], that is precisely the use of the Borel Tranform.

However, it seems to us that any direct attempt to get a rigorous proof of that fact (even for the σ model for finite N) needs a non perturbative use of the Dyson-Schwinger Equations and is a formidable program [26].

4) The mechanism described above which deals with I.R. renormalons is somewhat different to what happens for Instantons in quantum mechanics or in massive fields theories [27].

In the case of Instantons, one can write any quantity E as

$$E(g) = \sum_{n=0}^{\infty} E^{(n)}(g) \qquad (4.1)$$

where $E^{(n)}(g) \sim \mathcal{O}([E^{(1)}(g)]^n)$ is the sum of n-instantons contributions. However, each $E^{(n)}$ is ambiguous even at leading order $\mathcal{O}([E^{(1)}(g)]^n)$ (this is related physically to the instability of intantons-anti instantons configurations, and mathematically to Stockes Phenomenon, i.e. to the different ways to catch those configurations in the functional integral). This ambiguity in the definition of $E^{(n)}$ is raised only when precising an integration prescription around the corresponding singularity at $s = n\, S_o$ of the Borel Tranform of $\sum_{p=0}^{n-1} E^{(p)}(g)$ (S_o is the action of the 1-instanton).

In the Operator Expansion (3.32), the terms of order $n \sim \mathcal{O}([e^{2/\beta_2 g}]^n)$ given by operators $\mathcal{O}_k$ of dimension $d_k = 2n$, are ambiguous only at next order $\mathcal{O}([e^{2/\beta_2 g}]^{n+1})$, but the v.e.v. $\mathcal{O}_k(g) = \langle 0|\mathcal{O}_k|0\rangle$ are given by perturbation theory only up to a numerical factor. Indeed, the $\mathcal{O}_k$'s satisfy the R.G. equation*

$$[\mu \frac{\partial}{\partial\mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_n(g)]\, \mathcal{O}(g) = 0 \qquad (4.2)$$

---

*γ_n is an operator mixing all $\mathcal{O}_k$ with dimension 2n.

which determines the $\mathcal{O}_k$'s up to dimensionless factors C_k *.

$$\mathcal{O}_k(g) = C_k \, \mu^{2n} \exp \int^g du \, \frac{-2n + \gamma_n(g)}{\beta(g)} \qquad (4.3)$$

5) However, the cancellation of renormalons at $s = -\frac{2}{\beta_2}(n+1)$ between the term of order n and those of order < n fixes strong constraints on the C_k's. For instance, the existence of a renormalon at $s = -\frac{4}{\beta_2}$ fixes the value of the first perturbative term $<0|(\partial_\mu \vec{S})^2|0>$. An interesting question is : can all non perturbative quantities be fixed in that way ? In such a case one could say that (formally) perturbation theory contains enough information to recover the full theory. One could also imagine numerical estimations of the first non perturbative terms based on that scheme.

6) In section 3 we never precised the integrability at infinity of the Borel Transform. In fact we expect no problem order by order in the 1/N expansion but it is possible that in the full theory this condition breaks down, so that the Borel sum itself is only asymptotic [3]**. With this restriction this does not change our conclusions.

7) In the case of the non linear σ model, the existence of a "Spin Wave Condensate", $<0|(\partial_\mu \vec{S})^2|0> \neq 0$,is obviously related to the restoration of the $\mathcal{O}(N)$ symmetry. This is self evident in the limit $N = \infty$ since we have then***

$$<0|\partial_\mu \vec{S} \partial_\mu \vec{S}|0> = -(\text{physical Mass})^2 \qquad (4.4)$$

8) The presence of condensates is often justified by the existence of non perturbative effects such as Instantons [1]. It is interesting to note that the existence of Instantons is not necessary to get such condensates.

---

* The R G functions $\beta(g)$ and $\gamma_n(g)$ are Borel summable and so computable from perturbative theory. This may be checked in the 1/N expansion ; the reason why there are no I.R. renormalons, at least in the M.S. scheme, is that β and the γ's are not modified when there is an I.R. cut off.

** t'Hooft's argument for such a singularity is based on the existence of an infinite number of resonances at arbitrary large energy. This is not the case for the non linear σ model [28] .

*** $\partial_\mu \vec{S} \partial_\mu \vec{S}$ is subtracted according to some Normal Product Algorithm and may perfectly have a negative vacuum expectation value.

However, the problem of taking into account Instantons in the Operator Expansion in a mathematically consistent way has still to be understood [29] and is obviously related to a correct understanding of the dense Instanton gas problem [30].

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APPENDIX A

Borel Transforms

A convenient object to study non Borel summable functions is the (Modified) Borel Tranform [31]

$$\hat{f}(s) = \int_o^{cte} \frac{dg}{g^2} e^{s/g} f(g) \tag{A.1}$$

If f(s) is analytic in some disc $\mathrm{Re}(g^{-1}) > \rho^{-1}$ tangent to the origin $g = 0$ (and has an ad hoc behaviour as $g \to 0$, which we do not precise) $\hat{f}(s)$ is analytic away from the positive real axis $s \in \mathbb{R}^+$ and the following inverse representation holds*

$$f(g) = \int_C \frac{ds}{2i\pi} e^{-s/g} \hat{f}(s) \tag{A.2}$$

where the (anticlockwise) contour C encircles $\mathbb{R}^+$ (fig.8) (A.1) is nothing else than a Laplace Tranform with respects to $1/g$, or a Mellin Transform with respects to $\exp(-1/g)$.

If f(g) has an asymptotic expansion

$$f(g) = \sum_{k=0}^{\infty} g^k C_k \tag{A.3}$$

and is Borel summable (i.e. satisfies the Nevanlinna - Sokal Theorem [32]) the discontinuity at s=0 $\Delta_o\hat{f}$ given by

$$\Delta_o\hat{f}(s) = \frac{1}{2i\pi} [\hat{f}(s + i\varepsilon) - \hat{f}(s - i\varepsilon)] \tag{A.4}$$

is equal to the Ordinary Borel Transform $\tilde{f}(s)$ of f

$$\tilde{f}(s) = \sum_k s^{k-1} \frac{C_k}{\Gamma(k)} \tag{A.5}$$

where $\frac{s^{-1}}{\Gamma(0)}$ has to be understood as $\delta(s)$. (A.2) is nothing else than the usual inverse Borel Transform

$$f(g) = \int_o^{\infty} ds\, e^{-s/g} \tilde{f}(s) \tag{A.6}$$

---

*$\hat{f}(s)$ depends on the upper bound of integration in (A.1) but its discontinuity along $\mathbb{R}^+$ does not.

When f is not Borel summable, as in instanton problems, $\hat{f}(s)$ has other branch points on the positive real axis which have to be taken into account when one writes (A.2) as an integral over the discontinuities.

In this paper the term Borel Tranform denotes in general the transform (A.1) and we precise when dealing with the ordinary Borel Tranform (A.5),(A.6).

We finally recall that the Borel transform of a product $f_1.f_2$ is the Borel convolution product $\hat{f}_1 * \hat{f}_2$

$$\hat{f}_1 * \hat{f}_2(s) = \int_C \frac{du}{2i\pi} \hat{f}_1(u)\, \hat{f}_2(s-u) \qquad (A.7)$$

where the contour C encircles $\mathbb{R}^+$ (or $s-\mathbb{R}^+$) as in Fig. 9 .

APPENDIX B

The Operator Expansion in Perturbation Theory :

A simple way to get the terms of the (formal) Operator Expansion (I.2) in perturbation theory is to see it as an I.R. expansion when some I.R. cut off goes to zero. In the case of the 2-dimensional non linear σ model one may consider the theory in a finite volume V or put an external constant magnetic field H.

General technics of studying asymptotic estimates in perturbation theory [2,10,11] can be used to get the following result : within perturbation theory, the following expansion holds as V (resp. $1/H$) $\to \infty$ for any observable G :

$$G(V) \cong \sum_n G_n(V)\ \mathcal{O}_n(V) \qquad \text{(B.1)}$$

where the sum runs over all composites operators $\mathcal{O}_n$.

$\mathcal{O}_n(V)$ is equal to the vacuum expectation value of the renormalized operator $\mathcal{O}_n$ and is of the form

$$\mathcal{O}_n(V) = V^{-d_n/2} \sum_{k=0}^{\infty} g^k\ P_{(n),k}(\text{Ln } V) \qquad \text{(B.2)}$$

where d_n is the dimension of the operator $\mathcal{O}_n$ and each $P_{(n),k}$ a polynomial (of order $\leq k$) in Ln V.

The term dual to $\mathcal{O}_n$, $G_n(V)$ is analytic in V^{-1}, and more precisely

$$G_n(V) \simeq \sum_{k=0}^{\infty} g^k\ G_{n,k}(V^{-1}) \qquad \text{(B.3)}$$

where each $G_{n,k}(V^{-1})$ is a (formal) series in V^{-1} and corresponds to a sum of U.V. and I.R. *subtracted* * Feynman amplitudes.

The key of the proof of the I.R. finiteness of the non linear σ model [8] is the fact that, for invariant observables, among all operators of dimension 0, only the invariant one $\mathbb{1}$ survives in (B.1). Similarly one expects that at in general, only $\mathcal{O}(N)$ invariant operators $\mathcal{O}_n$ are present in (B.1). In fact, to the "naively" invariant operators one must add new ones involving

* In the sense of [4,12].

the operator $\frac{\Delta\sigma}{\sigma}$, which is the perturbative analog to the operator α(in the parametrization $\vec{S} = (\vec{\pi}, \sigma = \sqrt{1-\pi^2})$), namely may be written as the derivative of the action versus the constraint and so is related to the action $(\partial_\mu \vec{S} \partial_\mu \vec{S})$ via the equations of motion [9].

Taking $V = \infty$ the G_n are finite perturbative series (B.3) and the $\mathcal{O}_n$ are zero order by order in g (B.2). It is those $\mathcal{O}_n$ which are assumed to be non zero in the S.V.Z. Operator Expansion.

APPENDIX C

The complete Mellin (C.M.) representation of Feynman Amplitudes [23] provides a systematic tool to study its analytic properties. Indeed, an amplitude is written as an integral of the Mellin type, the C.M. integrand is a product of Γ functions and of linear powers of external invariants and of internal masses, its analytic structure allows the determination of any asymptotic expansion. Another advantage is that the renormalization according to the Zimmermann scheme takes a very simple form : it results in a modification of the integration path without any change in the integrand [23] . It is not difficult to extend the C.M. representation to our problem. Starting from the representation (3.4) we get for some convergent graph G :

$$I_G = \int_{C_o \cap \Delta_G} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_k S_k^{y_k} \Gamma(-y_k) \prod_a \alpha_c^{-\varphi_a} \Gamma_a(\varphi_a) \tag{C.1}$$

where the variables x_j and y_k are attached respectively to each one-tree j or two-trees k of G. S_k is the cut invariant corresponding to the two trees k (and is quadratic in external momenta). The linear function $\varphi_a(x,y)$ associated to the line a is

$$\varphi_a = \sum_j u_{aj} x_j + \sum_k u_{ak} y_k - \underline{\delta}_a \tag{C.2}$$

where u_{aj} (respectively u_{ak}) = 0 or 1 following the line a belongs or not to the one-tree j (respectively two-trees k). The terms $\Gamma_a(\varphi)$ are given by

$$\Gamma_a(\varphi) = \begin{cases} \hat{D}(-\varphi)/\Gamma(-\varphi-\underline{\delta}_a) = \Gamma(\varphi) & \text{if } a \in \mathcal{D}(G) \quad \text{(C.3.a)} \\ \hat{\hat{G}}(-\varphi)/\Gamma(-\varphi-\underline{\delta}_a) & \text{if } a \in \mathcal{G}(G) \quad \text{(C.3.b)} \end{cases}$$

The integration symbol means :

$$\int_{-i\infty}^{+i\infty} \frac{dx_j}{2i\pi} \frac{dy_j}{2i\pi} \cdot \delta(\sum_j x_j + \sum_k y_k + \frac{d}{2}) \tag{C.4}$$

and that (x,y) belongs to the intersection of the cell C_o

$$C_o = \{x,y \mid \operatorname{Re} x > 0 \ , \ \operatorname{Re} y > 0\} \tag{C.5}$$

and of the U.V. convergence domain Δ_G :

$$\Delta_G = \{x,y| \ \text{Re}\ \varphi_a > 0, \forall\ a \in G\} \tag{C.6}$$

The only change with [23] is in the φ_a's and the functions Γ_a which have now a more complex analytic structure (with cuts at negative $\varphi = -n$). Similarly, a renormalized amplitude according to the Zimmermann scheme (subtractions at zero momenta) I_G^R is

$$I_G^R = \sum_c \mu_c \int_{c \cap \Delta_G} J_G(x_i, y_j, S_k)\ \alpha_c^{-\sum_\alpha \varphi_a} \tag{C.7}$$

where the sum runs over cells c delimited by the singularities of the function $\prod_j \Gamma(-x_j) \prod_k \Gamma(-y_k)$ in (C.1). The multiplicity factor μ_c is an integer $\neq 0$ only for a finite number of cells c such that $c \cap \Delta_G \neq \emptyset$ and does not depend on the functions Γ_a. J_G is the same integrand as in (C.1). The Borel Tranform $\hat{I}_G^R$ is for Re $s < 0$:

$$\hat{I}_G^R(s) = \sum_c \mu_c \int_{c \cap \Delta_G} \delta(s + \sum_a \varphi_a)\ J_G(x_i, y_j, S_k) \tag{C.8}$$

and its analytic structure is obtained by translating s into the domain Re $s > 0$ (Re $\varphi < 0$). Without getting into a general study, one may deduce that the singularities are branch points at the same positions than in the case of usual propagators, and are the right limit of the series of poles occurring at $d = 2-\varepsilon$, as claimed in (3.C).

In order to recover the M.S. subtracted amplitude, one has to introduce finite counterterms. One has in fact

$$I_G^{MS}(p) = \sum_{\{S\}} I^R_{G/US}(p)\ \prod_S (I_S^{MS}) \tag{C.9}$$

where the sum runs over family of disjoint C.1 P.I. divergent subgraphs S. The vertex corresponding to S in(G/US) has to be oversubtracted according to the superficial degree of S ($\underline{\omega}(S) \geq 0$). The countertems I_S^{MS} are renormalized amplitudes taken at zero external momenta. They can depend on g only as

$$I_S^{MS}(g) = \alpha_c^{\underline{\omega}(S)} \times (\text{series in } g)$$

and so corresponds only to a discontinuity at $s = \underline{\omega}(S)$ in the Borel plane.

APPENDIX D

Let us apply the C.M. representation to study the Borel Transform of the 1/N order of the two points function Γ_2. It is expressible in terms of the two graphs of Fig.6. The second one does not depend on the external momenta and by homogeneity it gives a single cut at s = 1. So we are interested in the one loop graph S_1.

Using the C.M. representation for its amplitude subtracted at zero momenta, one gets for its Borel Transform the representation

$$\hat{I}^R(s) = \int_{-i\infty}^{+i\infty} \frac{du}{2i\pi} \frac{\Gamma(u-s)\,\Gamma(2-u)\,\Gamma(s-1)}{\Gamma(2-s)} (p^2)^{1-s} \Gamma_G(-u)\, \Gamma_D(u-s) \tag{D.1}$$

where the functions Γ_D and Γ_G are given respectively by (C.3.a) and (C.3.b) (D.1) holds for $-1 < \text{Re } s < \text{Re } u < 0$. The integrand has then double poles at $u = s-k$ ($k \in \mathbb{N}$) and branch points at $u = k \in \mathbb{N}$. Going into the domain $\text{Re } s < 0$, $\hat{I}^R$ has a singularity if the integration contour is pinched between two of those singularities (Fig. 10). So one gets immediately the branch points at $s = n \in \mathbb{N}$.

To get the M.S. subtracted amplitude, according to (C.3), one needs two counterterms which modify the discontinuities of $\hat{I}$ at s = 0 and s = 1, but not the next ones. So we start from (D.1) to study the singularities at $s = n \geq 2$ in the different sheets.

Moving the integration contour around the poles ($u = s-n$) we get for $\hat{I}^R(s)$ the (formal) sum

$$\hat{I}^R(s) = \sum_{n=0}^{\infty} (p^2)^{1-s} \frac{\Gamma(s-1)}{\Gamma(2-s)} \left(\frac{(-1)^n}{n!} d_n \frac{d}{ds} + b_n \right) . \Gamma_G(n-s)\Gamma(2-s+n) \tag{D.2}$$

where $d_n = \frac{(-1)^n}{n!}$ is the residue of Γ_D at n and b_n is irrelevant in the following.

The discontinuities of $\hat{I}(s)$ in the first sheet are given by Theorem A. The existence of an I.R. renormalon at s = 2 comes from the difference between that discontinuity at s = 2 and the discontinuity in the sheet reached by passing under s = 1 (Fig. 7).

The difference of $\hat{I}^R(s)$ between those two sheets comes from the discontinuity of Γ_G at $s = 1$; one gets for that difference

$$\hat{I}^R(s) - \hat{I}'^R(s) = (p^2)^{1-s} \frac{\Gamma(s-1)}{\Gamma(2-s)} \left(d_o \frac{d}{ds} + b_o\right) 2i\pi\, \Delta_1\Gamma_G(s).\Gamma(2-s)$$

$$= -2i\pi \frac{d_o}{p^2} \Delta_1\Gamma_G(2) \,.\, \Gamma(1) \frac{1}{s-2} + \text{regular function at } s = 2 \tag{D.3}$$

where $\Delta_1\Gamma_G(s) = \dfrac{\Delta_1 \hat{G}(s)}{\Gamma(s-1)}$ is the discontinuity of Γ_G at $s = 1$ and is analytic for Re $s > 1$.

From (D.3), the first renormalon at $s = 2$, (that is the first singularity of the discontinuity of $\hat{I}^R(s)$ at $s = 0$) is a *single pole* and is proportional to d_o/p^2, which corresponds graphically to the term dual to the operator $\alpha.\alpha$.

The same arguments hold for the next discontinuities at $n > 2$. At that order of $1/N$ expansion, one may check that the various renormalons are always single poles and are proportional to terms dual to composite operators involving only α field. For instance, we get renormalons at $s = 3$ in terms of the duals of the operators $\alpha.\ \alpha.\ \alpha$ and $\partial_\mu\alpha.\partial_\mu\alpha$. A closer look at (D.3) suggests that those singularities may be classified in terms of nests of essentials but we need to study diagrams with many loops in order to make this idea more precise and such a study becomes very complicated.

Figure Captions :

Fig.1. Elements of the 1/N expansion :(a) The $\vec{S}$ propagator D(p) ; (b) the $\tilde{\alpha}$ propagator G(p) ; (c) the interaction vertex ; general internal $\vec{S}$ loop (d) : the internal loops forbidden in the expansion (e) and (f).

Fig.2. The Operator Expansion of the $\vec{S}$ propagator.

Fig.3. Diagrammatic interpretation of the expansion of the $\tilde{\alpha}$ propagator (a) (Eq. 2.25) ; (b) expansion of the perturbative propagator G_o; (c) operator expansion of $\sum$ (2.28-30).

Fig.4. Analytic structure of the Borel Transform of G(p) at $d = 2-\varepsilon$ (a) and at $d = 2$ (b).

Fig.5. Analytic structure of a regularized amplitude in d : $d=2$ is an essential singularity with a cut at $d > 2$ and an accumulation of poles at $d < 2$.

Fig.6. Diagrammatic interpretation of the first terms of the expansion (3.22) for the irreducible two points function at order $(1/N)^1$. (a) the two graphs of 1/N expansion ; the terms (b), (c); (d) correspond to operators $<\mathbb{1}>$, $<(\partial_\mu\vec{S})>^2$ and $<\alpha>$ at order $N = \infty$; the term (e) to $<\alpha>$ at order $(1/N)^1$.

Fig.7. The two different cuts at $s = 2$ which give the first I.R. renormalon.

Fig.8. Integration contour for the inverse Borel Tranform (A.2).

Fig.9. Integration contour from the Borel convolution (A.7)

Fig.10. Analytic structure and integration contour for the integral (D.1).

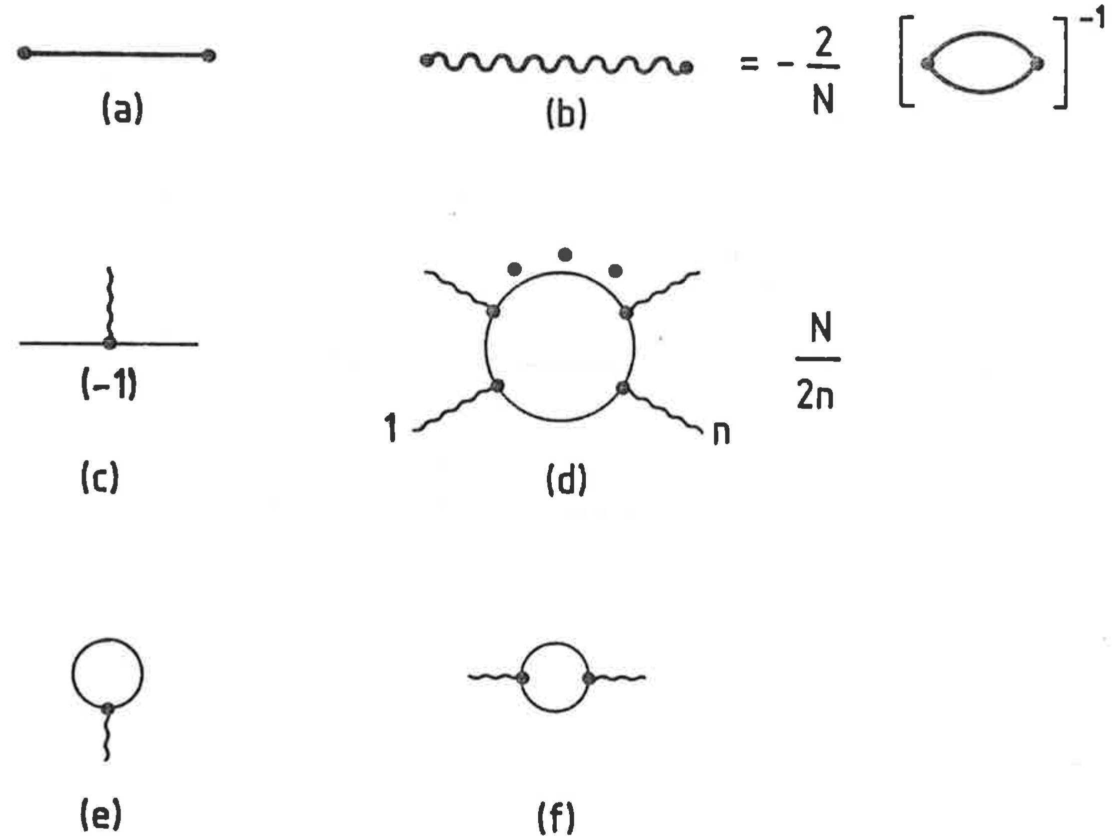

Fig. 1

Fig. 2

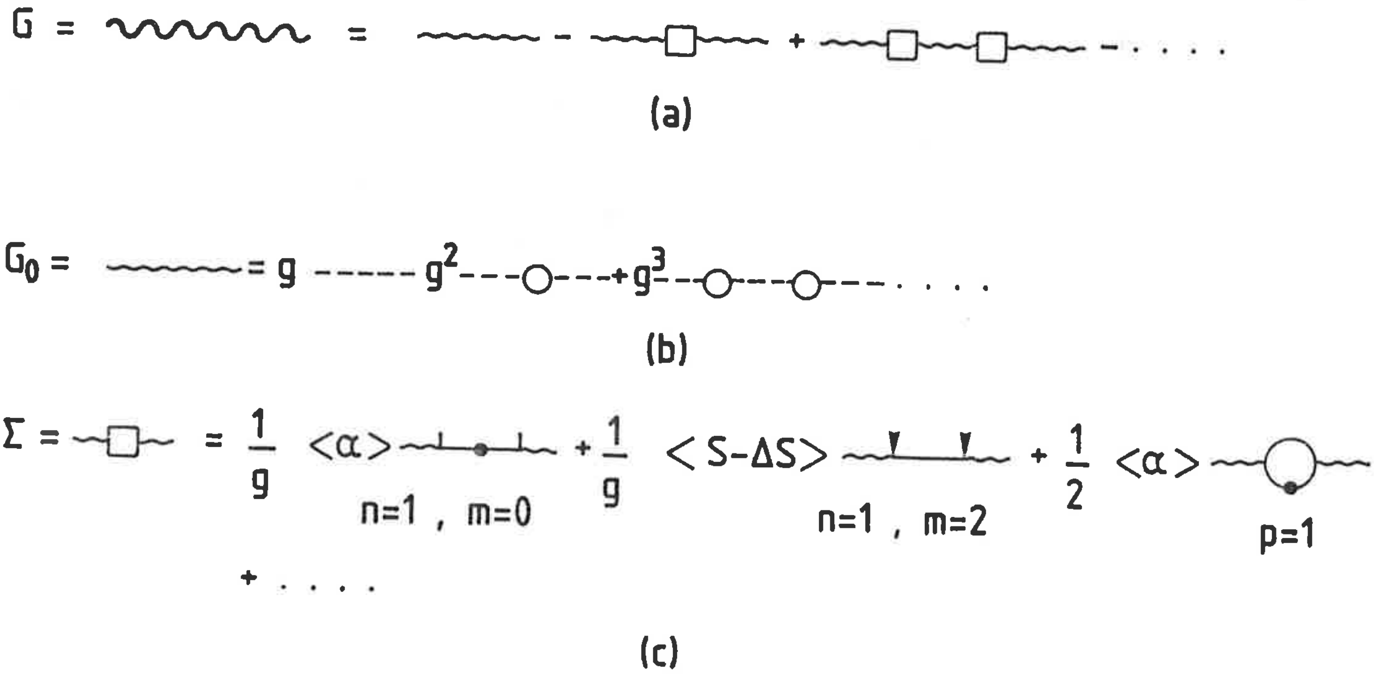

Fig. 3

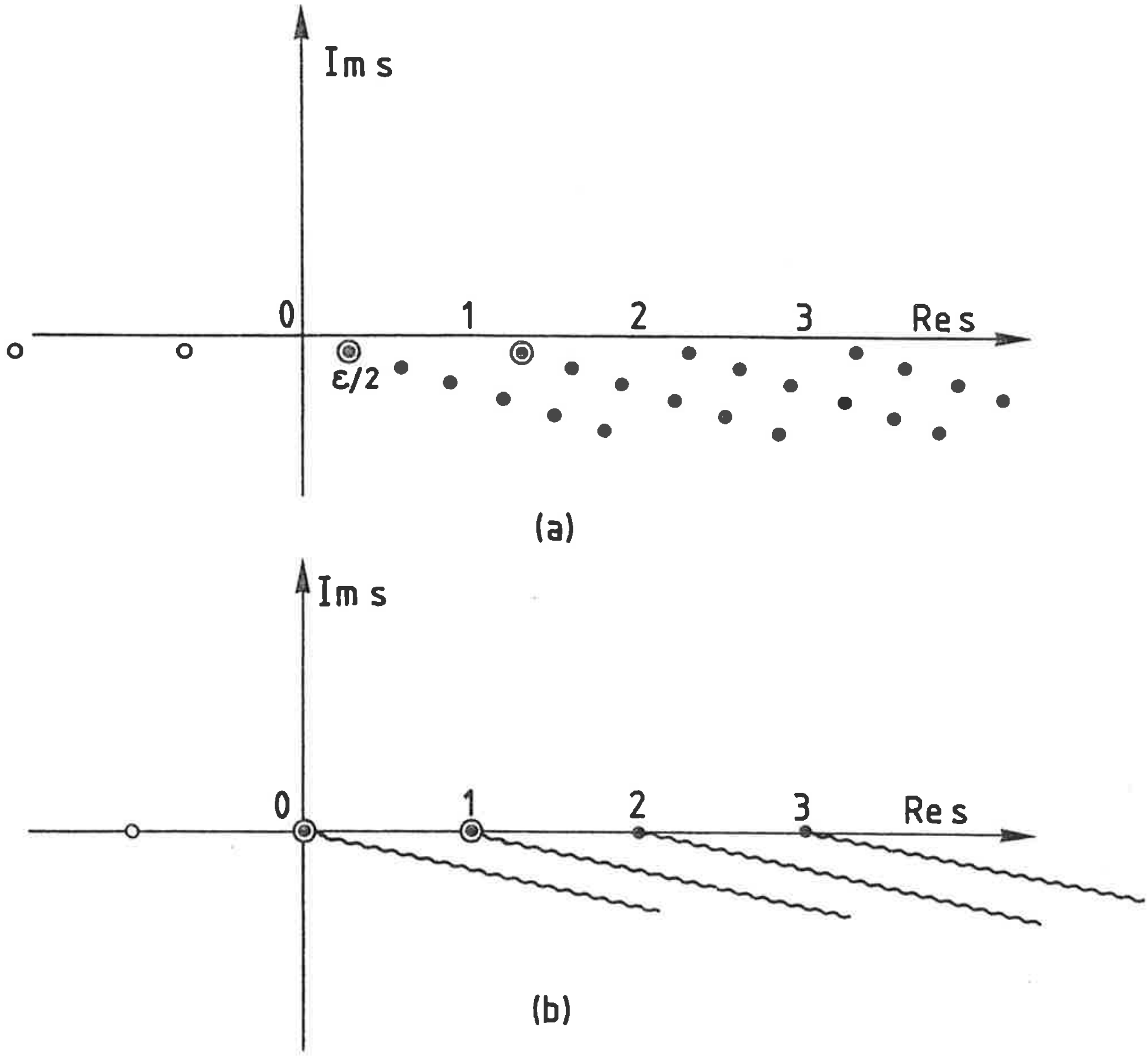

Fig. 4

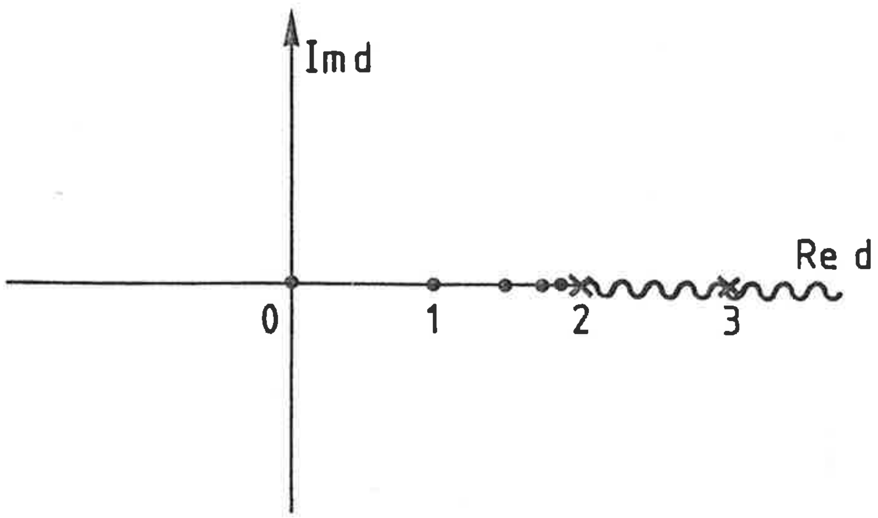

Fig. 5

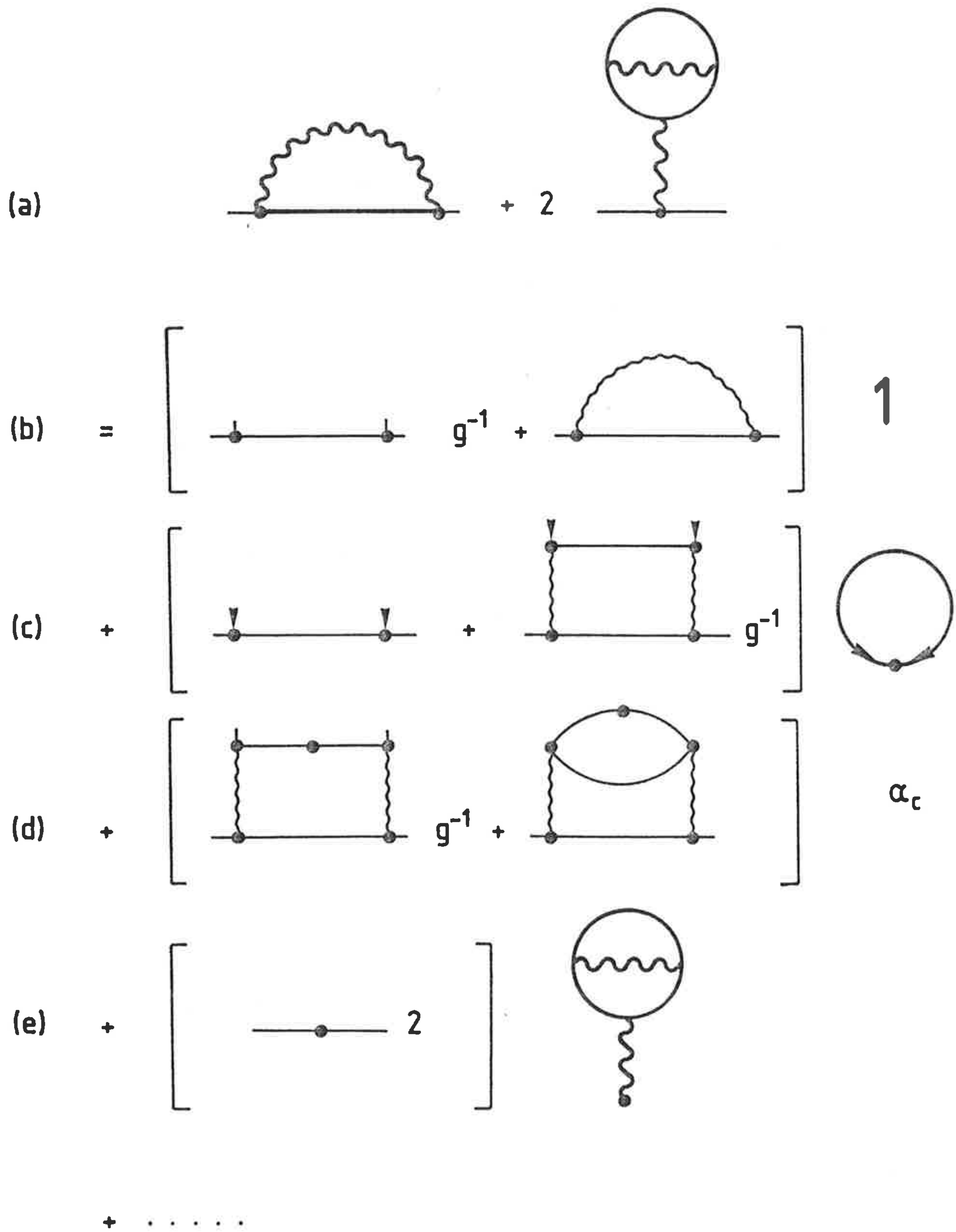

Fig. 6

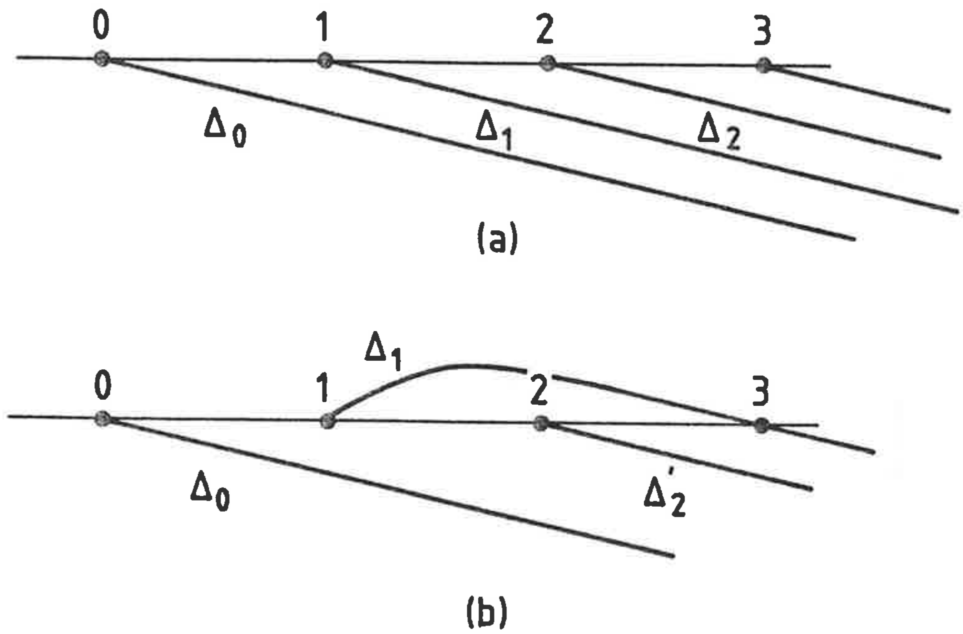

Fig. 7

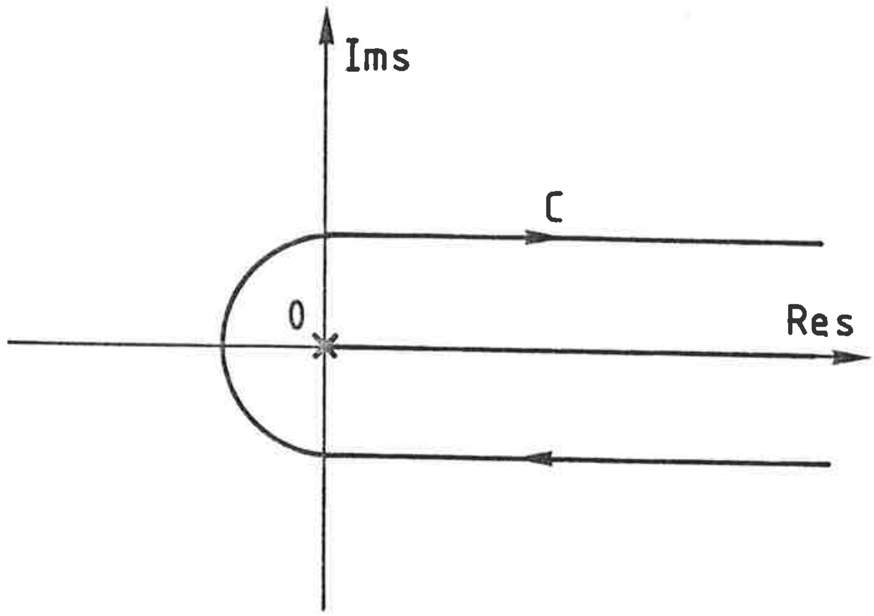

Fig. 8

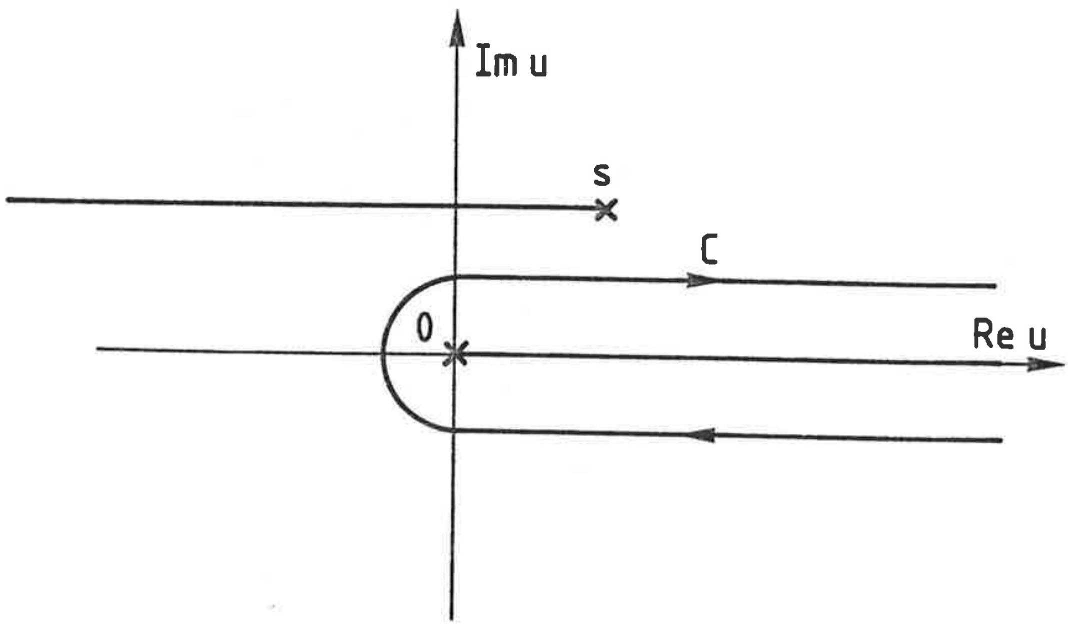

fig. 9

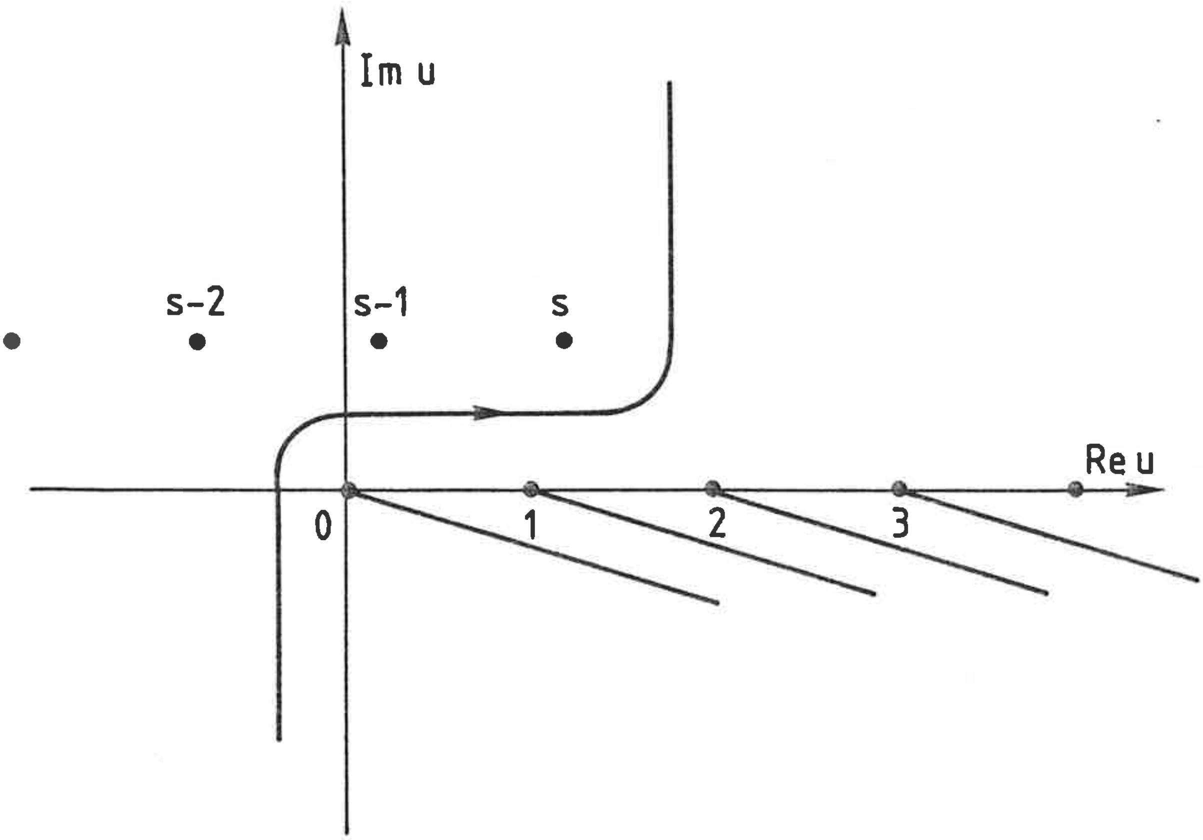

Fig. 10

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