Entanglement and Shannon entropies in low-dimensional quantum systems

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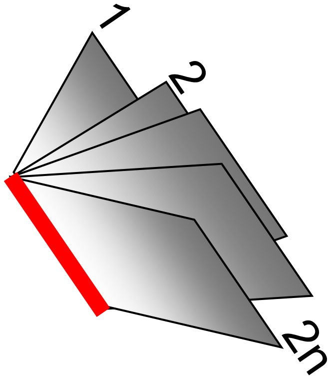

Soutenance d'habilitation à diriger des recherches
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Vendredi 20 juin 2014

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LPTMS, Orsay

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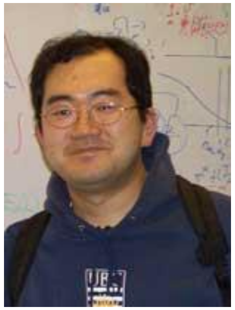

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Yuta Kumano

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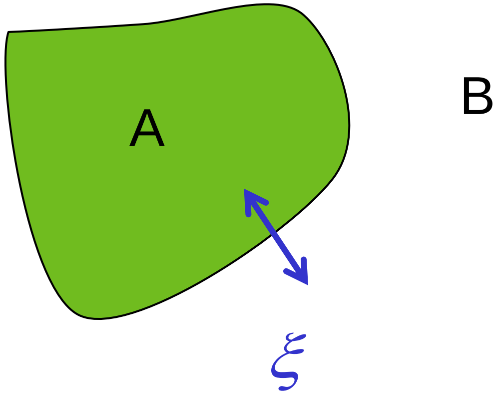
Introduction: entanglement and boundary law

- Why studying entanglement (in condensed matter) ?
  - A theoretical probe for quantum many-body wave-functions
  - The scaling of the entanglement entropy can reveals some long-distance properties of the system.
  - Understanding entanglement is useful to construct new algorithms to store efficiently quantum states in a computer, or construct some variational anstaz.



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❑ Definitions

Wave function

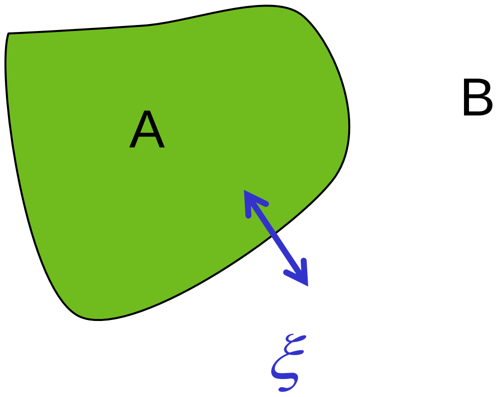
$$|\psi\rangle \in H_A \otimes H_B$$

Reduced density matrix

$$\rho_A = Tr_B |\psi\rangle\langle\psi|$$

Von Neumann entropy

$$S_A = -Tr_A\left[\rho_A \log \rho_A\right]$$



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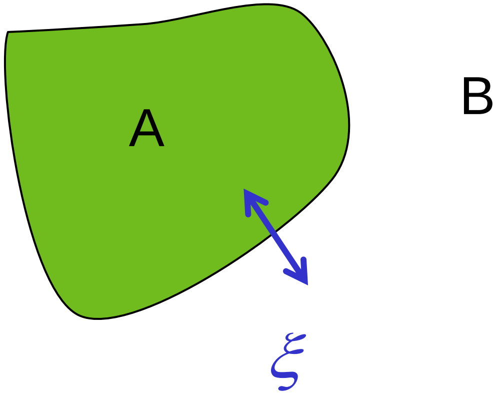

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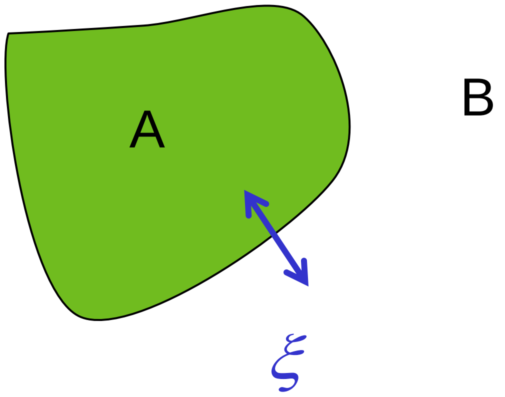

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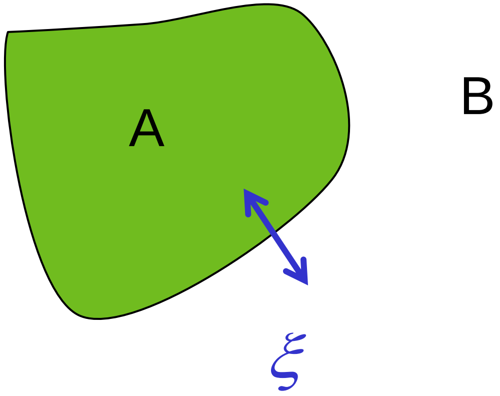

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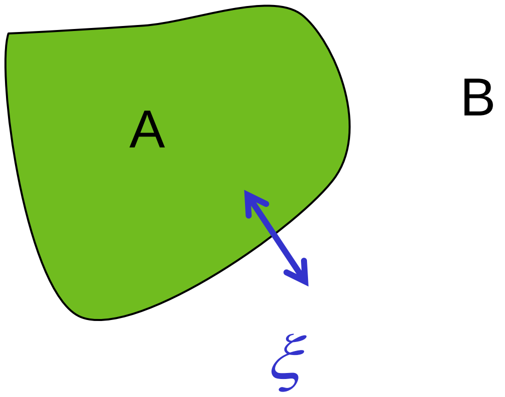

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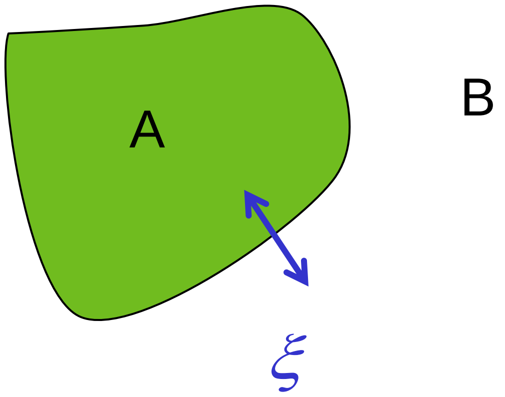
Area/Boundary law

Low-energy states of local Hamiltonians are "weakly" entangled (compared to random or high energy states)



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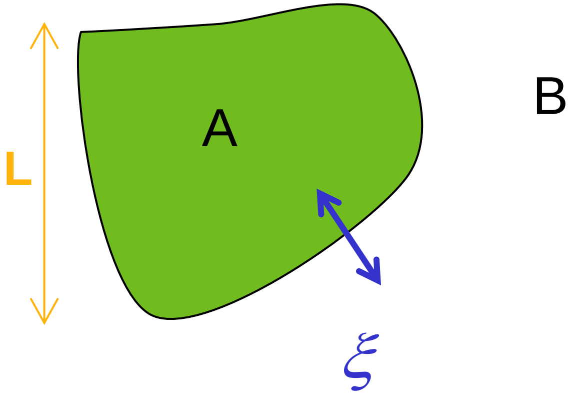
$$\boxed{S_A \propto L^{d-1}}$$

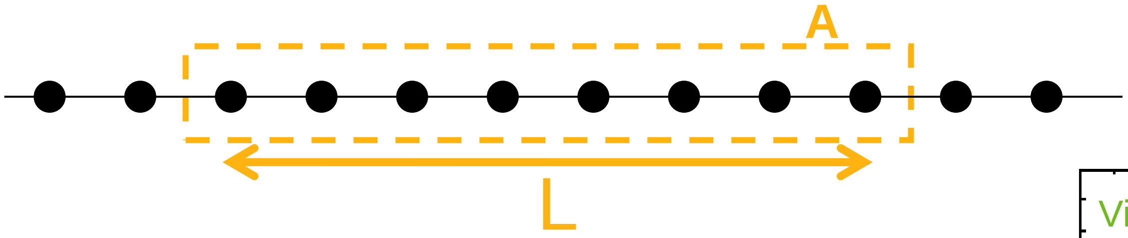


Eisert Cramer Plenio RMP 2010

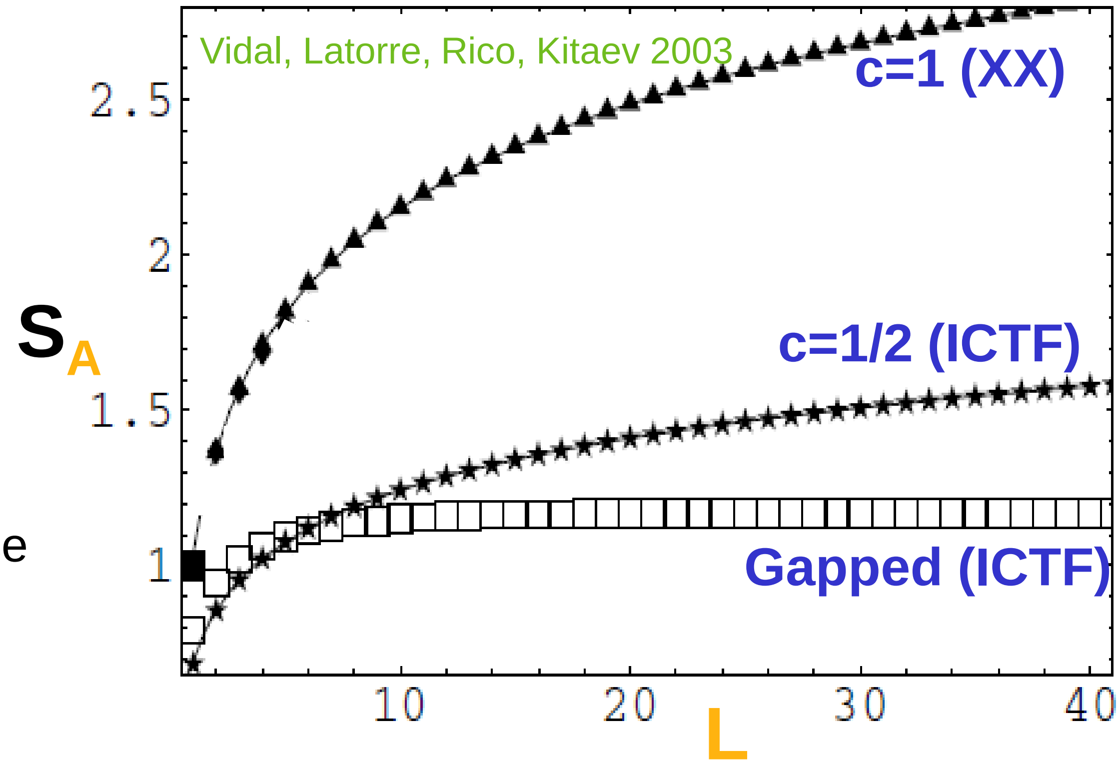
Introduction: violations of the boundary law

Holzhey,Larsen,Wilczek 1994
Calabrese, Cardy 2004

- ❑ Critical 1d systems



$$S_A(L) \approx \frac{c}{3}\log(L) + O(1)$$

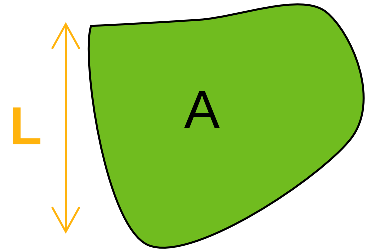

Vidal, Latorre, Rico, Kitaev 2003

- ❑ Systems with a Fermi surface in dimension d:

$$S_A \approx L^{d-1}\log(L)$$

Wolf 2006; Gioev & Klich 2006

Introduction: subleading corrections to the boundary law



Boundary law

$$S_A(L) \approx \alpha L + \begin{cases} \gamma \\ \beta \log(L) \end{cases} \qquad (d=2)$$

- Discrete symmetry breaking → $\gamma = \log(\text{deg})$
- Goldstone modes → $\beta \sim \frac{1}{2}\#\text{modes}$ Metlitsky & Grover 2011
- “Topologically ordered” states: $\gamma = -\log(D_{tot})$ Levin Wen 2006; Kitaev Preskill 2006
  - Gapped, no broken symmetry, no local order parameter, no “conventional order”
  - But… ground state degeneracy which depends on the topology
    The degenerate ground states cannot be distinguished by any local observable
  - *Fractionalized* excitations
  - Examples: **fractional quantum Hall** fluids, some “RVB” **spin liquids**, …
  - Example: FQHE Laughlin ν=1/3→ γ=-log(3)
- Critical states

Casini & Huerta 2007;
Metlitski, Fuertes, Sachdev 2009, Grover 2014
+ this work

How to compute the entanglement entropy ?

- 1+1d CFT

  Entanglement entropy of an interval → correlator of twist fields   Calabrese, Cardy 2004

- Free fermions / free bosons

  Reduced density is completely determined by two-point function:

$$\langle c_i^+ c_j \rangle_{ij\in A} \xrightarrow[\text{Wick}]{} \rho_A \sim \exp(\sum_{ij} h_{ij} c_i^+ c_j)$$

Peschel 2003

- Numerics

  DMRG in 1D or 2D

  Tensor network states in 2D (PEPS, …)

  Exact diagonalizations (lattice spin systems, FQHE, …)

  Quantum Monte Carlo (if no sign problem + integer Rényi index >1)

  Hastings et al. 2010
  Humeniuk & Roscilde 2012

- Rokhsar-Kivelson states

  This work

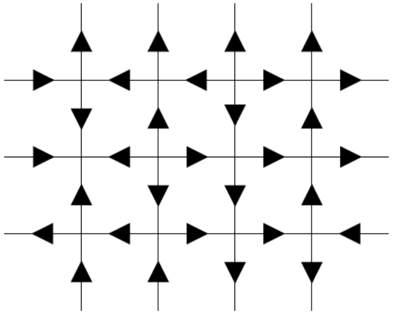
Rokhsar-Kivelson states

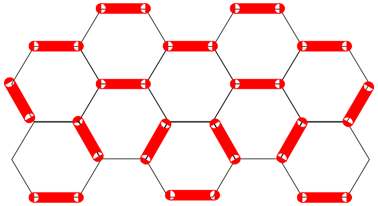
Rokhsar& Kivelson 1988; Henley 2004

- ❑ Start from a 2d classical (stat. mech) system with short-ranged interactions.

$$Z=\sum_c e^{-E(c)}$$

- ❑ Examples :
  - ❑ Ising model
  - ❑ (6-,8-) vertex models
  - ❑ hard-core dimer model
  - ❑ …





❑ Promote the classical configurations to orthogonal basis states, and the partition function above to a *wave-function :*

$$\left|\Psi\right\rangle=\frac{1}{\sqrt{Z}}\sum_c e^{-\frac{1}{2}E(c)}\left|c\right\rangle$$

-one can import some knowledge about 2D stat. mech to build tractable quantum models (observable which are diagonal in the “classical” basis have the same expectation values in the RK state and in the classical models)
- It is the ground-state of some local Hamiltonian
- some non-RK wave functions can be approximated by RK states
- Entanglement properties are simple !
(and make contact other classical entropies)

Rokhsar-Kivelson states: Schmidt decomposition

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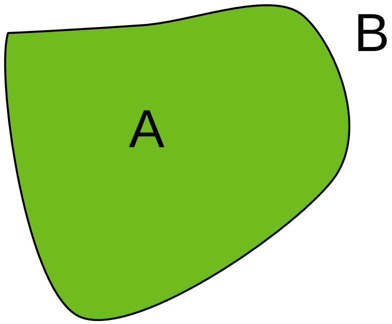

Furukawa & GM 2007; Stéphan *et al.* 2009.

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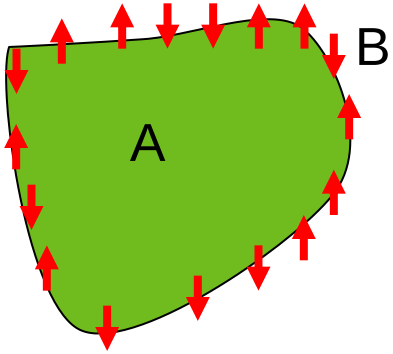
Boundary configuration **i**



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$$= \sum_i \sqrt{p_i}\underbrace{\left(\frac{1}{\sqrt{Z_i^A}}\sum_a e^{-\frac{1}{2}E^A(a,i)}\left|a\right\rangle\right)}_{\left|RK_i^A\right\rangle}\underbrace{\left(\frac{1}{\sqrt{Z_i^B}}\sum_b e^{-\frac{1}{2}E^B(b,i)}\left|b\right\rangle\right)}_{\left|RK_i^B\right\rangle}$$

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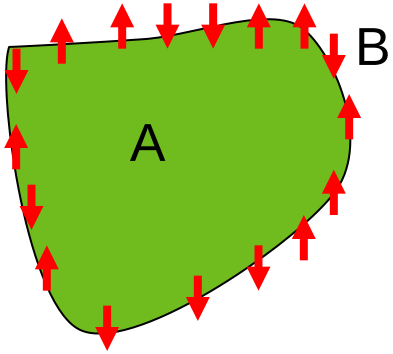

$$\boxed{p_i = \frac{Z_i^A Z_i^B}{Z}}$$

classical probability of **i**

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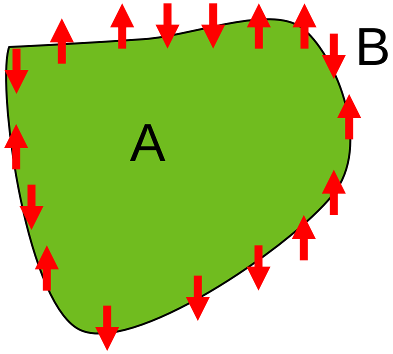

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=exact Schmidt decomposition
of a (constrained) RK state

$$\langle RK_i^A | RK_j^A\rangle = \delta_{ij}$$

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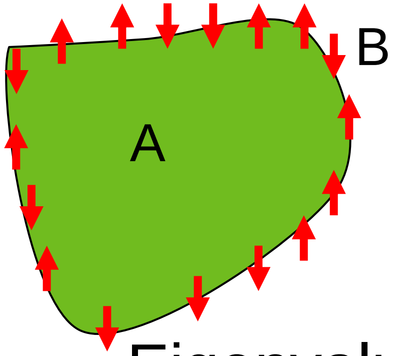

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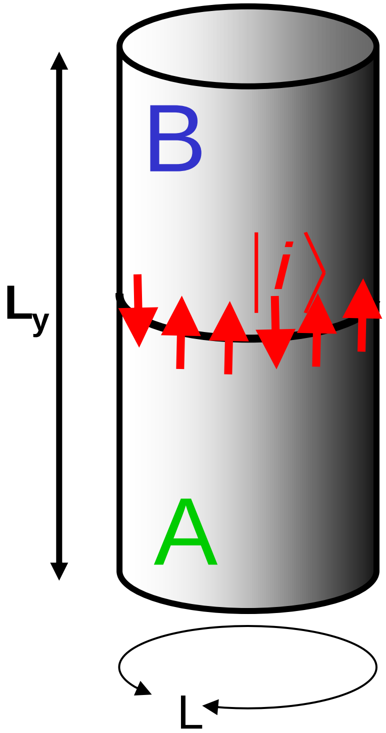
=exact Schmidt decomposition of a (constrained) RK state $\left\langle RK_i^A \middle| RK_j^A \right\rangle = \delta_{ij}$

Eigenvalues of ρ_A = classical boundary probabilities

The entanglement Hamiltonian is classical & acts on boundary degrees of freedom $S_A = -\sum_i p_i \log(p_i)$ → boundary law

Furukawa & GM 2007; Stéphan *et al.* 2009.

Rokhsar-Kivelson states: long cylinders & transfer matrix



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dominant eigenvector of the transfer matrix

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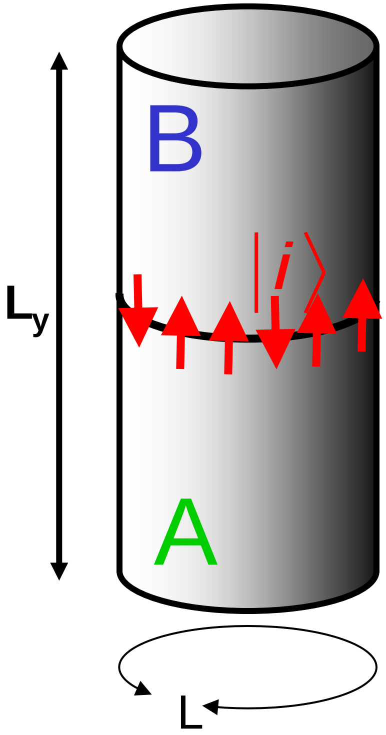

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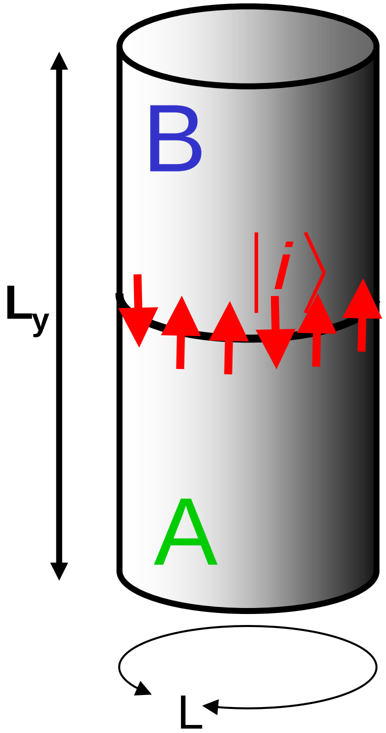

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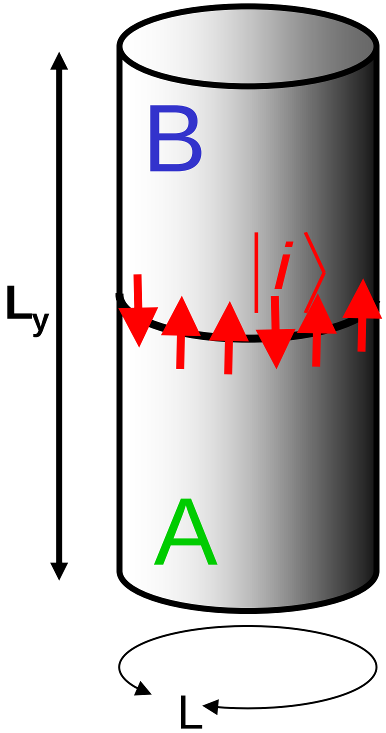

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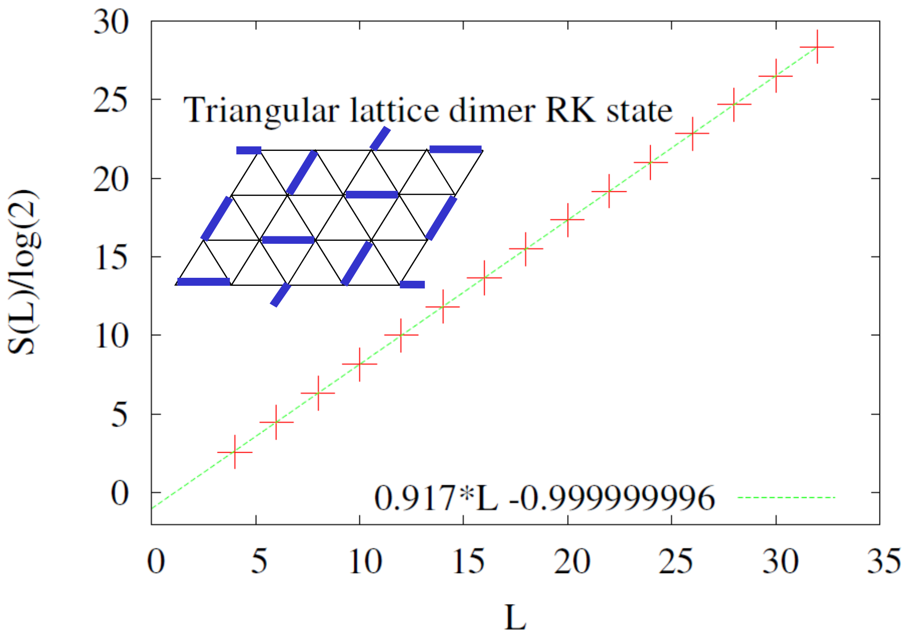

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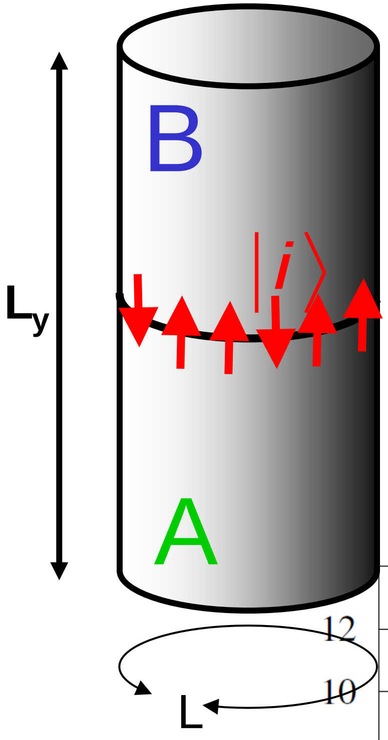

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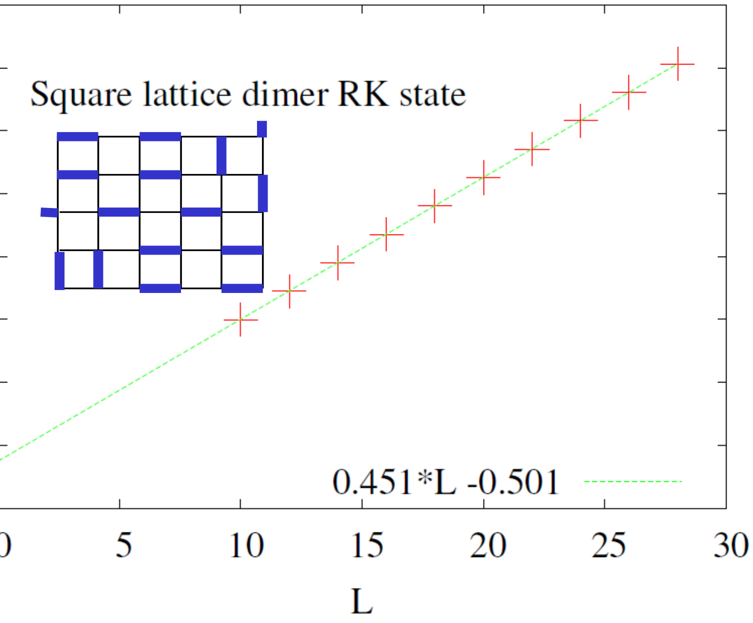
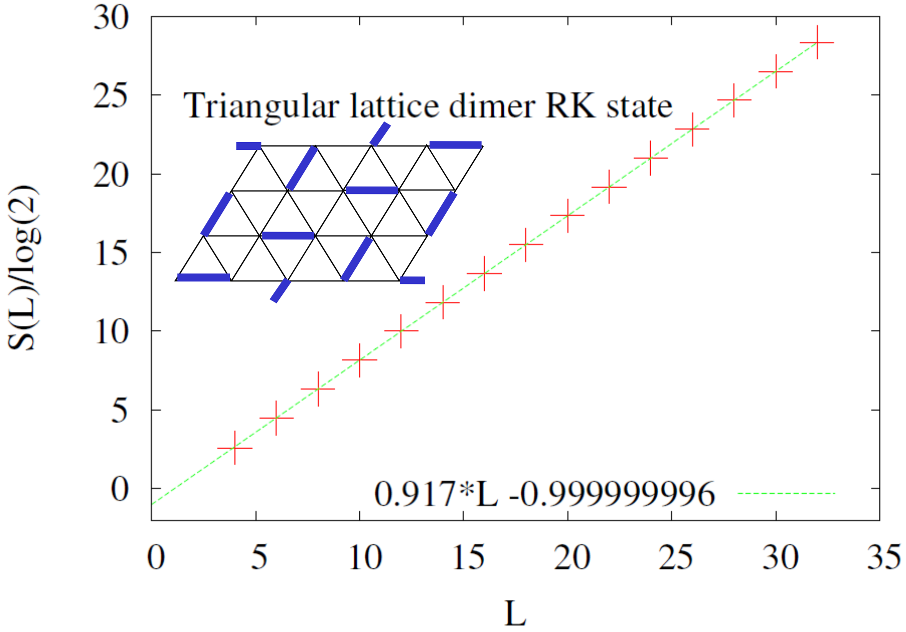
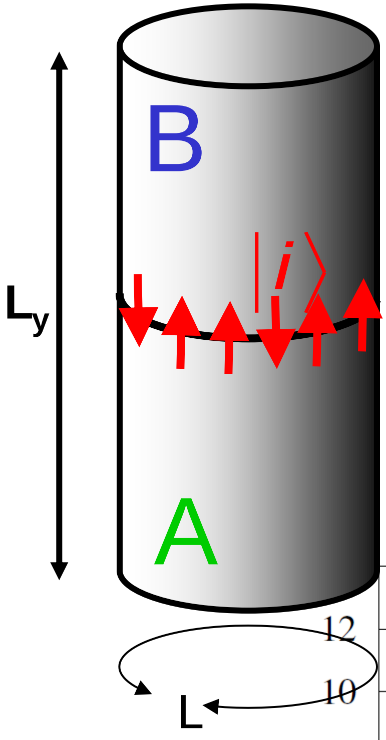
$$S(L) = -\sum_i p_i \log(p_i)$$





Rokhsar-Kivelson states: long cylinders & transfer matrix



$$p_i = \frac{\langle -\infty | T^{L_y/2} | i \rangle \langle i | T^{L_y/2} | +\infty \rangle}{\langle -\infty | T^{L_y} | +\infty \rangle}$$

$$T^{L_y/2} \approx \lambda^{L_y/2} | g \rangle \langle g |$$

dominant eigenvector of the transfer matrix

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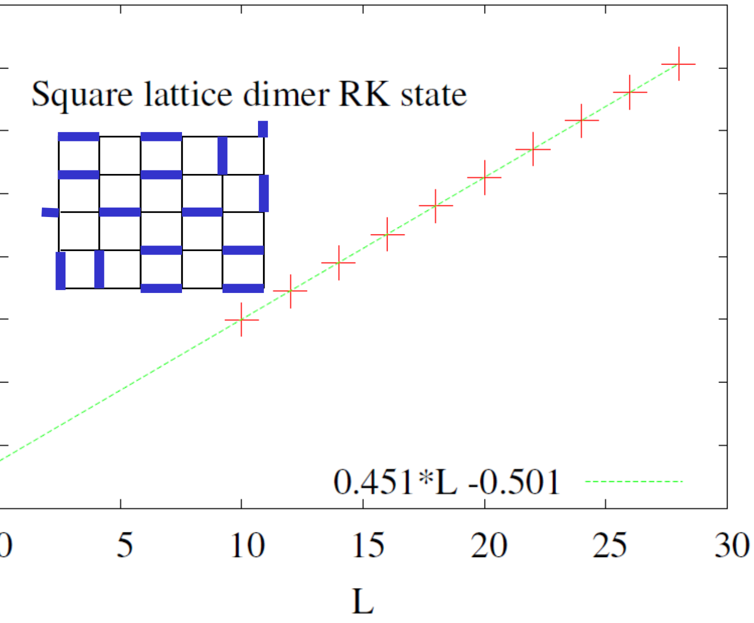

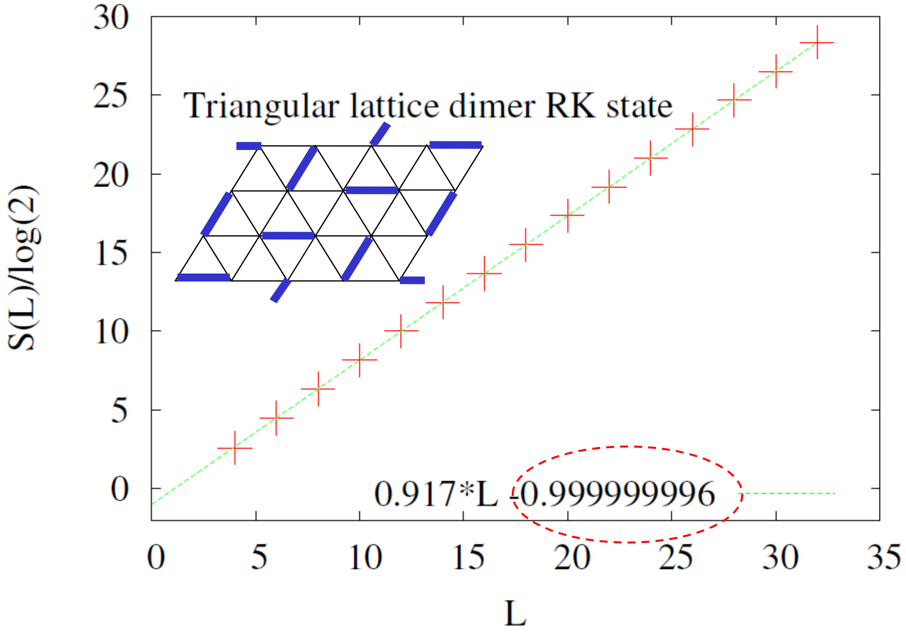

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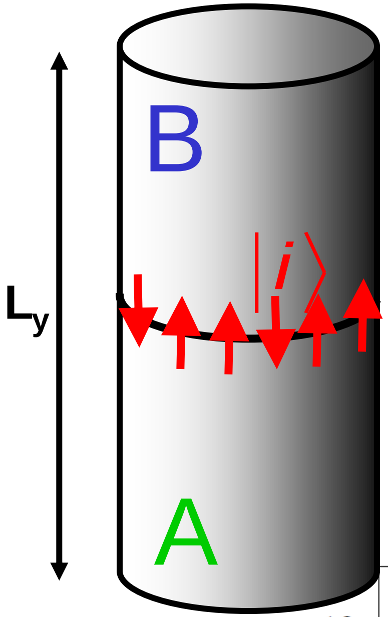

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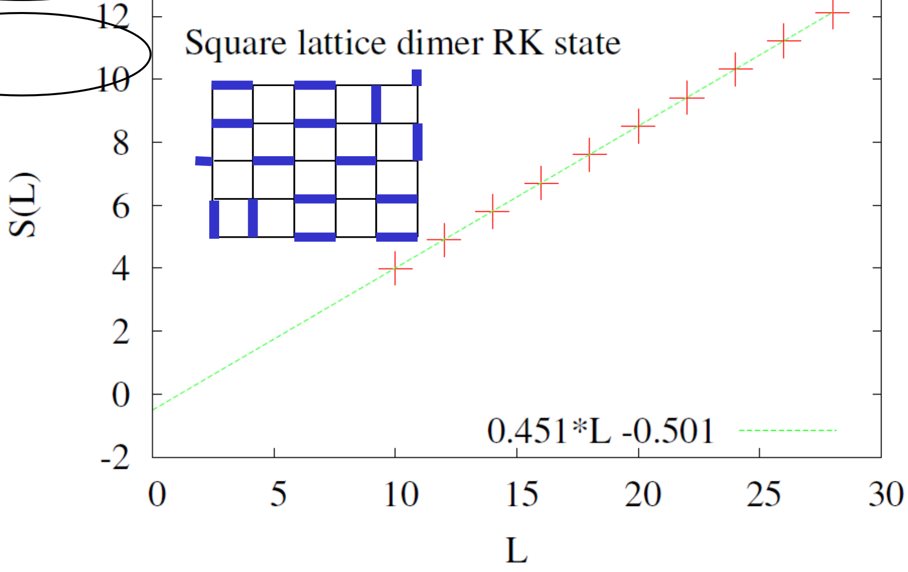
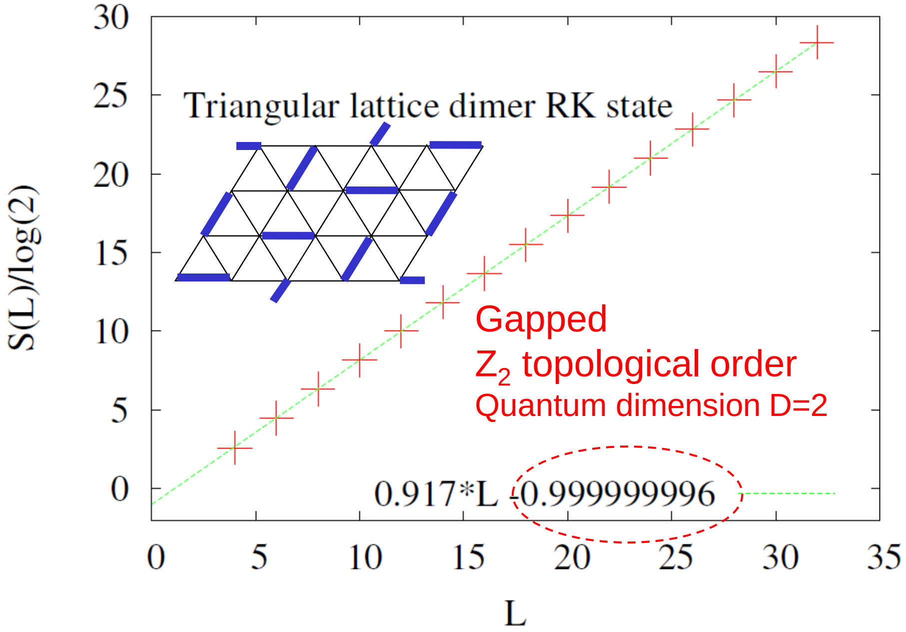
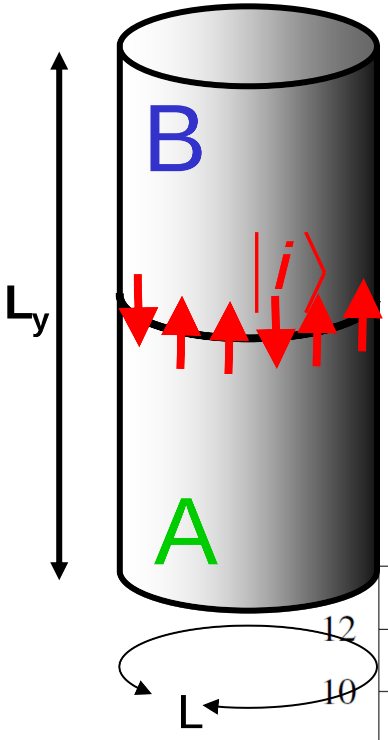




Gapped
Z_2 topological order
Quantum dimension D=2

Rokhsar-Kivelson states: long cylinders & transfer matrix



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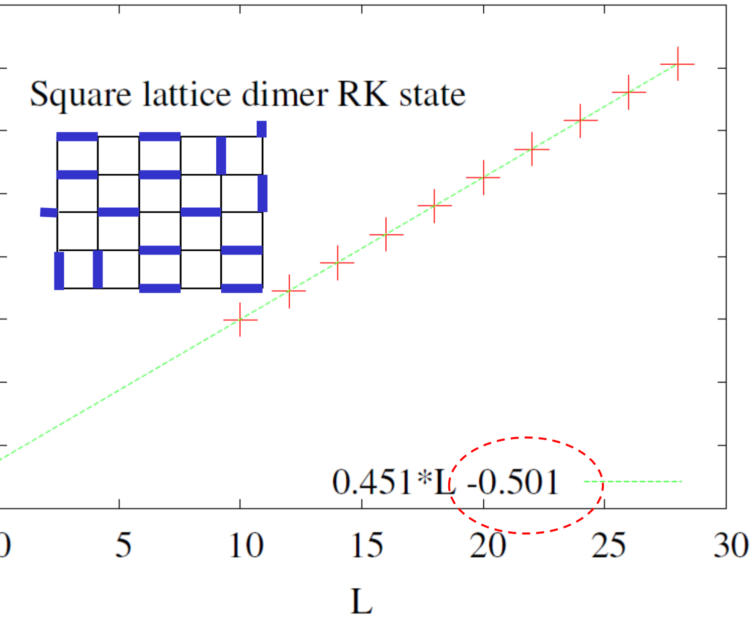

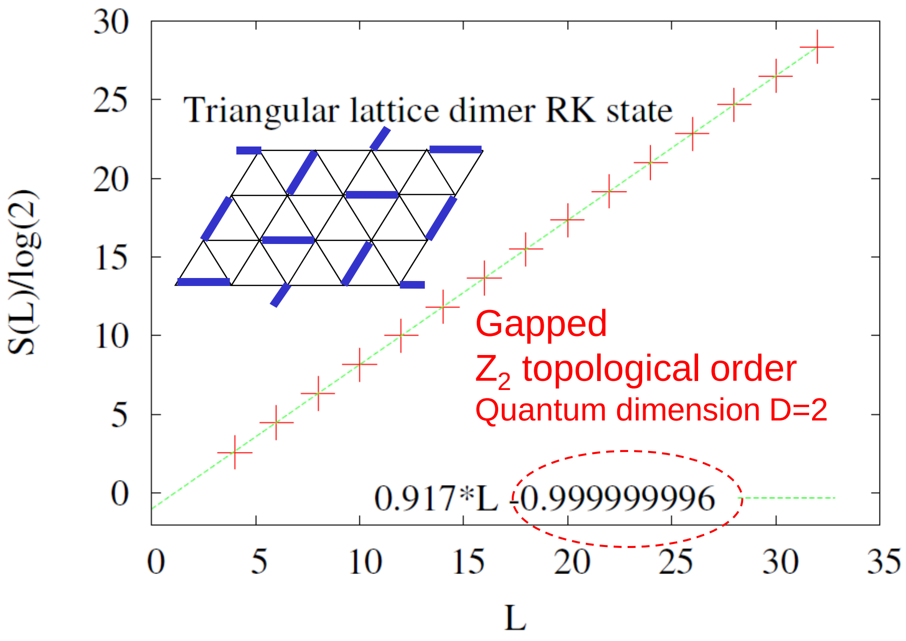

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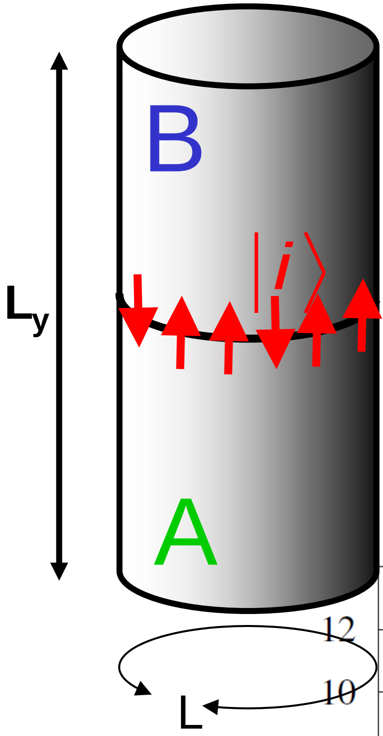

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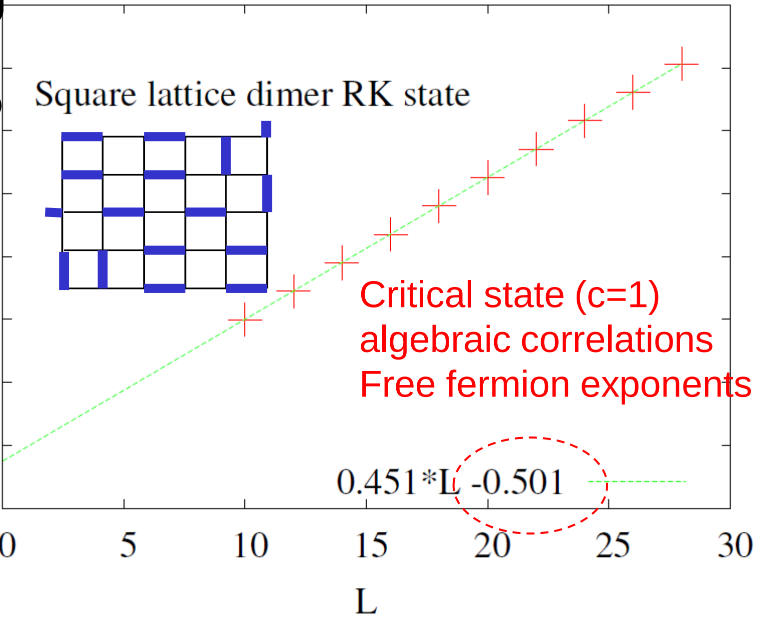
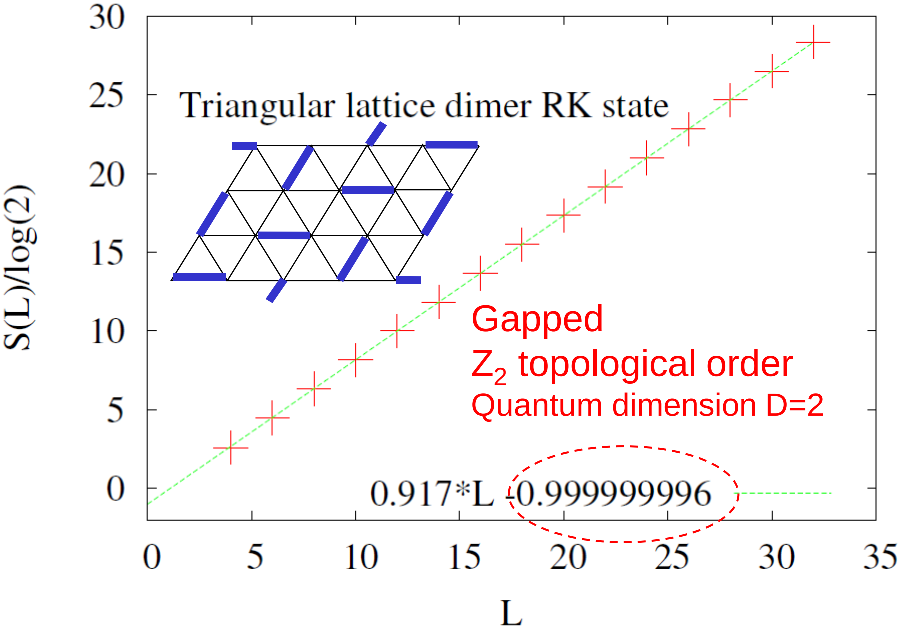


Critical state (c=1)
algebraic correlations
Free fermion exponents



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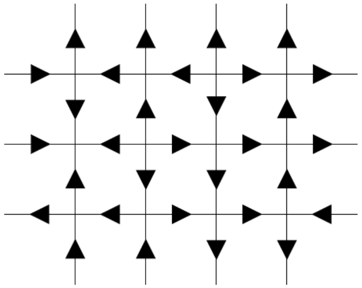
Von Neumann entropy, classical entropy, Shannon entropy



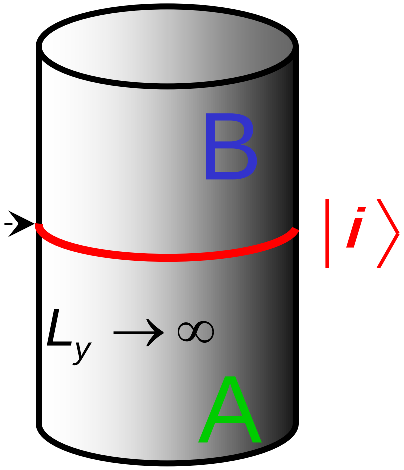
Classical model

$$Z = \sum_c e^{-E(c)}$$

$$\left|RK\right\rangle$$

Von Neumann entropy

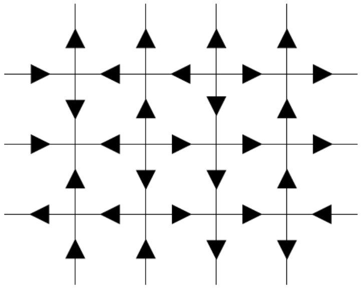
$$S_A = -Tr\left[\rho_A \log \rho_A\right]$$



Shannon entropy

$$-\sum_i p_i \log(p_i)$$

Von Neumann entropy, classical entropy, Shannon entropy



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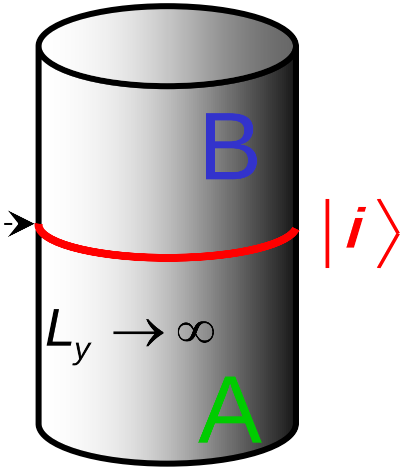
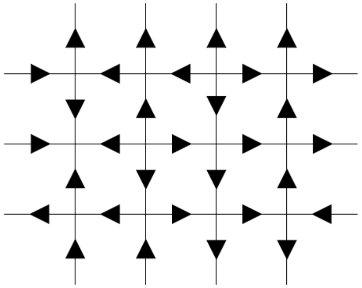


Transfer matrix

Shannon entropy

$$|g\rangle \dashrightarrow p_i = \left|\langle i|g\rangle\right|^2 \dashrightarrow -\sum_i p_i \log(p_i)$$

Von Neumann entropy, classical entropy, Shannon entropy



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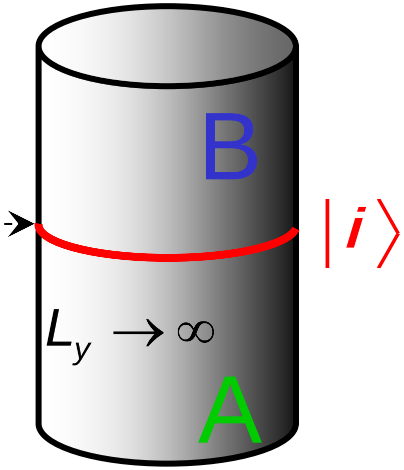

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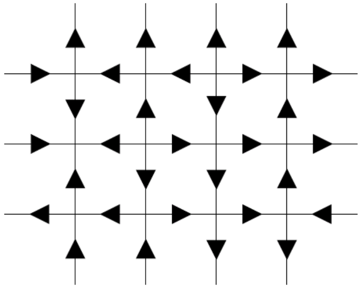
- ☐ Classical entropy of a line
- ☐ or classical mutual information

$MI^{class}=S_A^{cl}+S_B^{cl}-S_{AB}^{cl}$

(some models)

Von Neumann entropy, classical entropy, Shannon entropy



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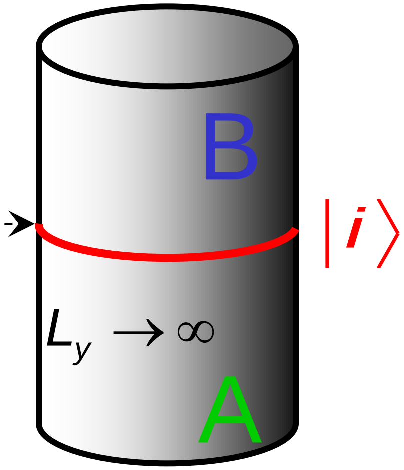

Transfer matrix

Shannon entropy

Quantum Hamiltonian
(spin chain, …)

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(some models)

Shannon entropies of wave-functions

- 1d XXZ spin chain & Luttinger liquids
- 1d quantum Ising chain
- 2d systems with Goldstone modes

Shannon entropy of an XXZ ground state – definition

$$|g\rangle = \sum_{i=1}^{2^L} \psi_i |i\rangle$$ ground-state of $H = -\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + \Delta \sum_i S_i^z S_{i+1}^z$

$|i\rangle = |\downarrow\uparrow\uparrow\downarrow \cdots\rangle, \cdots$ "Ising configurations" $S_i^z = \pm\frac{1}{2}$

$p_i = |\langle i|g\rangle|^2$ $\sum_i p_i = 1$ probabilities (normalized)

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$$S_n = \frac{1}{1-n} \log\left(\sum_i (p_i)^n\right)$$

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Remarks: $S_{n>1}$ is similar to participation ratios but here the sum (CPU time!) is exponential in L

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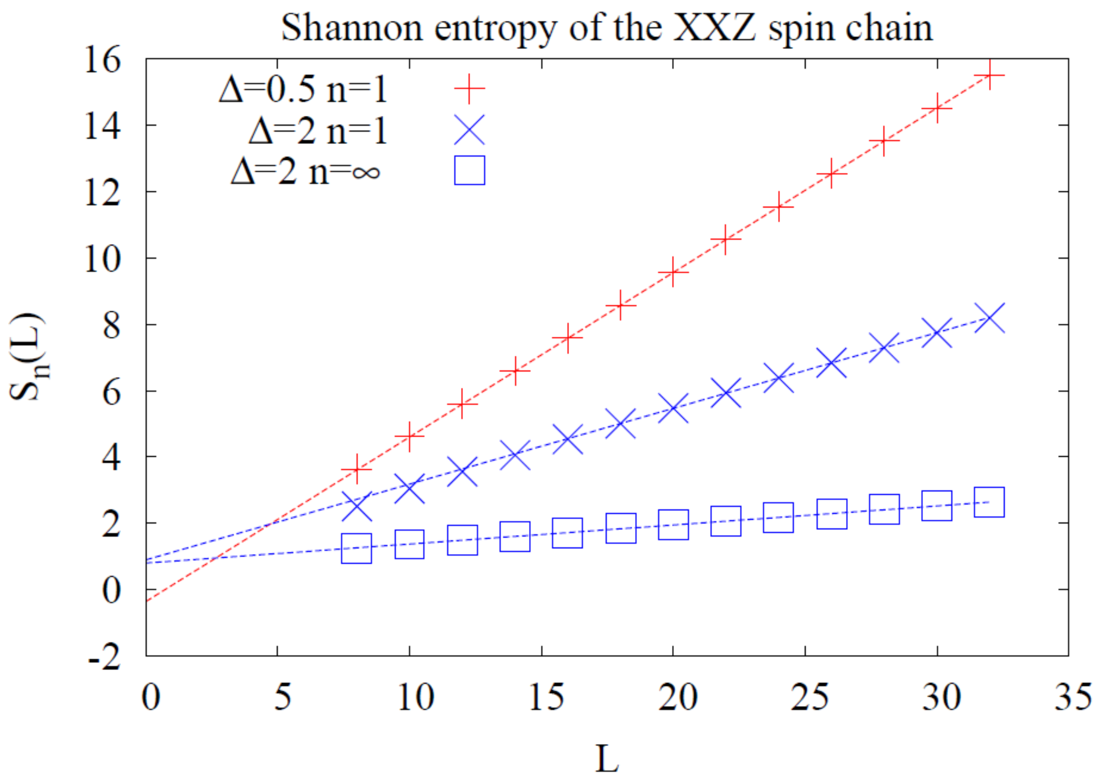

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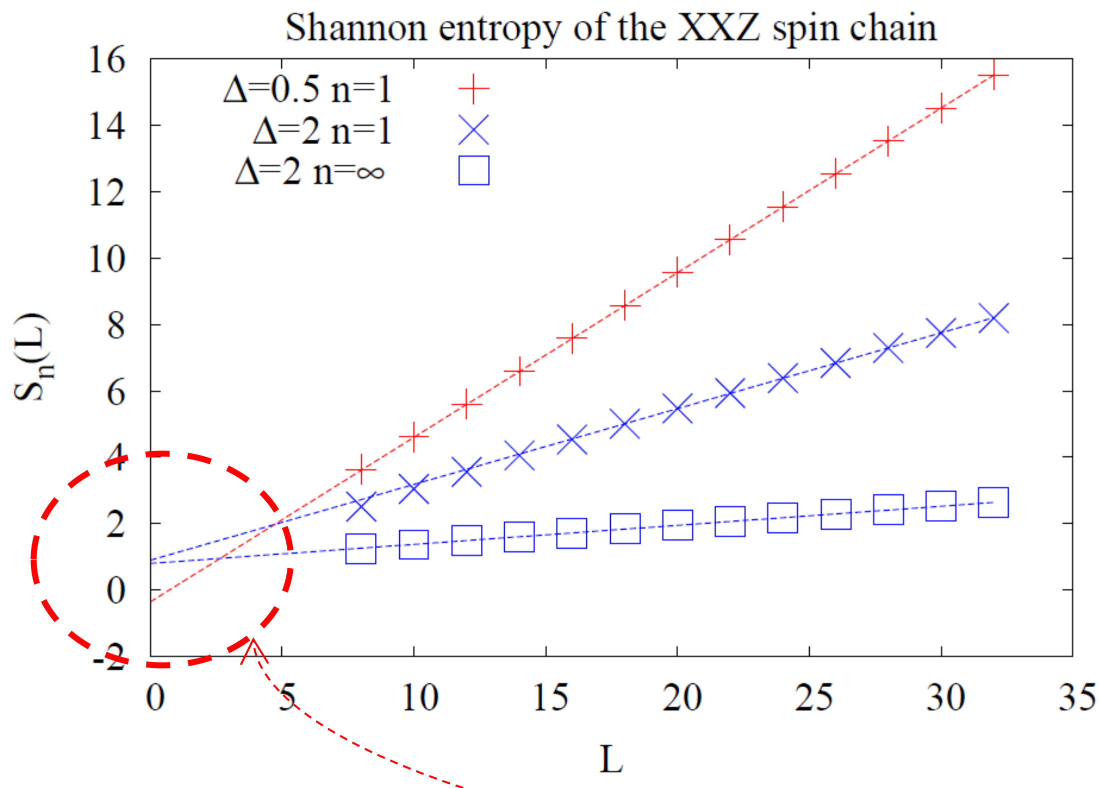

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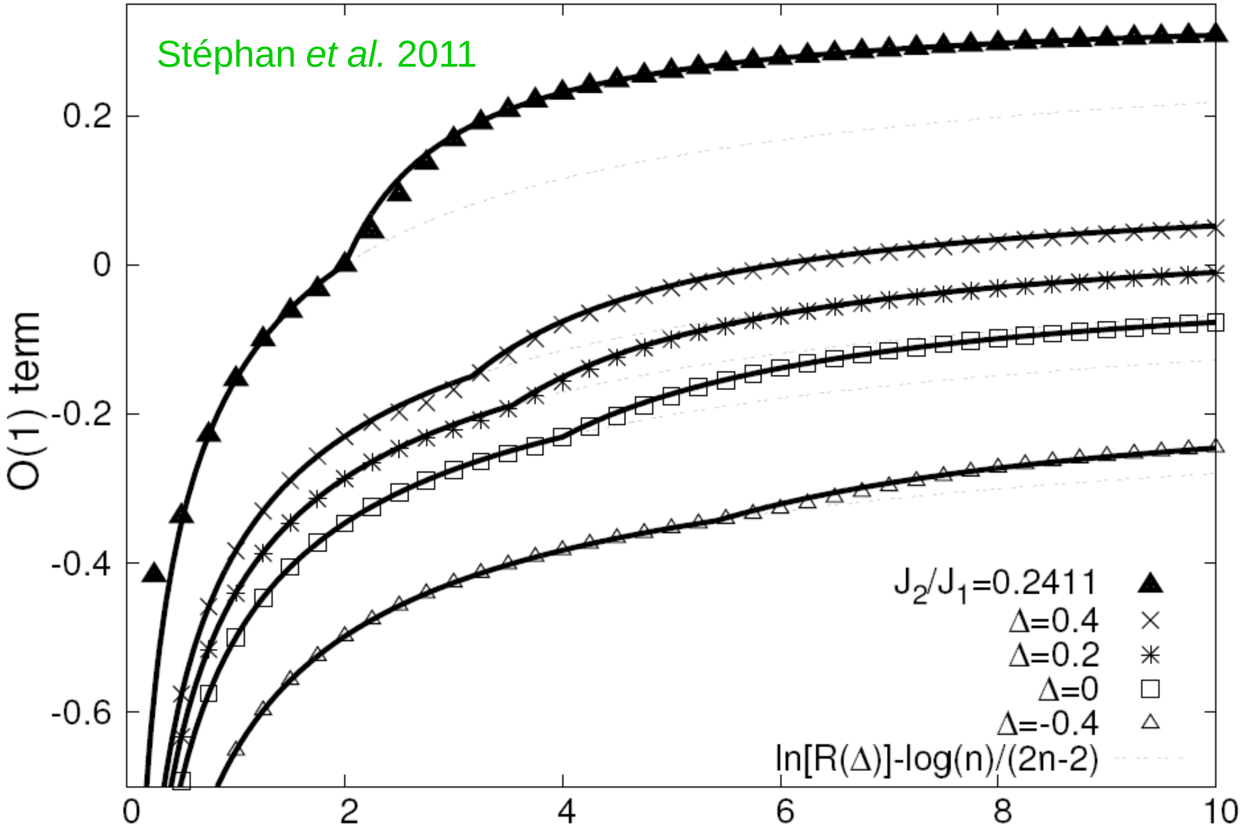
Volume law and **subleading constant**

$$S_n(L) \approx \boxed{a_n L} + \boxed{b_n} + \cdots$$

Shannon entropy of an XXZ ground state – numerics

Volume law and **subleading constant**

$$S_n(L) \approx \boxed{a_n L} + \boxed{b_n} \quad '$$



Stéphan *et al.* 2011

n (=Rényi index)

Theory for these subleading entropy constants ?

Luttinger liquid – Continuum limit

- Algebraic correlations & Luttinger parameter **K**

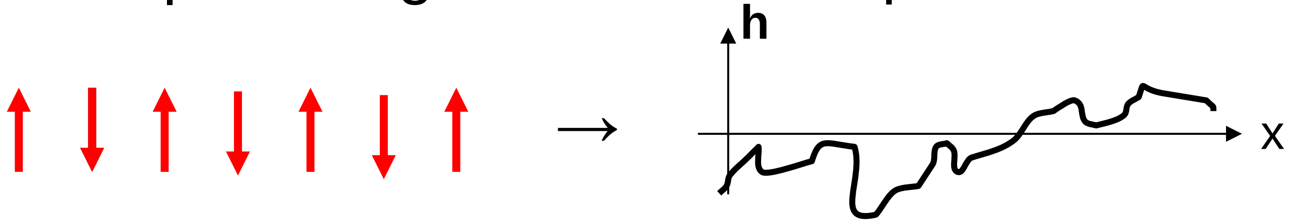
$$\langle S_0^z S_x^z\rangle \simeq a x^{-2} + b(-1)^x x^{-2K} + \cdots$$

$$\langle S_0^+ S_x^-\rangle \simeq c x^{-\frac{1}{2K}} + d(-1)^x x^{-2K-\frac{1}{2K}} + \cdots$$

$$2K = \left[1 - \frac{\arccos(\Delta)}{\pi}\right]^{-1}$$

- Bosonization:

  microscopic configurations → compactified scalar field



$$h(x,\tau) \equiv h(x,\tau) + 2\pi$$

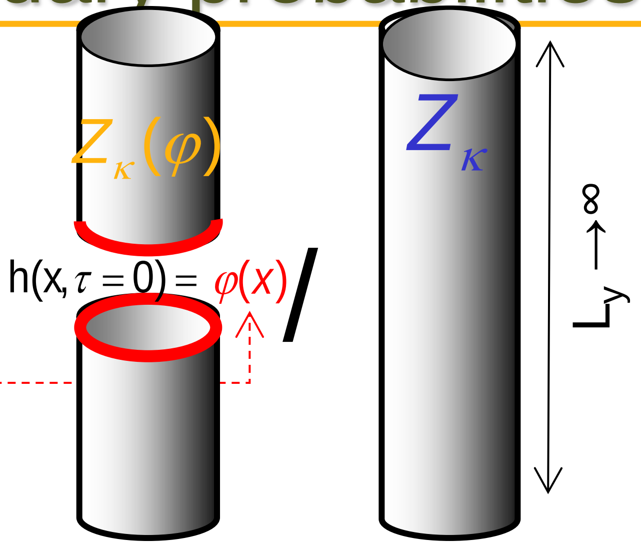
- Gaussian effective action

$$S_\kappa[h] = \frac{\kappa}{4\pi}\int \left(\vec{\nabla} h\right)^2 \underbrace{dx}_{\text{space}} \underbrace{d\tau}_{\text{Imaginary time}}$$

$$K = \frac{1}{2\kappa} = \frac{1}{R^2}$$

Gaussian Free field : boundary probabilities

$$p_\kappa(\varphi) = Z_\kappa(\varphi) / Z_\kappa \;=$$



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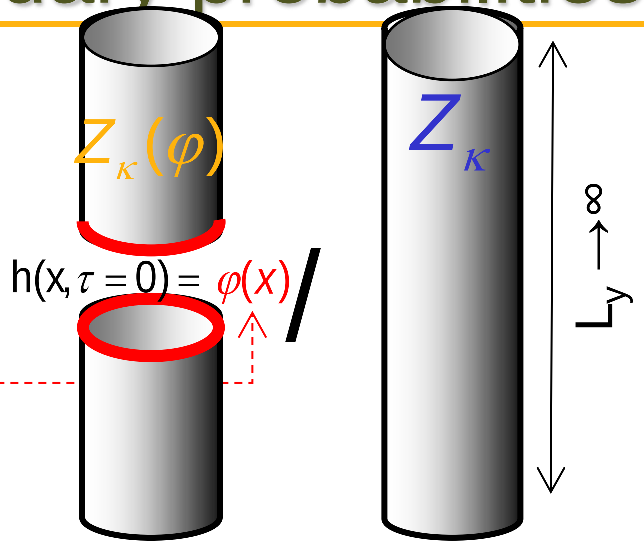

$$Z_\kappa(\varphi) = \exp(-\kappa A[\varphi]) \cdot Z_\kappa(0)$$

Gaussian factor

Partition function
with Dirichlet condition (φ=0)

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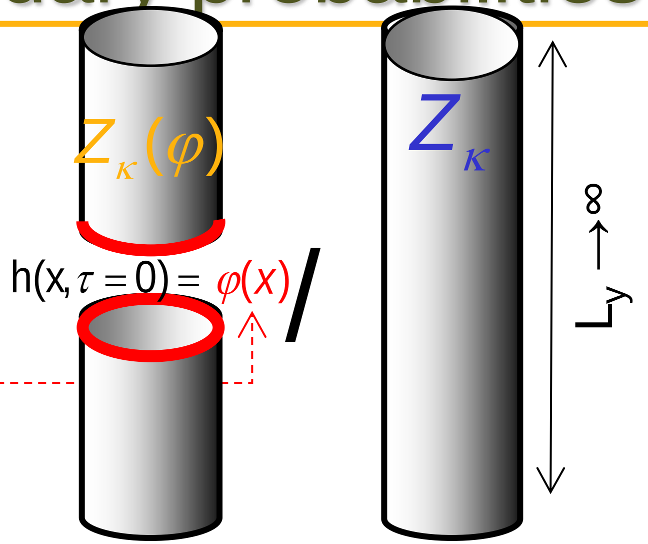

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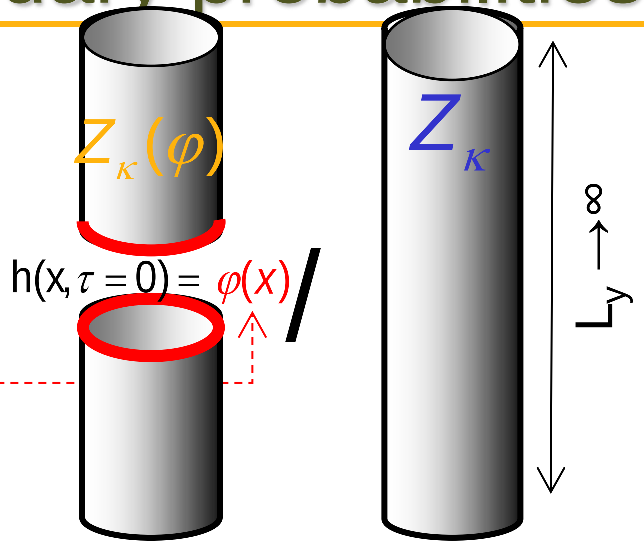

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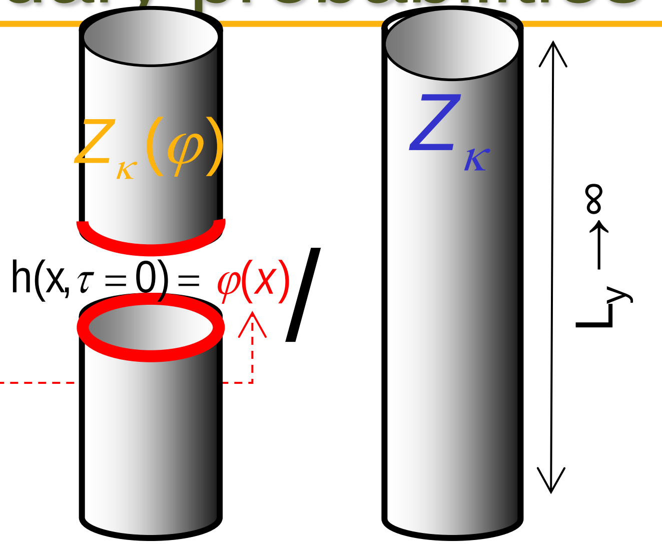

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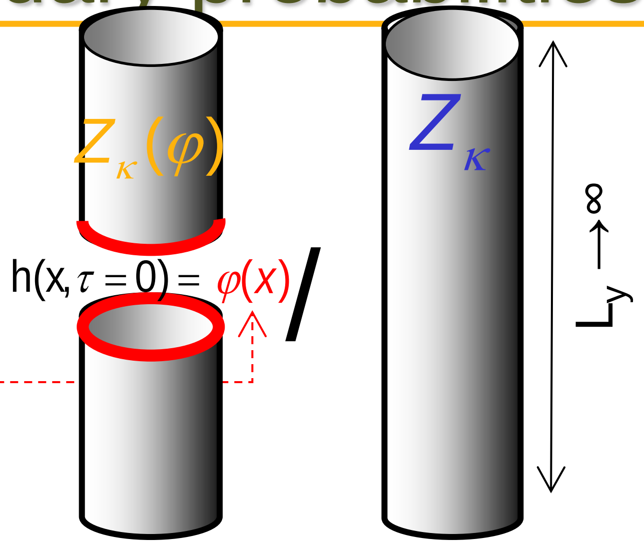

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$$= \frac{Z_{n\kappa}(\varphi)}{Z_{n\kappa}(0)}$$

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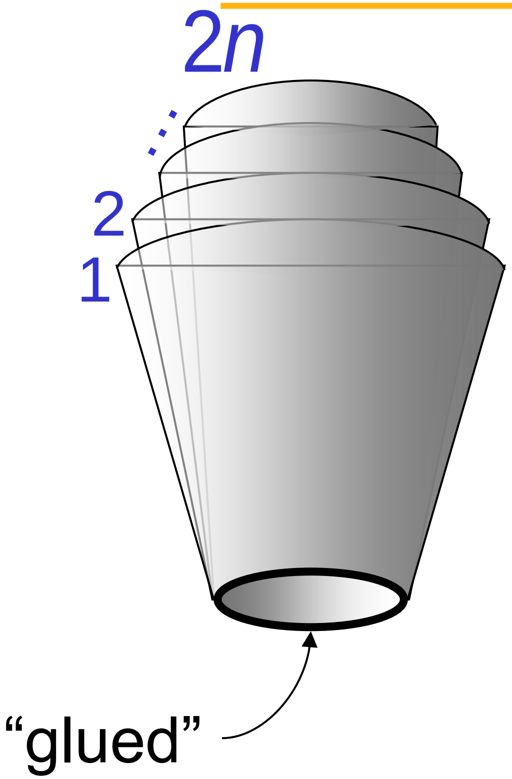
$$\boxed{\sum_\varphi \left(p_\varphi\right)^n = \frac{Z_{n\kappa}}{Z_{n\kappa}(0)} \times \left(\frac{Z_\kappa(0)}{Z_\kappa}\right)^n}$$

2n-sheeted partition function & Rényi entropy



$$= \sum_{\varphi} p_{\varphi}^{n}$$

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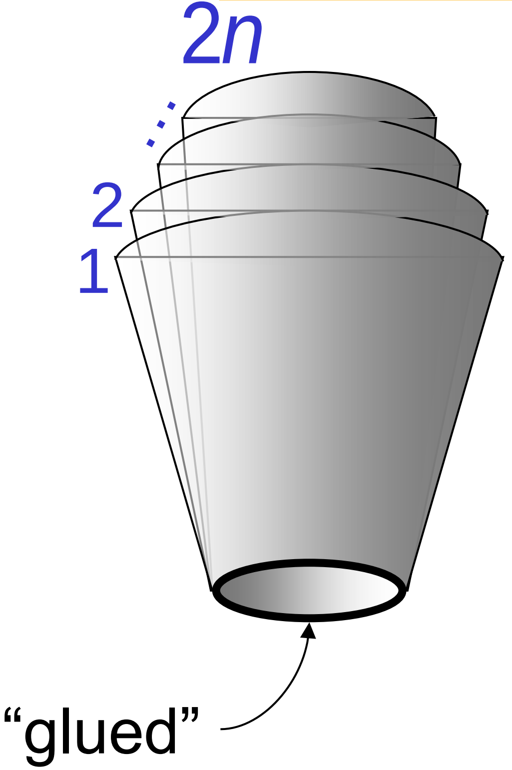

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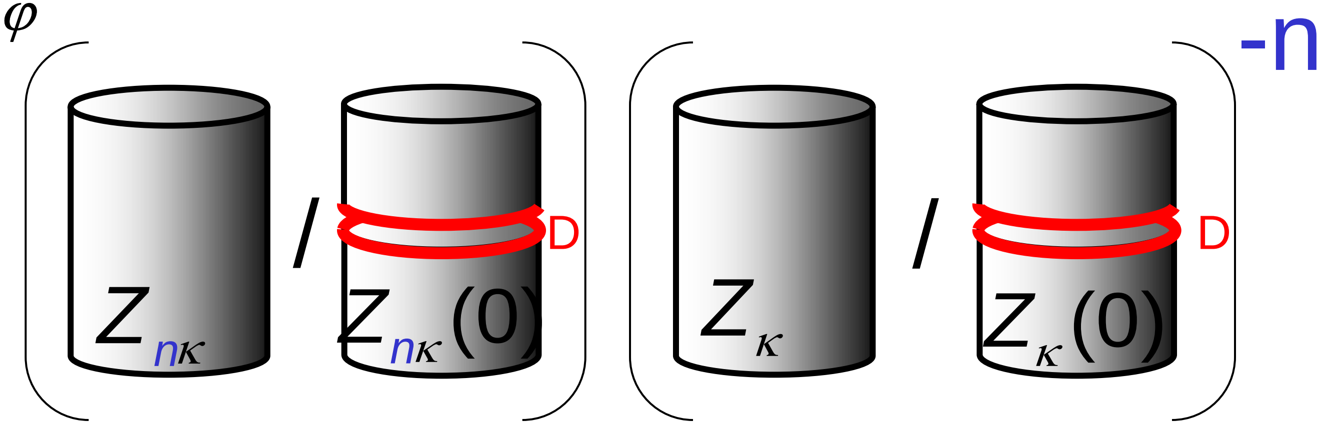

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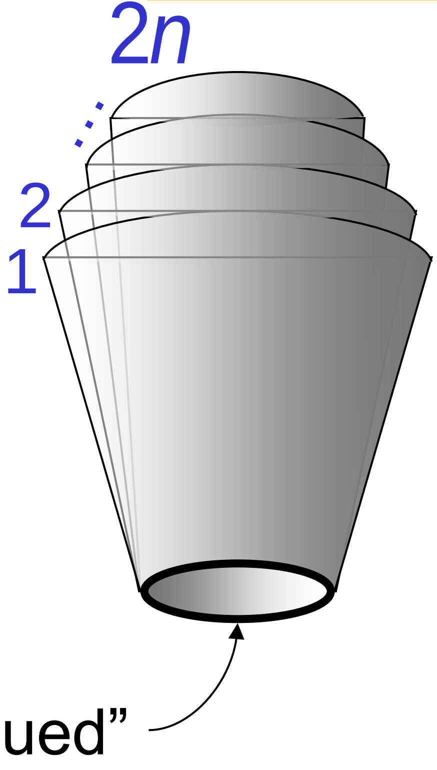

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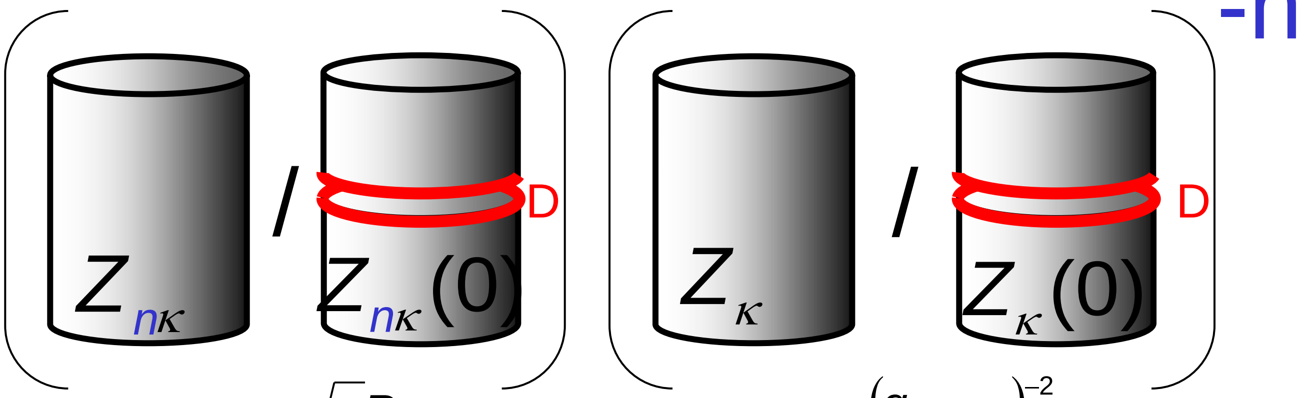


$$= \left(Z_{n\kappa} \,/\, Z_{n\kappa}(0) \right) \left(Z_\kappa \,/\, Z_\kappa(0) \right)^{-n}$$

$= \sqrt{n}R$

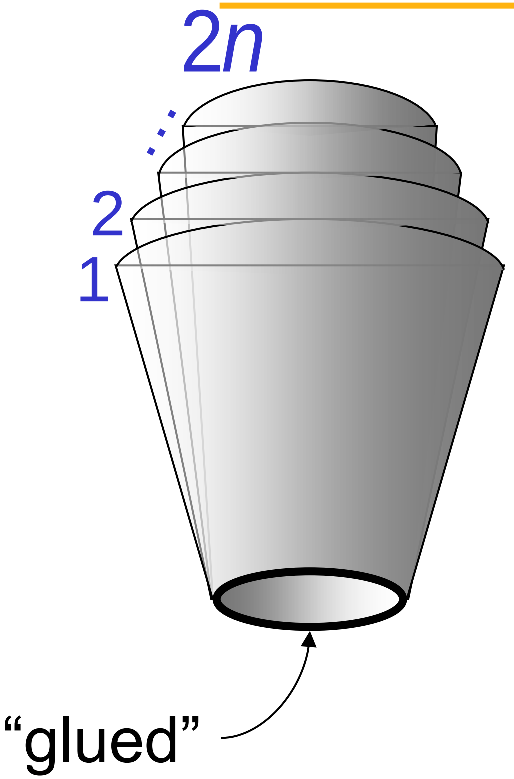
$= \left(g_{\text{Dirichlet}}\right)^{-2}$

$= \sqrt{2\kappa} = R$

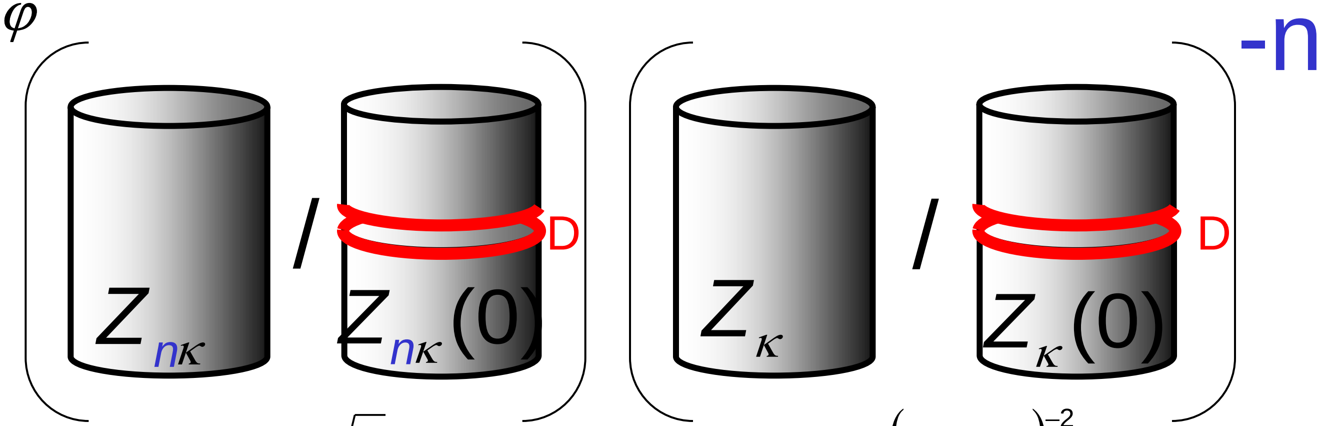
"g-factor" Affleck-Ludwig 1991
Fendley, Saleur, warner 1994

$$S_n(L) = \frac{1}{1-n} \log\left(\sum_\varphi \left(p_\varphi\right)^n \right)$$

2n-sheeted partition function & Rényi entropy



$$= \sum_{\varphi} p_{\varphi}^{n}$$



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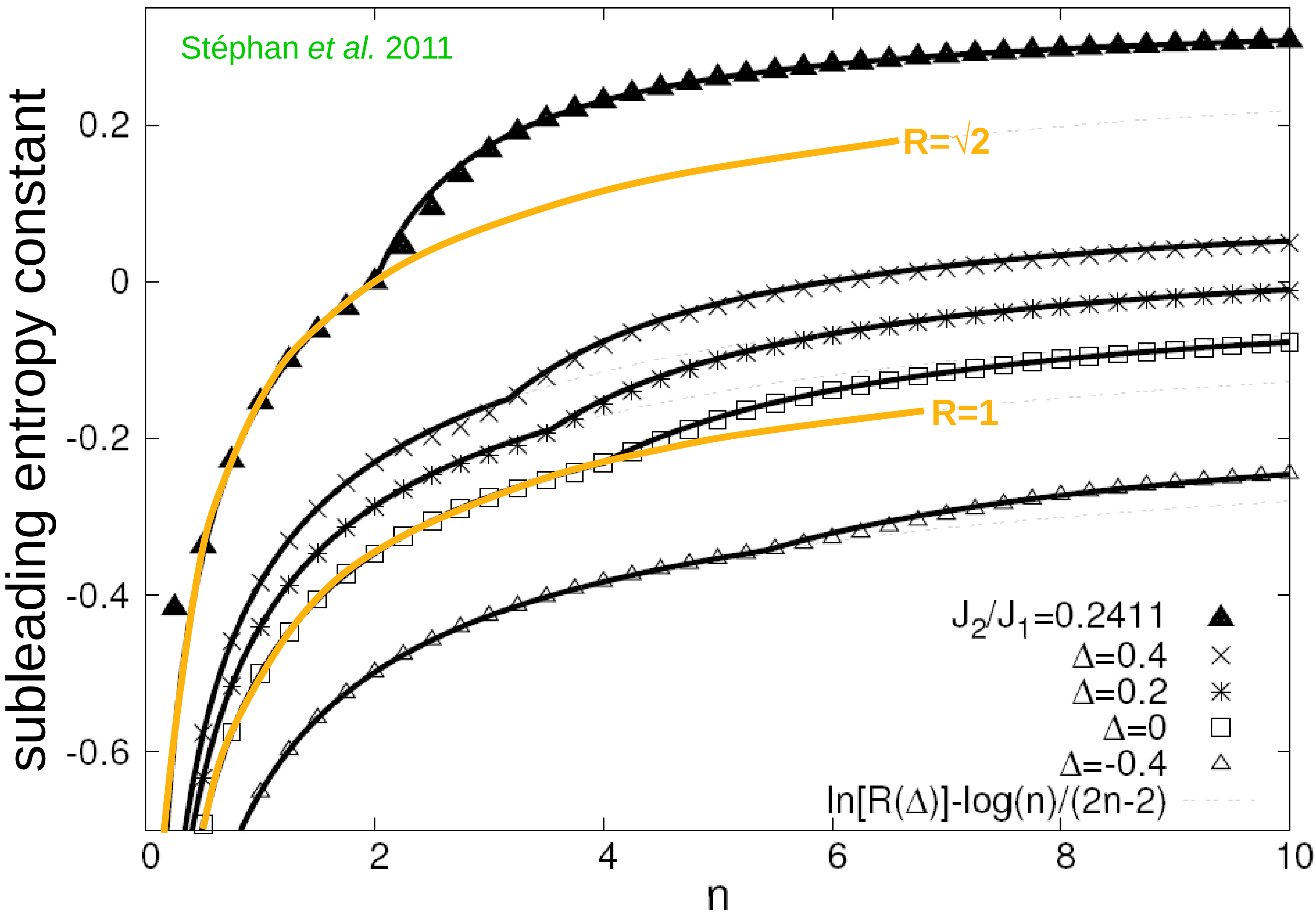
$$S_n(L) = \frac{1}{1-n}\log\left(\sum_{\varphi}\left(p_{\varphi}\right)^{n}\right) = (\cdots)L + \log(R) - \frac{1}{2}\frac{\log(n)}{n-1}$$

Shannon entropy of an XXZ ground state – numerics



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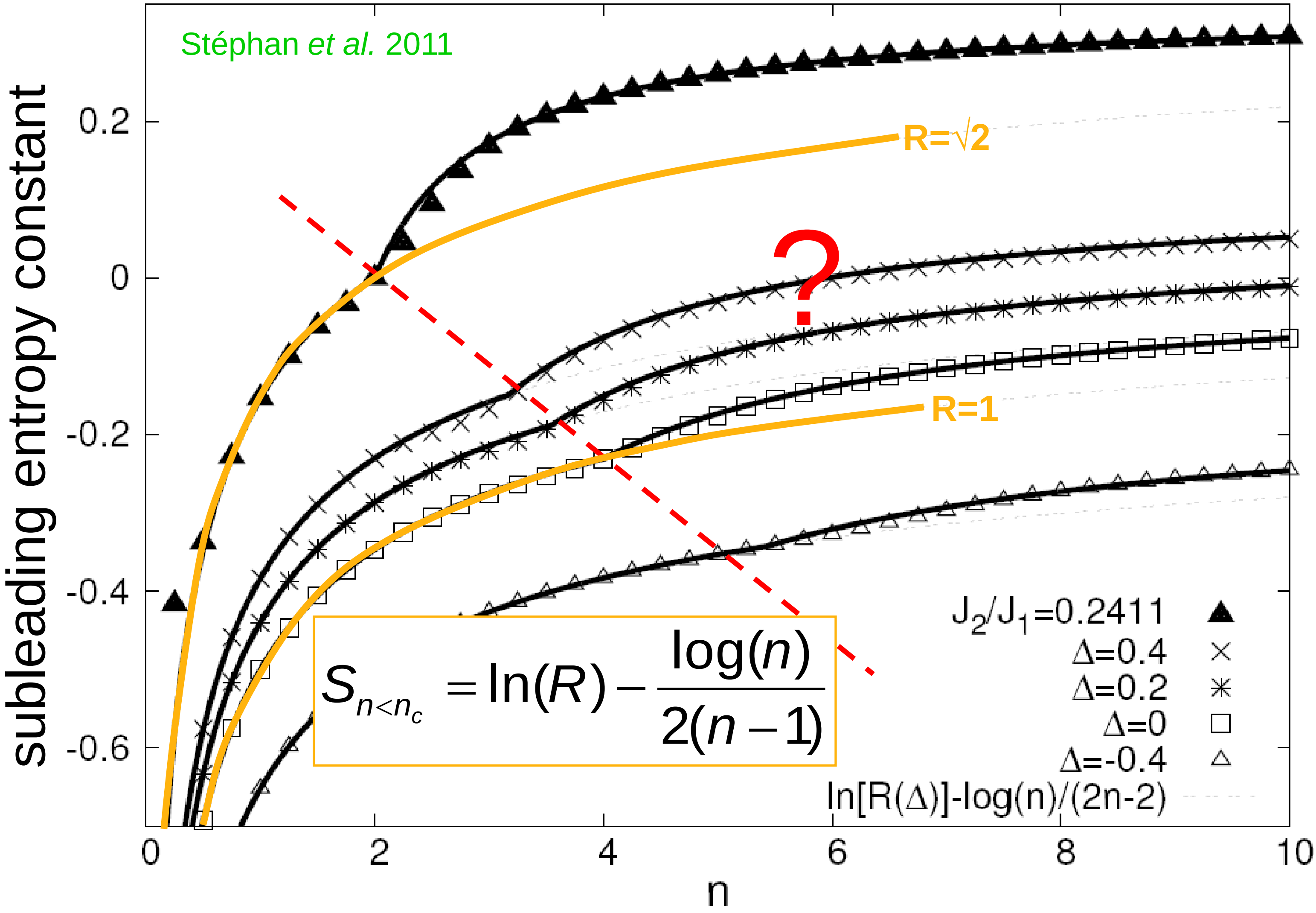

$$S_{n<n_c} = \ln(R) - \frac{\log(n)}{2(n-1)}$$

Boundary phase transition in the Shannon-Rényi entropy

- **Vertex operator** cos(2h(x,τ)) has dimension $x_{\text{boundary}}(\kappa) = \frac{4}{\kappa}$

  (relevant at the boundary if x<1)

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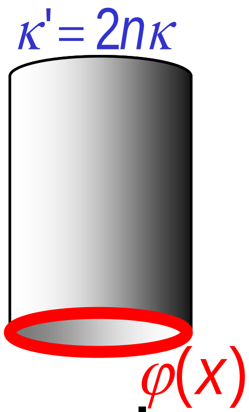
- Rényi index → **effective stiffness multiplied by 2n :**

$$p_\kappa(\varphi)^n \approx$$



- Beyond some critical n, the vertex operator cos(2h/r) become **relevant**

→ boundary phase transition at $n_c = \frac{2}{\kappa}$ (XXZ at zero magnetization)

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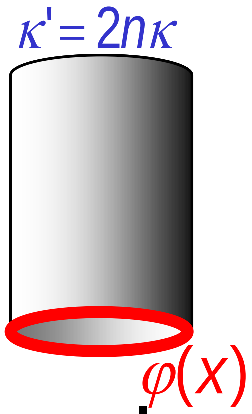
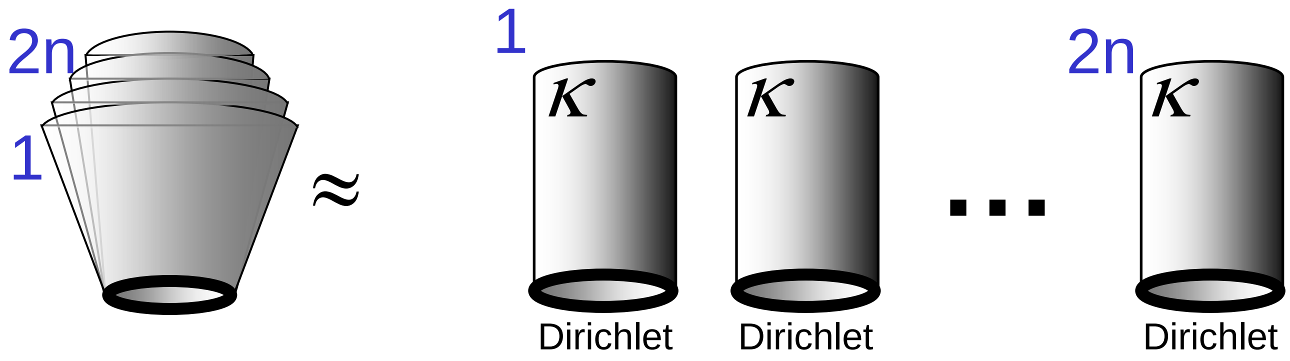


- Beyond some critical n, the vertex operator cos(2h/r) become **relevant**

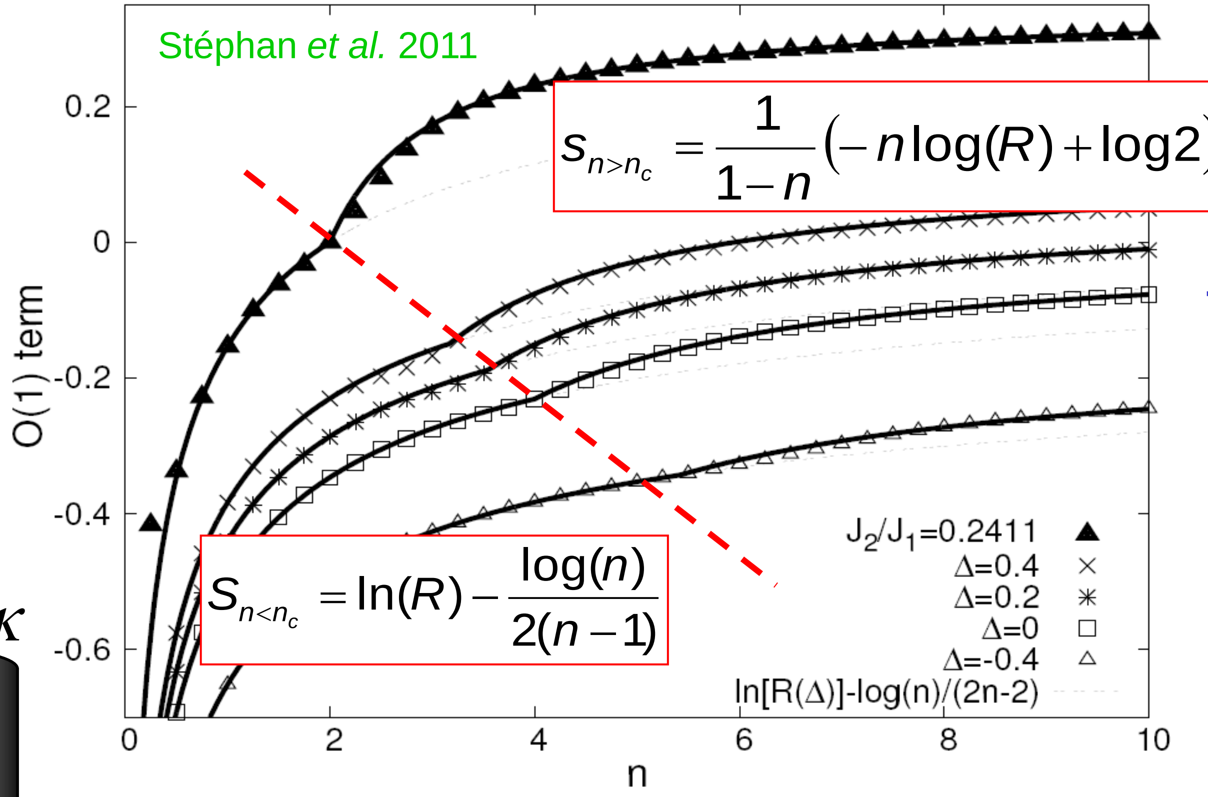
→ boundary phase transition at $n_c = \frac{2}{\kappa}$ (XXZ at zero magnetization)

- n>n$_c$ decoupled replicas

$$\sum_\varphi (p_\varphi)^n =$$

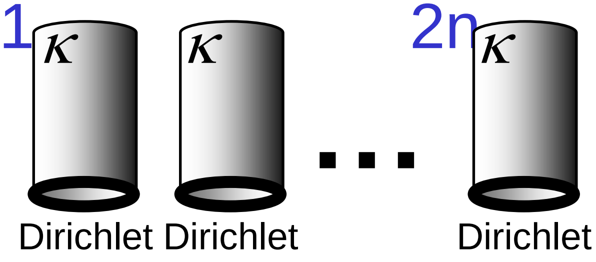


Shannon entropy of an XXZ ground state – summary

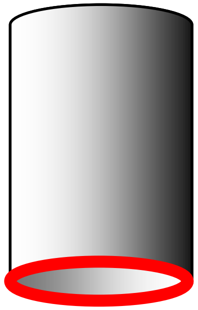


$$s_{n>n_c} = \frac{1}{1-n}\left(-n\log(R)+\log 2\right)$$

$$S_{n<n_c} = \ln(R) - \frac{\log(n)}{2(n-1)}$$



$\kappa' = 2n\kappa$

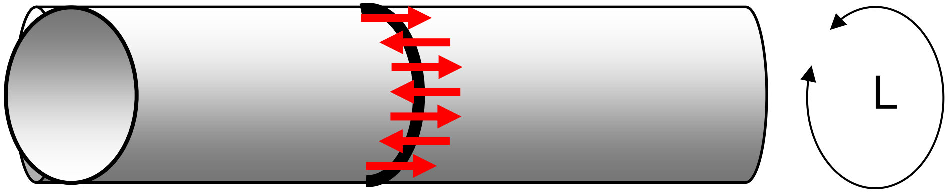


Remark: the scaling of the largest weight in the wave function gives the compactification radius:

$$S_{n=\infty} = -\log(p_{\max}) = a_\infty L + \log(R)$$

Ising model

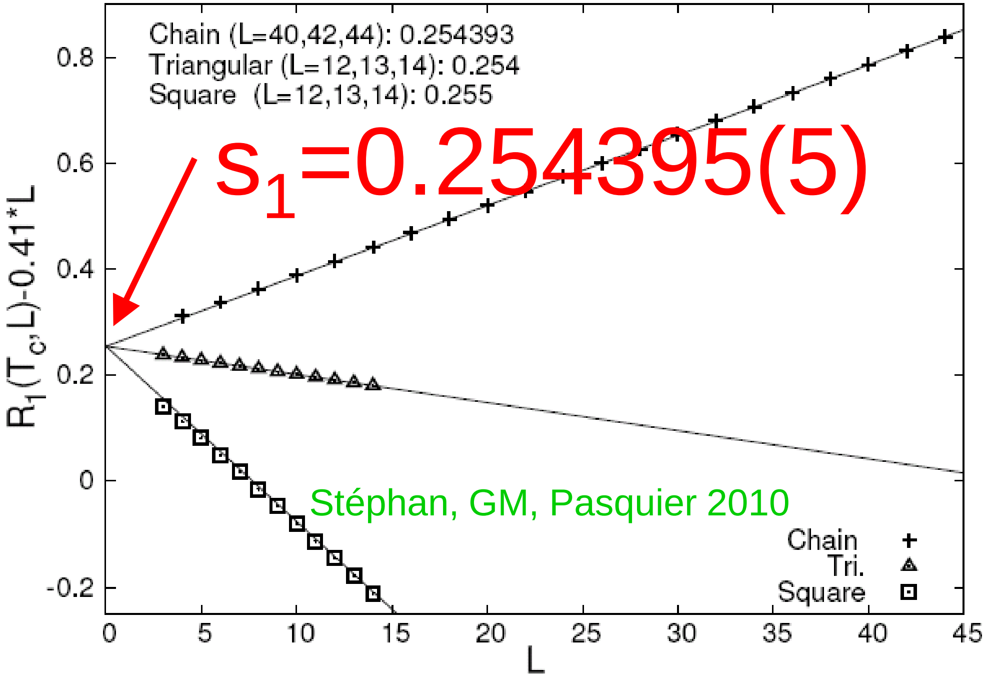
- Shannon entropy of the critical Ising chain in transverse field or classical entropy of a line in a 2d classical model



$$\mathcal{H} = -\mu \sum_{j=0}^{L-1} \sigma_j^x \sigma_{j+1}^x - \sum_{j=0}^{L-1} \sigma_j^z$$

- Boundary phase transition at $n_c=1$

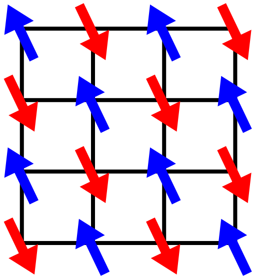
- Universal (and mysterious) Subleading entropy constant at n=1.
NB: a replica approach fails !



$s_1=0.254395(5)$

Stéphan, GM, Pasquier 2010

Shannon entropy in 2+1d & Goldstone modes



$$H_{\text{Heisenberg}} = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$H_{XY} = \sum_{\langle i,j \rangle} \left(S_i^x S_j^x + S_i^y S_j^y \right)$$

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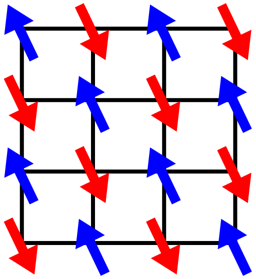

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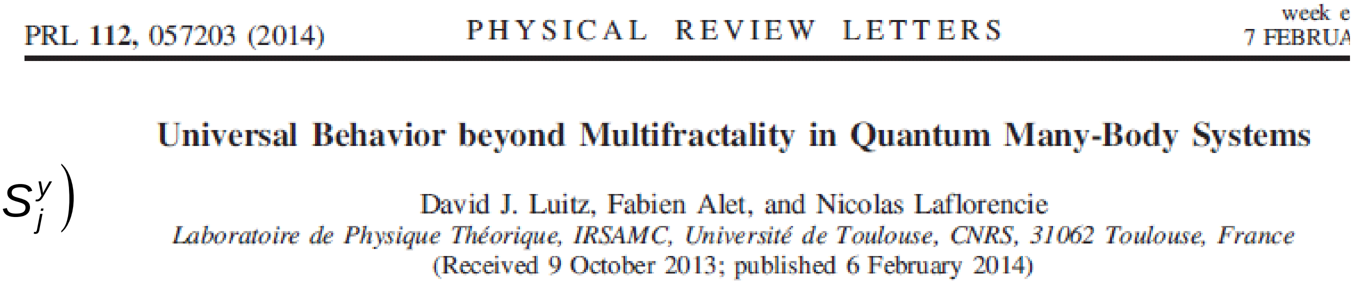

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Universal Behavior beyond Multifractality in Quantum Many-Body Systems

David J. Luitz, Fabien Alet, and Nicolas Laflorencie
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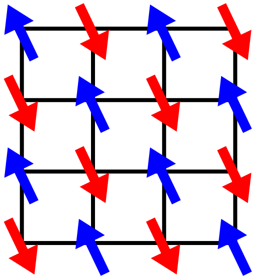
(Received 9 October 2013; published 6 February 2014)

$$S_n(N) \approx a_n N + \boxed{l_n \log(N)} + \cdots$$

n=2,3,4,∞ : accessible to quantum Monte-Carlo simulations

Shannon entropy in 2+1d & Goldstone modes



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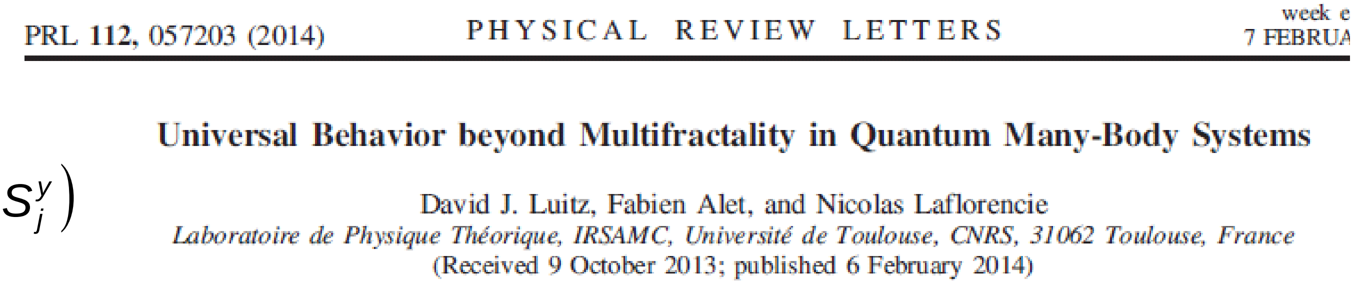

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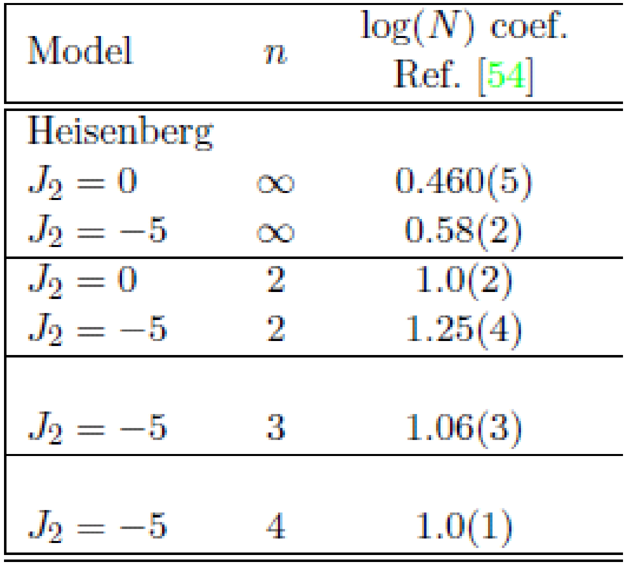
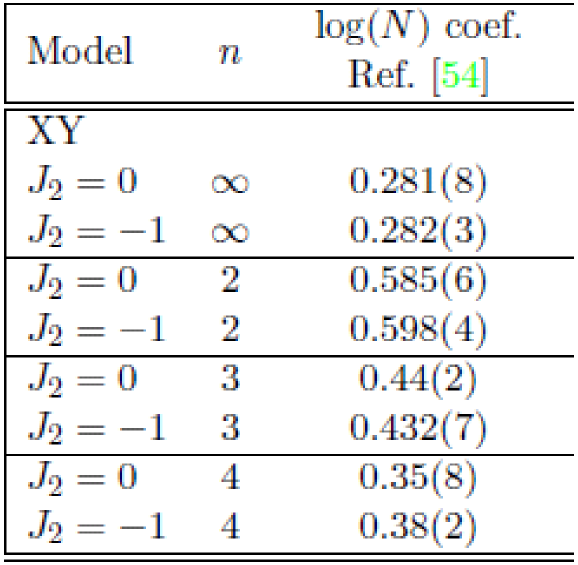


Model	n	$\log(N)$ coef. Ref. [54]
Heisenberg		
$J_2 = 0$	∞	0.460(5)
$J_2 = -5$	∞	0.58(2)
$J_2 = 0$	2	1.0(2)
$J_2 = -5$	2	1.25(4)
$J_2 = -5$	3	1.06(3)
$J_2 = -5$	4	1.0(1)



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$J_2 = 0$	3	0.44(2)
$J_2 = -1$	3	0.432(7)
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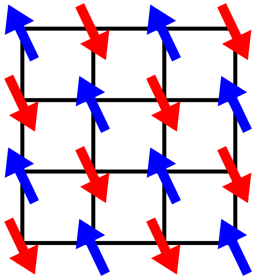
Shannon entropy in 2+1d & Goldstone modes



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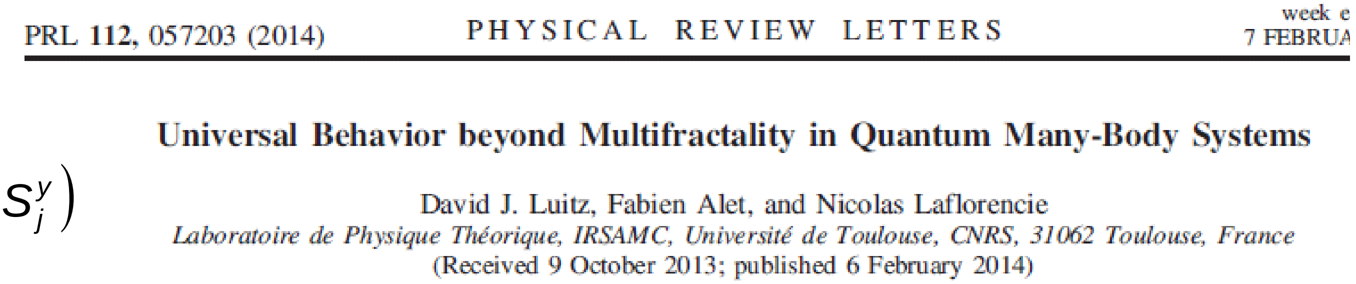

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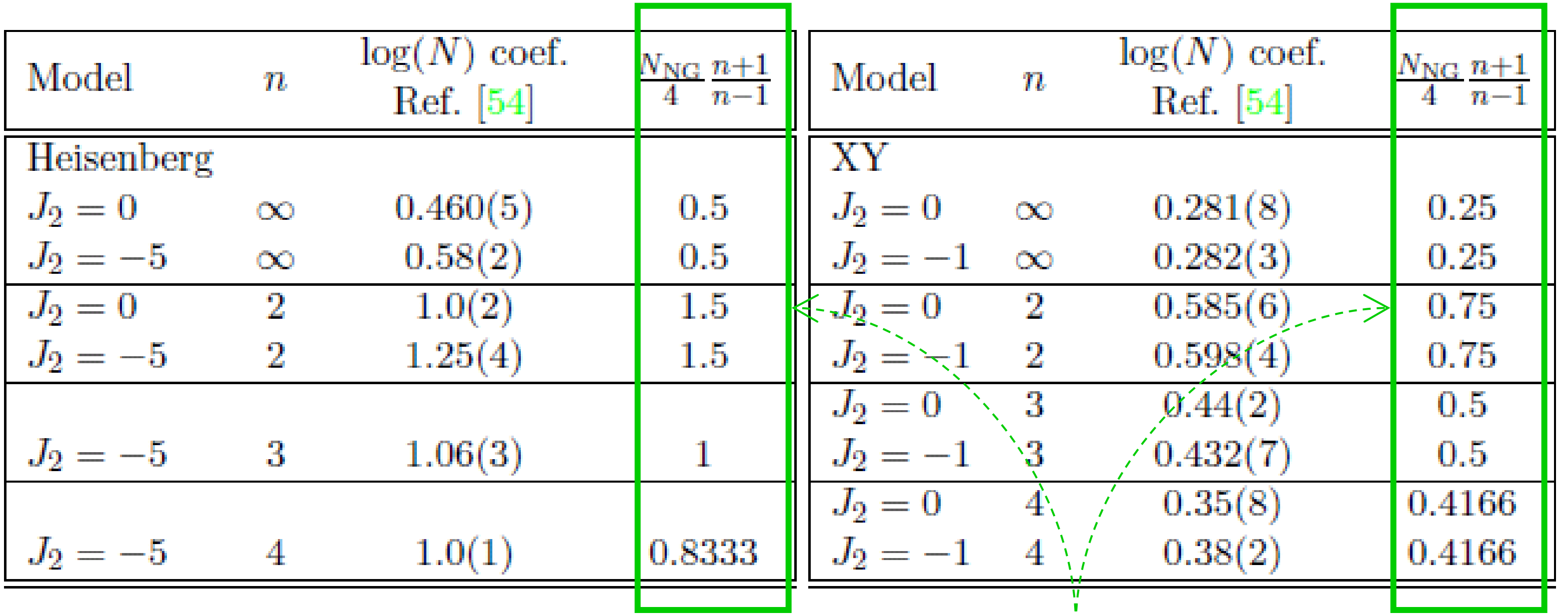

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XY			
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$J_2 = -1$	∞	0.282(3)	0.25
$J_2 = 0$	2	0.585(6)	0.75
$J_2 = -1$	2	0.598(4)	0.75
$J_2 = 0$	3	0.44(2)	0.5
$J_2 = -1$	3	0.432(7)	0.5
$J_2 = 0$	4	0.35(8)	0.4166
$J_2 = -1$	4	0.38(2)	0.4166

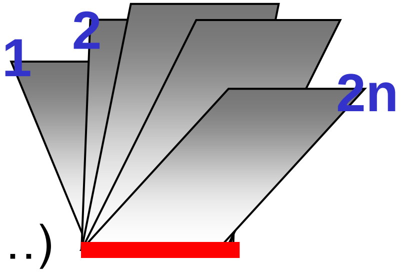
Work in progress, collaboration with V. Pasquier, M. Oshikawa & Toulouse

Conclusions

Shannon entropy: probing a wave function *directlty.*

Subleading terms reveal the long-distance properties of the state:

Compactification radius R in Luttinger liquids, discrete symmetry breaking, Goldstone modes, universality class (Ising), …



❑ A few things I have not talked about :

- Open chains
- Shannon entropy of a subsystem (a segment in 1d, a line in 2d, …)
- Basis dependence

❑ Some future directions:

- Better understanding of $n=n_c$ (c=1 ⇔ marginal boundary sine-Gordon)
- Goldstone modes in 2d: dependence on the geometry (corners, topology, …)
- Measure/characterize the correlations in “Renyified” states (1d or 2d) :

$$\left|\psi\right\rangle = \sum_c \psi_c \left|c\right\rangle \longrightarrow \left|\psi, n\right\rangle = \frac{1}{\sqrt{\cdots}} \sum_c \left(\psi_c\right)^n \left|c\right\rangle$$