
Quantum spin liquids and fractionalization

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Summary. This chapter discusses quantum antiferromagnets which do not break any symmetries at zero temperature – also called “spin liquids” – and focuses on lattice spin models with Heisenberg-like (i.e. $SU(2)$ -symmetric) interactions in dimensions larger than one. We begin by discussing the Lieb-Schultz-Mattis theorem and its recent extension to $D > 1$ by Hastings (2004), which establishes an important distinction between spin liquids with an integer and with a half-integer spin per unit cell. Spin liquids of the first kind, “band insulators”, can often be understood by elementary means, whereas the latter, “Mott insulators”, are more complex (featuring “topological order”) and support spin-1/2 excitations (spinons). The fermionic formalism (Affleck and Marston, 1988) is described and the effect of fluctuations about mean-field solutions, such as the possible creation of instabilities, is discussed in a qualitative way. In particular, we explain the emergence of gauge modes and their relation to fractionalization. The concept of the projective symmetry group (X.-G. Wen, 2002) is introduced, with the aid of some examples. Finally, we present the phenomenology of (gapped) short-ranged resonating-valence-bond spin liquids, and make contact with the fermionic approach by discussing their description in terms of a fluctuating Z_2 gauge field. Some recent references are given to other types of spin liquid, including gapless ones.

1 Introduction

The concept of “spin liquid” is due to P. W. Anderson, who observed in 1973 [1] that magnetically long-range-ordered (Néel) states were in principle not the only possible ground states for two-dimensional (2D) quantum (and frustrated) antiferromagnets. He explained that such systems could avoid all spontaneous symmetry-breaking, and thus remain “disordered” down to $T = 0$. The picture he provided for such states is the celebrated (short-range) “resonating valence-bond” (RVB) wave function, which is the linear and coherent superposition of a large number of short-range singlet coverings of the lattice. Since then, although our understanding of frustrated quantum spin systems has improved greatly, in general it remains quite incomplete.

First it is necessary to define precisely what is meant by the term “quantum spin liquid”. Depending on the context (experiment, theory, simulation, ...), these words are often applied with rather different meanings. In Sec. 2 we will discuss three possible definitions used frequently (and often implicitly) in the literature, and will comment on their implications. The third definition is the most restrictive, having probably no “overlap” with the more “common” states of matter in $D > 1$ magnetic systems. This is the definition we adopt in the remainder of this chapter. It requires the existence of *fractional* excitations, *i.e.* quasiparticles with quantum numbers (usually the total spin) which are *fractions* of the elementary local degrees of freedom. In spin models, this is essentially equivalent to the existence of spin- $\frac{1}{2}$ excitations (known as spinons). However, such excitations are not easy to realize, because in a system where the local degrees of freedom are spin-flipping processes which change S_{tot}^z by ± 1 , any excitation created in a finite region of the system can only have an *integer* spin. In a one-dimensional system (such as a spin chain), it is well known that domain-wall excitations (or kinks) can carry a half-odd-integer spin. This situation is, however, rather different in $D > 1$, where only some particular states of matter may sustain fractional excitations. We explain, at a qualitative level, in Sec. 2.3 how these fractional excitations interact with each other through emerging gauge fields, and that such spin liquids sustain a kind of hidden order, called “topological order”, a concept due to X.-G. Wen [2, 3, 4, 5] which is connected at a profound level to that of fractionalization.

Why is it that spin liquids should be “fractional” ? To answer, we will review in Sec. 2.4 the Lieb-Schultz-Mattis theorem [6] and its extension by Hastings [7, 8] to $D > 1$. Under certain physically reasonable assumptions, we will argue that this theorem implies essentially (if not mathematically) that a spin- $\frac{1}{2}$ system with an odd number of sites per unit cell (a genuine Mott insulator) and conserved total magnetization S_{tot}^z must either i) be ordered in a conventional way, meaning with a spontaneously broken symmetry, or ii) have some type of topological order.¹ Because topological order is connected intimately to the existence of fractional excitations, we conclude that a Mott insulator with conserved S_{tot}^z and no spontaneous symmetry-breaking supports topological order and fractional excitations.

In Sec. 3 we discuss fractionalized spin liquids in a more rigorous framework, by introducing the basics of the slave-particle formalism and by explaining (Sec. 3.4) how gauge fields arise when investigating the fluctuations around mean-field states. Section 4 describes phenomenologically the properties of the simplest *gapped* spin liquids, called \mathbb{Z}_2 liquids in modern terminology, which correspond essentially to *short-range* RVB states. Their excitations, spinons and visons, are discussed, and a number of realizations in frustrated 2D spin models are reviewed. Section 5 is devoted to gapless liquids in $D > 1$, also known as algebraic spin liquids. These states are more complex than the sim-

¹A third possibility for $D > 1$ is that the system has been fine-tuned to a critical point, but this does not correspond to a stable *phase*.

ple \mathbb{Z}_2 liquids, and we present an overview of their study using the mean-field approximations discussed in Sec. 3. These liquids are closely related to the *long-range* RVB theories of high-temperature superconductors [9, 10, 11, 12]). Sec. 6 mentions briefly some of the spin liquids which are not discussed elsewhere in this chapter.

2 What is a spin liquid ?

We focus on the zero-temperature properties of lattice quantum spin models with global $U(1)$ (conservation of S_{tot}^z) or $SU(2)$ symmetry (total spin $\mathbf{S}_{\text{tot}}^2 = S(S+1)$ conservation).

2.1 Absence of magnetic long-range order (definition 1)

Definition 1: *a quantum spin liquid is a state in which the spin-spin correlations, $\langle S_i^\alpha S_j^\beta \rangle$, decay to zero at large distances $|r_i - r_j| \rightarrow \infty$.*

This definition is very simple, but it suffers from several limitations. First, any system with continuous spin-rotation symmetry in $D \leq 2$ at finite temperature would satisfy this definition (Mermin-Wagner theorem), even if it is classical and/or ordered at $T = 0$. Secondly, a spin *nematic* [13] would satisfy this definition, despite the fact that it breaks spin rotation symmetry and has some long-range order in the four-spin correlations (see Refs. [14, 15, 16] for recent numerical studies of quantum spin nematics). The definition could be made more strict by requiring the global spin-rotation symmetry to be unbroken. In this case, the spin nematics would be excluded. However, a valence-bond crystal² (VBC, see Ref. [17]) would satisfy this definition, although it possesses certain features of conventional crystalline order.

2.2 Absence of spontaneously broken symmetry (definition 2)

Definition 2: *a quantum spin liquid is a state without any spontaneously broken symmetry.*

Such a definition excludes the VBC state discussed above, but still has some unsatisfactory features. Consider a spin- $\frac{1}{2}$ model where the lattice is composed of clusters with an *even* number of spins (for example 2 or 4). Inside each cluster, the exchange interactions J are strong and antiferromagnetic. By contrast,

² In a VBC, the spins group themselves spontaneously into small clusters (with an even number of sites) which are arranged spatially in a regular pattern. In the crudest approximation, the wave function would be simply a tensor product of singlet states (one for each cluster). Because a VBC wave function is a spin singlet (rotationally invariant) and has short-ranged spin-spin correlations, it is a spin liquid according to definition 1. However, it also possesses some order in the four-spin correlation functions and breaks some of the lattice symmetries.

the inter-cluster interactions J' are weak.³ A two-chain spin ladder, in which the rungs form 2-site clusters, realizes this type of geometry. Such models can be understood qualitatively by a perturbation theory around the decoupled limit $J'/J \rightarrow 0$. The (unique) ground state is a total-spin singlet (reasonably well approximated by a tensor product of singlets on each cluster) with gapped excitations and no broken symmetry. This state obeys definition 2. However, these systems do not realize new states of matter. These systems undergo a smooth evolution from the $T = 0$ limit (singlet ground state) to the $T = \infty$ limit (free spins), with a characteristic crossover temperature determined by the spin gap. From this point of view, this state would more appropriately be called a “ $T = 0$ paramagnet” or a “band insulator” rather than a spin liquid. To our knowledge, almost all gapped Heisenberg spin systems in $D > 1$ which have been observed experimentally belong in this category. These systems are quite similar to *valence-bond solids*, with the spin-1 Haldane chain or AKLT models [18] as standard examples.

2.3 Fractional excitations (definition 3)

Definition 3: *a quantum spin liquid is a state with fractional excitations.*

In the present context, these fractional excitations are usually “spinons”, carrying a half-odd-integer spin (normally $\frac{1}{2}$).

What is a fractional excitation ?

Operations involving any finite number of S_i^+ and S_j^- operators may only change the total magnetization S_{tot}^z by an *integer* (in units where $\hbar = 1$). Thus the creation of a spin- $\frac{1}{2}$ excitation (one “half” of a spin-flip) requires acting in a non-local way on the system. Strictly speaking, such a process is possible only in an infinite system. As a simple example, let us consider the spin- $\frac{1}{2}$ Heisenberg chain with first- and second-neighbor couplings, respectively J_1 and J_2 . For $J_1 = 2J_2 > 0$ (the Majumdar-Ghosh point [19]), the (two-fold degenerate) ground states are given exactly as

$$|a\rangle = \cdots \otimes |[01]\rangle \otimes |[23]\rangle \otimes |[45]\rangle \otimes \cdots, \quad (1)$$

$$|b\rangle = \cdots \otimes |[12]\rangle \otimes |[34]\rangle \otimes |[56]\rangle \otimes \cdots, \quad (2)$$

where $|[ij]\rangle = |\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle$ is a spin singlet state for sites i and j . This is an example of a VBC with spontaneous translational symmetry-breaking. Now we insert in $|a\rangle$ an “up” spin on site 2 by a non-local operation consisting of a shift of the rest of the configuration by one lattice constant to the right,

$$\cdots \otimes |[01]\rangle \otimes |\uparrow_2\rangle \otimes |[34]\rangle \otimes |[56]\rangle \otimes \cdots \quad (3)$$

³Here “weak” does not imply necessarily that the coupling is numerically small, but that the system can be understood qualitatively from a weak-coupling limit.

This state has $S_{\text{tot}}^z = \frac{1}{2}$ and contains a domain wall between two regions (of types “ a ” and “ b ”). It is not an exact eigenstate of the Hamiltonian, and is clearly a finite-energy excitation. From a variational point of view, it proves the existence of finite-energy spin- $\frac{1}{2}$ excitations in the thermodynamic limit. In a chain of finite length, one may act locally with S_2^+ on $|a\rangle$. The resulting state,

$$\cdots \otimes |[01]\rangle \otimes |\uparrow_2\rangle \otimes |\uparrow_3\rangle \otimes |[45]\rangle \otimes \cdots, \quad (4)$$

can be viewed as *two* spinons, with parallel spins, on sites 2 and 3. The spin flip has created a pair of spinons which may then propagate to large distances as two independent and elementary excitations.

Another very simple example of fractionalization in a 1D spin chain is the XY chain, which can be mapped exactly onto free, gapless, fermionic spinons by the Jordan-Wigner transformation [20]. Fractionalized (but interacting) spinons are also present in the spin- $\frac{1}{2}$ Heisenberg chain, and spin-charge separation is a general phenomenon in Tomonaga-Luttinger liquids. These examples illustrate that fractionalization is a rather common phenomenon in one dimension. However, ordered states do not in general support fractional excitations in $D > 1$. As an example, any attempt to “separate” two spinons in a $D > 1$ VBC will not lead to a two-spinon state when the entities are far apart. At sufficiently large distances, this results instead in two excitations with *integer* spins (see Fig. 2 in C. Lhuillier’s chapter). We will show that the mechanisms leading to fractionalization in $D > 1$ are very different from the “domain wall”, “soliton”, or “kink” picture valid in 1D.

In 2D, the most famous example of fractionalized systems is provided by fractional quantum Hall fluids. Here the elementary excitations carry an electric charge which is a fraction (for example $\frac{1}{3}$) of that of the electron. As above, local excitations may only have an integer charge. However, if an electron is added to a $\nu = \frac{1}{3}$ quantum Hall fluid, it will decay into three elementary quasiparticles of charge $+\frac{e}{3}$. The property that the system is fractional means that these quasiparticles can be placed far apart from each other with a *finite energy cost*. In the same way, a spin flip (changing S_{tot}^z from 0 to 1, as induced by a neutron scattering process) in the Majumdar-Ghosh chain would decay into two spinons, each carrying half a quantum of magnetization. There may (or may not) be some short-distance bound states between spinons, but the fact that the system is fractionalized means that one can separate two spin- $\frac{1}{2}$ excitations to infinite distances with only a finite energy cost. The spinons are then said to be deconfined. Spinons have the same spin as electrons, but do not carry an electric charge. In this sense, a spinon is a “fraction” of an electron.⁴ Removing an electron from a Mott insulator is equivalent to creating a charged hole and removing a spinon (spin). Magnets with deconfined spinons are thus closely related to the problem of spin-charge separation in doped Mott insulators. This said, it is not obvious how to describe effective

⁴There is, however, no *charge* fractionalization, as in the quantum Hall effect.

long-range spinon-spinon interactions in a model where the microscopic interactions are purely local. We will explain – at a rather qualitative level – that gauge theories provide a framework to deal with the question of deconfinement.

What is the connection between gauge theories and fractional spin liquids ?

To describe a quantum system with deconfined spinons, it is logical to seek a formalism including single-spinon creation and annihilation operators. On general grounds, such a formalism necessarily involves some *gauge fields*. A spinon creation operator changes the magnetization by $\pm\frac{1}{2}$, and thus cannot be written locally in terms of the spin operators \mathbf{S}_i . The usual choice is to decompose the spin operators into two spinon operators,

$$S_i^+ = c_{i\uparrow}^\dagger c_{i\downarrow} \quad , \quad 2S_i^z = c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow} \quad , \quad (5)$$

and to impose the constraint of one particle per site for all states in the physical Hilbert space,

$$c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} = 1 \quad \forall i. \quad (6)$$

In this review, we focus on the *fermionic*⁵ representation, $\{c_{i\sigma}^\dagger, c_{j\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}$. Acting with a single $c_{i\uparrow}^\dagger$ operator transforms a physical state into a non-physical one which violates the constraint above. To deal with this, physically it is clear that when inserting a spinon one must “shift” the spin state along some path on the lattice, ending at a point where another spinon is created or destroyed.⁶ Thus $c_{i\uparrow}^\dagger$ must “dress” with a “string” containing the path information. This will be the role of the gauge field.

Spin operators, and all physical states satisfying Eq. (6), are invariant under

$$c_{i\sigma}^\dagger \longrightarrow e^{-iA_i} c_{i\sigma}^\dagger \quad , \quad \sigma = \uparrow, \downarrow \quad , \quad A_i \in [0, 2\pi[. \quad (7)$$

In fact, invariance under this *gauge transformation* and Eq. (6) are equivalent.⁷ That $c_{i\uparrow}^\dagger$ transforms a physical state into a non-physical one arises

⁵To this point, the “bare” spinon operators can be chosen to be fermionic or bosonic. The actual statistics of the *physical* excitations should not depend on this arbitrary choice, which suggests that the fractionalized excitations are not always simply related to the bare creation operators introduced in Eq. (5).

⁶In a spin chain, there are only two ways to do this, to the right or to the left, but in $D > 1$, many paths are possible.

⁷The transformation of Eq. (7) can be implemented by the operator $\hat{U}(A) = \exp(i \sum_{i,\sigma=\uparrow,\downarrow} A_i c_{i\sigma}^\dagger c_{i\sigma})$. When applied on a state $|\psi\rangle$ satisfying Eq. (6), this operation gives only a global phase, $\hat{U}(A)|\psi\rangle = \exp(i \sum_i A_i)|\psi\rangle$. It is then convenient to redefine \hat{U} by $\hat{U}(A) = \exp(i \sum_i A_i (c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} - 1))$. Thus any state $|\psi\rangle$ obeying $\hat{U}(A)|\psi\rangle = |\psi\rangle$ (gauge invariance, for any A), must satisfy $(c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} - 1)|\psi\rangle = 0$ for any site i , which is precisely Eq. (6).

because it is not a gauge-invariant operator. The standard solution for this is to introduce⁸ a gauge-field operator A_{ij} on each bond of the lattice, which transforms according to $A_{ij} \rightarrow A_{ij} + A_i - A_j$, so that

$$c_{0\uparrow}^\dagger \exp(iA_{01} + iA_{12} + iA_{13} + \cdots + iA_{(n-1)n}) c_{n\uparrow} \quad (8)$$

is gauge-invariant. To understand the physical the meaning of this gauge field, from the discussion of spinons in valence-bond states one may anticipate that $\exp(iA_{01} + \cdots + iA_{(n-1)n})$ performs the “shift” operation required to insert or destroy a spinon at each end of the path connecting site 0 to site n .

At this stage, the gauge-field operators A_{ij} do not appear explicitly in the spin Hamiltonian [the Heisenberg model can be written using Eq. (5)]. Thus the transformations above do not yet deliver a gauge theory with a *dynamical* gauge field.⁹ However, for many purposes it is important only to derive an effective low-energy theory for the spinons, which is obtained by integrating over some high-energy degrees of freedom (such as gapped fluctuation modes). Such a procedure generally produces all of the local terms which are allowed by symmetry. The simplest terms involving the gauge field, and which are invariant under Eq. (7), are those of Maxwell type (*i.e.* analogs of the terms for magnetic and electric energy). The precise nature of the gauge field and its interaction terms depends on details of the spin model, and is (unfortunately) very difficult to predict from microscopic calculations. In some systems, the relevant gauge field will take angular values ($\in [0, 2\pi[$, known as a $U(1)$ gauge field) and in some other cases it is restricted to 0 or π (known as \mathbb{Z}_2). We refer the reader to Sec. 3 for more details, and to the review of Lee, Nagaosa and Wen [21] for a complete discussion.

From the example of electrodynamics, we know that gauge fields can mediate long-range interactions between electric charges (although the Hamiltonian is local). In the present context, the elementary “charges” are the spinons. Generally speaking, a gauge theory can have two kinds of phase: confined phases where excitations with non-zero charge cannot be spatially isolated from each other, and deconfined phases where isolated non-zero charges are finite-energy excitations. Confinement occurs when the flux B (defined as the circulation of A_{ij}) piercing the plaquettes of the lattice fluctuates strongly. In this case, the description of the spin system in terms of spinons interacting with a gauge field is formally correct but not of practical utility, because the gauge field generates an effective, long-range attraction (with a linear potential) between spinons, they are confined in gauge-neutral pairs (with an integer spin, like a magnon), and cannot be elementary excitations of the system.

The situation is qualitatively different if the gauge field is in a deconfined phase, which is realized when the flux fluctuations are small. In this case,

⁸A more formal construction is presented in Sec. 3.2.

⁹By comparison with the Maxwell term in electromagnetism, $\frac{1}{e^2} F^{\mu\nu} F_{\mu\nu}$, the Heisenberg model corresponds formally to infinite coupling, $e = \infty$, which is a non-trivial limit because the gauge-field fluctuations cost no energy and are therefore large.

the spinons (possibly “dressed” by interactions) are finite-energy states of the model, whose ground state is a fractionalized spin liquid.¹⁰ Thus the existence of fractionalized spin liquids may be formulated as a problem of confinement or deconfinement in certain types of lattice gauge theory coupled to spinons (we refer the reader again to Sec. 5 for details).

Topological order

For a conventional type of order associated with a discrete, spontaneous symmetry-breaking (as is the case for a VBC), several ground states $|1\rangle$, $|2\rangle$, \dots , $|d\rangle$ are degenerate in the thermodynamic limit. One may look for two (normalized) linear combinations $|a\rangle = \sum_{i=1}^d a_i|i\rangle$ and $|b\rangle = \sum_i b_i|i\rangle$ of the degenerate ground states, and for a local observable \hat{O} (acting on a finite number of sites) which acts to distinguish them, $\langle a|\hat{O}|a\rangle \neq \langle b|\hat{O}|b\rangle$. If such states $|a\rangle$, $|b\rangle$ and such an operator \hat{O} do exist (in the thermodynamic limit), \hat{O} is (by definition [22]) an order parameter for the broken symmetry.

If the topology of the lattice is non-trivial, as for a cylinder or torus, a gapped, fractionalized spin liquid will also exhibit a ground-state degeneracy, *even in the absence of any broken symmetry*. The crucial difference with conventional forms of order is that no *local* observable, \hat{O} , can distinguish the ground states in the thermodynamic limit (in $D > 1$). The ground-state degeneracy is suggestive of some form of order, but without an associated local order parameter. This type of non-local order has been named “topological order” in the pioneering works of X.-G. Wen [2, 3, 4]. This type of degeneracy is a consequence of fractionalization. The sequence of arguments is not mathematically rigorous, but rather simple and (hopefully) intuitive. We refer to Ref. [23] for a more precise discussion.

Consider a spin model with deconfined spinons as elementary excitations and periodic boundary conditions in one direction (taken to be x). Starting in a ground state $|1\rangle$, we i) create locally a pair of spinons, ii) move one of them around the cylinder, iii) annihilate this spinon with its partner, and iv) denote by $|2\rangle$ the resulting final state. Let us further define \hat{T} as the unitary operator describing this process, $|2\rangle = \hat{T}|1\rangle$. If the spectrum remains gapped during the (adiabatically slow) moving process, \hat{T} brings the system back to a ground state, which may however not be the same as the initial ground state. At each intermediate time step, the system contains two spinons. Such an intermediate state may be viewed as being obtained from a spinon-free state by applying some combination of “string” operators which connect the

¹⁰ *Confinement* should not be confused with the existence of *bound states*. As an example, protons and electrons have bound states, those of the hydrogen atom, but they are *not* confined by the electromagnetic gauge field: because they can be separated to infinite distance by only a finite input energy, they exist as isolated particles. The situation is different for quarks, which are confined by the QCD gauge field and cannot be observed as *isolated* particles at any energy.

two spinons, as in Eq. (8). In other words, even when they are far apart, the spinons remain “connected” by a gauge-field string. When the two spinons meet again and annihilate, there remains a non-contractible gauge-field “loop” winding around the cylinder. Intuitively, this is why the new ground state, $|2\rangle$, is generally *different* from $|1\rangle$. To make this schematic argument somewhat more precise, one introduces a second adiabatic process, in which a “twist” $\phi \in [0, 2\pi]$ is applied gradually to the system [24, 23]. This twist amounts to a modification of the boundary conditions for S_i^+ (and S_i^-): $S_{x+L_x, y}^+ \equiv S_{x, y}^+ e^{i\phi}$. Up to a unitary transformation U , the spectrum at $\phi = 0$ is identical to that at $\phi = 2\pi$: $\mathcal{H}_{\phi=2\pi} = U^\dagger \mathcal{H}_{\phi=0} U$. Again we assume that the operation of switching ϕ from 0 to 2π may be performed adiabatically without closing the excitation gap, in which case it defines a unitary operator \hat{F} which transforms ground states (of $\mathcal{H}_{\phi=0}$) into ground states (of $\mathcal{H}_{\phi=2\pi}$). Accordingly, $U\hat{F}$ acts in the ground-state manifold of $\mathcal{H}_{\phi=0}$. Finally, one can show that the two adiabatic processes satisfy $(U\hat{F})\hat{T} = -\hat{T}(U\hat{F})$ [23], because one spinon winding around the cylinder in the presence of a twist $\phi = 2\pi$ experiences an Aharonov-Bohm phase equal to $e^{i\phi/2} = -1$ (measured by the gauge-field loop mentioned above). Clearly, this relation cannot be satisfied in the ground-state manifold unless the degeneracy is at least 2.

2.4 Half-odd-integer spins and the Lieb-Schultz-Mattis-Hastings theorem

We consider a lattice spin system with periodic boundary conditions, short-range interactions, conserved S_{tot}^z (global U(1) symmetry) and a half-odd-integer spin (*e.g.* $1/2$) in the unit cell. The lattice dimensions L_1, L_2, \dots, L_D are taken to be such that each “section” perpendicular to direction 1 has an odd number ($L_2 \times \dots \times L_D$) of unit cells, and thus has a half-odd-integer spin. The theorem states that, in the thermodynamic limit, the spectrum cannot simultaneously satisfy the two conditions: i) unique ground state; ii) finite gap to all excitations. Although the proof is quite simple in 1D [6] its generalization to higher dimensions [25], due to Hastings, is quite involved [7, 8]. The argument proposed by Oshikawa [24] is less general,¹¹ but its simplicity offers deep insight into the LSM theorem for $D > 1$.

What is the relation with the above discussion of the QSL? A conventional reason for a degeneracy of the ground state is spontaneous symmetry-breaking (SBB).¹² However, ordered states do not generally support fractional excitations (consider a VBC), and are thus not QSLs according to the third defini-

¹¹It assumes that the gap does not close when twisting the boundary conditions.

¹²A magnetically ordered system satisfies the theorem because the spectrum is gapless due to the presence of a spontaneously broken continuous symmetry (Goldstone modes). A VBC has a gapped spectrum, but the ground state is degenerate in the thermodynamic limit, due to translational symmetry-breaking.

tion.¹³ Thus, in the absence of any SSB, one might conclude that the ground state is unique and should therefore sustain *gapless* excitations in order to satisfy the LSM theorem. This is indeed a possibility (the *algebraic* QSL of Sec. 5), but the LSM theorem allows another alternative: gapped excitations above a degenerate ground state *without SSB*. In such a case, the states in the degenerate ground-state manifold are locally identical (indistinguishable by any local order parameter) but globally different due to the topological order [4] discussed in the previous paragraph. The LSM theorem is useful because it provides a natural classification for the ground states. A half-odd-integer-spin system is either i) conventionally ordered (SSB), ii) a gapless QSL, or iii) a topologically ordered, gapped QSL. Only integer-spin systems have the additional possibility of being iv) “quantum paramagnets” (non-degenerate ground state and gapped excitations, as discussed in Sec. 2.2).

3 Mean fields and gauge fields

We review here a formalism for describing deconfined liquids in Heisenberg models, and discuss the possible emergence of gauge fields. The origin of this approach lies in the slave-particle approaches to the Hubbard and t - J models [21].

3.1 Fermionic representation of Heisenberg models

The group $SU(2)$ can act on the spinon operators of Eqs. (5-6) in two different ways, globally, in describing spin rotations, and locally, related to the (gauge) redundancy of the description of spin operators.

Spin rotations — A *global spin rotation* is effected by multiplying the doublet $d_1 \equiv [c_{i\uparrow} \ c_{i\downarrow}]$ to the right by an $SU(2)$ matrix V , $d_1 = [c_{i\uparrow} \ c_{i\downarrow}] \rightarrow d_1 V$. By taking the Hermitian transpose of d_1 and using $V^\dagger = V^{-1}$, one may show that $d_2 = [c_{i\uparrow}^\dagger \ -c_{i\downarrow}^\dagger]$ is also transformed by a right-multiplication: $d_2 \rightarrow d_2 V$. Thus d_1 and d_2 may be grouped into a 2×2 matrix which transforms under $SU(2)$ rotations by right-multiplication,

$$\psi_i = \begin{bmatrix} c_{i\uparrow} & c_{i\downarrow} \\ c_{i\uparrow}^\dagger & -c_{i\downarrow}^\dagger \end{bmatrix} \rightarrow \psi_i V. \quad (9)$$

From Eq. (9), $\psi_i \psi_j^\dagger$ is manifestly invariant. This allows one to introduce two rotation-invariant operators, χ_{ij} and η_{ij} , for each pair of sites in the system,

$$\psi_i \psi_j^\dagger \equiv \begin{bmatrix} c_{i\uparrow} c_{j\uparrow}^\dagger + c_{i\downarrow} c_{j\downarrow}^\dagger & c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \\ c_{i\downarrow} c_{j\uparrow}^\dagger - c_{i\uparrow} c_{j\downarrow}^\dagger & c_{i\downarrow} c_{j\downarrow} + c_{i\uparrow} c_{j\uparrow}^\dagger \end{bmatrix} = \begin{bmatrix} -\chi_{ij}^\dagger & -\eta_{ij}^\dagger \\ -\eta_{ij} & \chi_{ij} \end{bmatrix}. \quad (10)$$

¹³In principle there can be coexistence of some conventional order *and* fractional excitations; this possibility is ignored here.

The quantity ψ_i also gives a convenient expression of the spin operators and of the constraint,

$$S_i^a = \frac{1}{2} \text{Tr} \left[\psi_i^\dagger \psi_i (\sigma^a)^T \right], \quad a = x, y, z, \quad (11)$$

$$\text{Tr} \left[\psi_i^\dagger \sigma^z \psi_i \right] = c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow} = c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} - 1 = 0. \quad (12)$$

It is useful here to add two other constraints which are consequences of the first, $c_{i\downarrow} c_{i\uparrow} = 0 = \eta_{ii}^\dagger$ (no double site occupancy) and $c_{i\downarrow}^\dagger c_{i\uparrow}^\dagger = 0 = \eta_{ii}$ (no empty sites). Together, the three constraints can be written in the compact form

$$\text{Tr} \left[\psi_i^\dagger \sigma^a \psi_i \right] = 0, \quad a = x, y, z. \quad (13)$$

Gauge transformations — Because of Eq. (6), $c_{i\uparrow}$ and $c_{i\downarrow}^\dagger$ have the same physical effect, namely of decreasing S_i^z by one unit. They can be placed in a doublet, p_1 , upon which $SU(2)$ matrices act without changing the physical spin operators. Let W_i be a (site-dependent) $SU(2)$ matrix encoding this *gauge transformation*, $p_1 = \begin{bmatrix} c_{i\uparrow} \\ c_{i\downarrow}^\dagger \end{bmatrix} \rightarrow W_i p_1$. It is easy to verify that $p_2 = \begin{bmatrix} c_{i\downarrow} \\ -c_{i\uparrow}^\dagger \end{bmatrix}$ transforms by the same left-multiplication. Taken together, these two-column vectors p_1 and p_2 form once again the matrix ψ_i . From Eq. (11), it is evident that the spin operators are gauge-invariant,

$$\psi_i \rightarrow W_i \psi_i, \quad S_i^a \rightarrow S_i^a. \quad (14)$$

In summary, global spin rotations are described by right-multiplication of ψ and local gauge transformations by left-multiplication. As a specific example of a gauge transformation, we consider the $U(1)$ subgroup of $SU(2)$, which is parameterized by the phase $\Lambda(i)$ as $W_i = \exp(i\Lambda(i)\sigma^z)$, and which corresponds to Eq. (7). A spinon (“anti-spinon”) carries a charge +1 (-1) of this $U(1)$ gauge field. The spin-flip operator $S_i^+ = c_{i\uparrow} c_{i\downarrow}^\dagger$, is *gauge-neutral*.

3.2 Local $SU(2)$ gauge invariance

In a path-integral formulation based on the fermionic representation, the imaginary-time Lagrangian takes the form [11]

$$L = \sum_i \text{Tr} \left[\psi_i^\dagger (\partial_\tau + \mathbf{A}_i^0 \cdot \boldsymbol{\sigma}) \psi_i \right] - H, \quad (15)$$

where H is the Hamiltonian and $\mathbf{A}_i^0 = (A_{ix}^0, A_{iy}^0, A_{iz}^0)$ a real, three-component vector which plays the role of a Lagrange multiplier for the three constraints of Eq. (13). Consider a time-dependent gauge transformation

$$\psi_i(\tau) \rightarrow W_i(\tau) \psi_i(\tau). \quad (16)$$

To ensure the invariance of the action (15), \mathbf{A}_i^0 must transform as the time component of an $SU(2)$ gauge field,

$$\mathbf{A}_i^0 \cdot \boldsymbol{\sigma} \rightarrow W_i(\tau) (\partial_\tau + \mathbf{A}_i^0 \cdot \boldsymbol{\sigma}) \cdot W_i^\dagger(\tau). \quad (17)$$

From Eq. (11), the Heisenberg interaction can be written as

$$\mathbf{S}_i \cdot \mathbf{S}_j = -\frac{1}{8} \text{Tr} \left[\psi_i \psi_j^\dagger \psi_j \psi_i^\dagger \right], \quad (18)$$

a quartic term in fermionic operators which can be decoupled (Hubbard-Stratonovich procedure) by introducing a 2×2 complex matrix U_{ij} on each bond $\langle i, j \rangle$. The corresponding contribution to the Lagrangian is

$$-H = -\frac{8}{J} \sum_{\langle i, j \rangle} \text{Tr} \left[U_{ij}^\dagger U_{ij} \right] - \sum_{\langle i, j \rangle} \text{Tr} \left[\psi_i^\dagger U_{ij} \psi_j + \text{H.c.} \right], \quad (19)$$

whence a Gaussian integration over U_{ij} returns the spin-spin interaction of Eq. (18). From Eq. (9), it is clear that U is invariant under spin rotations, and from Eq. (14) one observes that U transforms as the spatial component of an $SU(2)$ gauge field under gauge transformation,

$$\psi_i(\tau) \rightarrow W_i(\tau) \psi_i(\tau), \quad U_{ij}(\tau) \rightarrow W_i(\tau) U_{ij}(\tau) W_j^\dagger(\tau). \quad (20)$$

3.3 Mean-field (spin-liquid) states

Mean-field Hamiltonian — Various mean-field approximations may be applied when the Heisenberg model is expressed in the form of Eqs. (15) and (19). As we will show, these can describe a large variety of spin-liquid states, and in particular the “RVB spin liquids.” The procedure is to replace the fluctuating fields $U_{ij}(\tau)$ and $\mathbf{A}_i^0(\tau)$ by time-independent, complex matrices U_{ij}^0 and complex vectors \mathbf{a}_i^0 , because the mean-field Hamiltonian is then quadratic, and hence soluble, in the fermion operators,¹⁴

$$\begin{aligned} H_{\text{MF}} = & \frac{8}{J} \sum_{\langle i, j \rangle} \text{Tr} \left[U_{ij}^{0\dagger} U_{ij}^0 \right] + \sum_{\langle i, j \rangle} \text{Tr} \left[\psi_i^\dagger U_{ij}^0 \psi_j + \text{H.c.} \right] \\ & + \sum_i \text{Tr} \left[\psi_i^\dagger (\mathbf{a}_i \cdot \boldsymbol{\sigma}) \psi_i \right]. \end{aligned} \quad (21)$$

The first term is a constant, the second describes spinon hopping and pairing, and the third term arises from the constraints. Minimizing the energy with respect to the parameters U_{ij}^0 and \mathbf{a}_i gives the self-consistency conditions

¹⁴This approximation is equivalent to particular large- N limits of the model, obtained when the spin-rotation symmetry group $SU(2)$ is generalized to $SU(N)$ [26, 27]. With bosonic operators instead of fermions, this type of mean-field approximation is closely related to “Schwinger-boson” approaches [28, 29, 30, 31, 32].

$$U_{ij}^0 = \frac{J}{8} \langle \psi_i \psi_j^\dagger \rangle = \begin{bmatrix} -\chi_{ij}^{0*} & -\eta_{ij}^{0*} \\ -\eta_{ij}^0 & \chi_{ij}^0 \end{bmatrix}, \quad \langle \text{Tr} [\psi_i^\dagger (\mathbf{a}_i \cdot \boldsymbol{\sigma}) \psi_i] \rangle = 0, \quad (22)$$

where we have used the notation $\chi_{ij}^0 \equiv \langle \chi_{ij} \rangle = \langle c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow} \rangle$ and $\eta_{ij}^0 \equiv \langle \eta_{ij} \rangle = \langle c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger \rangle$. From Eq. (20), the parameters U_{ij}^0 and \mathbf{a}_i are not gauge-invariant,¹⁵ so different mean-field parameters may lead to the same physical quantities. This will have important consequences in Sec. 3.4.

Ground state and excitations of H_{MF} — Equation (21) describes a system of free spinons. The ground state is obtained by calculating the spinon band structure and filling the negative-energy single-particle states. The resulting state satisfies the constraints [Eq. (13)] only on average, and is therefore not a valid spin- $\frac{1}{2}$ wave function. One way to obtain a spin state is to apply a Gutzwiller projection in order to remove configurations with empty or doubly occupied sites [33]. This can be performed numerically by using Monte Carlo methods [34, 33]. Another approach is to analyze the qualitative effects of fluctuations, based on symmetry arguments.

Because U_{ij} and \mathbf{A}^0 are invariant under spin rotations, H_{MF} does not have any preferred direction in spin space. The mean-field ground state is thus a total-spin singlet, without magnetic long-range order. It is already a spin-liquid state in the sense of definition 1 (Sec. 2.1). This type of mean-field approach is not appropriate to describe Néel-ordered phases; bosonic representations of the spin are more appropriate, because bosons may *condense*.

In addition, H_{MF} contains spin- $\frac{1}{2}$ excitations, obtained by adding or removing one spinon to or from the ground state. A crucial question is whether the existence of “deconfined” (in fact, free at this crude level of approximation) spinons is merely an artefact of the mean-field approximation. In such a case, the inclusion of fluctuations (in particular of gauge-field fluctuations) would confine the spinons. The other possibility is that the spinons remain deconfined in the presence of fluctuations (or Gutzwiller projection), in which case the mean-field approximation is indeed a useful starting point for an accurate description.¹⁶

¹⁵ $\langle \psi_i \psi_j^\dagger \rangle \neq 0$ is in apparent contradiction with Elitzur’s theorem, which states that non-gauge-invariant quantities should average to zero. Some slight abuse of notation has been committed here, as true expectation values in the mean-field theory should be averaged over all gauge-rotated copies of a given representative state.

¹⁶A similar question arises concerning the presence of a gap in the excitation spectrum. H_{MF} can be gapless, as in the π -flux example below. It is then important to understand whether fluctuations beyond the mean-field approximation can act to open a gap. In some cases, the spectrum is expected to remain gapless, although fluctuations will in general change the correlation exponents. This is the case if the terms which could potentially open a gap (terms relevant in the renormalization-group sense) are actually forbidden by gauge invariance or by symmetry (Sec. 3.4). Then the mean-field approximation is again a good starting point to describe a gapless spin liquid.

Example: the “ π -flux” state — Consider the mean-field state on the square lattice introduced by Affleck and Marston [27], which is equivalent to the “mixed $s+id$ ” RVB state [35] and labeled “SU2B n 0” in Wen’s classification [36]. This state has $\mathbf{a}_i = 0$, $\eta_{ij} = 0$, and a modulus $|\chi_{ij}| = \chi_0$ identical on all bonds. The phases $\theta_{ij} = \arg(\chi_{ij})$ are such that $\theta_{12} + \theta_{23} + \theta_{34} + \theta_{41} = \pi \pmod{2\pi}$ on every square plaquette. As in the case of hopping amplitudes for a charged particle in the presence of a uniform magnetic field, there is no gauge in which θ_{ij} is translation-invariant. The unit cell defined by θ_{ij} contains at least *two* sites. A possible choice is $\theta_{ij} = \frac{\pi}{4}(-1)^{i_x+j_y}$, corresponding to $U_{ij}^0 = i\chi_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for bonds $i \rightarrow j$ oriented as in Fig. 1. The corresponding mean-field Hamiltonian is

$$H_{\text{MF}} = 2\chi_0 \sum_{\langle i \rightarrow j \rangle, \sigma=\uparrow, \downarrow} \left(ic_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right) + \text{constant}, \quad (23)$$

and gives two bands of quasiparticles with dispersion relations [27]

$$E^\pm(k) = \pm 4\chi_0 \sqrt{\cos(k_x)^2 + \cos(k_y)^2} \quad (24)$$

(in the Brillouin zone defined by $|k_x| + |k_y| \leq \pi$). The Fermi energy is at $E = 0$,¹⁷ where the two bands meet at $\mathbf{k}_A = (\pi/2, \pi/2)$ and $\mathbf{k}_B = (\pi/2, -\pi/2)$. To describe the long-distance properties of the system, it is useful to focus on low-energy excitations and to linearize the spectrum in the vicinity of \mathbf{k}_A and \mathbf{k}_B . The corresponding Hamiltonian is that for *four* fermion flavors (two for the spin and two for the A - or B -“valley” index) of two-component¹⁸ Dirac fermions.

Because spinons are necessarily created in pairs, we show in Fig. 1 the energy of the two-spinon continuum, $E^+(\mathbf{k} - \mathbf{q}) + E^+(\mathbf{q})$, as a function of \mathbf{k} . These excitations are gapless and linearly dispersive around the four minima (located at $\mathbf{k} = (0, 0)$, $(\pi, 0)$, $(0, \pi)$, and (π, π)). As we will discuss in Sec. 3.4, the presence of gapless fluctuation modes around this mean-field state means that the stability of the mean-field approximation is *a priori* not at all clear. A somewhat involved analysis suggests, however, that it could indeed represent a stable spin-liquid phase with gapless magnetic excitations (remnants of the excitations discussed above) and algebraic correlations [37].

Dimerized mean-field states — Among all the different self-consistent mean-field states, the “dimerized” states have the lowest energy at the mean-field level for a large class of lattices [38]. Such a mean-field solution can be viewed as a hard-core dimer covering of the lattice (Fig. 2): $\chi_{ij}^0 = J_{\text{max}}/2$ on the bonds occupied by a “dimer” and $\chi_{ij}^0 = 0$ otherwise, while $\eta_{ij}^0 = 0$ everywhere. The configuration of “dimers” is such that each site is touched by exactly one dimer, and only occupies the bonds (ij) with the strongest

¹⁷In agreement with Eq. (22), the system is half-filled.

¹⁸There are *two* zero-energy single-particle states when $E^+ = E^-$.

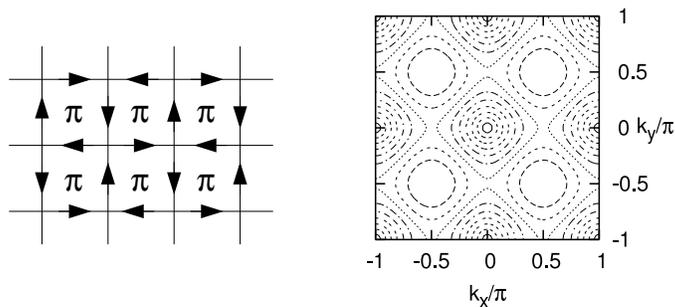


Fig. 1. Left: bond orientations used to define the mean-field π -flux state on the square lattice ($\chi_{i \rightarrow j} = -\chi_{j \leftarrow i} = i\chi_0 \in i\mathbb{R}$) [27]. Right: minimum energy for a pair of spinons with total momentum $\mathbf{k} = (k_x, k_y)$.

antiferromagnetic exchange, $J_{ij} = J_{\max}$. Because the number of such dimer coverings is (usually) an exponential function of the number of sites, these mean-field solutions are massively degenerate. However, we expect that the *fluctuations* of the field U will lift this degeneracy. Indeed, if these fluctuations are treated perturbatively ($1/N$ expansions), one obtains an effective model [39, 40] in the subspace of dimerized states, known as a quantum dimer model [41]. Fluctuations may also lower the energy of some other (undimerized) mean-field solutions, and certain types of solution may also be stabilized by other interactions (notably by ring-exchange terms). For these reasons, it is very important to study mean-field states which are *locally stable* even if they are not global energy minima at the mean-field level; on this point we comment that it is much more significant to compare energies *after* Gutzwiller projection.

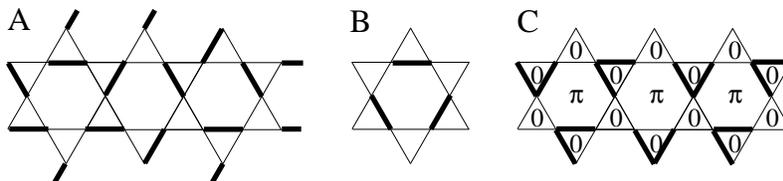


Fig. 2. Examples of mean-field states on the kagome lattice. A: dimerized state with $\chi_{ij}^0 = J/2$ on the thick bonds and $\chi_{ij}^0 = 0$ on all other bonds. Such states are the (degenerate) lowest-energy states at the mean-field level [38]. B: state including fluctuations $\delta\chi_{ij}$ of the bond field χ_{ij} at order $(\delta\chi_{ij})^3$ lifts this degeneracy in favor of configurations maximizing the number of hexagons with three “dimers” [40]. C: $[0, \pi]$ -flux phase on the kagome lattice [42, 43]. Thin (thick) bonds have $\chi_{ij} = +\chi_0$ ($\chi_{ij} = -\chi_0$) $\in \mathbb{R}$, such that the flux is 0 on triangles and π on hexagons.

3.4 Gauge fluctuations

For a given mean-field solution, its stability must be verified by investigating the low-energy fluctuation modes. In a large N formalism the mean-field solution is exact at $N = \infty$ and the question to be addressed is: does the phase found at $N = \infty$ survive at large but *finite* N , that is when the fluctuations modes of the fields $\mathbf{A}_i^0(\tau)$ and $U_{ij}(\tau)$ are introduced. In other words, do we have a (quantum) phase transition between the (mean-field) state at $N = \infty$ and $1 \ll N < \infty$. If any mode or modes drive(s) the system to an instability, the mean-field approximation is not an appropriate starting point and the physical properties of the system (presence/absence of a gap, broken symmetries, excitations) cannot be those of the mean-field Hamiltonian. On the other hand, if no modes lead to a divergence of physical quantities, the sign of an instability, the mean-field solution may describe, to some extent, a real phase of the spin model (at least for large enough N). Of course, it is difficult to examine all possible fluctuations, as this would be equivalent to solving the original spin model. As a first approximation, gapped degrees of freedom may be integrated over (or simply ignored), as they are expected to play no qualitative role at low energies. By contrast, gapless modes are potential sources of instability, and thus are likely to influence the low-energy, long-distance physics of the system.

Fermion density fluctuations may be gapped or gapless, depending on the spectrum of H_{MF} . When these modes are gapped, the fermions form an incompressible state at the mean-field level, and the density fluctuations are expected to have no effect on the long-distance properties of the system. There are three simple cases [4] where such a thing happens: i) the bond parameters U_{ij}^0 break the translational invariance in such a way that the ground state of H_{MF} is a band insulator (Fermi level between a completely filled and a completely empty band), as in the dimerized solutions of Ref. [38]; ii) H_{MF} contains a pairing term ($\eta \neq 0$) so that its ground state is a BCS-like gapped “superconductor” [4]; iii) H_{MF} contains a non-trivial flux (*i.e.* different from 0 or π) piercing some of the plaquettes, and its ground state is analogous to a set of completely filled Landau levels, as in the integer Hall effect [44].

Gauge excitations with a continuous gauge group are also natural candidates for gapless excitations. The reason for this is that the gauge invariance forbids “naïve” mass terms, such as $(A^\mu)^2$, for the gauge field in the same way that it preserves the masslessness of ordinary photons; however, more elaborate mechanisms, such as Anderson-Higgs, may still open a gap. Gauge excitations are also important because they can mediate long-range interactions between the spinons. The importance of gauge modes in slave-boson mean-field theory was first put forward by Baskaran and Anderson [45].

The spin models of interest here have a local $SU(2)$ gauge invariance (Sec. 3.2). Thus one may ask why the fluctuations are not always described by an $SU(2)$ gauge field, or alternatively, why the nature of the gauge field does actually depend on the particular mean-field state. The answer is that the

mean-field parameters U_{ij}^0 break partially the local $SU(2)$ gauge invariance (unless all the U_{ij}^0 are diagonal matrices on all bonds), leaving only a lower invariance symmetry. This is in a sense analogous to global symmetries, where the number of Goldstone modes depends on the number of broken continuous symmetries. Following Wen [4, 36], we will discuss how to construct the gauge fields describing fluctuations about a given mean-field state.

Projective symmetry group and invariant gauge group

Because the mean-field parameters U_{ij}^0 are not gauge-invariant, two apparently different solutions may be physically identical. A set of parameters U_{ij}^0 which are not translationally invariant may describe a (mean-field) QSL with no broken symmetry.

As an example, consider the π -flux state defined by Eq. (23). Under any translation by one lattice constant (in the x or y direction), U_{ij}^0 is changed into $\tilde{U}_{ij}^0 = -U_{ij}^0$. However, the gauge transformation associated with $W_i = (-1)^{i_x+i_y}\mathbb{I}$ maps \tilde{U}^0 back to U^0 , and thus both U^0 and its translation, \tilde{U}^0 , label the same mean-field state.¹⁹ This result illustrates that the physical symmetries are encoded in a non-trivial way in the mean-field parameters U_{ij}^0 . In fact this is a fundamental property, inherent to any description of the system in terms of fractional excitations (in this case spinons): the Hamiltonian describing the hopping of the spinons requires a gauge choice and is apparently less symmetric than the original spin model. This led X.-G. Wen [36] to introduce the concept of Projective Symmetry Group (PSG).

Definition of the PSG — Let $T : i \mapsto T(i)$ be a lattice symmetry of the original spin model and W be a (time-independent) gauge transformation [Eq. (20)]. The PSG associated with the mean-field parameters $\{U_{ij}^0, \mathbf{a}_i\}$ is defined as the set of all the pairs (T, W) satisfying

$$U_{ij}^0 = W_i U_{T(i)T(j)}^0 W_j^\dagger, \quad \mathbf{a}_i \cdot \boldsymbol{\sigma} = W_i (\mathbf{a}_{T(i)} \cdot \boldsymbol{\sigma}) W_i^\dagger \quad \forall i, j. \quad (25)$$

An element of the PSG is thus a lattice symmetry followed by a gauge transformation, such that the mean-field parameters U_{ij}^0 and \mathbf{a}_i are unchanged. In the “ π -flux” example above, the PSG contains (among other elements) the translations by one unit cell associated with $W_i = \exp(i\Lambda_i \sigma^z)$.

The invariant gauge group (IGG) [36] is a special subgroup of the PSG, containing all the elements (T, W) where T is the identity. As we will show, *the IGG determines the gauge group, and therefore the nature of the gauge fluctuations around the mean-field solution.*

We consider a mean-field state and denote by \mathcal{I} its IGG. We first assume for simplicity that \mathcal{I} is isomorphic to $U(1)$. In such a case, the gauge transformations $W^\theta \in \mathcal{I}$ can be parameterized as

$$W^\theta : i \mapsto W_i^\theta = \exp(i\theta \mathbf{n}_i \cdot \boldsymbol{\sigma}), \quad (26)$$

¹⁹They give the same spin- $\frac{1}{2}$ wave function after Gutzwiller projection.

where W_i^θ is an $SU(2)$ rotation of angle $\theta \in [0, 2\pi[$ about the axis defined by the (spatially varying) unit vector \mathbf{n}_i . At each site, we rotate \mathbf{n}_i to the z axis,

$$V_i W_i^\theta V_i^\dagger = \exp(i\theta \sigma^z). \quad (27)$$

These elements $V_i \in SU(2)$ define a new gauge, in which U^0 becomes \tilde{U}^0 , defined by

$$\tilde{U}_{ij}^0 = V_i U_{ij}^0 V_j^\dagger, \quad (28)$$

and certain types of fluctuation of \tilde{U} about \tilde{U}^0 may be parameterized by a real field A ,

$$\tilde{U}_{ij} = \tilde{U}_{ij}^0 e^{iA_{ij}\sigma^z}. \quad (29)$$

We will now show that A is the spatial component of a $U(1)$ gauge field, and is thus potentially important in describing the low-energy excitations of the system. We consider a particular family of gauge transformations, $i \mapsto \exp[i\theta(i)\sigma^z]$, where the angle θ parameterizing the elements of the IGG [Eq. (27)] has been promoted to a *local* variable $\theta(i)$. The bond field \tilde{U}_{ij} transforms according to $\tilde{U}_{ij} \rightarrow e^{i\theta(i)\sigma^z} \tilde{U}_{ij} e^{-i\theta(j)\sigma^z}$, whence only a few short algebraic manipulations are required²⁰ to cast \tilde{U}_{ij} in the form of Eq. (29), with the replacement $A_{ij} \rightarrow A_{ij} + \theta(i) - \theta(j)$. Thus these phase fluctuations of the bond variables are those of a $U(1)$ gauge field. In some cases [36], several subgroups of the IGG are isomorphic to $U(1)$. Each one can be parameterized as in Eq. (26), but with different directions $\mathbf{n}_i^1, \mathbf{n}_i^2, \dots$. Repeating this construction for each subgroup leads to the same number of $U(1)$ gauge fields. We note finally that the IGG always contains the group \mathbb{Z}_2 , because the gauge transformation $W : i \mapsto -\mathbb{I}$ leaves all U_{ij}^0 unchanged. The construction of the associated \mathbb{Z}_2 gauge field, $A_{ij} \in \{0, \pi\}$, is identical to the $U(1)$ case, except for the restriction $\theta(i) \in \{0, \pi\}$.

²⁰ One uses Eqs. (27), (28), and (29) to transform the right-hand side,

$$e^{i\theta(i)\sigma^z} \tilde{U}_{ij} = e^{i\theta(i)\sigma^z} \tilde{U}_{ij}^0 e^{iA_{ij}\sigma^z} \quad (30)$$

$$= e^{i\theta(i)\sigma^z} V_i U_{ij}^0 V_j^\dagger e^{iA_{ij}\sigma^z} \quad (31)$$

$$= V_i W_i^{\theta(i)} U_{ij}^0 V_j^\dagger e^{iA_{ij}\sigma^z}, \quad (32)$$

and then employs the fact that, for any angle θ , W^θ belongs to the IGG of the mean field U^0 . Setting $\theta = \theta(i)$ yields $W_i^{\theta(i)} U_{ij}^0 = U_{ij}^0 W_j^{\theta(i)}$, and finally

$$\tilde{U}_{ij} \rightarrow V_i U_{ij}^0 W_j^{\theta(i)} V_j^\dagger e^{iA_{ij}\sigma^z} e^{-i\theta(j)\sigma^z} \quad (33)$$

$$= V_i U_{ij}^0 V_j^\dagger e^{i\theta(i)\sigma^z} e^{iA_{ij}\sigma^z} e^{-i\theta(j)\sigma^z} \quad (34)$$

$$= \tilde{U}_{ij}^0 e^{i\theta(i)\sigma^z} e^{iA_{ij}\sigma^z} e^{-i\theta(j)\sigma^z}. \quad (35)$$

Two simple examples of IGG

As a first exercise, one may determine the IGG of the π -flux state on the square lattice, where $U_{ij}^0 = \pm i\chi_0$ is proportional to the identity. By definition, an element of the IGG is a set of matrices $W_i \in SU(2)$ satisfying $W_i U_{ij}^0 W_j^\dagger = U_{ij}^0$ on all bonds. From the particular form of U^0 , this equality becomes $W_i W_j^\dagger = \mathbb{I}$, showing that W_i must be the same (arbitrary) $SU(2)$ matrix on every site. We have thus shown that the IGG of this spin-liquid state is isomorphic to $SU(2)$.

As a second example, one may consider the spin liquid proposed by Hastings [42] and Ran *et al.* [43] for the kagome-lattice Heisenberg model. At the mean-field level, the bond field takes the values $U_{ij}^0 = \pm\chi_0 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \pm\chi_0 \sigma^z$, with the signs chosen to produce a flux 0 for each triangle and π for each hexagon (Fig. 1). From this particular form of U^0 , the condition on $W \in \mathcal{I}$ may be expressed as $W_j = \sigma^z W_i \sigma^z$. Thus if W_i is specified at any point i , all the other matrices are fixed. By propagating this condition around any triangle of the lattice, one finds $W_0 = (\sigma^z)^3 W_0 (\sigma^z)^3 = \sigma^z W_0 \sigma^z$, *i.e.* W_0 must commute with σ^z , and therefore has the form of Eq. (26) with $\mathbf{n} = [0, 0, 1]$. Thus $\mathcal{I} = U(1)$.

PSG beyond the mean-field approximation

Thus far, we have defined the PSG as the symmetry of the mean-field Hamiltonian (through U_{ij}^0 ; the time component \mathbf{a} is omitted hereafter for simplicity). The utility of the PSG is, however, that it is robust to fluctuations, at least at the perturbative level [36]. We denote by $\mathcal{L}(\psi, U)$ the exact Lagrangian of the spin model, in terms of the fermions $\psi_i(\tau)$ and bond fields $U_{ij}(\tau)$ introduced in Sec. 3.2. \mathcal{L} is invariant under any gauge transformation W and lattice symmetry T : $\mathcal{L}(\psi, U) = \mathcal{L}(P\psi P^{-1}, PUP^{-1})$, where $P = (W, T)$ need not be in the PSG of U_0 . However, if $P \in \text{PSG}$, then $U_0 = PU_0 P^{-1}$, and the Lagrangian \mathcal{L}^0 describing the fluctuations, defined by $\mathcal{L}^0(\psi, \delta U) = \mathcal{L}(\psi, U_0 + \delta U)$, is invariant: $\mathcal{L}^0(\psi, \delta U) = \mathcal{L}(\psi, U_0 + \delta U) = \mathcal{L}(P\psi P^{-1}, P(U_0 + \delta U)P^{-1}) = \mathcal{L}(P\psi P^{-1}, U_0 + P(\delta U)P^{-1}) = \mathcal{L}^0(P\psi P^{-1}, P\delta U P^{-1})$. In addition, the mean-field ground state and the mean-field Hamiltonian are also symmetric under the PSG, because the expectation values $\langle \psi_i \psi_j^\dagger \rangle = U_0$ are by definition PSG-invariant. Thus *the theory for the fluctuations about a given mean-field U_0 has the symmetry group of its PSG.*

We appeal now to the general property that, unless a phase transition occurs, broken or unbroken symmetries do not change their nature under the inclusion of perturbations. The ground state will receive some corrections once fluctuations are included but, on the assumption that the mean-field state exists within a *stable phase*, the symmetries will not change. The PSG is not only a property of the mean field, but is a property of the whole *phase*. Different spin liquids may be distinguished and hence classified by the symmetry

group of the Hamiltonian driving their spinon dynamics, which can be read from the PSG of the appropriate mean-field starting point, H_{MF} .

This symmetry principle has important consequences. To obtain an effective description of $\mathcal{L}^0(\psi, \delta U)$, some high-energy modes can be integrated out formally. One may, for example, eliminate the fermionic modes which are far from the Fermi level. In the example of the π -flux state (Eq. (23)), such a procedure leads to Dirac fermions with a linear dispersion relation. One may also integrate out the amplitude fluctuations of the bond variables U_{ij} , retaining only the phase fluctuations (which in the example of Eq. (23) is equivalent to neglecting the fluctuations of $|\chi_0|$). Under the assumption that no change of symmetry occurs when including these fluctuations, any term (even if it is invariant under all lattice symmetries) which is not PSG-invariant cannot appear in the effective action \mathcal{L}^0 .

Consider the example of a mean-field Hamiltonian where the spinons are gapless at N_f points in the Brillouin zone, and are described by $2N_f$ flavors of two-component Dirac fermions after linearization (the factor of 2 given by the spin \uparrow, \downarrow). The corresponding fermion operators, $\Psi_i^{\alpha=1 \dots 2N_f}$, are linear combinations of the microscopic spinon operators, $c_{i,\sigma}$, and each element of the PSG is equivalent to a particular transformation of the Ψ operators (a detailed example is presented in Ref. [46]). The effective action for the Dirac fermions (and their associated gauge field) is obtained by a formal integration over higher-energy degrees of freedom. During this process, terms which are quadratic in Ψ may *a priori* be generated and open a gap (which would spoil the algebraic nature of the spin correlations). However, from the discussion above, such terms are constrained to be invariant under the PSG. In certain cases [37, 43, 46], one may show that none of the possible terms arising this way is PSG-invariant. Such terms cannot be generated by integrating out the fluctuations (particularly of U_{ij}) perturbatively, and the system may remain gapless.

In such cases, the PSG “protects” the gapless spectrum. This can lead to stable critical states, even when the Hamiltonian has not been tuned to a critical point. For a recent example on the kagome lattice, see Refs. [43, 46]. We note in concluding this section that the PSG analysis does not provide any information about *non-perturbative* effects caused by fluctuations. This is the case, in particular, for the proliferation of magnetic monopoles in a $U(1)$ gauge field [29], which can lead to spinon confinement (but, however, is not expected to occur if N_f is sufficiently large [37]).

4 \mathbb{Z}_2 spin liquids

The simplest spin-liquid states (according to definition 3) for a two-dimensional spin- $\frac{1}{2}$ system are the \mathbb{Z}_2 liquid states, which have gapped spinons. The name is taken from the fact that the gauge group (IGG) relevant for describing its elementary excitations is \mathbb{Z}_2 . All the excitations are gapped, and the spin

correlations are short-ranged. The magnetic (spin- $\frac{1}{2}$) excitations are deconfined spinons, which may be fermions or bosons. In addition, the system has singlet (total spin $S = 0$, *i.e.* non-magnetic) excitations which are fluxes (or vortices) of the \mathbb{Z}_2 gauge field. These vortices were discussed in the early days of RVB theories [47, 48, 4] and have more recently been christened “visons” [49]. Although such spin liquids can be discussed within the slave-fermion formalism [36], below we describe the approach based on the short-range RVB framework.

4.1 Short-range RVB description

An RVB wave function can be written as $|\phi\rangle = \sum_c \phi(c)|c\rangle$, where c labels a valence-bond covering of the lattice. This state is manifestly a spin singlet, but the nature of the spin correlations depends on the weights $\phi(c)$. In particular, if this state has a sufficient weight $\phi(c)$ for configurations c with long singlet bonds, $\langle \phi | \mathbf{S}_i \cdot \mathbf{S}_j | \phi \rangle$ can even be long-range-ordered (Néel order) [50]. Here we focus instead on states where the weight $\phi(c)$ can be neglected if the valence-bond exceeds a finite length $\xi \sim \mathcal{O}(1)$. In this case, spin correlations are expected to decay exponentially.

However, this condition is not sufficient to guarantee a *liquid*, as a VBC wave function can also be written using short-range valence-bonds. In a VBC, one may define “parent” configurations c_i ($i = 1 \cdots d$, where d is the degeneracy) which have the spatial periodicity of the crystal. In a columnar VBC on the square lattice, the parent states would be the 4 columnar configurations. If $|\phi\rangle$ is a crystalline state, each covering c can be compared to its “closest” parent, from which it will differ only by collection of small *loops*.²¹ These loops represent fluctuations around the maximally ordered configurations [17].

If $|\phi\rangle$ describes an RVB *liquid*, there is no parent configuration to which to compare c , but one may still consider the transition graphs between two *typical* configurations c and c' . Such loops can be visualized as resulting from a process where two neighboring spinons are created out of c , propagate along a closed loop, and are annihilated to form again a short-range valence-bond in c' . On the assumption that these virtual processes occurring within the ground state contain some information about elementary *excitations*, the characteristic size of the resonance loops in the ground state represents the typical distance between excited spinons. If this lengthscale is finite, it would indicate spinon confinement. Because the short-range RVB liquids are by contrast deconfined, the associated resonance loops should be large, with their size described by a scale-invariant (critical) distribution. We find here an interesting situation where the spectrum is gapped, and local observables are short-ranged, but some critical phenomena are “hidden” in the loop distribution of the ground-

²¹We employ the standard notion of the *transition graph* to compare different valence-bond configurations. By overlaying two configurations c and c' , one obtains closed loops by following alternately a valence bond of c and a valence bond of c' .

state wave function. These loops are related to Wilson loop operators in a gauge-theory description.

With periodic boundary conditions, one may choose a closed loop Δ_1 on the dual lattice, which winds around the torus in the direction 1. Short-range valence-bond configurations may then be sorted according to the parity $P_1 = \pm 1$ of their number of valence-bonds crossing Δ_1 . P_1 is a topological invariant, in that it cannot be changed by any local operator (for precise statements see Ref. [51]). However, moving a spinon around the system in direction 2 *does* change P_1 , which is the analog of the operator $U\hat{F}$ discussed in Sec. 2.3. P_1 defines two *topological sectors*, while P_1 and P_2 together define four. These sectors are *locally* equivalent: if a valence-bond configuration is known only over a finite region of the lattice, it is not possible to decide to which sector it belongs [52]. Because conventional liquids are insensitive to their boundary conditions (compared to solids), a short-range RVB liquid, where all local observables have short-range correlations, can reach the lowest ground-state energy equally well in all sectors.²² Thus a \mathbb{Z}_2 liquid has as many ground states as it has topological sectors.

4.2 \mathbb{Z}_2 gauge theory, spinon deconfinement, and visons

In a VBC, the confining potential experienced by the spinons arises from the ordered background. It is thus *plausible* that valence-bond liquids do not generate such a confinement force. To show that spinons truly are deconfined requires a deeper analysis. One possibility is to derive an effective \mathbb{Z}_2 gauge theory [53], which is known to have a deconfined phase, by analyzing the structure of the gauge fluctuations about an appropriate mean-field state [4].²³ This approach has also been described in the context of an $Sp(N)$ generalization [57] of a frustrated Heisenberg model on the square lattice (large- N limit) [31], and is similar to the mean-field theory of Sec. 3 except in that it has *bosonic* spinons. In particular, it has been shown that the phase fluctuations of the bond variables are described by a $U(1)$ or a \mathbb{Z}_2 gauge theory, depending on whether the short-range spin correlations are collinear or non-collinear, respectively (these two cases have different IGGs). Short-range RVB

²² It is instructive to compare with the case of a VBC: for a general VBC covering, invariant under two translations T_1 and T_2 , choose a lattice size and geometry such that the periodicity vectors are an even multiple of T_1 and an even multiple of T_2 . The directions of T_1 and T_2 are also taken to define the cuts $\Delta_{1,2}$ required to define parity sectors. It is easy to verify that these choices guarantee that all ordered parent configurations, and thus the degenerate ground states in the VBC phase, belong to the same “even \times even” topological sector. The lowest states in the other sectors will lie higher by an energy proportional to the linear system size.

²³ An alternative is to describe short-range RVB liquids by effective quantum dimer models [41], such as those considered in Refs. [54, 55], which in turn can be mapped (sometimes exactly [55]) onto \mathbb{Z}_2 gauge theories [56].

spin liquids correspond to the \mathbb{Z}_2 case, where the presence of competing interactions is essential to produce the non-collinear spin structures responsible for the emergence of \mathbb{Z}_2 gauge degrees of freedom and spinon deconfinement. The large- N description of gapped \mathbb{Z}_2 liquids has been extended to several 2D frustrated models [58, 59, 60].

We consider a mean-field state $\{U_{ij}^0, \mathbf{a}_i\}$ with a gapped spinon spectrum [obtained from Eq. (21)] and $\text{IGG} = \mathbb{Z}_2$. From the discussion of Sec. 3.4, the relevant gauge modes can be parameterized with a \mathbb{Z}_2 gauge field $A_{ij} \in \{0, \pi\}$ by $U_{ij} = U_{ij}^0 e^{iA_{ij}}$. Because the spinons are gapped at the mean-field level, they can be integrated out, which generates short-range interactions for A_{ij} . Given that $\sigma_{ij}^z \equiv e^{iA_{ij}} = \pm 1$, these interactions must be invariant under the \mathbb{Z}_2 gauge transformations $\sigma_{ij}^z \rightarrow \eta_i \sigma_{ij}^z \eta_j$, where $\eta_i = \pm 1$ may take an arbitrary value at each lattice site. A product $B_p = \sigma_{12}^z \sigma_{23}^z \cdots \sigma_{n1}^z$ around any plaquette p of the lattice is such an invariant: this is the \mathbb{Z}_2 “magnetic” (or gauge) flux. On each bond we denote by σ_{ij}^x the operator which changes $\sigma_{ij}^z = 1$ to $\sigma_{ij}^z = -1$ (and vice versa); this is the “electric” field. The \mathbb{Z}_2 gauge invariance requires that the Hamiltonian commutes with any $G_{i_0} = \prod_{\alpha=1}^p \sigma_{i_0 i_\alpha}^x$, where i_1, \dots, i_p are the neighbors of site i_0 , because G_{i_0} generates the elementary gauge transformation defined by $\eta_{i_0} = -1$ and $\eta_{j \neq i_0} = 1$.²⁴ Thus the “electric” field is also gauge-invariant, and hence is an allowed term in the effective Hamiltonian for the gauge fluctuations.

To discuss the typical phenomenology of such a \mathbb{Z}_2 gauge theory, we consider the simplest Hamiltonian

$$H_{\mathbb{Z}_2} = -\Gamma \sum_{(ijkl)=\square} \sigma_{ij}^z \sigma_{jk}^z \sigma_{kl}^z \sigma_{li}^z - J \sum_{\langle i,j \rangle} \sigma_{ij}^x, \quad (36)$$

where Γ controls the “magnetic” energy term and J the “electric” one, and the sums run over all plaquettes and bonds respectively. Creating a pair of (infinitely heavy) “test” spinons at sites i_0 and i_n can be effected by the gauge-invariant operator $c_{i_0 \uparrow}^\dagger \sigma_{i_0 i_1}^z \cdots \sigma_{i_{n-1} i_n}^z c_{i_n \uparrow}$ [as in Eq. (8)], which changes (anticommutes with) σ^x on all the bonds along the path $i_0 \cdots i_n$. For $\Gamma/J \ll 1$ we may ignore the “magnetic” term, and the ground state has $\sigma^x = 1$ everywhere. Because any bond with $\sigma_{ij}^x = -1$ introduces a high energy penalty, the spinons experience a potential which grows linearly with their separation, *i.e.* they are *confined*. By contrast, the spinons are essentially free in the limit $\Gamma/J \gg 1$, where the model can be studied perturbatively from the $J = 0$ limit.

At $J = 0$, the ground state has $\sigma_{ij}^z = 1$, and the elementary excitation is a gapped and localized \mathbb{Z}_2 flux, for example on plaquette p_0 , correspond-

²⁴States must also be gauge-invariant. Fermions transform according to $\psi_i \rightarrow \eta_i \psi_i$, which corresponds to the gauge generator $F_i = e^{i\pi(c_{i \uparrow}^\dagger c_{i \uparrow} + c_{i \downarrow}^\dagger c_{i \downarrow})}$. Physical states should therefore satisfy $G_i F_i |\phi\rangle = |\phi\rangle$. However, $F_i = 1$ because of the constraint [Eq. (6)], and thus $G_i |\phi\rangle = -|\phi\rangle$, which is the origin of the term *odd \mathbb{Z}_2 gauge theory* [56].

ing to $B_{p_0} = -1$, whereas $B_p = 1$ elsewhere (we specialize the discussion to dimension $D = 2$). Such vortices (“visons” [49]) can be created in pairs by applying to the ground state a product $\prod_l \sigma_l^x$ of electric-field operators involving all the bonds l cutting some path on the dual lattice (by considering the operator G , one observes that the resulting state is unaffected by local deformations of the path). The cores of the two visons are located in the plaquettes at the two ends of the path, and the energy is independent of their separation. Away from $J = 0$, visons acquire a finite bandwidth and non-trivial short-range interactions. However, if short-distance effects close to the vortex core are neglected, the vison creation operator is essentially the product of \mathbb{Z}_2 “electric-field” operators as defined above.

To see what this means in the RVB description, we note that $e^{iA_{ij}}$ can be viewed as an operator which shifts the valence bonds by one lattice constant (Sec. 2.3). Because the electric field σ_{ij}^x anticommutes with $\sigma_{ij}^z = e^{iA_{ij}}$, it can be interpreted as an operator measuring the presence or absence of a valence-bond between sites i and j . Thus the vison creation operator *counts the parity of the number of valence bonds crossing a path ending at the vortex core* (the other end may be at the boundary of the system or at another vortex core). In addition to local modifications close to the core, a vison excitation is obtained from the ground state by changing the *sign* of the valence-bond amplitude $\phi(c)$ if the number of valence-bonds crossing the path is odd.

As suggested by the names “electric” and “magnetic”, a spinon winding around a vison experiences a long-range Aharonov-Bohm effect, corresponding to a phase factor -1 . In the approach discussed here, spinons are fermionic, and thus if a bound state of a spinon and a vison happens to be energetically favorable, the resulting composite spin- $\frac{1}{2}$ excitation would be a boson.

4.3 Examples

In this section we review a (not exhaustive) selection of lattice models with a gapped, fractionalized \mathbb{Z}_2 phase. The Ising-like model introduced by Kitaev [61] is quite possibly the simplest example. It contains four-site interactions between Ising spins, and can be solved exactly, but has no continuous symmetry and is not microscopically related to the RVB states expected in frustrated, Heisenberg-like magnets. Still, it provides a very simple realization of spin systems with “spinon”- and “vison”-type excitations. A related model was introduced in Ref. [62]. The bosonic models discussed by Motrunich and Senthil [63] also have a \mathbb{Z}_2 phase and are somewhat closer to the types of magnets discussed here, in that they possess a global $U(1)$ symmetry. The model of Balents, Fisher, and Girvin [64] (see also [65]) is a spin- $\frac{1}{2}$ model on the kagome lattice, with easy-axis, Heisenberg interactions between 1st, 2nd, and 3rd neighbors. It is one of the simplest known Heisenberg-like models with a well-characterized \mathbb{Z}_2 liquid phase.

Several numerical studies have also found indications of possible gapped QSL phases in $SU(2)$ -symmetric spin- $\frac{1}{2}$ models. These systems are, however,

hard to simulate, and the theoretical understanding of the candidate QSL states which emerge remains rather incomplete. Here we mention also models with four-spin “ring” exchange on the triangular lattice [66], with $J_1 - J_2 - J_3$ interactions on the honeycomb lattice [67], and with $J_1 - J_2$ [68] or $J_1 - J_3$ [69] interactions on the square lattice.

Although a dimer may be viewed as a pair of nearest-neighbor spins coupled in a singlet state, quantum dimer models are not related exactly to simple $SU(2)$ magnets. However, they do provide simple realizations of \mathbb{Z}_2 liquids [54, 55], and can also be used to construct “by hand” $SU(2)$ spin models with QSL ground states [70].

4.4 How to detect a gapped \mathbb{Z}_2 liquid

In this section we consider some *observables* which can be used to investigate whether a system is a gapped QSL. First, the system should not develop any SSB upon cooling. If the system fulfils the conditions of the LSM theorem (energy gap and half-odd-integer spin per unit cell, Sec. 2.4), the absence of SSB at $T = 0$ is in fact sufficient to guarantee the existence of fractionalized excitations. In this case, the detection of an energy continuum in the dynamical spin structure factor (accessible through inelastic neutron scattering), as opposed to the single peak characteristic of a long-lived spin-1 excitations, is a signature of spinon deconfinement. In the case of a \mathbb{Z}_2 liquid, short-range vison correlations are a further necessary condition (for example Ref. [65]), and a possible experimental technique for the detection of visons in a doped \mathbb{Z}_2 liquid was proposed in Ref. [49]. Theoretically, another test is to search for a ground-state degeneracy and to verify that the ground state cannot be distinguished by any local observable in the thermodynamic limit [22, 52]. We mention finally that the topological order can also be detected from the wave function itself, by analyzing its bipartite entanglement entropy [71, 72].

5 Gapless (algebraic) liquids

The mean-field theory described in Sec. 3 can lead to states with a gapless spinon and/or a gapless gauge-excitation spectrum. In some cases, these gapless QSLs have been argued to be stable with respect to fluctuations. The gapless excitations mean that these new states of matter are *a priori* quite “fragile”. If the spinons are gapless at the mean-field level, there are possible “mass” terms which could be generated when including fluctuations (even if these are weak), and which could open a spin gap (*i.e.* cause an instability towards a gapped QSL or some type of VBC). However, as noted in Sec. 3.4, these terms are sometimes forbidden by the PSG. One must then consider the effect of gapless gauge modes, for example if the IGG is $U(1)$. With *gapped* matter fields, which here are the spinons, such compact lattice gauge theories are generically in a *confined phase* in 2D due to the proliferation of “magnetic

monopoles” (particular space-time configurations of the gauge field [73]), and the mean-field theory is unstable to gauge fluctuations [29, 30]. However, this result does not always apply in the presence of gapless spinons with a linear dispersion relation (one or more “Dirac cones”, as in the π -flux state), in which case the gaplessness of the mean-field state may survive fluctuations [74, 37, 75]. The resulting QSLs are known as “ $U(1)$ ”, “algebraic”, or “long-range RVB” spin liquids. The rich physics of these critical *states* is closely related to that of “deconfined critical points” [76], because in both cases the monopoles are irrelevant for the low-energy properties. The “ π -flux” state [27, 37] on the square lattice and an analog on the kagome lattice [42, 43] are two examples of mean-field states with Dirac-type spinon spectra which have been argued to survive fluctuations and to give rise to algebraic spin liquids. It should also be stressed that spinons are not free quasiparticles after fluctuations are taken into account, even at very low energy. Because of the strong interactions with the gauge modes, many correlation functions (including spin correlations) show an algebraic decay with non-trivial exponents (different from the mean-field ones) [74, 37, 75]. To our knowledge, there is as yet no lattice *spin* model for which clear evidence of such an algebraic QSL has been found.

6 Other spin liquids

We have included several families of QSL in the present review, but have also omitted several important ones. Here we provide a brief list of some of these. Chiral spin liquids [77, 44], which have spontaneous breaking of time-reversal symmetry (and therefore do not obey definition 2), have deconfined spinons. The mechanism by which these systems can escape confinement is the existence of a Chern-Simons term, allowed because of the time-reversal symmetry-breaking, in the effective action, which gaps the gauge modes. We also mention possible QSL states with rich topological structures, including fractional excitations with *non-Abelian* statistics [78, 79, 80, 81]. A further class of quantum spin liquid is the set of “algebraic vortex liquids” [82], proposed in certain frustrated models with easy-plane interactions; their description is based on a mapping of the vortices to fermionic degrees of freedom.

7 Conclusion

We have introduced some theoretical ideas for describing disordered ground states in Mott insulators (by which is meant that the total spin is a half odd integer per unit cell). Using the fermionic representation of the spin operators, we have discussed how gauge fields emerge as fluctuation modes around given mean-field solutions of the Heisenberg model. The stability and hence validity of such a mean-field approximation depends on whether the gauge field

mediates a confining interaction between the spinons (instability) or whether the spinons remain deconfined (stability). In the specific case of short-range resonating-valence-bond liquids, the fluctuating Z_2 gauge field is simply the valence-bond background in which the spinons propagate.

This type of approach is very useful in shedding light on the low-energy properties of spin liquids. It also allows a classification of the different possible phases and the extraction of certain universal properties. Deciding whether or not a given frustrated spin model has a spin-liquid ground state remains a difficult task, because the approaches discussed here are not easy to apply as quantitatively accurate calculations for microscopic Hamiltonians. However, the concepts we have reviewed, including gauge fluctuations, fractionalization, and topological order, are crucial elements guiding the search for and characterization of these new states of matter.

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