

Ising anyons with a string tension

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We consider the string-net model on the honeycomb lattice for Ising anyons in the presence of a string tension. This competing term induces a nontrivial dynamics of the non-Abelian anyonic quasiparticles and may lead to a breakdown of the topological phase. Using high-order series expansions and exact diagonalizations, we determine the robustness of this doubled Ising phase, which is found to be separated from two gapped phases. An effective quantum dimer model emerges in the large tension limit, giving rise to two different translation symmetry-broken phases. Consequently, we obtain four transition points, two of which are associated with first-order transitions whereas the two others are found to be continuous and provide examples of recently proposed Bose condensation for anyons.

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More than 20 years after its discovery [1–3], topological quantum order remains one of the most fascinating fields of condensed matter physics. Topologically ordered systems are characterized by several features such as, e.g., the topological degeneracy, exotic braiding statistics, or long-range entanglement (see Ref. [4] for a recent review). Contrary to conventional phases, topological phases cannot be described by a local order parameter so that the Landau-Ginzburg theory cannot be used to investigate transitions between them. In this context, new tools have been developed to understand transition mechanisms. Among them, an appealing approach relying on condensation of bosonic quasiparticles [5] has been proposed to determine some possible connections between different phases. We refer the interested readers to Refs. [6–10] for concrete examples in lattice models and to Refs. [11–14] for more mathematical considerations. However, a complete description of topological phase transitions is still missing and, in particular, a classification of universality classes for the critical properties is still to be established. From that respect, it seems essential to study microscopic models in order to explore possible scenarios.

In this Rapid Communication, we analyze the zero-temperature phase diagram of the string-net model [15] defined on the honeycomb lattice with Ising anyons in the presence of a string tension. First, we give some properties of the unperturbed Ising string-net model and we discuss several limiting cases, allowing for a qualitative understanding of the phase diagram. To go beyond, we compute high-order series expansions of the low-energy spectrum in two limiting cases that we compare with exact diagonalization (ED) results. Apart from a trivial (polarized) phase and the doubled Ising (DIsging) topological phase, we find two different translation

symmetry-broken phases emerging from an effective quantum dimer model whose analysis is given in Ref. [16]. Furthermore, our results also suggest the possibility of universality classes associated with the condensation of Ising quasiparticles.

Hilbert space. Microscopic degrees of freedom of the string-net model are defined on the edges of a trivalent graph [15]. For the Ising theory considered thereafter, they can be in three different states $|1\rangle$, $|\sigma\rangle$, and $|\psi\rangle$. The Hilbert space \mathcal{H} is then defined by the set of states that satisfy the so-called branching rules (at each vertex) stemming from the $SU(2)_2$ fusion rules

$$1 \times a = a \times 1 = a, \quad \forall a \in \{1, \sigma, \psi\}, \quad (1)$$

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \psi \times \sigma = \sigma, \quad \psi \times \psi = 1. \quad (2)$$

For any trivalent graph with N_v vertices, the dimension of the Hilbert space is then given by [17]

$$\dim \mathcal{H} = 2^{N_v+1} + 2^{N_v/2}. \quad (3)$$

Model. Let us consider the following Hamiltonian,

$$H = -J_p \sum_p \delta_{\Phi(p),1} - J_e \sum_e \delta_{l(e),1}, \quad (4)$$

first introduced in Refs. [17,18]. The first term is the string-net Hamiltonian introduced by Levin and Wen [15]. It involves the projector $\delta_{\Phi(p),1}$ onto states with no flux $\Phi(p)$ through plaquette p . The second term is diagonal in the canonical basis introduced above since $\delta_{l(e),1}$ is the projector onto state $|1\rangle$ on edge e . This latter term is a string tension since it breaks the topological properties of the ground state described in Ref. [15]. Without loss of generality, we set $J_p = \cos \theta$ and $J_e = \sin \theta$.

For $\theta = 0$, the system is, by construction, in a doubled (achiral) Ising topological phase, dubbed DIsging in the following [9,19]. Consequently, the degeneracy of the eigenstates depends on the graph topology. For instance, the degeneracy

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of the k th energy level ($E_k = -N_p + k$) on a torus with N_p plaquettes is [20]

$$\mathcal{D}_k = \binom{N_p}{k} [1 + 6(-1)^k + 2 \times 3^k], \quad (5)$$

where the binomial coefficient results from the different ways to choose k plaquettes carrying the flux excitations among N_p . In particular, one finds $\mathcal{D}_0 = 9$ ground states that are labeled by the (trivalued) flux contained in each of the two noncontractible loops of the torus [9]. Note that, using the Euler-Poincaré relation for a trivalent graph on this genus-one surface ($N_v = 2N_p$), it is easy to check that $\dim \mathcal{H} = \sum_{k=0}^{N_p} \mathcal{D}_k$ matches with Eq. (3).

One goal of the present work is to analyze the stability of this topological phase when J_e is switched on as well as to characterize the transition between various phases. Indeed, for $\theta = \pi/2$, the (unique) ground state is the state where all edges are in the state $|1\rangle$. Thus, there must be at least one phase transition in the range $[0, \pi/2]$. For $\theta = 3\pi/2$, the ground state is infinitely many degenerate in the thermodynamical limit so that (at least) one phase transition is expected in the range $[3\pi/2, 2\pi]$. Finally, for $\theta = \pi$, the ground-state degeneracy (on a torus) is given by \mathcal{D}_{N_p} so that transitions must also occur in the range $[\pi/2, 3\pi/2]$. In the following, we consider the simplest two-dimensional trivalent graph, namely, the honeycomb lattice.

The “simple” case: $\theta \in [0, \pi]$. To determine the boundaries of the DIsing topological phase (around $\theta = 0$), we computed the ground-state energy as well as the quasiparticle gaps by means of high-order series expansions in powers of J_e/J_p (lengthy expressions are given in Ref. [16]) using various techniques [21–23]. In the vicinity of $\theta = 0$, one must make the distinction between two different low-energy gaps corresponding to quasi- σ and quasi- ψ excitations. These excitations are usually referred to as $\sigma_L \sigma_R$ and $\psi_L \psi_R$ in the literature (see, for instance, Ref. [9]) but, for simplicity, we will adopt here a quasiparticle language, keeping in mind that these are achiral objects. Contrary to quasi- ψ excitations, a single quasi- σ excitation cannot exist on a compact surface such as the torus because of the branching rules [20]. Using standard extrapolation methods, we determined the points where these gaps vanish and thus established the stability range of the DIsing phase. However, if level crossings due to higher-energy levels are present, a first-order transition may also arise and cannot be captured by our perturbative approach that only deals with low-energy states. To check the validity of the conclusions drawn from the series expansions, we performed ED of H using periodic boundary conditions and systems with unit vectors of equal norms forming an angle of $\pi/3$. On the torus, the Hamiltonian H can be split into topological different sectors that must not be confused with the nine flux sectors discussed previously for $\theta = 0$. Indeed, for any θ , branching rules impose that a $|\sigma\rangle$ link is always connected to a single $|\sigma\rangle$ link. In particular, there exist noncontractible loops of $|\sigma\rangle$ links enclosing the torus. Fusion rules impose that H only conserves the parity of the number of such loops and, since there are two independent noncontractible loops on the torus, one has 2×2 different sectors for any θ .

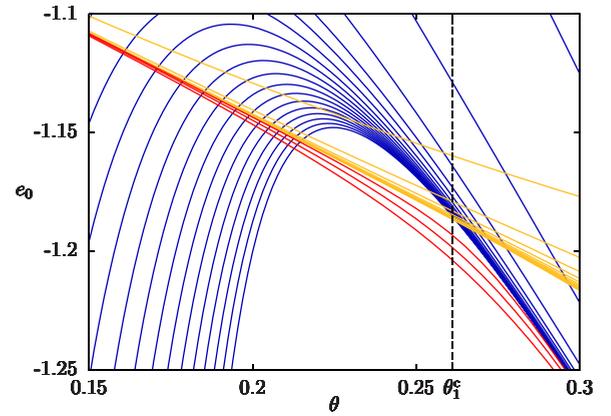


FIG. 1. (Color online) From top to bottom, ground-state energy per plaquette e_0 computed from series expansions at order 1–11 around $\theta = 0$ (yellow) and at order 1–18 around $\theta = \pi/2$ (blue). For comparison, ED results (red) are also displayed for $N_p = 7, 9, 13$ (from bottom to top).

We display in Fig. 1 a comparison between the ED results and the series expansions for the ground-state energy performed around $\theta = 0$ (red) and $\theta = \pi/2$ (blue). As can be seen, at each order, the series intersect at two different points. Similarly, we show in Fig. 2 the results for the low-energy gap that intersect in a unique point. After extrapolations, using the same analysis as in Ref. [24], we found that all these crossing points converge towards a unique value defining a second-order transition point at $\theta_1^c \simeq 0.261$. This point also matches with the position of the infinite-size extrapolation of the gap minimum as well as the minimum of $\partial_\theta^2 e_0$ computed from ED. This critical point separates the DIsing phase

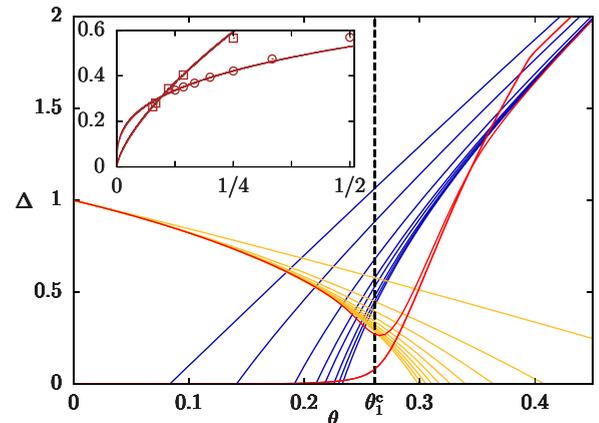


FIG. 2. (Color online) From top to bottom, low-energy gap Δ computed from series expansions at order 1–10 around $\theta = 0$ (yellow) and at order 1–8 around $\theta = \pi/2$ (blue). For comparison, the first nine excitation energies obtained from ED (red) are shown for $N_p = 13$. The first excited level is eightfold degenerate. Inset: Minimum of ninth excitation energy as a function of N_p^{-1} computed from ED (squares); minimum of Δ as a function of n^{-1} (circles) computed at the crossing point between order n series performed around $\theta = 0$ and $\theta = \pi/2$. Lines are power law fits consistent with $\lim_{n, N_p \rightarrow 0} \Delta = 0$.

originating from $\theta = 0$ from the polarized (nontopological) phase near $\theta = \pi/2$.

We stress that the relevant low-energy gap in the ED must be interpreted as a single quasiparticle gap associated to quasi- ψ excitations. The corresponding energy level is indeed adiabatically connected to $E_1 = E_0 + 1$ at $\theta = 0$. Within the topological symmetry-breaking formalism proposed in Ref. [5], if quasi- ψ excitations condense while quasi- σ excitations remain gapped, one should switch towards another topological phase, which is not the case here. In this framework, the only possibility to enter a nontopological phase is that the quasi- σ gap also vanishes at θ_1^c . As explained above, this gap cannot be observed in ED on a torus but it can be computed perturbatively since this approach is independent of the surface topology in the thermodynamical limit. Series expansions of both quasiparticle gaps (quasi- ψ and quasi- σ) are given in Ref. [16]. As can be checked explicitly, they are strictly identical up to order 4 and differ beyond. However, the sign of the (tiny) difference between both gaps changes at each order, which is compatible with a simultaneous vanishing of these gaps at θ_1^c . Note that we performed similar calculations for the ladder geometry and we found that both gaps are identical at all orders we computed [25]. For $\theta \in [\pi/2, \pi]$, we did not find any indication of a transition but, as for the Fibonacci theory [24], the first derivative of the ground-state energy per plaquette $\partial_\theta e_0$ displays a jump at $\theta = \pi$, indicating a first-order transition. Thus, the trivial phase originating from $\theta = \pi/2$ extends from θ_1^c (second-order transition point) to π (first-order transition point).

The “original” case: $\theta \in [3\pi/2, 2\pi]$. For $\theta = 3\pi/2$, the ground-state manifold is spanned by all states minimizing the number of edges in state $|1\rangle$. Interestingly, for the Ising theory, fusion (branching) rules allow some states without any $|1\rangle$ bond provided each vertex touches exactly one $|\psi\rangle$ and two $|\sigma\rangle$. These constraints are nothing but those of hard-core dimer coverings of the hexagonal lattice if the state $|\psi\rangle$ is viewed as a bond occupied by a “dimer.”

The exponential ground-state degeneracy at $\theta = 3\pi/2$ prevents a simple series expansion around this point. In addition, the alternating signs in the series around $\theta = 0^-$ prevents from an analysis similar to the one used in the range $[0, \pi/2]$. However, a close inspection of the gap series expansion around $\theta = 0$ indicates a transition point near $\theta_2^c \simeq 5.57$. As can be seen in Fig. 3, ED results are consistent with a unique phase transition in the range $[3\pi/2, 2\pi]$, but accessible sizes are definitely too small to characterize properly the phase for $\theta \in [3\pi/2, \theta_2^c]$ as well as the nature of the transition. To gain a deeper understanding of this region, we derived the low-energy effective theory near $\theta = 3\pi/2$, at leading order, by considering the effect of the string-net Hamiltonian on the infinitely many degenerate ground-state manifold of the unperturbed ($J_p = 0$) problem. The effective Hamiltonian can be written in the following form:

$$H_{\text{eff}} = -\frac{J_p}{4} \sum_p [t(|\star\rangle\langle\star| + |\star\rangle\langle\star|) + v|\star\rangle\langle\star| - \mathbb{1}] \quad (6)$$

with $t = 1/2$ and $v = 1$. In this representation, a red (blue) link corresponds to a $|\sigma\rangle$ ($|\psi\rangle$) state. We emphasize that H_{eff} is only

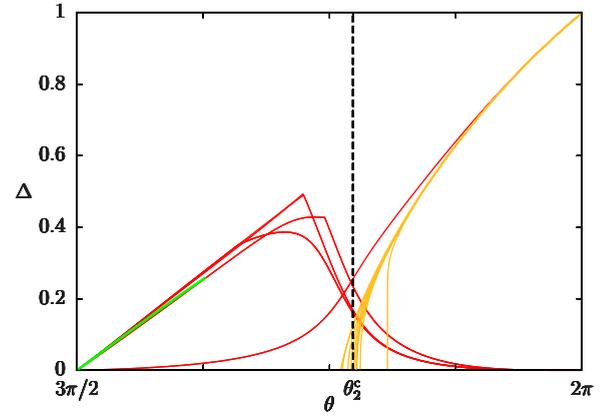


FIG. 3. (Color online) Low-energy gap Δ computed from resummed series expansions around $\theta = 0$ [$d\log$ Padé approximants $[m, n]$ for $m, n \geq 3$ are displayed (yellow)] and from the expression (8), which is a good approximation of the gap around $\theta = 3\pi/2^+$ (green). The first nine excitation energies obtained from ED (red) are shown for $N_p = 12$.

valid to compute the correction of order 1 ($\propto J_p$) to the spectrum of H near $\theta = 3\pi/2$. This effective Hamiltonian consists of two terms: a kinetic term t acting on “flippable” plaquettes and a potential term v proportional to the number of empty plaquettes (without dimers). This model looks very similar to the famous quantum dimer model introduced by Rokhsar and Kivelson on the square lattice [26] and later studied on the honeycomb lattice [27]. The only difference between both models comes from the potential term that, in the Rokhsar-Kivelson model, is proportional to the number of flippable plaquettes. To our knowledge, the Hamiltonian (6) has yet to be studied and cannot be solved exactly for arbitrary couplings. However, for $J_p > 0$, it is possible to infer its low-energy properties by considering the limit $t/v \ll 1$ while keeping in mind that $t/v = 1/2$ in our problem. For $t = 0$ (and $J_p > 0$), the energy is minimized by maximizing the number of empty plaquettes (without dimers). In the thermodynamical limit, there are three possible ground states satisfying this constraint so that the system is in the so-called $\sqrt{3} \times \sqrt{3}$ star crystal (SC) phase [28] (see Ref. [16] for an illustration). First excited states are obtained by flipping one plaquette in one of these ground states.

The kinetic term t induces quantum fluctuations that can be captured using perturbation theory. At order 2 in t/v , the ground-state energy per plaquette is given by

$$e_0 = \frac{J_p}{4} \left(-1 - \frac{v}{3} - \frac{2t^2}{9v} \right), \quad (7)$$

whereas the low-energy gap reads

$$\Delta = \frac{J_p}{4} \left(3v - \frac{4t^2}{3v} \right). \quad (8)$$

Of course, one could reach higher orders in this t/v expansions but it is not of crucial importance for the present study. Indeed, these expressions already provide a very good approximation of e_0 and Δ for $t/v = 1/2$ since, as shown in Ref. [16], they only differ from the infinite-size values extrapolated

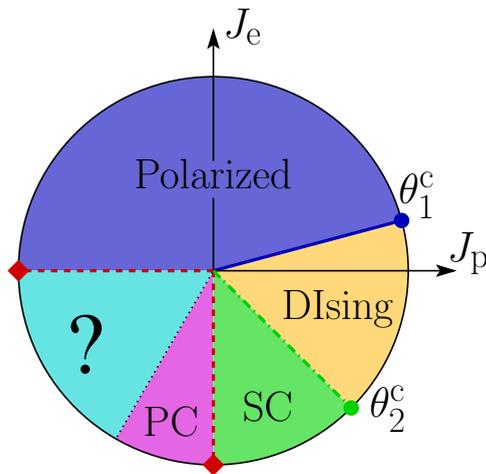


FIG. 4. (Color online) Sketch of the phase diagram. Four different gapped phases are identified among which one is topologically ordered (DIsing) and two break the translation symmetry (SC and PC). Diamonds (circles) indicate first-order (second-order) transition points (see text for more details).

from ED results by less than 0.3% and 1.5%, respectively. In the range $[3\pi/2, \theta_2^c[$, we thus find a gapped translation symmetry-broken phase with a threefold degenerate ground state. Furthermore, the three momenta (center and the two corners of the hexagonal Brillouin zone) of the ground states in the SC coincide with the locations of the minima of the excitation gap in the DIsing phase. This suggests that the transition from the topological phase to the crystal also corresponds to a simultaneous condensation of the anyonic quasiparticles.

Furthermore, momenta of these SCs also minimize the dispersion in the range $\theta \in [0, \theta_2^c[$ which is compatible with a second-order transition. This is an example of a continuous phase transition between a non-Abelian topological phase and a nontopological (translation) symmetry-broken phase.

The “tricky” case: $\theta \in [\pi, 3\pi/2]$. Obviously, the effective model (6) is also valid for $\theta = 3\pi/2^-$ but, contrary to the case $J_p > 0$ where a SC is favored (see discussion above),

the ground state is infinitely many degenerate at $t = 0$ so that it is difficult to consider a perturbative t/v expansion. Consequently, we performed ED of H_{eff} up to relatively large system sizes ($N_p = 63$) and we found that the point $t = 1/2$, $v = 1$ lies in the same phase as the point $t = 1/2$, $v = 0$ (see Ref. [16]). As discussed in Ref. [27] for $v = 0$, the ground state displays the so-called plaquette order that breaks the translational symmetry. Thus, for $\theta = 3\pi/2^-$, we expect a nontopological ordered (gapped) plaquette crystal (PC) phase with a threefold degenerate ground state. However, in the absence of perturbative analysis near $\theta = \pi$ and because of important variations with the system size of the ED results (at least up to $N_p = 13$ which is our current limit), we did not succeed in characterizing the whole interval $[\pi, 3\pi/2]$. However, we observe that $\partial_\theta e_0$ displays a jump for $\theta = \pi$ and $\theta = 3\pi/2$ so that first-order transitions occur at these points.

Summary and outlook. A sketch of the phase diagram gathering all information discussed throughout this Rapid Communication is given in Fig. 4. One of the main results is the possibility to condense simultaneously quasi- ψ and quasi- σ excitations at the critical point θ_1^c , unveiling a likely new universality class. Unfortunately, in the absence of an alternative description, it is difficult to predict the associated critical exponents. Setting $\theta_1^c = 0.261$, a standard ED data collapse analysis gives results that obey the hyperscaling relation for $z = 1$ and $\nu \simeq 0.39$, and the resulting specific-heat exponent is $\alpha = 2 - \nu(2 + z) \simeq 0.83$. One can also compute directly the exponent $z\nu$ using different d log Padé approximants of the gap series and we found values in the range $[0.35, 0.5]$. This rather broad range clearly indicates the lack of precision of such an approach for order 10 series, but suggests that this phase transition may belong to a new universality class. We hope that the present work will stimulate further studies to deepen our understanding of the topological phase transitions.

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