Distance statistics in planar graphs

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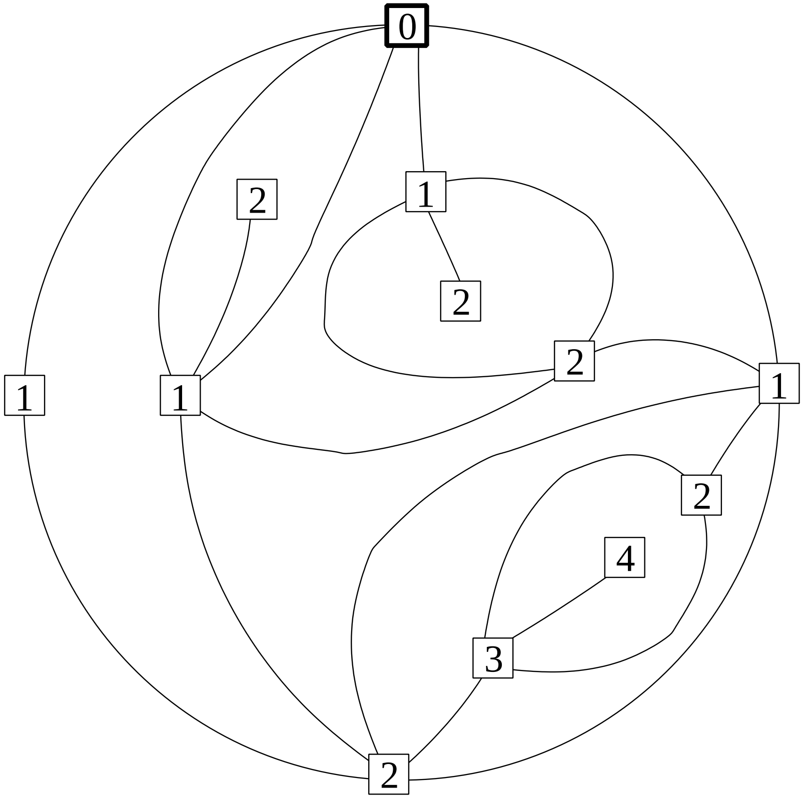
From graphs to labeled trees

Start with a planar quadrangulation with an origin



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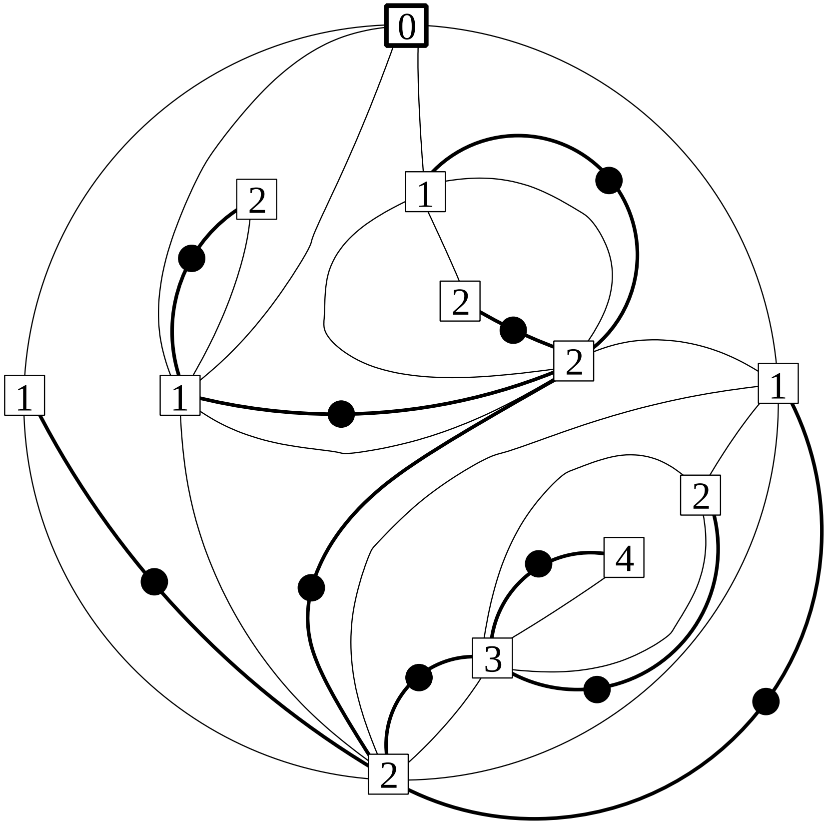

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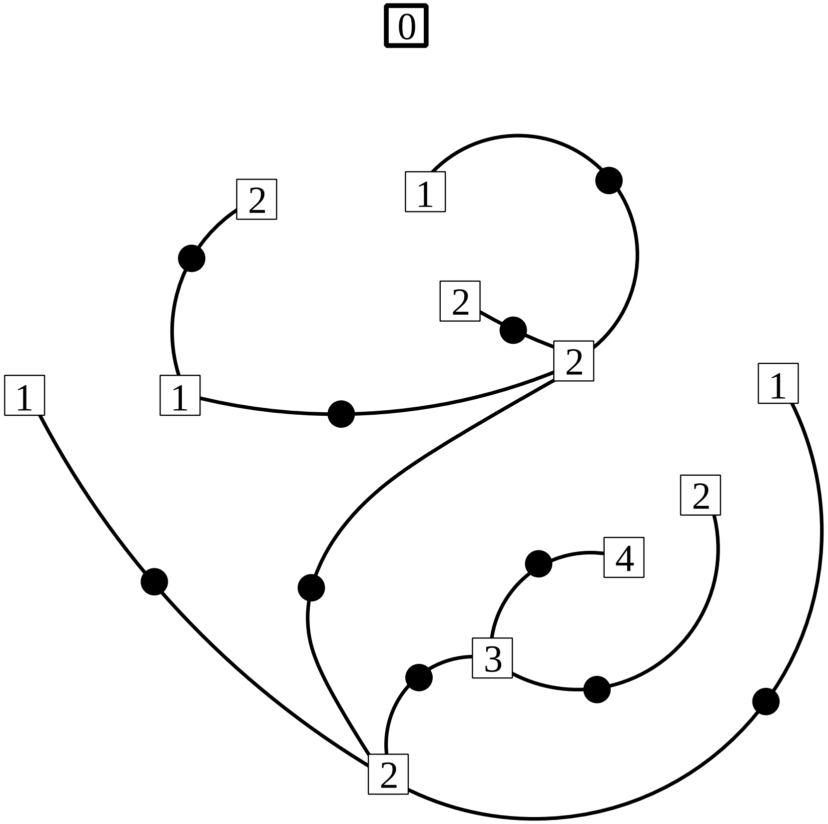

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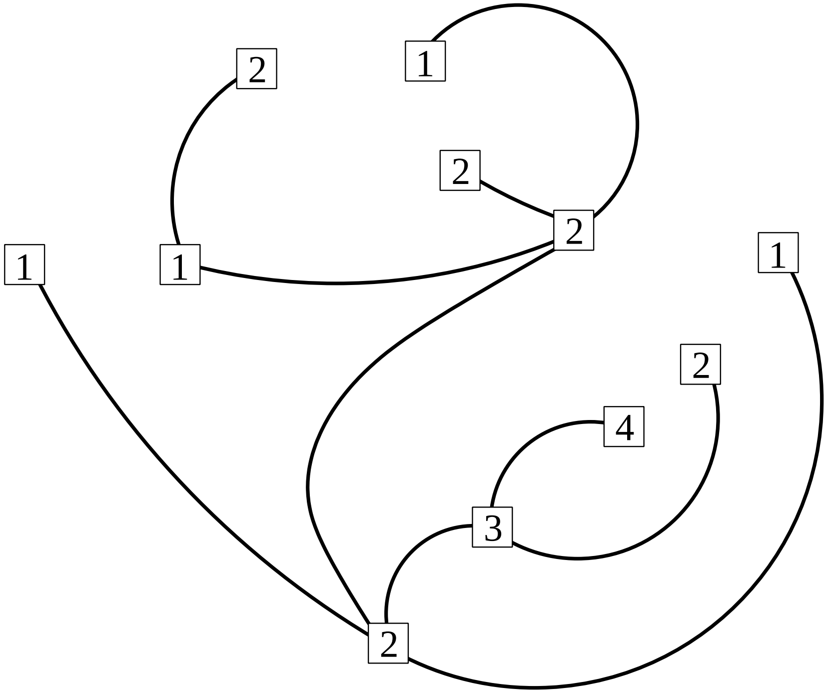
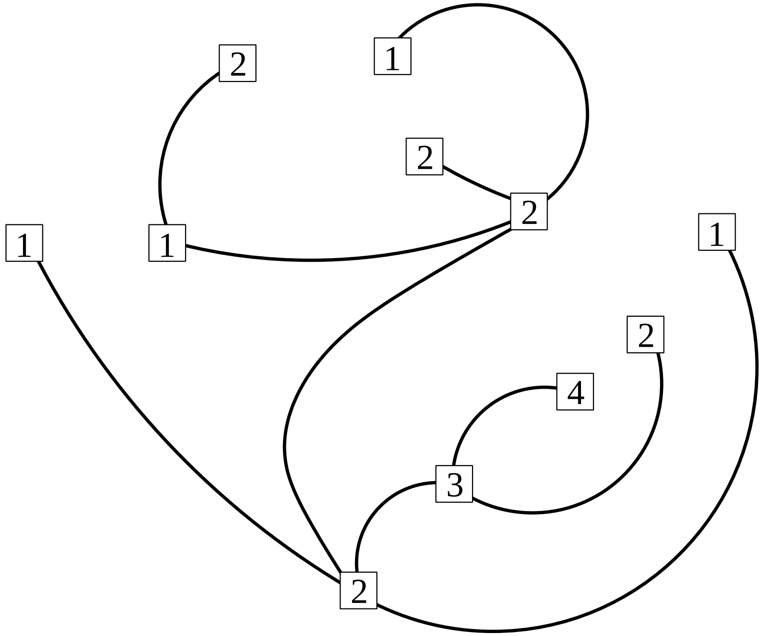


End up with a planar well-labeled tree

Well-labeled trees

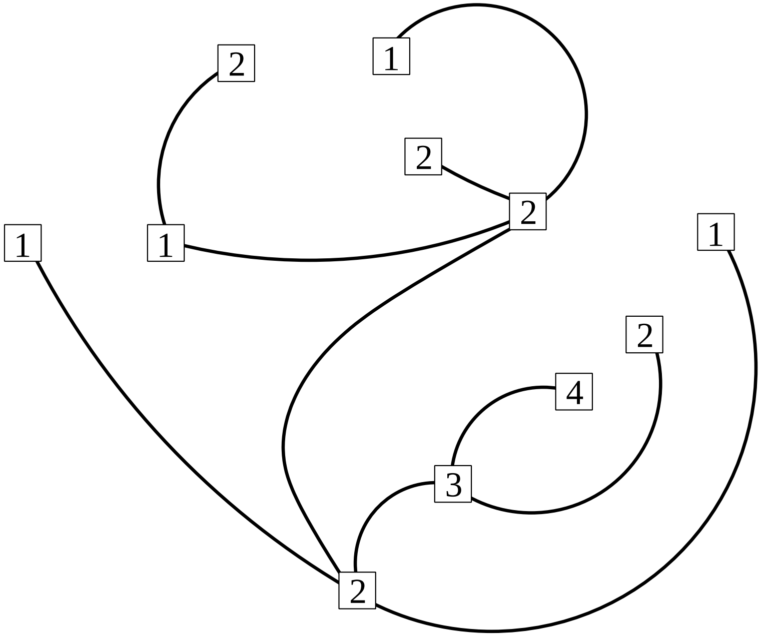
Well-labeled:



Well-labeled trees

Well-labeled:

(i) positive integer labels;



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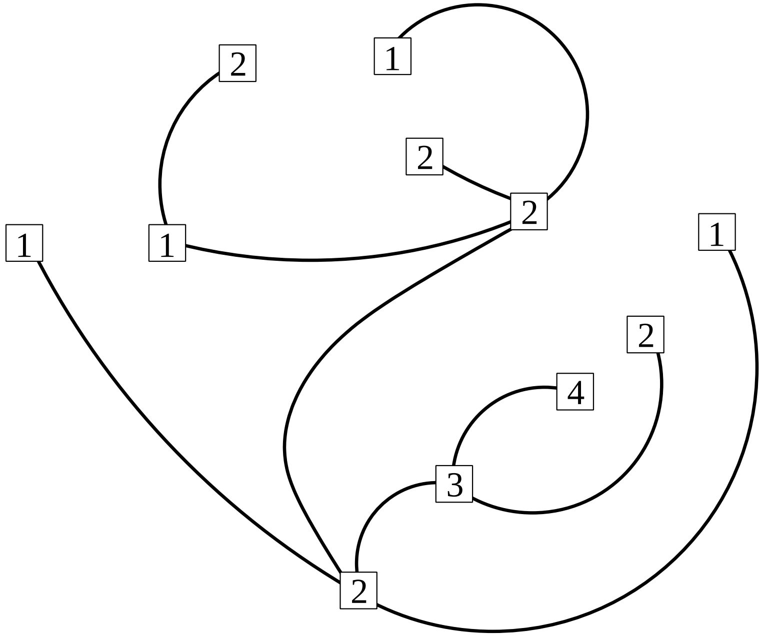

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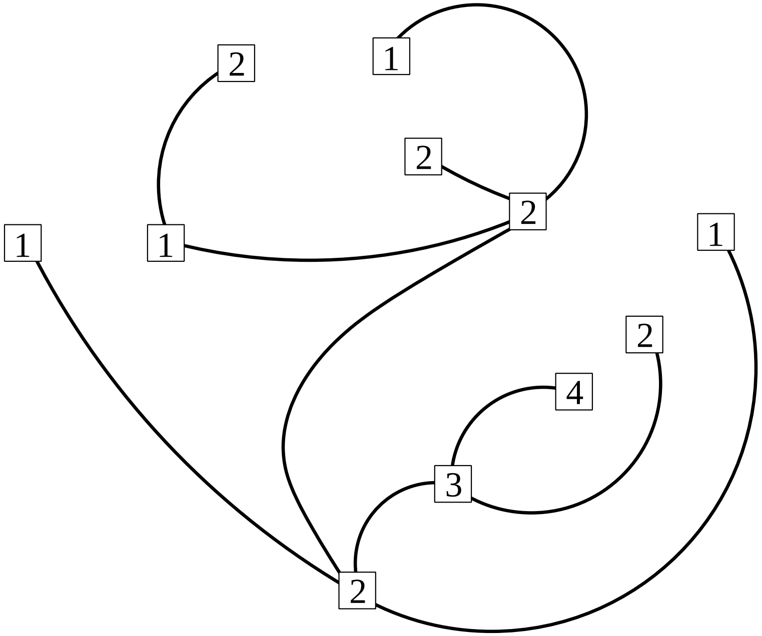

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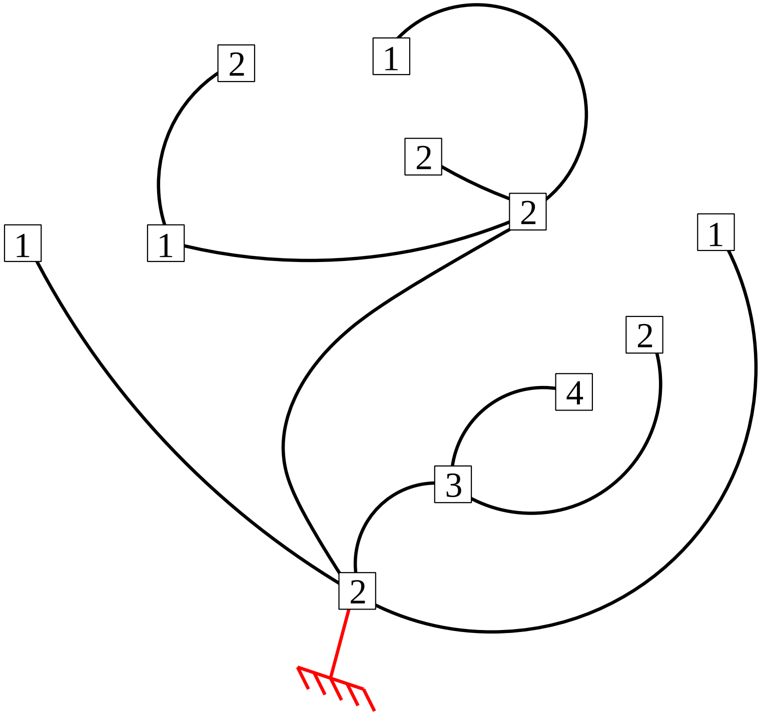

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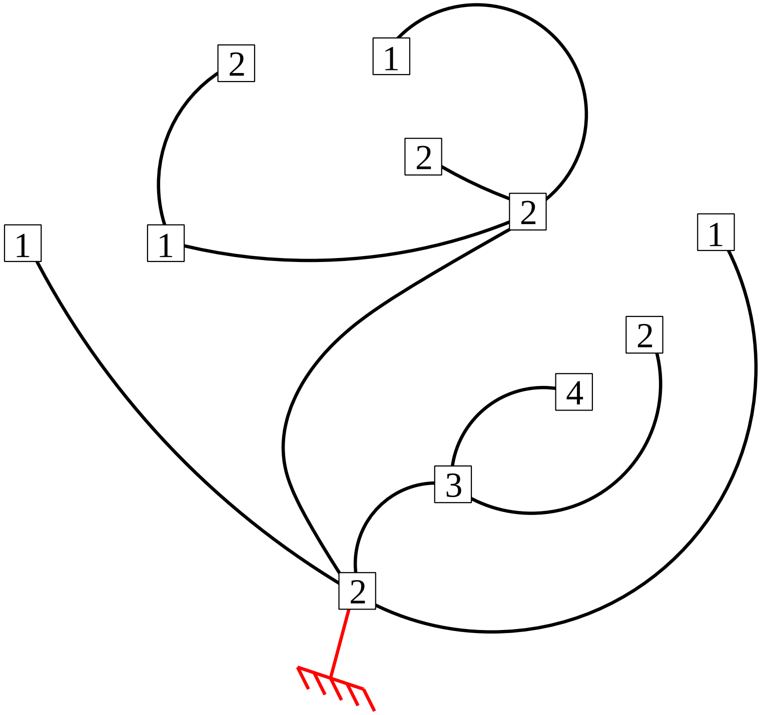
Generating function for rooted tree:

- $G_n(g)$ with a weight g per edge and a root labeled n

Well-labeled trees



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between neighbors;

(iii) ~~there is at least one label 1~~ ;

Generating function for rooted tree:

- $G_n(g)$ with a weight g per edge and a root labeled n
- $R_n(g)$ $\qquad G_n(g) = R_n(g) - R_{n-1}(g), \quad R_0 \equiv 0$

Graph-tree correspondence

planar quadrangulation
with an origin vertex

well-labeled tree

Graph-tree correspondence

planar quadrangulation with an origin vertex	well-labeled tree
vertices at geodesic distance n from the origin	vertices labeled n

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edges $(n-1) \leftrightarrow n$

half-edges incident to
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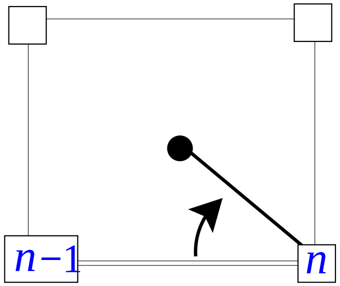
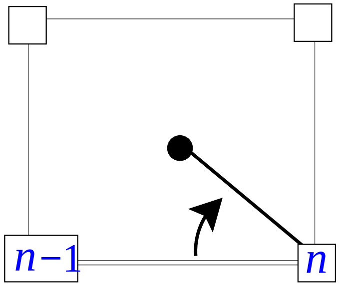


Graph-tree correspondence

planar quadrangulation with an origin vertex	well-labeled tree
vertices at geodesic distance n from the origin	vertices labeled n
edges $(n-1) \leftrightarrow n$	half-edges incident to vertices labeled n
marked edge $(n-1) \leftrightarrow n$	rooting at a vertex labeled n



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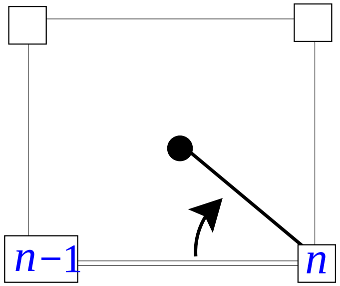

R_n is the g.f. for quadrangulations with an origin and a marked edge $(m-1) \leftrightarrow m$ with $m \leq n$, and weight g per face

Recursion relations

$$R_n = \frac{1}{1 - g(R_{n+1} + R_n + R_{n-1})}$$

with $R_0 = 0$ and $R_n \overset{n\to\infty}{\longrightarrow} R$
Here R is the “combinatorial" solution of $R = 1/(1 - 3\,g\,R)$, namely

$$R = \frac{1 - \sqrt{1 - 12\,g}}{6g} = \sum_{N \geq 0} \frac{3^N}{(N+1)} \binom{2N}{N} g^N$$

R is the g.f. of quadrangulations with an origin and a marked edge

Solution

$$R_n = R\,\frac{u_n u_{n+3}}{u_{n+1} u_{n+2}}, \qquad u_n \equiv 1 - x^n, \qquad x + \frac{1}{x} = \frac{1 - 4g\,R}{g\,R}$$

Solution

$$R_n = R\,\frac{u_n u_{n+3}}{u_{n+1} u_{n+2}}, \qquad u_n \equiv 1 - x^n, \qquad x + \frac{1}{x} = \frac{1 - 4g\,R}{g\,R}$$

Integral of motion: $F(R_{n+1}, R_n) = F(R_n, R_{n-1})$

$$F(X, Y) \equiv X\,Y\,(1 - g(X + Y)) - X - Y$$

Solution

$$R_n = R\,\frac{u_n u_{n+3}}{u_{n+1}u_{n+2}}, \qquad u_n \equiv 1 - x^n, \qquad x + \frac{1}{x} = \frac{1-4g\,R}{g\,R}$$

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In particular:

$$F(R_1, R_0 = 0) = F(R,R) \quad \to \quad R_1 = R - g\,R^3$$

$$R_1|_{g^N} = \frac{2}{(N+2)} R|_{g^N} = 2\frac{3^N}{(N+1)(N+2)}\binom{2N}{N}$$

number of rooted *(marked oriented edge)* quadrangulations with $(N+2)$ vertices

Average properties

The average number $\langle e_n \rangle$ of edges at geodesic distance n *(i.e. $n-1 \leftrightarrow n$)* in infinite quadrangulations is given by

$$\frac{\langle e_n \rangle}{\langle e_1 \rangle} = \lim_{N \to \infty} \frac{(R_n - R_{n-1})|_{g^N}}{R_1|_{g^N}}$$

with $\langle e_1 \rangle = 4$ from Euler's relation,

$$\langle e_n \rangle = \frac{6}{35} \frac{(n^2+2n-1)(5n^4+20n^3+27n^2+14n+4)}{n(n+1)(n+2)}$$

$$\overset{n \to \infty}{\sim} \frac{6}{7} n^3$$

→ fractal dimension $d_F = 4$

The average number $\langle v_n \rangle$ of vertices at geodesic distance n in infinite quadrangulations is given by

$$\langle v_n \rangle = \frac{3}{35}\Big((n+1)(5n^2+10n+2)+\delta_{n,1}\Big)$$

$$\underset{\sim}{n\to\infty}\ \frac{3}{7}n^3$$

first values:

$$\langle e_1 \rangle = 4 \qquad \langle e_2 \rangle = 19 \qquad \langle e_3 \rangle = \frac{1234}{25}$$

$$\langle v_1 \rangle = 3 \qquad \langle v_2 \rangle = \frac{54}{5} \qquad \langle v_3 \rangle = \frac{132}{5}$$

Neighbor statistics

Beyond averages, what are the probabilities?

$$P_N(e_1, e_2, \cdots, e_k; v_1, v_2, \cdots, v_l)$$

of having exactly e_i edges at distance i and v_j vertices at distance j from the origin.

Introduce:
- weight α_i per edge $(i-1) \leftrightarrow i$
- weight ρ_j per vertex j

On the well-labeled tree:
- weight α_i per half-edge incident to i
- weight ρ_j per vertex labeled j

$$R_n(g; \{\alpha_i\}, \{\rho_j\})$$

The g.f. for the probabilities are

$$\Gamma_N\ (\{\alpha_i\},\{\rho_j\}) \equiv \sum_{\{e_i\},\{v_j\}} \prod_{i=1}^{k} {\alpha_i}^{e_i} \prod_{j=1}^{l} {\rho_j}^{v_j} P_N\ (\{e_i\},\{v_j\})$$

$$= \frac{\int_0^{\alpha_1} \frac{d\alpha}{\alpha} R_1(g;\{\alpha,\alpha_2,\cdots\alpha_k\},\{\rho_j\})|_{g^N}}{\int_0^1 \frac{d\alpha}{\alpha} R_1(g;\{\alpha,1,\cdots 1\},\{1\})|_{g^N}}$$

$\frac{d\alpha}{\alpha}$ to get rid of the marking of a $0 \leftrightarrow 1$ edge (rooting of the tree)

for quadrangulations of size N

The g.f. for the probabilities are

$$\Gamma_{\infty}\left(\{\alpha_i\},\{\rho_j\}\right) \equiv \sum_{\{e_i\},\{v_j\}} \prod_{i=1}^{k} {\alpha_i}^{e_i} \prod_{j=1}^{l} {\rho_j}^{v_j} P_{\infty}(\{e_i\},\{v_j\})$$

$$= \frac{\int_0^{\alpha_1} \frac{d\alpha}{\alpha} R_1(g;\{\alpha,\alpha_2,\cdots\alpha_k\},\{\rho_j\})|_{sing}}{\int_0^{1} \frac{d\alpha}{\alpha} R_1(g;\{\alpha,1,\cdots 1\},\{1\})|_{sing}}$$

$\frac{d\alpha}{\alpha}$ to get rid of the marking of a $0 \leftrightarrow 1$ edge (rooting of the tree)

for infinite quadrangulations

$$R_n = \frac{\rho_n}{1 - g\alpha_n(\alpha_{n+1}R_{n+1} + \alpha_n R_n + \alpha_{n-1}R_{n-1})}$$

or, upon changing from R_n to $\tilde{R}_n = \alpha_n R_n$:

$$\tilde{R}_n = \frac{\alpha_n \rho_n}{1 - g\alpha_n(\tilde{R}_{n+1} + \tilde{R}_n + \tilde{R}_{n-1})} \qquad (\star)$$

Neighbors at a finite distance: $\alpha_n = \rho_n = 1$ for $n > L$
- use the above equations $(\star)$ for $n \leq L$ only
- complete by the integral of motion

$$F(\tilde{R}_L, \tilde{R}_{L+1}) = F(R, R)$$

$L + 1$ equations → algebraic equation for $\tilde{R}_1$ (or for R_1)

Nearest neighbors

Immediate neighbors ($L = 1$): $\alpha_1 \to \alpha$, $\rho_1 \to \rho$

$$(R_1 - \rho)(R_1(\alpha - 1) + \rho - 1) + 2g^2\,\alpha\,R^3\,R_1$$
$$= g\alpha\,R_1(R_1^2\alpha(\alpha - 1) + \alpha\,\rho R_1 + R(R-2))$$

$\to$ cubic equation for Γ_∞:

$$\alpha\Big(2\Gamma_\infty(1 + 4\Gamma_\infty + \Gamma_\infty^2) + 3\rho(1 + \Gamma_\infty)^2(2 + \Gamma_\infty)\Big)$$
$$= 6\Gamma_\infty(1 + \Gamma_\infty)(3 + \Gamma_\infty)$$

with $\Gamma_\infty = 1$ for $\alpha = \rho = 1$.

$$\Gamma_\infty(1;\rho) = \frac{2}{\sqrt{4 - 3\rho}} - 1 = \sum_{v \geq 1} \rho^v \left(\frac{3}{16}\right)^v \binom{2v}{v}$$

$$P_\infty(v) = \left(\frac{3}{16}\right)^v \binom{2v}{v}, \quad \langle v_1 \rangle = 3, \quad \langle v_1^2 \rangle = \frac{33}{2}, \quad \langle v_1^3 \rangle = \frac{579}{4}$$

Incident edges:

$$\Gamma_\infty(\alpha; 1) = \frac{1}{2}\left(\sqrt{\frac{3(2+\alpha)}{6-5\alpha}} - 1\right) = \frac{1}{3}\alpha + \frac{1}{6}\alpha^2 + \frac{13}{108}\alpha^3 + \cdots$$

$$P_\infty(e=1) = \frac{1}{3}, \quad P_\infty(e=2) = \frac{1}{6}, \quad P_\infty(e=3) = \frac{13}{108}, \cdots$$

$$\langle e_1 \rangle = 4, \quad \langle e_1^2 \rangle = \frac{100}{3}, \quad \langle e_1^3 \rangle = \frac{1372}{3}$$

In general, there are multiple nearest neighbors: $e \geq v$

We may impose $e = v$ by considering

$$\Pi(t) = \lim_{\alpha \to 0} \Gamma_\infty(\alpha, \frac{t}{\alpha}) = \frac{1}{2}\left(\sqrt{\frac{18-t}{2-t}} - 3\right)$$

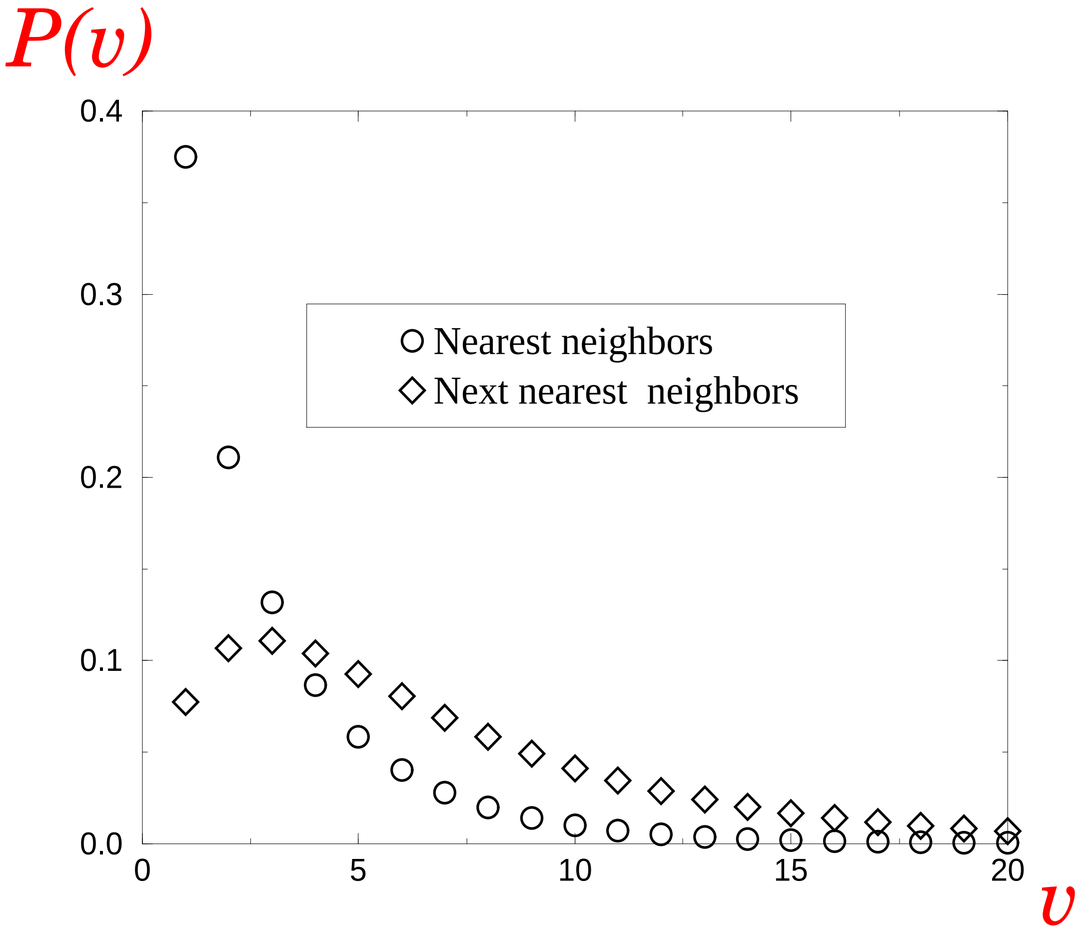
g.f. for the probability of having v neighbors, all simple.

The probability for having no multiple neighbors is

$$\Pi(1) = \frac{\sqrt{17}-3}{2}$$

Next-nearest neighbors

Probabilities for next-nearest neighbor vertices



Scaling limit (1)

In the g.f. language

$$R_n = R\frac{u_n u_{n+3}}{u_{n+1}u_{n+2}}, \; R = \frac{1}{1-3gR}, \; u_n = 1 - x^n, \; x + \frac{1}{x} = \frac{1-4gR}{gR}$$

scaling limit: $g = \frac{1}{12}(1-\epsilon^4), \qquad n = \frac{r}{\epsilon}$

$$gR = \frac{1}{6}(1-\epsilon^2), \qquad x = e^{-\sqrt{6}\epsilon} + \mathcal{O}(\epsilon^2)$$

$\longrightarrow$ continuum generating function $\mathcal{F}(r)$ for graphs with two marked point at distance larger to r

$$\mathcal{F}(r) = \lim_{\epsilon \to 0} \frac{R - R_n}{\epsilon^2 R} = \frac{3}{\sinh^2(\sqrt{3/2}\, r)}$$

Scaling limit (2)

In the probability language (fixed size N)

$$R_n|_{g^N} = \oint \frac{dg}{2i\pi g^{N+1}} R_n(g)$$

for N large and $n = \alpha N^{1/4}$

Upon changing variable from g to $V \equiv gR$, we get

$$R_n|_{g^N} = \oint \frac{dV(1-6V)}{2i\pi(V(1-3V))^{N+1}} R_n(g)$$

$\rightarrow$ saddle point $V = \frac{1}{6}\left(1 + i\frac{\xi}{\sqrt{N}}\right)$

$$\epsilon = \sqrt{-i\xi}/N^{1/4}$$

$$g = \frac{1}{12}\left(1+\frac{\xi^2}{N}\right), \quad gR_n = V\left(1+\frac{i\xi}{\sqrt{N}}\mathcal{F}(\alpha\sqrt{-i\xi})\right)$$

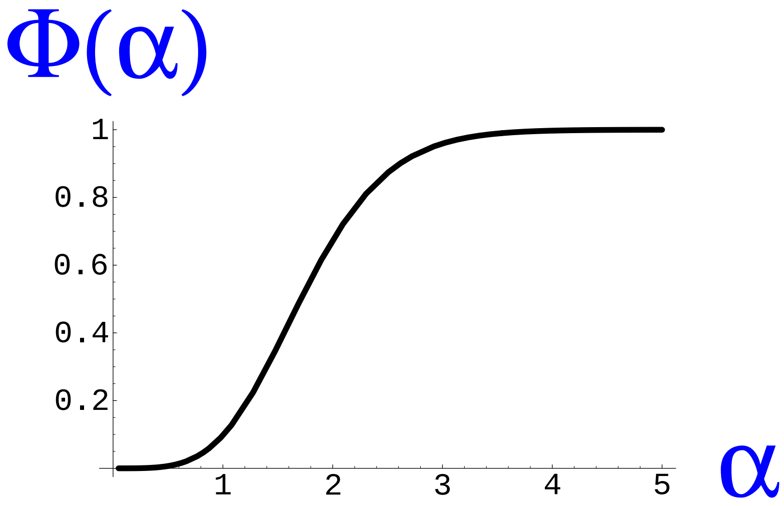
$$R_n|_{g^N} \sim 4\frac{12^N}{\pi N^{3/2}}\int_{-\infty}^{\infty} d\xi\xi^2 e^{-\xi^2}\left(1+\mathcal{F}\left(\alpha\sqrt{-i\xi}\right)\right)$$

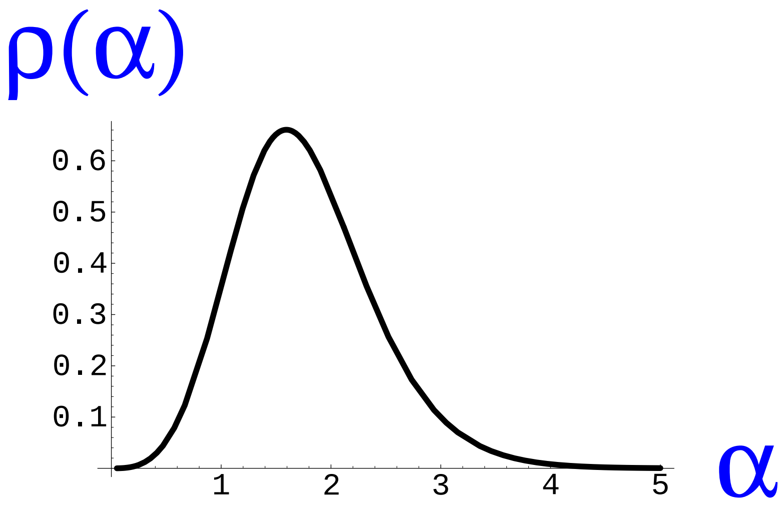
Probability $\Phi(\alpha)$ for a point (vertex or edge) to be at geodesic distance less than α:

$$\Phi(\alpha) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} d\xi\xi^2 e^{-\xi^2} \times$$

$$\frac{\cosh(2\alpha\sqrt{3\xi}) + \cos(2\alpha\sqrt{3\xi}) + 8\cosh(\alpha\sqrt{3\xi})\cos(\alpha\sqrt{3\xi}) - 10}{\left(\cosh(\alpha\sqrt{3\xi}) - \cos(\alpha\sqrt{3\xi})\right)^2}$$

$\rho(\alpha)$: point at distance equal to α





$$\rho(\alpha) \overset{\alpha\to 0}{\sim} \frac{3}{7}\alpha^3, \qquad \rho(\alpha) \overset{\alpha\to\infty}{\sim} \exp\left(-3(3/8)^{2/3}\alpha^{4/3}\right)$$

in agreement with $\langle v_n \rangle$ and with Fisher's law $\delta = \frac{4}{3} = \frac{1}{1-\nu}$ with $\nu = \frac{1}{4} = \frac{1}{d_F}$

Generalization

Graphs with faces of even valences $4, 6, \cdots 2(m+1)$,
weights g_k per face of valence $2k$ → well-labeled mobiles

g.f. for mobiles with a root-label n

$$R_n = R\frac{U_n(w_1,...,w_m)U_{n+3}(w_1,...,w_m)}{U_{n+1}(w_1,...,w_m)U_{n+2}(w_1,...,w_m)}$$

with $R = 1/(1 - \sum_{k=1}^{m} g_{k+1}\binom{2k+1}{k+1}R^k)$

$$U_n(w_1,...,w_m) \equiv \det\left[U_{n+2j-2}(w_i)\right]_{1\le i,j\le m}$$

in terms of Chebyshev polynomials

$w = \sqrt{x} + \frac{1}{\sqrt{x}}$ roots of $\sum_{k=0}^{m} g_{k+1}R^k \sum_{l=0}^{k} \binom{2k+1}{l} U_{2k-2l}(w) = 1$

Multicritical points

Fine tuning: $g_k = g^{k-1}(-1)^k \frac{\frac{1}{m+1}\binom{m+1}{k}}{\binom{2k-1}{k-1}} \left(\frac{6}{m}\right)^{k-1}$ with $g = g_2$

approaching the critical value $g_c = \frac{m}{6(m+1)}$

Scaling function: Wronskian determinant

$$\mathcal{F}(r) = -2\frac{d^2}{dr^2}\mathrm{Log}\,\mathcal{W}\left(\sinh\left(a_1\frac{r}{2}\right), \sinh\left(a_2\frac{r}{2}\right), \ldots, \sinh\left(a_m\frac{r}{2}\right)\right)$$

$$r \equiv \frac{n}{\epsilon}, \quad \epsilon = \left(\frac{g_c - g}{g_c}\right)^{\nu}, \quad \sum_{l=0}^{m}(-a^2)^l \frac{l!}{(2l+1)!}\frac{m!}{(m-l)!} = 0$$

with $\nu = 1/d_F$ and $d_F = 2m + 2$

Probability distribution for a point to be at *rescaled* geodesic distances less than α with $\alpha = n/N^{\frac{1}{2(m+1)}}$

$$\Phi(\alpha) = \frac{(m+1)^2}{\cos\left(\frac{\pi(m-1)}{2(m+1)}\right)\Gamma(\frac{1}{m+1})} \int_0^\infty d\xi \xi^{m+1} e^{-\xi^{m+1}}$$

$$\mathrm{Re}\left(e^{-i\frac{\pi(m-1)}{2(m+1)}}(1 + \mathcal{F}(\alpha e^{i\frac{\pi}{2(m+1)}}\sqrt{\xi}))\right)$$

distances equal to α: $\rho(\alpha) = \Phi'(\alpha)$

$$\rho(\alpha) \overset{\alpha \to 0}{\sim} \alpha^{2m+1}, \quad \rho(\alpha) \overset{\alpha \to \infty}{\sim} \exp(-C\alpha^{2(m+1)/(2m+1)})$$

Tree vs labels

In the simple critical case, the fractal dimension 4 for the graph is the product of:

- the dimension 2 for the tree
- the dimension 2 for the labels

\- mass N *(number of edges of the tree* $\equiv$ *number of faces of the graph)*
\- generation T along the tree
\- position n *(label* n *on the tree* $\equiv$ *distance on the graph)*

Tree vs labels

In the simple critical case, the fractal dimension 4 for the graph is the product of:

- the dimension $\textcircled{2}$ for the tree
- the dimension 2 for the labels

- mass N *(number of edges of the tree $\equiv$ number of faces of the graph)*
- generation T along the tree
- position n *(label n on the tree $\equiv$ distance on the graph)*

$$T \sim N^{1/2}$$

Tree vs labels

In the simple critical case, the fractal dimension 4 for the graph is the product of:

- the dimension 2 for the tree
- the dimension ② for the labels

- mass N *(number of edges of the tree* $\equiv$ *number of faces of the graph)*
- generation T along the tree
- position n *(label* n *on the tree* $\equiv$ *distance on the graph)*

$$T \sim N^{1/2}, \quad n \sim T^{1/2}$$

Tree vs labels

In the simple critical case, the fractal dimension $\bigcirc\!\!\!\!4$ for the graph is the product of:

- the dimension 2 for the tree
- the dimension 2 for the labels

- mass N *(number of edges of the tree $\equiv$ number of faces of the graph)*
- generation T along the tree
- position n *(label n on the tree $\equiv$ distance on the graph)*

$$T \sim N^{1/2}, \quad n \sim T^{1/2}, \quad n \sim N^{1/4}$$

multicritical case:

both the tree and the labels are multicritical

$$T \sim N^{\frac{m}{m+1}}, \quad n \sim T^{\frac{1}{2m}}, \quad n \sim N^{\frac{1}{2(m+1)}}$$

$$d_F = 2(m+1) = \frac{m+1}{m} \times 2\,m$$

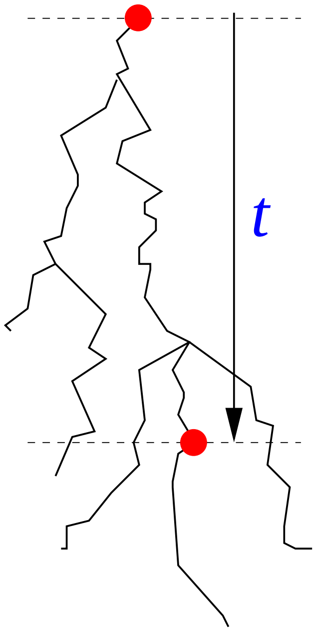
tree labels

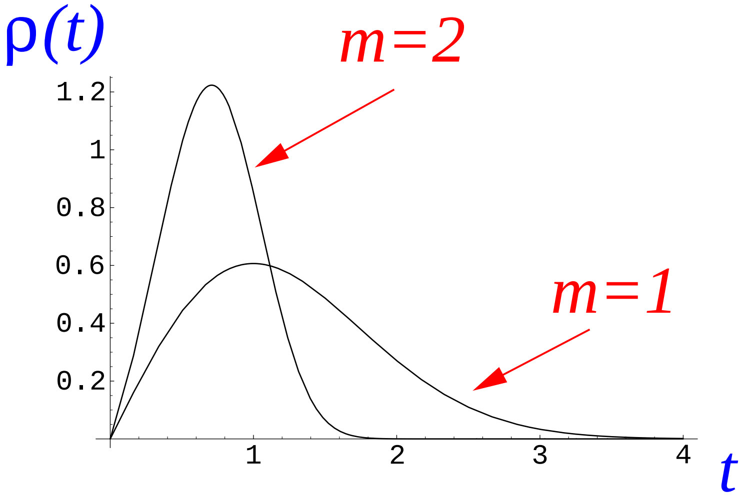
continuum limit → multicritical continuous random tree (CRT)

Generation $T = t/N^{\frac{m}{m+1}}$

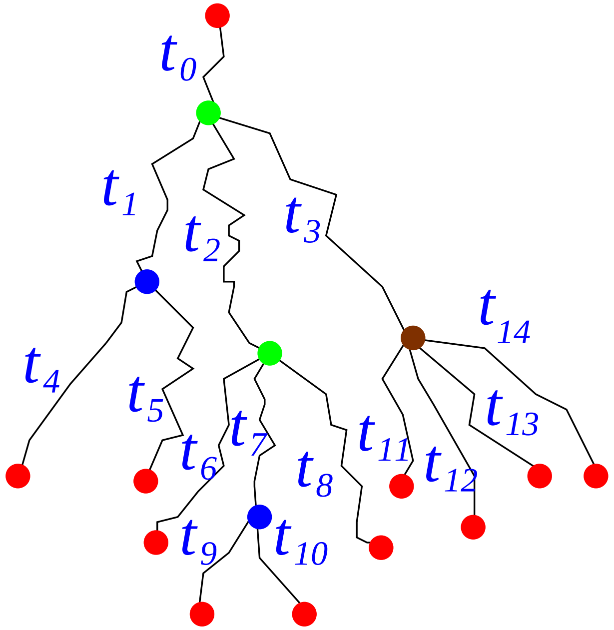
Density profile (density of points at generation t)

$$\rho(t) = A_m \mathrm{Im}\left[\int_0^\infty d\xi \xi^m e^{\frac{\xi^{m+1}}{m+1} + \omega \xi^m t}\right], \qquad \omega = e^{\frac{i\pi}{m+1}}$$





The multicritical CRT has vertices of valence $1, 2, \cdots$, up to $(m+2)$



with fine tuned couplings

with both signs and derivatives

$$\rho_{p_3,\cdots,p_{m+2}}(\{t_l\}) = \prod_{i=3}^{m+2} \left[\left(-\frac{d}{dt} \right)^{\frac{m+2-i}{m}} (-1)^{i-1} \frac{\binom{m+1}{i-1}}{m+1} \right]^{p_i} \rho(t)$$

with $t = \sum t_l$

Branching processes

A random graph is the “superposition" of

Branching processes

A random graph is the “superposition" of

- a random planar tree

Branching processes

A random graph is the “superposition" of

- a random planar tree
- integer labels on the tree

Branching processes

A random graph is the “superposition" of

- a random planar tree
- integer labels on the tree
- boundary condition (*positive labels*)

Branching processes

A random graph is the “superposition" of

- a random planar tree → genealogical tree
- integer labels on the tree
- boundary condition (*positive labels*)

A parent individual gives rise to k children with probability $p(k) = (1-p)p^k, \quad (average\ number\ of\ children\ \frac{p}{1-p})$

Branching processes

A random graph is the “superposition" of

- a random planar tree → genealogical tree
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The child of a parent at position n lives at position $n, n \pm 1$

Branching processes

A random graph is the “superposition" of

- a random planar tree → genealogical tree
- integer labels on the tree → diffusion process in 1D
- boundary condition (*positive labels*) → walls, forbidden zone

A parent individual gives rise to k children with probability

$$p(k) = (1-p)p^k, \quad (average\ number\ of\ children\ \tfrac{p}{1-p})$$

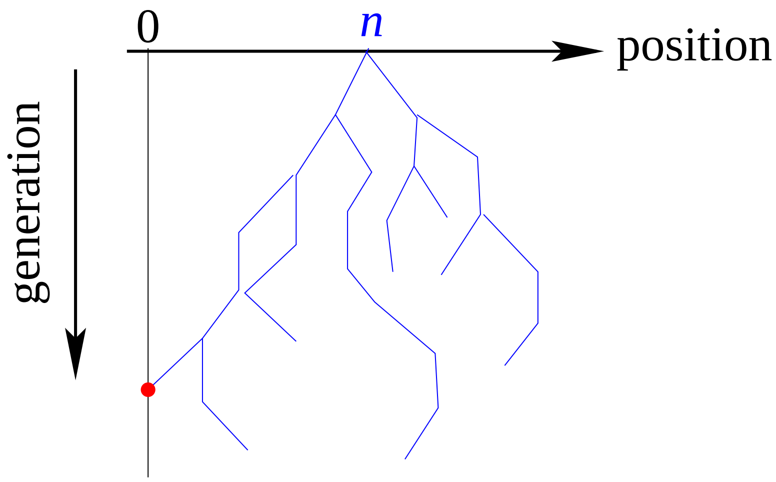
The child of a parent at position n lives at position $n, n \pm 1$

What is the probability $\mathcal{P}_n(p)$ for the population whose germ is at position n to reach position 0 ?

$\mathcal{P}_n(p) = 1 - (1-p)R_n(g)$ with $g = \frac{p(1-p)}{3}$

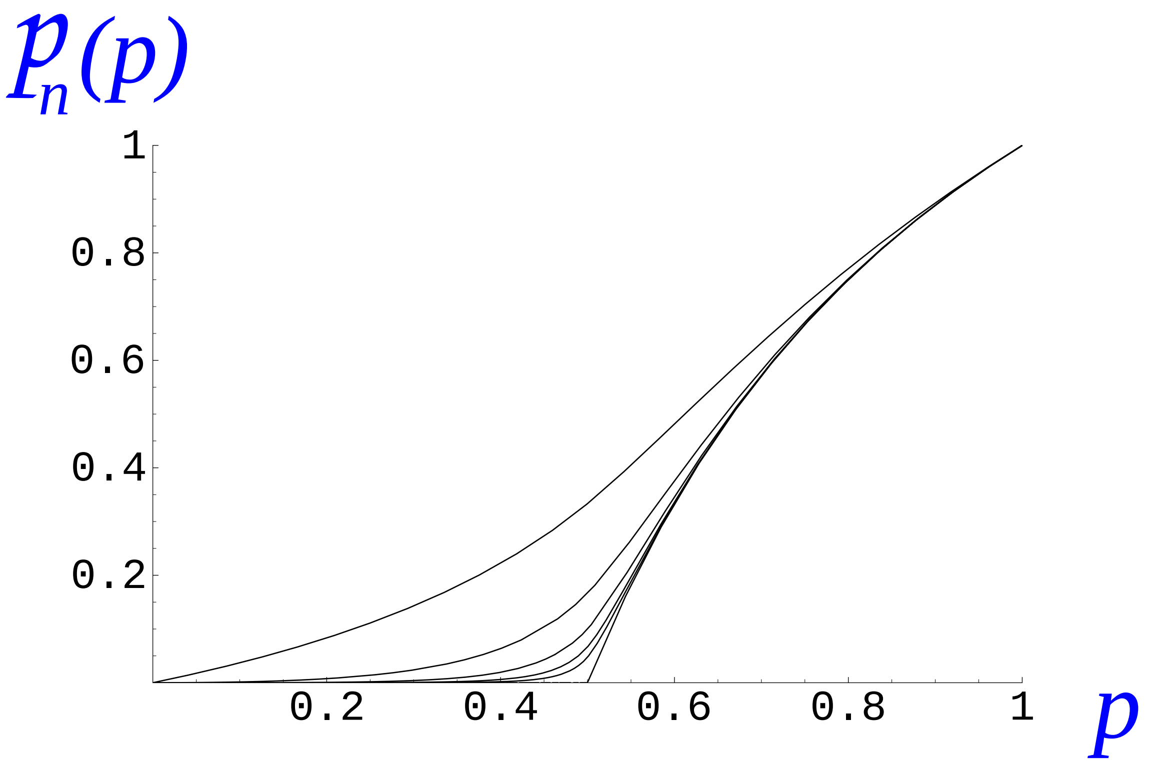
$$\mathcal{P}_n(p) = 1 - \frac{1-|2p-1|}{2p}\frac{(1-x^n)(1-x^{n+3})}{(1-x^{n+1})(1-x^{n+2})}$$

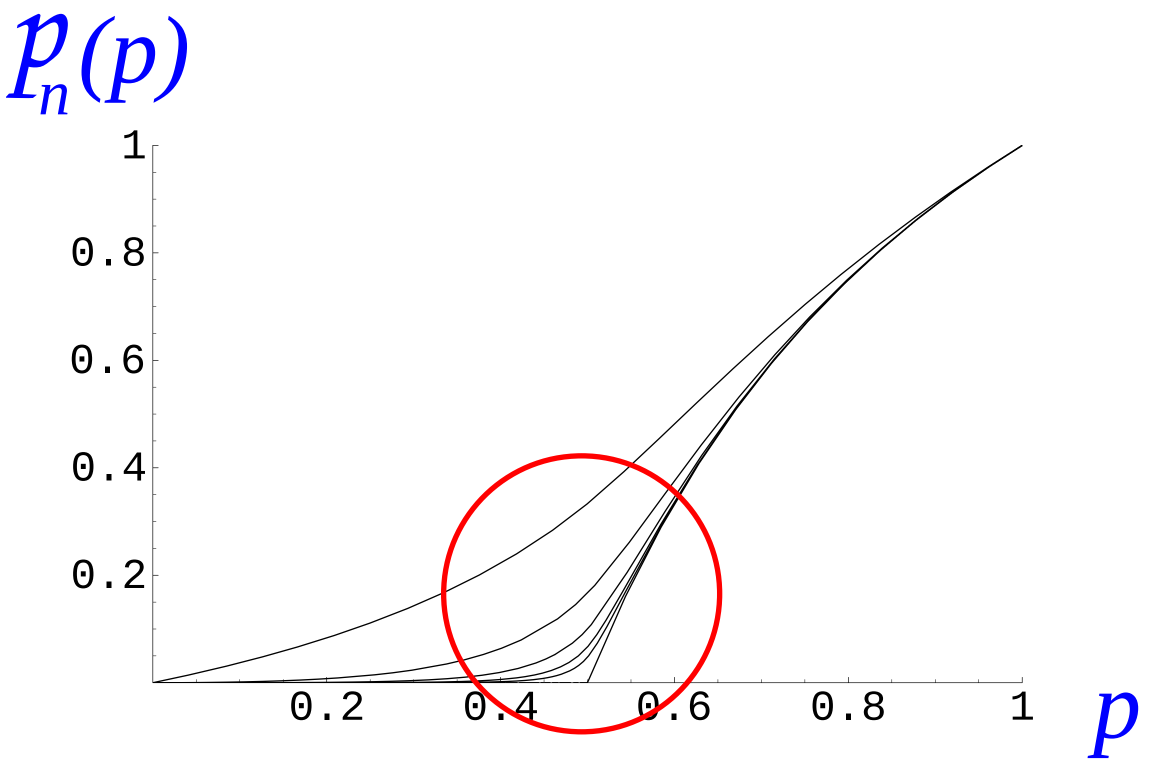
with $x = \frac{1+2|1-2p|-\sqrt{3|1-2p|}\sqrt{2+|1-2p|}}{1-|1-2p|}$



$\mathcal{P}_n(p) \overset{n\to\infty}{\sim} S(p)$: survival probability

$$S(p) = 1 - \frac{1-|2p-1|}{2p} = \begin{cases} 0 & p \le \frac{1}{2} \\ \frac{2p-1}{p} & p \ge \frac{1}{2} \end{cases}$$


$p_n(p)$
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
p



scaling behavior around $p = \frac{1}{2}$:

$$\mathcal{P}_n(p) \sim |2p-1| \left(\frac{3}{\sinh^2(\sqrt{3/2}n|2p-1|^{1/2})} + 1 \right) + (2p-1)$$

Summary

- Quadrangulations as well labeled trees
- Statistics of distances
- Probabilities for immediate neighbors
- Scaling limit
- Generalization to multicritical points
- (Multicritical) continuous random trees
- Application to branching processes

More to come:

Ising model, hard objects, ...